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Cartan's constructions, the homology of $K(\pi,n)$'s, and some later developments

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In order to explain partially the role played by H. Cartan's calculation of the homology of $K(\pi,n)$'s in some of the subsequent development of homotopy theory and homological algebra it seems worthwhile to recall some things about the state of homotopy theory in 1950 and early 1951.

For the purposes of this article, space will mean topopological space having the homotopy type of a CW complex and a compactly generated topology. At the time being considered, homotopy groups had been around for almost two decades, the Hurewicz isomorphism theorem was fairly well understood, as was the Hopf classification theorem for the maps of an n-dimensional space into the n-sphere. The Hopf construction had been studied by several people, and at least its early consequences were familiar to most people interested in homotopy theory. However, though the Freudenthal suspension theorem was well known, knowledge about the homotopy groups of spheres and other spaces was rudimentary. Indeed the (n+2)'nd homotopy group of the n-sphere had only recently been computed ([19]).

The study of spaces with a single non-vanishing homotopy group was introduced by Eilenberg and Mac Lane and spaces of this type had

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been named for these authors ([6]). Such a space was and is called a $\mathcal{K}(\pi, n)$ if its single non-vanishing homotopy group is the group π and is in the dimension n. Since higher homotopy groups are abelian, π must be abelian if $n > 1$. An explicit simplicial set called $K(\pi, n)$ had been introduced by Eilenberg and Mac Lane, and it was known that this simplicial set had the same homology groups as the singular complex of any space of type $\mathcal{K}(\pi, n)$. Indeed it was known that the singular complex of any space had a minimal subcomplex, that this minimal subcomplex was unique up to isomorphism, and had the same homology groups as the original complex. Further it was known that the minimal subcomplex of the singular complex of any space $\mathcal{K}(\pi, n)$ was isomorphic with the standard $K(\pi, n)$. Nowadays one also knows about the geometric realization of a simplicial set ([10]), and the standard space $K(\pi, n)$ is the geometric realization of the simplicial set of the same name. For π abelian this space has canonically the structure of an abelian topological group. At the time spaces $\mathcal{K}(\pi, n)$ were constructed by attaching cells to bouquets of spheres in the manner of J. H. C. Whitehead, and such spaces were known to have properties of the standard $K(\pi, n)$ up to homotopy canonically.

In 1950/51 the spaces $\mathcal{K}(\pi, n)$ were thought to be important building blocks for other spaces, but there was little understanding as to how this might take place, and their role vis a vis cohomology operations was not appreciated. This was in part because one still thought of fibrations (other than fibre bundles) as being somewhat exotic, and indeed did not have a good definition of such a notion. Further, though spectral sequences had been introduced by Leray and had proved useful in studying some problems, there were no spectral sequences for singular theory and the algebraic apparatus which existed was understood by only a few people. Nevertheless, computations about the homology of $K(\pi, n)$'s were believed to be important and computations in degrees less than about $n + 10$ for the case π cyclic had been made by Eilenberg and Mac Lane. The laboriousness of these computations led one to believe that obtaining much further information might be of the same order of difficulty as computing homotopy groups of spheres.

At this time the situation began to change with great rapidity. Serre's thesis appeared giving a simple and useful definition of fibration and proving that a singular spectral sequence existed for these fibrations. Moreover it illustrated the power of the techniques introduced by showing that the homotopy groups of spheres are all finite except for the groups $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$, $n > 0$ ([17]), and it showed that for every prime p an element of order p appeared in the homotopy groups of spheres and that the first element which so appeared was stable and in stable degree $2p-3$. In addition shortly thereafter, the fact that all spaces could be constructed up to homotopy type using towers of fibrations with $K(\pi, n)$'s was discovered ([5], [16], [20], and later [14]), and many calculations were made using singular spectral sequences of fibrations. These gave much new information about both the homotopy groups of spheres and the homology of $K(\pi, n)$'s.

Next the homology of $K(\pi, n)$'s was calculated functorially in degrees less than $n+6$ by Eilenberg and Mac Lane, the relationship between the cohomology of $K(\pi, n)$'s and cohomology operations was understood, and the cohomology modulo 2 of these spaces was calculated by Serre ([6], [18]). Thus it was understood that the problem of obtaining further homological knowledge about Eilenberg-Mac Lane spaces was of paramount importance for the advance of homotopy theory, and that this knowledge was more readily obtainable than information concerning the homotopy groups of spheres.

At this stage there was a giant step forward. It was the complete calculation of the homology of $K(\pi, n)$'s by H. Cartan ([1], [3]). In carrying out this work Cartain introduced the notion of construction, which may be viewed as an algebraic idea paralleling the geometry of fibrations.

Now turn to homological algebra. A few years earlier than the time considered above, the work of Eilenberg-Mac Lane and others on the homology of groups, problems related to the Künneth theorem in algebraic topology, and other algebraic work led H. Cartan and S. Eilenberg to formally found the subject of homological algebra. Some years later differential homological algebra emerged as a variant of homological algebra often useful in studying the homological properties of fibrations.

Its methods and applications stem from Cartan's constructions, explicit properties of the "bar" and "W" constructions of Eilenberg-Mac Lane, the notion of resolution of Cartan-Eilenberg, and algebraic versions of the Serre spectral sequence ([7],[4],[8],[15], and [9]).

The applications of Cartan's calculation of the homology of $\mathcal{K}(\pi, n)$'s have been numerous. He himself used this work to give a new derivation of the Adem relations for mod 2 Steenrod operations and to give an independent derivation of the relations on odd primary Steenrod operations. Further he proved his own cup product formula for Steenrod operations in such a general manner that it provided the basic information necessary for Milnor's proof of the fact that the Steenrod algebra has a natural diagonal which gives it the structure of a Hopf algebra ([1],[11]). This information is of course necessary for Milnor's description of the Steenrod algebra and its dual.

The remainder of this article will be devoted primarily to describing some part of Cartan's methods and computations together with a few related facts. It will conclude with an appendix making a calculation with a special type of Hopf algebra, and then showing how quickly Cartan's work leads via Newton's formulas to a description of the Steenrod algebra as a Hopf algebra.

In concluding this introduction I would like to say that the privilege of coming to know H. Cartan and his work a little has been one of the great pleasures of my life. I was honored to participate in the conference in his honor held at Orsay in June 1975. The contents of this article differ in some respects from the talk which I gave on that occasion. I hope the changes are in the interest of clarity.

Conventions. In the body of this article it will be assumed that R is a commutative ring fixed as ground ring. Graded modules will be assumed to be positively graded. Familiarity with the basic elementary properties of the category of differential graded modules will be taken for granted. Algebra will mean supplemented differential graded algebra. Thus every algebra A comes equipped with a morphism of algebras $\varepsilon(A): A \longrightarrow R$. Familiarity with properties of morphisms between (differential graded) modules over such algebras including the notion of homotopy will also be assumed (cf. [9], [15]). If A is an algebra, then $A^{\#}$ will denote the algebra obtained from A by setting the differential of A equal to zero (i.e. forgetting the differential), and if M is an A module $M^{\#}$ will denote the $A^{\#}$ module obtained by setting the differential equal to zero.

In addition to the preceding, sometimes familiarity with properties of (supplemental differential graded) coalgebras and comodules over such coalgebras will also be presupposed even though these were not studied at the time of Cartan's work on $K(\pi, n)$'s ([9], [15]). The reader familiar only with older conventions (cf. [4], [7], [17]) should be able to follow the major part of the discussion.

§1. Constructions.

The standard example of an algebra is the (normalized) singular chains of a topological group with coefficients in R. Note that if π is a discrete group, the algebra of singular chains of π with coefficients in R is just the group algebra $R(\pi)$.

The standard example of a module over an algebra is the singular chains of a space on which a group acts, the module being left or right according as the group acts on the left or right of the space. Here module will mean left module unless otherwise stated.

If A is an algebra, recall that an extended A-module is one of the form $A \otimes X$ where X is a differential graded R-module.

Definition. An A-module M is split # projective if $M^{\#}$ is a direct summand of an extended $A^{\#}$ module.

One thinks that an extended A-module is the analogue of the product of a group and a space, while a split # projective A-module is the analogue of the total space of a principal fibre bundle.

Theorem. Suppose that A is an algebra, and that

1) M is a split # projective A-module,

2) $f: M' \longrightarrow M''$ is a morphism of A-modules which viewed over R is a homotopy equivalence which neglecting differentials is a split epimorphism, and

3) $g: M \longrightarrow M''$ is a morphism of A modules.

Under these conditions there exists a

morphism of A-modules $\tilde{g}: M \longrightarrow M'$ such that $f\tilde{g} = g$, and $\tilde{g}$ is unique up to homotopy (over A).

A slight variant of the preceding for the case $M'' = R$ was proved by Cartan ([3], exp. 2) and is the basis of the most elementary properties of his constructions.

Definitions. An augmented A-module M is an A-module M together with a morphism of A-modules $\varepsilon(M): M \longrightarrow R$. An augmented A-module M is acyclic if viewed over R, the morphism $\varepsilon(M)$ is a homotopy equivalence.

Theorem. If M' is a split # projective augmented A-module and M'' is an acyclic augmented A-module, then there is a morphism of augmented A-modules $f: M' \longrightarrow M''$ and f is unique up to homotopy.

This theorem is an immediate corollary of the preceding one.

If M is an A-module, let $\overline{M} = R \otimes_A M = M/I(A)M$ where $I(A)$ is the augmentation ideal of A. The module $\overline{M}$ is called the base of M and there is a morphism of A-modules $\pi(M): M \longrightarrow \overline{M}$ such that if $f: M \longrightarrow N$ is a morphism of A-modules and $I(A)N=0$ (i.e. A acts trivially on N), then there is a unique morphism of A-modules $f': \overline{M} \longrightarrow N$ such that $f'\pi(M) = f$. If $g: M' \longrightarrow M''$ is a morphism of A-modules, let $\overline{g}: \overline{M}' \longrightarrow \overline{M}''$ be the morphism such that $\pi(M'')g = \overline{g}\pi(M')$. Observe that if $g, h: M' \longrightarrow M''$ are homotopic, then so are $\overline{g}, \overline{h}: \overline{M}' \longrightarrow \overline{M}''$. The full subcategory of the category of A-modules generated by the A-modules on which A acts trivially is just the category of differential graded R-modules.

Observe that if M is an augmented A-module, then $\varepsilon(M)\colon M \to R$ is the composite $M \xrightarrow{\pi(M)} \overline{M} \xrightarrow{\varepsilon(\overline{M})} R$.

<u>Theorem</u>. If M is a # projective augmented acyclic A-module then there is a morphism of A-modules $\Delta\colon M \to M \otimes \overline{M}$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\Delta} & M \otimes \overline{M} \\ \Big\downarrow{\scriptstyle \pi(M)} & & \Big\downarrow{\scriptstyle \varepsilon(M) \otimes \overline{M}} \\ \overline{M} & \xrightarrow{\overline{M}} & \overline{M} = R \otimes \overline{M} \end{array}$$

is commutative and this morphism is unique up to homotopy.

Notice that A operates naturally on $M \otimes \overline{M}$ on the left. Further in the notation employed the identity morphism of an object and the object itself are denoted by the symbol.

The theorem follows at once from the first theorem of this section noting that viewed over R, $\varepsilon(M) \otimes \overline{M}$ is a homotopy equivalence.

Now $\overline{\Delta}\colon \overline{M} \to \overline{M} \otimes \overline{M}$, and $\overline{\Delta}$ together with $\varepsilon(\overline{M})$ furnish $\overline{M}$ with the structure of a coalgebra up to homotopy. The preceding theorem says that up to homotopy M has a unique right $\overline{M}$ comodule structure.

Combining the results of this paragraph one has that if A is an algebra, and if there exists an A-module M satisfying the condition of the preceding theorem, it is unique up to homotopy, its base has the structure of a coalgebra uniquely up to homotopy, and it has the structure of a right comodule over its base uniquely up to homotopy.

Suppose that X is an augmented differential graded R-module, then the extended module $A \otimes X$ is an augmented A-module with augmentation

$$\varepsilon(A) \otimes \varepsilon(X) : A \otimes X \longrightarrow R \otimes R .$$

Definition. A construction consists of

1) an algebra,

2) an augmented A-module M, and

3) a morphism of augmented R-modules $2 : \overline{M}^{\#} \longrightarrow M^{\#}$ such that

   1) $\pi(M)^{\#} 2$ is the identity of $\overline{M}^{\#}$, and

   2) the natural morphism $A^{\#} \otimes \overline{M}^{\#} \longrightarrow M^{\#}$ induced by 2 is an isomorphism of augmented $A^{\#}$ modules.

Frequently one speaks of a construction (A, N, M). In this notation N is the base of the construction, i.e. $N = \overline{M}$ and it is understood that $M^{\#} = A^{\#} \otimes N^{\#}$.

Definition. A construction (A, N, M) satisfies the condition (B) if

1) $\varepsilon_0(N)_0 : N_0 \xrightarrow{\approx} R$,

2) $d(M)_{q+1} : N_{q+1} \xrightarrow{\approx} Z(M)_q$ for $q > 0$, and

3) $N_1 \xrightarrow{d(M)_1} M_0 \xrightarrow{\varepsilon(M)_0} R \longrightarrow 0$

is exact.

Notice that in the preceding an element $x \in N_q$ is considered to be the element $1 \otimes x$ of M_q.

Theorem. If A is an algebra, there exists a construction (A, N, M) satisfying the condition (B), and if (A, N′, M′) is a construction there is a unique morphism of A-modules $f: M' \longrightarrow M$ such that $f(N') \subset N$.

In the preceding theorem of Cartan ([3], exp. 3) once again neglecting differentials N has been identified with $1 \otimes N$ sitting inside M. Clearly this theorem guarantees the uniqueness of a construction satisfying the condition (B). In fact the base of this construction for the algebra A is just the well-known "bar construction" B(A) of Eilenberg and Mac Lane ([6], [7]). Observe also that the total module of this construction is a split # projective acyclic A-module.

When Eilenberg and Mac Lane introduced the bar construction there was no algebraic way of viewing it as the base of a "universal bundle" module for the algebra A. This had to wait for Cartan. The freedom of being able to use the base of any acyclic construction with initial algebra A for computational purposes was of basic importance in the calculation of the homology of $K(\pi, n)$'s, and was the foundation of much of the later study of algebras and modules of the type considered here.

Suppose that C is a coalgebra, A an algebra and τ a twisting morphism from C to A ([9], [15]), then the twisted tensor product $A \otimes_\tau C$ gives rise to a construction $(A, C, A \otimes_\tau C)$. Though many constructions arise in this way, it is not the case that all of those needed for efficient computation can be so obtained.

§2. Some examples of constructions and some relations with geometry.

Let n be a strictly positive integer, $E(x, 2n-1)$ the exterior algebra with one generator x of degree 2n-1, and $S'(y, 2n)$ the coalgebra which is connected free an R-module and has basis for primitive elements y in degree 2n. If $\tau: S'(y, 2n) \longrightarrow E(x, 2n-1)$ is the twisting morphism such that $\tau(y) = x$ and τ is zero in degrees other than 2n, then $E(x, 2n-1) \otimes_{\tau} S'(y, 2n)$ is a construction with initial algebra $E(x, 2n-1)$ and final coalgebra $S'(y, 2n)$. This construction satisfies the condition (B), and it is essentially the only small construction which satisfies the condition (B).

With n as above, let $S(x, 2n)$ be the free commutative algebra with one generator x of degree 2n, and $E(y, 2n+1)$ the exterior coalgebra with one primitive basic element y of degree 2n+1. If $\tau: E(y, 2n+1) \longrightarrow S(x, 2n)$ is the twisting morphism such that $\tau(y) = x$, then $S(x, 2n) \otimes_{\tau} E(y, 2n+1)$ is an acyclic construction with initial algebra $S(x, 2n)$ and final coalgebra $E(y, 2n+1)$. It is much smaller than the construction satisfying the condition (B) with initial algebra $S(x, 2n)$ (also called the polynomial algebra with one generator x of degree 2n).

If X is an R-module (graded), then there is an algebra $S(X)$ with zero differential and a morphism of R-modules $\beta(X): X \longrightarrow S(X)$ such that if A is any strictly commutative algebra and $f: X \longrightarrow A$ any morphism of (differential graded) R-modules such that $\varepsilon(A) f = 0$ (i.e. $f(X) \subset Z(A) \cap \operatorname{Ker}(\varepsilon(A))$), then there is a unique morphism of algebras $\bar{f}: S(X) \longrightarrow A$

such that $\overline{f}\beta(X) = f$. The algebra $S(X)$ is called the free strictly commutative algebra generated by X ([9]). If $X = (x, 2n)$ the free module with one basis element x in degree $2n$, then $S(x, 2n)$ is as in the preceding paragraph. If $X = (x, 2n-1)$, then $S(X) = E(x, 2n-1)$ the exterior algebra with one generator x of degree $2n-1$.

If X is a strictly positive projective R-module, there is a strictly commutative coalgebra $S'(X)$ and a morphism of R-modules $\alpha(X)\colon S'(X) \longrightarrow X$ such that if C is any strictly commutative coalgebra and $g\colon C \longrightarrow X$ a morphism of R-modules, then there is a unique morphism of coalgebra $\overline{g}\colon C \longrightarrow S'(X)$ such that $\alpha(X)\overline{g} = g$. The coalgebra $S'(X)$ is connected. If X is of finite type and X^* is the dual of X, it is the coalgebra which is the dual of the algebra $S(X^*)$ which is its dual algebra ([9]).

Suppose X is a projective R-module, Y is its suspension and $\tau\colon S'(Y) \longrightarrow S(X)$ is the twisting morphism which is the composite

$$S'(Y) \xrightarrow{\alpha(Y)} Y \xrightarrow{n} X \xrightarrow{\beta(X)} S(X).$$

In this case $S(X) \otimes_\tau S'(Y)$ is an acyclic construction with initial algebra $S(X)$ and final coalgebra $S'(Y)$. The constructions of both the first and second paragraphs of this section are special cases of this construction.

If X is as above, there is a unique morphism $S(X) \longrightarrow S(X \otimes X) = S(X) \otimes S(X)$ induced by the diagonal of the module X, and $S(X)$ together with this morphism is an abelian Hopf algebra. If Y is as above there is a unique morphism $S'(Y \otimes Y) = S'(Y) \otimes S'(Y) \longrightarrow S'(Y)$ induced by the codiagonal (addition) of the module Y, and $S'(Y)$ together with this morphism is an abelian Hopf algebra. These Hopf algebra structures extend in a natural way to a Hopf algebra structure on $S(X) \otimes_{\tau} S'(Y)$ which neglecting the differential is just the tensor product structure. If R should happen to be a field of characteristic zero the Hopf algebras $S(Y)$ and $S'(Y)$ are isomorphic. For any R, if Y is of finite type then the Hopf algebra $S(Y^{*})$ and $S'(Y)$ are the duals of each other.

Suppose $Y = (y, 2n)$, then $S'(Y)_j = 0$ for $j \not\equiv 0 \bmod 2n$, $S'(Y)_{2jn}$ is free with basis $\{\gamma_j(y)\}$; $\gamma_1(y) = y$, $\gamma_0(y) = 1$, and a multiplication table for the multiplication of $S'(Y)$ is given by $\gamma_i(y)\gamma_j(y) = (i, j)\gamma_{i+j}(y)$ where (i, j) denotes the appropriate binomial coefficient. The image of $\gamma_r(y)$ under the diagonal is $\sum_{i+j=r} \gamma_i(y) \otimes \gamma_j(y)$. In $E(x, 2m-1) \otimes_{\tau} S'(y, 2n)$, $d(\gamma_{j+1}(y)) = x \otimes \gamma_j(y)$, when $\tau(y) = x$.

Let A be the algebra $S(x, 2n)$ modulo the ideal generated by x^r where r is an integer strictly greater than two. Let $\tau: E(y, 2n+1) \longrightarrow A$ be the twisting morphism such that $\tau(y) = x$. Now $A \otimes_{\tau} E(y, 2n+1)$ has naturally the structure of an algebra, and its homology is an exterior algebra with one generator, the homology class of $x^{r-1} \otimes y$. Let $\tau': S'(z, 2rn+2) \longrightarrow A \otimes_{\tau} E(y, 2n+1)$ be the twisting morphism such that $\tau'(z) = x^{r-1} \otimes y$, and let $M = (A \otimes_{\tau} E(y, 2n+1)) \otimes_{\tau'} S'(z, 2rn+2)$. Now $(A, E(y, 2n+1) \otimes S'(z, 2rn+2), M)$ is an acyclic construction with initial algebra A, final coalgebra $E\ (y, 2n+1) \otimes S'(z, 2rn+2)$, and there is no acyclic construction having the same initial algebra and final coalgebra obtained by taking a twisting morphism from the final coalgebra to the initial algebra.

Now before turning to more explicit facts concerning the computation of the homology of $K(\pi, n)$'s a few facts concerning the relation of constructions to topology will be considered briefly. Suppose that $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibration sequence with fibre F, contractible total space E and base B. In this case one may suppose without loss that the fibre F is a topological group G and that the fibration is principal. Taking A to be the singular chain of G, one wants to know that the base of an acyclic construction with fibre A is equivalent to the chain of B. The first result in this direction was the result of Eilenberg and Mac Lane ([6],[7]) to the effect that the "bar construction" is homotopy equivalent with the "W construction." This result though formulated without the general idea of a construction, and without the intervention of either the bundle module or the bundle space of the problem nevertheless showed that if $G = K(\pi, n)$, then the base of Cartan's construction satisfying the condition (B) is homotopy equivalent with the chains of $K(\pi, n+1)$. The elementary results obtained from iteration were also derived in the same work.

A second method of dealing with the problem is to show that if (A, N, M) is any construction, then there is up to homotopy a unique morphism of A-modules $f: M \longrightarrow C_*(E)$. This morphism is compatible with the filtrations used to define the Serre spectral sequences abutting to $H_*(M)$ and $H_*(E)$, and hence in the case that construction is acyclic a homotopy equivalence $N \longrightarrow C_*(B)$ is obtained using a spectral comparison theorem ([3], exp. 3, 12, 13). This method can be modified using a little differential homological, as has been done by Eilenberg and Moore ([4]) to deal with the case where the bundle space is not acyclic.

A third method due to E. H. Brown is to show that if $G \longrightarrow E \longrightarrow B$ is a principal fibration, then there is a twisting morphism $\tau: C_*(B) \longrightarrow C_*(G)$ such that $C_*(G) \otimes_\tau C_*(B)$ is in an appropriate sense equivalent with $C_*(E)$.

§3. $H_*(\pi, 1)$ <u>and algebras with divided powers</u>.

Let π be a free group with 1-generator x, and let τ: $S'(y,1) = E(y,1) \longrightarrow R(\pi)$ be the twisting morphism such that $\tau(y) = x - 1$. Now $R(\pi) \otimes_\tau E(y,1)$ is an acyclic construction with initial algebra $R(\pi)$ and final coalgebra $E(y,1)$. Further the differential of the total module of this construction is compatible with the tensor product multiplicative structure, and one may say that it has final or base Hopf algebra $E(y,1)$. The total module may be given the structure of a Hopf algebra by setting $\Delta(1 \otimes y) = (1 \otimes y) \otimes (x \otimes 1) + (1 \otimes 1) \otimes (1 \otimes y)$. Note $\Delta(x) = x \otimes x$, $\epsilon(x) = 1$, and that the diagonal of the Hopf algebra $R(\pi) G_\tau E(y,1)$ is not commutative, in particular it is not the tensor product diagonal. One could of course use the opposite of the diagonal given above.

Let π'' be a cyclic group of order $n > 1$ with 1-generator $\bar{x}$ and let $\tau'': E(y,1) \longrightarrow R(\pi'')$ be the twisting morphism such that $\tau''(y) = \bar{x} - 1$.

The construction $R(\pi'')\otimes_{\tau''}E(y,1)$ is in a natural way a quotient of the one of the preceding paragraph. It is not acyclic. Indeed its homology is an exterior algebra with one generator degree one. This may be taken to be the homology class of the cycle $\sum_{j=0}^{n-1}\bar{x}^j\otimes y$. Let $\tau_1 : S'(z,2)\longrightarrow R(\pi'')\otimes_{\tau''}E(y,1)$ be the twisting morphism such that $\tau_1(z)=\sum_{j=0}^{n-1}\bar{x}^j\otimes y$.

Now one has an acyclic construction $(R(\pi''),N,M)$, where $M=(R(\pi'')\otimes_{\tau''}E(y,1))\otimes_{\tau_1}S'(z,2)$, and $N=E(y,1)\otimes_{\bar{\tau}_1}S'(z,2)$ with $\bar{\tau}_1(z)=ny$. Though M and N are algebras with multiplication the tensor product of the indicated multiplications, M does not admit a diagonal which makes it a Hopf algebra and the diagonal of N is not the tensor product one. Indeed letting $\bar{x},y,z$ denote the appropriate elements of M and y,z the corresponding elements of N, a morphism $M\longrightarrow M\otimes N$ satisfying the conditions of section 2 is given by $y\longrightarrow y\otimes 1+1\otimes y$, and $z\longrightarrow z\otimes 1+1\otimes z+\sum_{j=0}^{n-2}(n-1-j)\bar{x}^j y\otimes y$. Thus an appropriate diagonal for N is given by $y\longrightarrow y\otimes 1+1\otimes y$, and $z\longrightarrow z\otimes 1+1\otimes z+\frac{n(n-1)}{2}y\otimes y$. This diagonal makes N into a Hopf algebra with non-commutative diagonal of $n(n-1)\neq 0$ in R. The diagonal is homotopy commutative.

Suppose that π is a free abelian group and $\{x_j\}_{j\in J}$ is a basis for π. Let Y be the free R-module concentrated in degree one with basis $\{y_j\}_{j\in J}$, and let $\tau\colon S'(Y)=E(Y)\longrightarrow R(\pi)$ be the twisting morphism such that $\tau(y_j)=x_j-1$. Now $R(\pi)\otimes_{\tau}S'(Y)$ is an acyclic construction, and if one sets $\Delta(y_j)=y_j\otimes x_j+(\otimes y_j$ for $j\in J$, the total module is a Hopf algebra. It is the tensor product indexed on J of copies of the Hopf algebra of the first paragraph of this section.

Suppose that $1 \longrightarrow \pi' \longrightarrow \pi \longrightarrow \pi'' \longrightarrow 1$ is an exact sequence of abelian groups with π as just above. Let $\tau'' : S'(Y) \longrightarrow R(\pi'')$ be the composite

$$S'(Y) \xrightarrow{\tau} R(\pi) \longrightarrow R(\pi'').$$

Now $R(\pi'') \otimes_{\tau''} S'(Y)$ is a construction whose total module is a Hopf algebra. Its homology is an exterior algebra on generators of degree one. It is in fact $\mathrm{Tor}^{R(\pi')}(R, R)$. Suppose that $\{x_i'\}_{i \in I}$ is a basis for π'. For $i \in I$, $x_i' - 1 = \sum_{j \in J} \lambda_{i,j}(x_j - 1)$ where $\lambda_{i,j} \in R(\pi)$ and $\lambda_{i,j} = 0$ for all but a finite number of indices j. The cycles $\{\sum_{j \in J} \bar{\lambda}_{i,j} y_j\}_{i \in I}$ of $R(\pi'') \otimes_{\tau''} S'(Y)$ (where $\bar{\lambda}_{i,j}$ is the image of $\lambda_{i,j}$ in $R(\pi'')$) are such that these images form a basis of the homology of degree one. Let Z be the free R-module concentrated in degree two with basis $\{z_i\}_{i \in I}$, and let $\tau_1 : S'(Z) \longrightarrow R(\pi'') \otimes_{\tau''} S'(Y)$ be the twisting morphism such that $\tau_1(z_i) = \sum_{j \in J} \bar{\lambda}_{i,j} y_j$ and which is zero in degrees other than two. Now one has an acyclic construction $(R(\pi'')N, M)$, where $M = (R(\pi'') \otimes_{\tau''} S'(Y)) \otimes_{\tau_1} S'(Z)$, $N = S'(Y) \otimes_{\bar{\tau}_1} S'(Z)$ where $\bar{\tau}_1$ is the composite

$$S'(Z) \longrightarrow R(\pi'') \otimes_{\tau''} S'(Y) \longrightarrow S'(Y).$$

This construction is multiplicative with multiplication the tensor product of the indicated multiplications. For $(i, j) \in I \times J$ choose $u_{i,j} \in (R(\pi'') \otimes_{\tau''} S'(Y))_1 \subset M$, so that $du_{i,j} = \bar{\lambda}_{i,j} - \epsilon(\bar{\lambda}_{i,j})$ and $u_{i,j} = 0$ if $\lambda_{i,j} = 0$. An appropriate morphism $M \longrightarrow M \otimes N$ is given by $y_j \longrightarrow y_j \otimes 1 + 1 \otimes y_j$ for $j \in J$, and $z u_i \longrightarrow z_i \otimes 1 + 1 \otimes z_i + \sum_{j \in J} u_{i,j} \otimes y$, for

$i \in I$. Letting $\overline{u}_{i,j}$ be the images of $u_{i,j}$ in N, a diagonal for N is given by $z_i \longrightarrow z_i \otimes 1 + 1 \otimes z_i + \sum_{j\in J} \overline{u}_{i,j} \otimes y_j$ for $i \in I$.

If π'' is a coproduct of cyclic groups, and $\pi \longrightarrow \pi''$ is well chosen so that the basis of π maps according to a cyclic decomposition of π'', the construction of the preceding paragraph reduces to a tensor product of those for cyclic groups considered earlier.

Now it is time to turn to the consideration of algebras with divided powers. Indeed some of the preceding considerations are incomplete without this notion for diagonals have been given by saying what they do to generators as algebras with divided powers.

Definition. An algebra A with divided powers is a strictly commutative algebra A together with for each positive integer n, functions $\gamma_j(\ ): A_{2n} \longrightarrow A_{2nj}$ where j runs through the positive integers such that

1) $\gamma_0(x) = 1$ for $x \in A_{2n}$,

2) $j!\ \gamma_j(x) = x^j$ for $x \in A_{2n}$,

3) $\gamma_j(x+y) = \sum_{i'+i''=j} \gamma_{i'}(x)\gamma_{i''}(y)$ for $x, y \in A_{2n}$,

4) $\gamma_j(rx) = r^j \gamma_j(x)$ for $r \in R$, $x \in A_{2n}$,

5) $\gamma_i(x)\gamma_j(x) = (i,j)\gamma_{i+j}(x)$ for $x \in A_{2n}$,

6) $\gamma_i(\gamma_j(x)) = (j, j-1)(2j, j-1) \dots ((i-1)j, j-1)\gamma_{ij}(x)$ for $x \in A_{2n}$,

7) $\gamma_j(x'x'') = 0$ for $x' \in A_{n'}, x'' \in A_{n''}$, $n' + n'' = 2n$
n', n'' odd and $j \geq 2$, and

8) $d\gamma_{j+1}(x) = (dx)\gamma_j(x)$ for $x \in A_{2n}$.

Suppose that A is a commutative algebra, then the multiplication of A induces a morphism of algebras $A \otimes A \longrightarrow A$. Thus if B( ) denotes the functor which assigns to each algebra A its classifying coalgebra (i.e. the base of the construction with initial algebra A which satisfies the condition (B)), there is a natural morphism $B(A \otimes A) \longrightarrow B(A)$ in the case where A is commutative. Composing this with the natural morphism of coalgebras $B(A) \otimes B(A) \longrightarrow B(A \otimes A)$, one has that B(A) is a Hopf algebra with commutative multiplication ([7], [9]). One of the basic technical results of Cartan used in his computation of the homology of $K(\pi, n)$'s is to the effect that if A is a strictly commutative algebra, then the algebra B(A) is an algebra with divided powers ([3], exp. 7). A slight extension of Cartan's result says that B(A) is a Hopf algebra with divided powers.

Suppose that X is a differential graded module which is projective as a graded module, and such that $X_0 = 0$. Let A be the algebra such that its augmentation ideal has zero multiplication and is isomorphic with the desuspension of X as a differential graded module. There is a natural morphism of differential graded modules $X \longrightarrow B(A)$. Using Cartan's theorem let $i: X \longrightarrow \Gamma(X)$ be the morphism of differential graded modules induced by the above where $\Gamma(X)$ is the algebra with divided powers generated by the image of X in B(A). One can show that if A'' is an algebra with divided powers and $f: X \longrightarrow A''$ is a morphism of differential graded modules, then there is a unique morphism of algebras with divided powers $\bar{f}: \Gamma(X) \longrightarrow A''$ such that $\bar{f}i = f$, i.e. $\Gamma(X)$ is the algebra with divided powers generated by X. For a more general X'' ($X''_0 = 0$), the algebra $\Gamma(X'')$ is defined by representing X'' as the quotient of an X as above, and then dividing $\Gamma(X)$ by the appropriate ideal with divided powers.

Another result of Cartan is to the effect that if A' and A'' are algebras with divided powers, then $A' \otimes A''$ has divided powers, i.e. the category of algebras with divided powers has finite coproducts ([3], exp. 7). Given this fact, if X is as above then $\Gamma(X) \otimes \Gamma(X) = \Gamma(X \otimes X)$ is an algebra with divided powers, and the diagonal of X induces a morphism of algebras with divided powers $\Gamma(X) \longrightarrow \Gamma(X) \otimes \Gamma(X)$ making $\Gamma(X)$ into a Hopf algebra with strictly commutative diagonal. If one divided the augmentation ideal of $\Gamma(X)$ by the ideal with divided powers generated by the square of the augmentation ideal of $\Gamma(X)$ one obtains X. Thus there is a natural morphism of differential graded modules $\Gamma(X) \longrightarrow X$, and this induces a morphism of coalgebras $\Gamma(X) \longrightarrow S'(X)$. One verifies easily that this morphism is in fact an isomorphism of Hopf algebras, the multiplication in $S'(X)$ being induced by the codiagonal or addition of X as explained in Section 2. If one desires, one may use the functor $S'(\ \ )$ to define algebras with divided powers.

Suppose X'' is a graded R-module, $X''_0 = 0$, $d(X'') = 0$. Choose an exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ with X positive, $X_0 = 0$, X_n projective for all n, and $d(X) = 0$. Let Y be the suspension of X' and let $\tau: S'(Y) \longrightarrow \Gamma(X)$ be the twisting morphism which is the composite

$$S'(Y) \longrightarrow Y \longrightarrow X' \longrightarrow X \longrightarrow \Gamma(X).$$

It is not difficult to verify that $\Gamma(X'')$ is the image of $\Gamma(X)$ in $H_*(\Gamma(X) \otimes_\tau S'(Y))$.

In addition to introducing the notion of divided powers, and proving the results cited above, Cartan proved the existence of divided powers in some constructions of a more general type than those satisfying the condition (B). He also observed that if A is a commutative algebra over a field of characteristic zero, then divided powers are defined by setting $\gamma_j(x) = \frac{1}{j!} x^j$ for $x \in A_{2n}$ and j a positive integer.

Divided powers are known to exist in another situation. Unpublished work of Mac Lane shows that if A is the algebra of chains of a commutative ring complex, then divided powers exist in A ([7]). They are obtained by appropriately iterating the "↓ product" of Eilenberg-Mac Lane. These results imply that if A is the chains of $K(\pi, n)$, then A has divided powers, and the canonical morphisms from B(A) to the chains of $K(\pi, n+1)$ is a morphism of Hopf algebras with divided powers.

§4. The suspension, multiplicative constructions, and the transpotence.

Suppose that (A, N, M) is an acyclic construction. If $\dot{H}_*(A)$ denotes the reduced homology of A, there is a morphism of degree one

$$\sigma_*: \dot{H}_*(A) \longrightarrow H_*(N)$$

called the suspension of the acyclic construction ([3], exp. 6). It may be defined in the following way: There is up to homotopy a unique morphism of A-modules $i: A \longrightarrow M$ such that $\varepsilon(M)i = \varepsilon(A)$. If $x \in A_q, dx = 0$, $\varepsilon(x) = 0$, then there exists $y \in M_{q+1}$ such that $dy = i(x)$; $\bar{y} = \pi(M)y$ is a (q+1) cycle of N and its homology class depends only on the homology class, there results a function $\sigma_q: \dot{H}_q(A) \longrightarrow H_{q+1}(N)$. Verifying that this is a morphism of R-modules, the q'th component of the suspension is defined.

It was observed earlier that the homology of the base of an acyclic construction with initial algebra A depends only on A and not the choice of acyclic construction. Similarly the suspension depends only on A and not the choice of acyclic construction. Further it behaves functorially with respect to change of algebras. In addition it annihilates decomposable elements of the ideal $\dot{H}_*(A)$ and its image is contained in the primitive submodule of $H_*(N)$. When the acyclic construction corresponds to a topological situation as in Section 2, the suspension corresponds to the usual one in singular theory ([17]).

For any algebra A, let Q(A) denote the quotient of the augmentation ideal by its square. If $A = R(\pi)$, π abelian, a classical calculation shows $Q(A) = R \otimes \pi$. A short look at the acyclic construction of Section 3 shows that $\sigma: R\otimes\pi \longrightarrow H_1(\pi, 1; R)$ is an isomorphism, where for any R module G, $H_*(\pi, n; G)$ denotes the homology of $K(\pi, n)$ with coefficients in G. The preceding goes back to the origins of the homology of groups.

Given two constructions (A', N', M'), (A'', N'', M''), one easily defines their tensor product. It is a construction (A, N, M), where $A = A' \otimes A''$, $N = N' \otimes N''$, and $M = M' \otimes M''$ ([3], exp. 4).

<u>Definition.</u> A <u>multiplicative construction</u> is a construction (A, N, M) together with the structure of an algebra on M and also on N such that $A^{\#} \otimes N^{\#} \longrightarrow M^{\#}$ is an isomorphism of algebras.

One could give a more inclusive definition of multiplicative construction, but the original one ([3], exp. 4) is sufficient for the purposes considered here even though it demands that the algebra of the total modules be split when the differential is neglected. It is easy to verify that the tensor product of multiplicative construction is multiplicative. Another result of Cartan is to the effect that if A is a commutative algebra, the construction $(A, B(A), M(A))$ satisfying the condition (B) is a multiplicative construction. It has already been noted that in this case he also proved that $B(A)$ is an algebra with divided powers. All of the constructions considered in Section 3 are multiplicative constructions in a natural way. Notice also that if (A, N, M) is a multiplicative construction of commutative algebras, there is a natural morphism of the tensor product with itself back to the original construction. Indeed the category of multiplicative constructions of commutative algebras is a pointed category with coproducts, the finite coproducts being tensor products.

The remainder of this section will be devoted to describing some of the properties of the "transpotence" of Cartan ([3], exps. 6, 7, 8). Hence it will be assumed that $p \cdot 1 = 0$ in the ground ring R. For A a commutative algebra and q a positive integer let ${}_p(A_{2q})$ denote the subset of the cycles of A such that an element a is in this subset if $(a - \varepsilon(a))^p = 0$.

Proposition. If (A, N, M) is a multiplicative construction of commutative algebras which is acyclic, then there is a function

$$\psi\colon {}_p(A_{2q}) \longrightarrow H_{2pq+2}(N)$$

for each positive integer q such that

1) if $a \in {}_p(A_{2q})$, $x \in M_{2q+1}$, $dx = a - \varepsilon(a)$, $y \in M_{2pq+2}$, and $dy = (a - \varepsilon(a))^{p-1}x$, then the image of y in N_{2pq+2} is a representative cycle for $\psi(a)$,

2) if $a \in {}_p(A_{2q})$, $a = be$, $\varepsilon(b) = 0$, and $e^p = 0$, then $\psi(a) = 0$,

3) if p is odd, then ψ is additive,

4) if $p = 2$, $a', a'' \in {}_p(A_{2q})$, then $\psi(a' + a'') = \psi(a') + \psi(a'') + \sigma(a')\sigma(a'')$.

This proposition is a summary of the technical considerations leading to the definition of the transpotence ([3], exp. 6).

Given the preceding proposition, one would like that ψ was defined on all of the cycles of A of even degree, and that it passed to homology. One would then have the transpotence. If A is a connected algebra with divided powers, then if $a \in A_0$, $a = \varepsilon(a)$, and if a is an element of strictly positive degree, then $a^p = 0$ for if the degree of a is odd $a^2 = 0$, and if even $a^p = p!\ \gamma_p(a) = 0$.

Definition. A transpotence algebra is a strictly commutative algebra A such that

1) if $a \in A_q$ $(a - \varepsilon(a))^p = 0$, for q a positive integer,

and 2) if $b \in A_{2q+1}$, there exists $e \in A_{2pq+2}$ such that $de = (db)^{p-1}b$ for q a positive integer.

Observe that this notion is only defined over a ground ring of characteristic p, and that condition 2 is trivial for algebras satisfying condition 1 and having zero differential.

Proposition. If C is a connected algebra which is strictly commutative and such that if $c \in C_q$, $q > 0$, $c^p = 0$, and if there exists an acyclic multiplicative construction (A, C, M) with A commutative and such that $a \in A_q$, $q > 0$, $a^p = 0$, then C is a transpotence algebra.

Though the proof of the preceding proposition is essentially trivial it nevertheless guarantees that if A is a commutative algebra concentrated in degree 0, then $B(A)$ is a transpotence algebra. If A is a commutative algebra, so is $B(A)$. Hence one may define $B^0(A) = B(A)$, and $B^{n+1}(A) = B(B^n(A))$ by induction. The preceding also guarantees that for any commutative algebra A, $B^n(A)$ is a transpotence algebra for $n \geq 2$. Note that all of this is only in view of what has preceded. Finally observe that the preceding also guarantees that the bases of any of the acyclic constructions of Section 3 are transpotence algebras when of course the ground ring is of characteristic p.

Combining the preceding with the first proposition of this section, one has that for q a positive integer, (A, N, M) a multiplicative construction of commutative algebras which is acyclic and A a transpotence algebra, there is a function

$$\varphi_p : \dot{H}_{2q}(A) \longrightarrow H_{2pq+2}(N)$$

obtained from ψ by passing to quotients, this function is additive if p is odd, it always annihilates decomposables, and has the property that if $r \in R$, $a \in \dot{H}_{2q}(A)$, $\varphi_p(ra) = r^p \varphi_p(a)$. This is Cartan's transpotence. Observe that one may consider the composite $H_{2q}(A) \xrightarrow{\varphi_p} H_{2pq+2}(N) \longrightarrow Q(H_*(N))_{2pq+2}$ and that this is always additive even for $p = 2$.

§5. The Hopf algebras $H_*(\pi, n; k)$ for k a field.

First suppose that the ground ring R is a field of characteristic zero which will now be called k. One may as well suppose that $k = Q$ for if not $H_*(\pi, n; k) = H_*(\pi, n; Q) \otimes_Q k$. The computation of $H_*(\pi, n; k)$ for k the field of rational numbers is basically due to Eilenberg and Mac Lane ([6], II) who expressed the result in a slightly different form since at the time it was not usual to express things in terms of Hopf algebras at the time.

Theorem. The n-fold suspension

$$\sigma^n : \pi \otimes k \longrightarrow H_n(\pi, n; k)$$

extends to an isomorphism of Hopf algebras

$$\Gamma(\pi\otimes k, n) \longrightarrow H_*(\pi, n; k)$$

for π an abelian group, k a field of characteristic zero, and $n > 0$.

For $n = 1$, and π a cyclic group the result follows from the calculations of Section 3. Hence by the Künneth theorem, it follows for $n = 1$ and π finitely generated. Taking colimits the case $n = 1$ is proved.

Suppose X is a graded vector space over k, $X_0 = 0$, and X the suspension of X. Recall (Section 2) that the natural twisting morphism $S'(sX) \xrightarrow{\tau} S(X)$ gives rise to a construction $S(X) \otimes_\tau S'(sX)$ which is acyclic. Further this construction is a multiplicative construction of commutative algebras. Indeed of primitively generated commutative Hopf algebras. Further since k is a field of characteristic zero, the Hopf algebras $S(Y)$, $S'(Y)$, and $\Gamma(Y)$ coincide for any graded vector space Y, $Y_0 = 0$.

Suppose the theorem is true for some integer $n > 0$. Choose a morphism of Hopf algebras $f: \Gamma(\pi\otimes k, n) \longrightarrow C_*(K(\pi, n))$ which on passage to homology is the given isomorphism. Now $\Gamma(\pi\otimes k, n) \otimes_\tau \Gamma(\pi\otimes k, n+1)$ is a multiplicative acyclic construction of abelian Hopf algebras, where τ is as in the preceding paragraph, and this construction maps naturally to the construction $C_*(K(\pi, n)) \otimes_{f\tau} \Gamma(\pi\otimes k, n+1)$ which is acyclic. Hence the theorem is proved by induction.

It now remains to make the desired calculation for k a field of finite characteristic, and for this it suffices to suppose that k is a prime field. Thus suppose that k is the field with p-elements. For π an abelian group let π denote the subgroup of elements of order p of π and π_p denote π modulo the subgroup generated by p'th powers of elements of π.

It has already been observed in a slightly different form that $\sigma\colon \pi_p \longrightarrow H_1(\pi, i; k)$ is an isomorphism. Notice also that if A is the algebra $k(\pi)$, then ${}_p\pi \subset {}_p(A_0)$ (Section 4) and so the transpotence $\varphi_p\colon {}_p\pi \longrightarrow H_2(\pi, 1; k)$ is defined. Further the transpotence is linear for p odd.

Notice that there is a natural morphism ${}_p\pi \longrightarrow \pi_p$ which is the composite of reduction mod p with the inclusion ${}_p\pi \subset \pi$. For π cyclic of order p^j, this is an isomorphism if $j = 1$ and is trivial for $j > 1$. Denote this morphism by β_p. Also denote by $\beta_p\colon H_{n+1}(N) \longrightarrow H_n(\mathbb{N})$ the Bockstein morphism of any differential graded k module which is given as the reduction mod p of one flat over $\mathbb{Z}$.

<u>Proposition</u>. If $x \in {}_p\pi$, then $\beta_p\varphi_p(x) = \sigma\beta_p x$,

and 1) if p is odd, then $\varphi_p(x)$ is primitive, and

2) if $p = 2$, then the image of $\varphi_p(x)$ under the diagonal is $\varphi_p(x)\otimes 1 + 1\otimes\varphi_p(x) + \sigma\beta_p(x)\otimes\sigma\beta_p(x)$.

The result of Cartan can be checked easily for the case π cyclic using the construction for $k(\pi)$ for this case of Section 3. It suffices to check it in this case.

Proposition. If $n, q > 0$, then

$$\varphi_p : H_{2q}(\pi, n; k) \longrightarrow H_{2pq+2}(\pi, n+1; k)$$

$$\sigma : H_{2pq}(\pi, n; k) \longrightarrow H_{2pq+1}(\pi, n+1; k),$$

and 1) if $x \in H_{2q}(\pi, n; k)$; $\beta_p \varphi_p(X) = \sigma \gamma_p(X)$,

2) for p odd, $\varphi_p(x)$ is primitive,

and 3) for $p = 2$, $\Delta(\varphi_p(x)) = \varphi_p(x) \otimes 1 + 1 \otimes \varphi_p(x) + \sigma(x) \otimes \sigma(x)$

The proof of this proposition may be found in the Cartan seminar ([3], exp. 6, 7, 8).

Suppose that w is a word written in terms of the letters $\sigma, \varphi_p, \gamma_p$. The height of w is the number letters equal to σ or γ_p. The degree of w is defined inductively by the degree of the empty word in 0,

$$\text{degree } (\sigma w) = 1 + \text{degree } w,$$

$$\text{degree } (\gamma_p w) = p \cdot \text{degree } w,$$

and

$$\text{degree } (\varphi_p w) = 2 + p \text{ degree } w.$$

Now consider only words w such that

1) the last letter is φ_p or σ,

2) the number of σ's to the right of a φ_p or γ_p is even.

There are two such words of height 1, σ and φ_p. To σ corresponds $\sigma : \pi_p \longrightarrow H_1(\pi, 1; k)$, and to φ_p corresponds $\varphi_p : {}_p\pi \longrightarrow H_2(\pi, 1; k)$. Extend this so that to every word w satisfying 1) and 2) there corresponds a function to $H_*(\pi, n; k)$ where $n > 0$ is the height of w, the domain is π_p

if the last letter is σ, and the domain is ${}_p\pi$ if the last letter is φ_p, and the image is contained in $H_q(\pi, n; k)$ where q is the degree w. If σw is such a word, then the function for σw is the composite of that for w followed by the suspension, if $\gamma_p w$ is such a word, then the function for $\gamma_p w$ is the composite of that for w followed by the divided power γ_p, and the function for $\varphi_p w$ is the composite of φ_p with the function for w. The condition 2) assures that these are defined.

A <u>Cartan word</u> w is a word w is the letters $\sigma, \varphi_p, \gamma_p$ satisfying the conditions 1) and 2) above and such that the first letter is σ or φ_p. To a Cartan word w of height $n > 0$ and degree q, corresponds a function $\pi_p \longrightarrow H_q(\pi, n; k)$ if the last letter is σ or a function ${}_p\pi \longrightarrow H_q(\pi, n; k)$ if the last letter is φ_p. For p an odd prime these functions are linear, and their images are primitive elements. For $p = 2$, the functions are linear and have their image contained in the primitive elements if the first letter is σ.

For w a Cartan word, let [w] denote the graded vector space such that $[w]_j = 0$ if $j \neq$ degree w, $[w]_j = \pi_p$ for j = degree w and the last letter of w being σ, $[w]_j = {}_p\pi$ for j = degree w and φ_p the last letter of w. There is given a graded function $w\colon [w] \longrightarrow H_*(\pi, n; k)$ where $n > 0$ is the degree of w. It is linear for p odd or the first letter of w equal to σ, and under these conditions its image is primitive.

Suppose that p is an odd prime. For $n > 0$, let $C(\pi, n)$ be the coproduct of the vector spaces [w] where w ranges over the Cartan words of height n. Note $C(\pi, 1)_j = 0$ for $j \neq 0, 1$, $C(\pi, 1) = \pi_p$, $C(\pi, 1) = {}_p\pi$.

There is given a morphism of graded vector spaces $C(\pi, n) \longrightarrow H_*(\pi, n; k)$ explained above on the components $[w]$ of $C(\pi, n)$. The image of this morphism is contained in the primitive elements of $H_*(\pi, n; k)$ which is a Hopf algebra with divided powers. Thus there is a morphism of Hopf algebras $\alpha(\pi, n)\colon \Gamma(C(\pi, n)) \longrightarrow H_*(\pi, n; k)$. This is in fact a morphism of functors defined on the category of abelian groups.

Theorem. For p an odd prime, and $n > 0$ the morphism $\alpha(\pi, n)\colon \Gamma(C(\pi, n)) \longrightarrow H_*(\pi, n; k)$ is an isomorphism.

The preceding theorem is Cartan's calculation of $H_*(\pi, n; k)$ for k the field with p-elements and π an abelian group. It is proved by induction on n. For $n = 1$, and π a cyclic group, it follows by looking at the construction given for $k(\pi)$ in Section 3. Applying the Künneth theorem it is then true for $n = 1$ and π finitely generated. Since both functors preserve filtering colimits, the theorem follows for $n = 1$. It remains to prove the inductive step.

Suppose X is a graded vector space, $X_0 = 0$. Choose a morphism $f\colon Q(\Gamma(X)) \longrightarrow I(\Gamma(X))$ such that the composite $Q(\Gamma(X)) \to I(\Gamma(X)) \to Q(I(\Gamma(X)))$ is the identity. Since the composite $X \to I(\Gamma(X)) \to Q(\Gamma(X))$ is a monomorphism, one may also suppose that f extends $X \to I(\Gamma(X))$. Let $\tau\colon S'(sQ(\Gamma(X)) \longrightarrow \Gamma(X)$ be the twisting morphism which is obtained by composing f with the canonical morphism of degree -1, $S'(sQ(\Gamma(X)) \longrightarrow Q(\Gamma(X))$. Now $\Gamma(X) \otimes_\tau S'(sQ(\Gamma(X)))$ is a multiplicative construction. It is not however acyclic, its homology is an exterior algebra $E(Y)$ where $Q(\Gamma(X))_{2j} \simeq Y_{2pj+1}$ and $Y_i = 0$ for $i \not\equiv 1 \bmod 2p$. Choose a morphism $Y \longrightarrow \Gamma(X) \otimes_\tau S'(sQ(\Gamma(X))$ such that if $E(Y) \longrightarrow \Gamma(X) \otimes_\tau S'(sQ(\Gamma(X)))$ is the resulting morphism of algebras, then one obtains an isomorphism of homology. Let $\tau_1 : S'(sY) \longrightarrow \Gamma(X) \otimes_\tau S'(sQ(\Gamma(X))$ be the resulting twisting morphism. If one composes this with the projection to $S's\,Q(\Gamma(X)))$, one obtains the trivial twisting morphism. Now $(\Gamma(X), S'(sQ\Gamma(X) \otimes Y), M)$ is an acyclic construction of commutative algebras, where $M = (\Gamma(X) \otimes_\tau S'(sQ(\Gamma(X)))) \otimes_{\tau_1} S'(sY)$. If one takes $X = C(\pi, n)$, it is easy to verify that $sQ\Gamma(X) \oplus sY$ is $C(\pi, n+1)$.

Suppose that the theorem is proved for some $n > 0$. Since $C_*(K(\pi, n))$ is a Hopf algebra with divided powers, one may choose a morphism of algebras with divided powers $h: \Gamma(C(\pi, n)) \longrightarrow C_*(K(\pi, n))$ which is $\alpha(\pi, n)$ on the homology level, and up to homotopy if differential graded k-modules is a morphism of Hopf algebras. Combining this with what has preceded, it follows easily that $\alpha(\pi, n+1): \Gamma(C(\pi, n+1)) \longrightarrow H_*(\pi, n+1; k)$ is an isomorphism, and the theorem is proved.

The situation for $p = 2$ is slightly more complicated than that for p odd, when considered in the terms used here. Suppose $p = 2$. For $n > 0$, let $C'(\pi, n)$ be the coproduct of the vector spaces [w] where w ranges over words of height n and first letter σ. There is a natural linear map $C'(\pi, n) \longrightarrow H_*(\pi, n)$ whose image is contained in the primitive elements. This extends to a morphism of Hopf algebras $\Gamma(C'(\pi, n)) \xrightarrow{\alpha'(\pi, n)} H_*(\pi, n;k)$. Let $H_*(\pi, n;k) \longrightarrow D(\pi, n)$ be the cokernel of $\alpha'(\pi, n)$ as a morphism of Hopf algebras. For w a word of height n and first letter φ_2, the composite $[w] \longrightarrow D(\pi, n)$ is linear and has its image contained in the primitive elements. Thus if $C''(\pi, n)$ is the coproduct of the vector spaces $[w]$ where w ranges over words of height n and first letter φ_2, there is a natural morphism of Hopf algebras with divided powers $\Gamma(C''(\pi, n)) \longrightarrow D(\pi, n)$. One may verify inductively that $\alpha'(\pi, n)$ is a monomorphism, and that $\Gamma(C''(\pi, n)) \longrightarrow D(\pi, n)$ is an isomorphism. Thus there results a short exact sequence of abelian Hopf algebras with divided powers

$$k \longrightarrow \Gamma(C''(\pi, n)) \xrightarrow{\alpha'(\pi, n)} H_*(\pi, n;k) \xrightarrow{\alpha''(\pi, n)} \Gamma(C''(\pi, n)) \longrightarrow k$$

functorial on abelian groups π. This sequence is split as a sequence of algebras, but not as a sequence of coalgebras. The coalgebra extension may be characterized using properties of φ_2. The underlying coalgebra of $H_*(\pi, n;k)$ is injective commutative, but not strictly commutative for $n > 1$. In this case Cartan characterized the algebras $H_*(\pi, n;k)$ by introducing divided powers defined in odd degrees in characteristic 2 ([3]).

In addition to the work described above, Cartan also characterized the integral homology of $K(\pi, n)$'s. These groups are of such a nature that they can only be described succintly in very special cases, e.g. π free abelian and $n = 1$ or 2. If one chooses the ground ring to be a ring like the integers localized at a prime, the situation is a little simpler, but still not very manageable on an invariant level (e.g. [13]).

Appendix. Some remarks concerning the Steenrod algebra.

For a prime p, the Steenrod algebra $\mathcal{A}^*(p)$ is the algebra of stable cohomology operation with coefficients in the field k with p-elements. Multiplication in $\mathcal{A}^*(p)$ corresponds to composition of cohomology operations.

Using his calculation of $H_*(\pi, n; k)$, Cartan proved the existence of the Steenrod operations, and gave an explicit basis for $\mathcal{A}^*(p)$ where each basis element is an iterate of Steenrod operations and the standard Bockstein (which is a Steenrod operation for $p=2$) ([3], exp. 14, 15, 16). This showed even without knowing the relations on Steenrod operations that the algebra $\mathcal{A}^*(p)$ is generated by the Steenrod operations and the Bockstein as an algebra. Cartan also derived the relations on the Steenrod operations using his calculations ([2]).

In addition to preceding Cartan derived his cup product formula for the Steenrod operation together with some surrounding facts about stable cohomology operations ([3], exp. 16 bis), which led to the fact that $\mathcal{A}^*(p)$ is a Hopf algebra ([11]). In particular once it is known that $\mathcal{A}^*(p)$ is a Hopf algebra, then for an odd prime $\Delta(\mathcal{P}^n) = \sum_{i+j=n} \mathcal{P}^i \otimes \mathcal{P}^j$ where $\mathcal{P}^n$ is the standard Steenrod n-th power which is an element of degree $2n(p-1)$, and $\mathcal{P}^0 = 1$. The Bockstein is of course primitive. For

$p = 2$, the diagonal is given by $\Delta(Sq^n) = \Sigma_{i+j=n} Sq^i \otimes Sq^j$. These formulas determine the diagonal of $\mathcal{Q}^*(p)$ since they specify it on algebra generators. They are just a form of Cartan's cup product formula.

The aim of this section is to obtain further information about $\mathcal{Q}^*(p)$ using what has been stated above, a few Hopf algebra techniques, and one of Cartan's methods ([2]).

Definitions. Suppose that A is a connected Hopf algebra over R a set of elements $(a_j)_{j \in \mathbb{Z}^+}$ of A is comultiplicative if

1) $a_0 = 1$,

2) degree $a_j = j$ degree a_1 and either the characteristic of R is 2 or degree a_1 is even,

3) $\Delta(a_r) = \Sigma_{i+j=r} a_i \otimes a_j$, where Δ is the diagonal of A.

If $(a) = (a_j)_{j \in \mathbb{Z}^+}$ is a comultiplicative set of elements of A, the left Newton elements of (a) are the elements $(\ell_n(a))_{n \geq 1}$ defined recursively by

1) $\ell_1(a) = a_1$,

and 2) $\ell_{n+1}(a) = \Sigma_{j=1}^{n} (-1)^{j+1} \ell_{n+1-j}(a) a_j + (-1)^{n+2} (n+1) a_{n+1}$;

the right Newton elements of (a) are the elements $(r_n(a))_{n \geq 1}$ defined recursively by

1) $r_1(a) = a_1$,

and 2) $r_{n+1}(a) = \Sigma_{j=1}^{n} (-1)^{j+1} a_j r_{n+1-j}(a) + (-1)^{n+2} (n+1) a_{n+1}$.

Notice that if the multiplication in A is commutative, the left and right Newton elements coincide and the recursion formulas which define them are just Newton's formulas.

Proposition. If A is a connected Hopf algebra over R and $(a) = (a_j)_{j \geq 0}$ is a comultiplicative set of elements of A, then the left and right Newton elements of (a) are primitive.

The proof of this proposition is completely straightforward, somewhat tedious, and follows exactly the lines used in studying Hopf algebras of the type $H_*(BU)$ ([4]).

Given a fixed prime p, let $\alpha_p(n) = \sum_{j=0}^{n} p^j$. When the prime p is clear from the context, the subscript p will be omitted.

In any algebra A over a field of characteristic $p \neq 0$, let $\xi(x)$ be the p'th power of x for x an element of A.

Theorem. If p is an odd prime, $(\mathcal{P}) = (\mathcal{P}^n)$ is a comultiplicative set of elements of $\mathfrak{a}^*(p)$, $(\mathcal{P}^n, \delta^*)_{n \geq 1}$ generater $\mathfrak{a}^*(p)$ or an algebra, and

1) $\ell_j(\mathcal{P}) = 0$ for $j \geq 2$,

2) $r_i(\mathcal{P}) = 0$ unless $i = \alpha(j)$ for some j,

and 3) $(r_{\alpha(i)}(\mathcal{P}), \delta^*, [r_{\alpha(j)}(\mathcal{P}), \delta^*])_{i,j \geq 0}$ is a basis for the primitive elements of $\mathfrak{a}^*(p)$.

Proof. Cartan showed that in order to determine whether an element of the Steenrod algebra is zero or not it suffices to check it on finite products of 1 and 2 dimensional classes in an algebra of the form $H^*(\pi, 1; k)$ where k is the field with p elements and π is a finite dimensional vector space over k. Since an element x of $\mathcal{A}^*(p)$ is primitive if and only if it acts as a derivation on cohomology algebras of spaces, in order to see if a primitive element is zero or not it suffices to test it on i and $\delta^* i$ where $i \in H^1(k, 1; k)$ is the fundamental class. An element of $\mathcal{A}^*(p)$ takes primitive elements in the cohomology of an H-space with coefficients in k into primitive elements. A basis for the primitive elements of $H^*(k, 1; k)$ is $(i, \xi^j \delta^* i)_{j \geq 0}$. Thus a primitive element of $\mathcal{A}^*(p)$ is zero unless its degree is 1, or of one of the forms $2\alpha(j)(p-1)$ or $2\alpha(j)(p-1)+1$, and for these degrees the vector space of primitive elements has dimension at most 1. Now one makes an elementary inductive calculation showing $\ell_j(\mathcal{P})(\delta^* i) = 0$ for $j \geq 2$. $r_{\alpha(j)}(\mathcal{P})(i) = \xi^{j+1}(\delta^* i)$, $[r_{\alpha(j)}(\mathcal{P}), \delta^*](i) = \xi^{j+1}(\delta^* i)$, and the theorem follows.

Theorem. $(Sq) = (Sq^i)$ is a comultiplicative set of elements of $\mathcal{A}^*(2)$ which generates $\mathcal{A}^*(2)$ as an algebra, and

1) $\ell_j(Sq) = 0$ for $j \geq 2$

2) $r_i(Sq) = 0$ unless $i = \alpha(j)$ for some j,

and 3) $(r_{\alpha(i)}(Sq))_{i \geq 0}$ is a basis for the primitive elements of $\mathcal{A}^*(2)$.

The proof is as the proof of the preceding theorems. These theorems characterize the Steenrod algebras $\mathcal{A}^*(p)$, and one may obtain Milnor's description of the dual Hopf algebras $\mathcal{A}_*(p)$ algebraically from the data they furnish.

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