Astérisque

JOHN N. MATHER

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Astérisque, tome 32-33 (1976), p. 165-172

<http://www.numdam.org/item?id=AST_1976__32-33__165_0>

Société Mathématique de France
Astérisque 32-33 (1976)

Infinite Dimensional Group Actions

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Let N and P be smooth finite dimensional manifolds. Let $\mathcal{C}(N,P)$ denote the space of smooth mappings of N into P. Let $\mathcal{D}(N)$ denote the group of smooth diffeomorphisms of N. The orbit structure of the action $\mathcal{D}(N) \times \mathcal{D}(P)$ on $\mathcal{C}(N,P)$ given by $(h,h',f) \to h'fh^{-1}$ has certain properties in common with the orbit structure of an algebraic action of a real algebraic group on a real algebraic manifold, at least as far as the orbits of finite codimension in $\mathcal{C}(N,P)$ are concerned.

Here, we formulate a precise result of this type. Detailed proofs will be given elsewhere.

First, we formulate a result concerning real algebraic actions. Let α be an algebraic action of an algebraic group G on an algebraic manifold X. By this we understand the following: G and X are regular algebraic subsets of real number space, and the graphs of the mappings $G \times G \to G:(g,h) \to gh^{-1}$ and $G \times X \to X:(g,x) \to gx$ are regular algebraic sets.

<u>Theorem 1</u>. <u>There is a filtration of</u> X <u>by closed semi-algebraic invariant subsets</u>

$X = X_0 \supset X_1 \supset X_2 \supset \ldots \supset X_k \supset \ldots \supset \emptyset$ <u>such that</u>

(1) $X_i - X_{i+1}$ <u>is regular</u>.

(2) $(X_i - X_{i+1})/G$ <u>has a natural structure of a real analytic manifold such that the mapping</u> $X_i - X_{i+1} \to (X_i - X_{i+1})/G$ <u>is a real analytic function</u>.

(3) $\operatorname{cod} X_i \geq i$.

Before sketching the proof, we need a definition and a lemma.

<u>Definition</u>. Let U be a smooth (i.e., C^∞) manifold, and for each $u \in U$, let V_u be a smooth submanifold of X (not necessarily closed). We say the family $\{V_u\}_{u \in U}$ is <u>smooth</u> if the set $V = \bigcup_u u \times V_u \subset U \times X$ is a smooth manifold, and the restriction of $\pi: U \times X \to U$ to V is a smooth locally trivial fibration.

<u>Lemma 1</u>. <u>There is a filtration of</u> X <u>by closed semi-algebraic invariant subsets such that</u> (1) <u>and</u> (3) <u>in Theorem 1 hold and the family of orbits</u> $\{Gx : x \in X_i - X_{i+1}\}$ <u>is smooth</u>.

It is an exercise in differential topology to prove that the Lemma implies Theorem 1. For, it follows from the smoothness of the family that each small transversal intersects each nearby orbit exactly once, and that the quotient space has the Hausdorff property. □

We deduce Lemma 1 from the following result.

Lemma 2. *Let Y be a closed semi-algebraic subset of X. Then there is a closed semi-algebraic invariant subset Z of Y such that*

(1) $Y - Z$ *is regular.*

(2) Z *is nowhere dense in* Y.

(3) $\{Gx : x \in Y - Z\}$ *is a smooth family.*

The fact that Z is nowhere dense in Y implies $\dim Z < \dim Y$. Then Lemma 1 follows by decreasing induction. □

Sketch of the Proof of Lemma 2. Let $\mathcal{R} = \{(y,y') \in Y \times Y : \exists g \in G, gy = y'\}$. Embed $\mathbb{R}^n$ in $\mathbb{RP}^n$ in the standard way. Let $\bar{Y}$ be the closure of Y in $\mathbb{RP}^n$. Let $\bar{\mathcal{R}}$ be the closure of $\mathcal{R}$ in $Y \times \bar{Y}$. Since $\bar{\mathcal{R}}$ is semi-algebraic, we may find a Whitney stratification $\mathcal{S}$ of $\bar{\mathcal{R}}$ by semi-algebraic sets such that $\mathcal{R}$ and $\bar{\mathcal{R}} \cap (Y \times Y)$ are unions of strata. Moreover, we may suppose that the set of non-singular points of $\mathcal{R}$ forms a stratum. Let π be the restriction to $\bar{\mathcal{R}}$ of the projection of $Y \times \bar{Y}$ on Y. Let Z_0 be the set of points $y \in Y$ such that either

(a) y is a singular point of Y, or

(b) y is a critical value of $\pi | U : U \to Y$, for some stratum U of $\mathcal{S}$.

By the Tarski-Seidenberg theorem, Z_0 is semi-algebraic. By Sard's theorem it is nowhere dense (a semi-algebraic subset of measure 0 is nowhere dense). Let $Z = \bar{Z}_0$. By Thom's second isotopy lemma $(\bar{\mathcal{R}}, \pi)$ is locally trivial over $Y - Z$; moreover, the local trivialization can be taken to preserve strata and be smooth on strata.

We have $\pi^{-1}\pi(\mathcal{R}_{\text{sing}}) = \mathcal{R}_{\text{sing}}$. For, let G act on $Y \times Y$ by

$$G \times Y \times Y \to Y \times Y : (g, y, y') \to (y, gy') .$$

This action preserves $\mathcal{R}$, and hence also preserves $\mathcal{R}_{\text{sing}}$. Since $\pi^{-1}\pi(\mathcal{R}_{\text{sing}}) = \mathcal{R}_{\text{sing}}$, we obtain that $\pi(\mathcal{R}_{\text{sing}})$ is nowhere dense in Y. It follows that $\pi(\mathcal{R}_{\text{sing}}) \subset Z$.

Since $\mathcal{R}|Y - Z$ is entirely contained in the stratum of regular points of $\mathcal{R}$, it follows that $(\mathcal{R}|Y - Z, \pi)$ is a locally trivial fibration over

$Y - Z$. This means that $\{Gx : x \in Y - Z\}$ is a locally trivial family. □

Now we consider the action of $\mathcal{D}(N) \times \mathcal{D}(P)$ on $\mathcal{C}(N,P)$ described at the beginning of this section. If $f \in \mathcal{C}(N,P)$, we have the identification $T_f\mathcal{C}(N,P) = \Gamma(f^*TP)$, where Γ means the smooth sections of the vector bundle in parenthesis. Likewise $T_1(\mathcal{D}(N)) = \Gamma(TN)$. We let $\alpha_f : \mathcal{D}(N) \times \mathcal{D}(P) \to \mathcal{C}(N,P)$ be defined by

$$\alpha_f(h,h') = h'fh^{-1} .$$

We consider the Fréchet derivative

$$d\alpha_f : T_1\mathcal{D}(N) \oplus T_1\mathcal{D}(P) \to T_f\mathcal{C}(N,P) .$$

We say the orbit through f has <u>finite codimension</u> if the image of $d\alpha_f$ has finite codimension (note that $d\alpha_f$ is a linear mapping of real vector spaces).

At least when $(\dim N, \dim P)$ is in the nice range of dimensions [2], there are many orbits of finite codimension, as is indicated by Theorem 2 below.

<u>Definition</u>. Let U be a smooth manifold. A mapping $U \to \mathcal{C}(N,P)$ will be said to be <u>smooth</u> if the associated mapping $U \times N \to P$ is smooth. We let $\mathcal{C}(U,\mathcal{C}(N,P))$ denote space of smooth mappings of U

into $\mathcal{C}(N,P)$. Clearly, $\mathcal{C}(U,\mathcal{C}(N,P)) = \mathcal{C}(U \times N,P)$. By the C^∞ topology on $\mathcal{C}(U,\mathcal{C}(N,P))$, we mean the C^∞ topology on $\mathcal{C}(U \times N,P)$.

Definition. A subset K of $\mathcal{C}(N,P)$ will be said to be of infinite codimension if for every smooth mapping of a finite dimensional manifold U into $\mathcal{C}(N,P)$, there is an arbitrarily close approximation in the C^∞ topology whose image does not intersect K.

In what follows we let $\mathcal{U}$ denote the set of elements of $\mathcal{C}(N,P)$ of finite codimension.

Theorem 2. *$\mathcal{U}$ is open in $\mathcal{C}(N,P)$. If $(\dim N, \dim P)$ is in the nice range of dimensions, the complement of $\mathcal{U}$ has infinite codimension, and $\mathcal{U}$ contains orbits of arbitrarily high codimension.*

The first statement is essentially known (cf. Baas [1]). We omit the proof of the second, which requires lengthy detailed calculations and most of the theory of C^∞ stable mappings.

A smooth mapping $\varphi: U \to \mathcal{C}(N,P)$ has a Frechet derivative $d\varphi_u : TU_u \to T\mathcal{C}(N,P)_{\varphi(u)}$. We say φ is transverse to the orbit through $\varphi(u)$ if

$$\operatorname{im} d\varphi_u + \operatorname{im} d\alpha_{\varphi(u)} = T\mathcal{C}(N,P)_{\varphi(u)} \quad .$$

An invariant subset K of $\mathcal{U}$ will be said to be <u>pseudo-algebraic</u> if for any $f \in \bar{K}$ and any smooth mapping $\varphi: U \to \mathcal{C}(N,P)$ such that $\varphi(u) = f$ for some $u \in U$ and such that φ is transverse to the orbit through f, we have that $\varphi^{-1}(K)$ is semi-algebraic in a neighborhood of u, with respect to a suitable smooth coordinate system. In addition, K will be said to be of codimension $\geq i$, if $\varphi^{-1}(K)$ is of codimension $\geq i$ at u.

<u>Theorem 3</u>. <u>There exists a filtration</u> $\mathcal{U} = \mathcal{U}_0 \supset \mathcal{U}_1 \supset \ldots \supset \mathcal{U}_k \supset \ldots$ <u>by invariant closed pseudo-algebraic subsets such that</u> $\operatorname{cod} \mathcal{U}_i \geq i$, <u>and for each</u> i <u>an analytic manifold</u> Y_i <u>and a locally trivial fibration</u> $\pi: \mathcal{U}_i - \mathcal{U}_{i+1} \to Y_i$ <u>such that</u>

(1) $Y_i = (\mathcal{U}_i - \mathcal{U}_{i+1})/\mathcal{D}(N) \times \mathcal{D}(P)$ <u>as topological spaces, and</u> π <u>is the quotient mapping</u>.

(2) <u>The analytic structure on</u> Y_i <u>is uniquely determined by the condition that if</u> $\varphi: U \to \mathcal{U} - \mathcal{U}_{i+1}$ <u>is an analytic minimal unfolding of</u> $\varphi(u)$, <u>in the sense defined below, then</u> $(\pi\varphi)^{-1}Y_i$ <u>is an analytic submanifold of</u> U, <u>and</u> $\pi\varphi: (\pi\varphi)^{-1}Y_i \to Y_i$ <u>is a local analytic diffeomorphism at</u> u.

<u>Definition</u>. Let U be a finite dimensional manifold, and let φ be a smooth mapping of U into $\mathcal{U}$. We say φ is an <u>unfolding</u> of $\varphi(u)$ if its transverse to the orbit through $\varphi(u)$ at u. We say that it is a <u>minimal</u> unfolding if the mapping

$$TU_u \to T\mathcal{C}(N,P)_{\varphi(u)}/\mathrm{im}\ d\alpha_{\varphi(u)}$$

is an isomorphism.

Since $f = \varphi(u)$ has finite codimension in $\mathcal{C}(N,P)$, there is a finite set $\Lambda \subset N$ such that for any finite subset $S \subset N$ such that $S \cap \Lambda = \emptyset$, we have that f is infinitesimally stable at S. We say the unfolding φ is <u>analytic</u> if we can choose coordinates about Λ and $f\Lambda$ such that the resulting mapping $U \times N \to P$ is analytic in $U \times W$ for a suitable open neighborhood W of Λ in N.

REFERENCES

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John N.MATHER
Fine Hall, Princeton University
PRINCETON, N.J. 08540, USA