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HILBERT MODULAR SURFACES AND CLASS NUMBERS

by F. HIRZEBRUCH

To a real quadratic field is associated a theory of modular forms in two variables. We want to discuss how this theory leads to

- interesting examples of algebraic and non-algebraic surfaces
- results in number theory
- theorems and conjectures relating to modular forms in one variable.

1. Surfaces

A connected, compact complex surface X has a certain number of invariants :

Topological invariants

Euler characteristic $\quad e(X) = 2 - 2b_1 + b_2$

$(b_i = \dim_{\mathbf{C}} H^i(X ; \mathbf{C}) = i^{\text{th}}$ Betti number$)$

Signature $\qquad \mathrm{Sign}(X) = b^+ - b^-$

($b^{\pm}$ = number of $\pm$ signs in a diagonalized version of the intersection form on $H_2(X ; \mathbf{R})$)

Analytic invariants

Arithmetic genus $\qquad \chi(X) = 1 - q + p_g$

$(q = \dim_{\mathbf{C}} H^1(X, \mathcal{O}_X)\ ,\ p_g = \dim_{\mathbf{C}} H^2(X, \mathcal{O}_X))$

$K^2 = K \cdot K \quad$ (K any canonical divisor) .

These invariants are related by

$$\chi(X) = \frac{e(X) + \mathrm{Sign}(X)}{4}$$

$$\chi(X) = \frac{1}{12}(K^2 + e(X))$$

$$\mathrm{Sign}(X) = \frac{1}{3}(K^2 - 2e(X))$$

(thus $\chi(X)$ and K^2 are in fact topological invariants of X). One also has :

b_1 even $\Rightarrow$ $b_1 = 2q$ and $b^+ = 2p_g + 1$,

b_1 odd $\Rightarrow$ $b_1 = 2q - 1$ and $b^+ = 2p_g$,

(the second case cannot occur for algebraic surfaces X).

Kodaira in his program of classifying surfaces studies surfaces with first Betti number $b_1 = 1$ and shows that then $p_g = 0$. Since $q = 1$, we have also $\chi = 0$. We give examples of such surfaces later.

2. Modules in quadratic fields

Let K be a real quadratic field, $K = \mathbb{Q}(\sqrt{D})$,

$M \subset K$ a module (= free $\mathbb{Z}$-module of rank 2) ,

and let $U_M^+ = \{\varepsilon \mid \varepsilon$ a unit of K , $\varepsilon M = M$, $\varepsilon \gg 0\}$ be the group of totally positive units preserving M (the sign $x \gg 0$ means that x and its conjugate x' are positive). The group U_M^+ is infinite cyclic. Let

$$G(M) = \{\begin{pmatrix}\varepsilon & \mu \\ 0 & 1\end{pmatrix} \mid \varepsilon \in U_M^+ , \mu \in M\} \subset GL_2(K) .$$

The group $G(M)$ acts freely on $\underline{H} \times \underline{H}$ ($\underline{H}$ denotes the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$) via

$$\begin{pmatrix}\varepsilon & \mu \\ 0 & 1\end{pmatrix} \circ (z_1, z_2) = (\varepsilon z_1 + \mu , \varepsilon' z_2 + \mu') .$$

The orbit space $\underline{H}^2/G(M)$ is a non-compact complex surface. We shall not compactify it, but shall "make it a little more compact" by adding a single point as follows :

Since $\varepsilon\varepsilon' = 1$ for $\begin{pmatrix}\varepsilon & \mu \\ 0 & 1\end{pmatrix} \in G(M)$, the function $\underline{H} \times \underline{H} \to \mathbb{R}_+$ defined by $(z_1, z_2) \longmapsto y_1 y_2$ (where $z_j = x_j + iy_j$, $x_j , y_j \in \mathbb{R}$) is invariant under the operation of $G(M)$. We add a point ∞^+ to $\underline{H}^2/G(M)$ and topologize $\underline{H}^2/G(M) \cup \{\infty^+\}$ by taking the sets

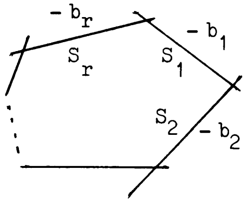
$$\{(z_1, z_2) \in \underline{H} \mid y_1 y_2 > C\}/G(M) \cup \{\infty^+\}$$

(C large) as neighbourhoods of the point ∞^+ . This point ∞^+ is in a natural way a singularity. We can introduce the local ring of holomorphic functions at ∞^+ and find that ∞^+ is a normal complex singularity. This singularity can then be resolved by a finite number of curves. It turns out that these curves are all non-singular rational and intersect one another in a cyclic configuration, i.e. we can number the curves $S_1,\ldots,S_r$ such that

$$S_1 \circ S_2 = S_2 \circ S_3 = \ldots = S_{r-1} \circ S_r = S_r \circ S_1 = 1 ,$$

all other intersection numbers $S_i \circ S_j$ $(i \neq j)$ equal 0 . (If $r = 1$, then S_1 has a double point.)

Let $S_i \circ S_i = -b_i$; then $b_1,\ldots,b_r$ are integers ≥ 2 , not all equal to 2 . In this way we have associated to the module $M \subset K$ a cycle $((b_1,\ldots,b_r))$ of integers. It turns out that this gives a <u>bijection</u>



$$\left\{\begin{array}{l}\text{all classes of modules}\\ \text{in all real quadratic}\\ \text{fields}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{primitive cycles}\\ ((b_1,\ldots,b_r)) ,\\ b_i \in \mathbb{Z} \text{ , } b_i \geq 2 \text{ , some } b_i \geq 3\end{array}\right\} ,$$

where "equivalence" of modules is defined by $M_1 \sim M_2$ if $M_1 , M_2 \subset K$ and $M_2 = \lambda M_1$ for some $\lambda \in K$, $\lambda \gg 0$, a "primitive" cycle is one where we only run through the smallest period once (e.g. $((3,5,2,3,5,2))$ is not primitive), and the notation $((b_1,\ldots,b_r))$ indicates that a cycle is only considered up to cyclic permutation (e.g. $((3,5,2)) = ((5,2,3)) = ((2,3,5))$).

By a result of Laufer, the analytic type of the singularity having the given resolution is completely determined by the cycle $((b_1,\ldots,b_r))$.

3. <u>Inoue's surfaces</u> (cf. Inoue's lecture at the Vancouver International Congress).

Let $G(M)$ act on $\underline{H} \times \mathbb{C}$, the action being given by the same formula as before. We now have a map

$$(\underline{H} \times \mathbb{C})/G(M) \rightarrow \mathbb{R} ,$$

$$(z_1,z_2) \bmod G(M) \mapsto y_1 y_2 ,$$

so we compactify by adding <u>two</u> points ∞^+ and ∞^- with the obvious neighbourhoods. The map $(z_1,z_2) \rightarrow (\sqrt{D}\, z_1 ,-\sqrt{D}\, z_2)$ interchanges $\underline{H} \times \underline{H}$ and $\underline{H} \times \underline{H}^-$ where H^-

is the lower half-plane of $\mathbb{C}$. Therefore ∞^+ has the resolution given above but ∞^- is resolved by the cycle $((c_1,\ldots,c_s))$ corresponding to the equivalence class of the module $\sqrt{D}\,M$, which is in general not equivalent to M because $\sqrt{D}$ is not totally positive.

Thus the resolution of $\{\infty^-\} \cup (\underline{H} \times \mathbb{C})/G(M) \cup \{\infty^+\}$ is a compact non-singular complex surface X containing two cycles of curves, and having a natural projection π onto $\{\infty^-\} \cup \mathbb{R} \cup \{\infty^+\} = I$ (closed interval) which over $\mathbb{R}$ is a fibre bundle with a certain 3-manifold as fibre. (This 3-manifold is a torus bundle over a circle.) The exceptional "fibres" $\pi^{-1}(\infty^+)$ and $\pi^{-1}(\infty^-)$ are cycles of rational curves $S_1, S_2, \ldots, S_r$ and $T_1, \ldots, T_s$ belonging to $((b_1,\ldots,b_r))$ and $((c_1,\ldots,c_s))$ respectively.

For this surface X one can show :

$$\begin{aligned} q &= 1 \\ b_1 &= 1 \\ b^+ &= 0 \\ b_2 = b^- &= r + s \\ \mathrm{Sign} &= -(r+s) \\ e &= r + s \ . \end{aligned}$$

The $r + s$ curves $S_1,\ldots,S_r$ and $T_1,\ldots,T_s$ occuring in the cycles of the resolutions of ∞^+ and ∞^- are linearly independent and give all of $H_2(X, \mathbb{R})$ (and they have negative definite intersection matrix, which explains why $b^+ = 0$). In fact, these $r + s$ curves are the only curves on X. In particular, there are no meromorphic functions on X except constants (a meromorphic function f would give infinitely many level curves $f^{-1}(t)$ on X).

By the signature theorem,

$$\frac{1}{3}\,(K^2 - 2e(X)) - \mathrm{Sign}(X) = 0 \ .$$

We calculate this expression separately for the neighbourhoods U^+ and U^- of $\pi^{-1}(\infty^+)$ and $\pi^{-1}(\infty^-)$ given by $y_1y_2 \geq 0$ or $y_1y_2 \leq 0$ respectively and check whether the sum gives 0 :

One can easily calculate that

$$-\,(\text{sum of all curves on } X) = -\,(S_1 + \dots + S_r) - (T_1 + \dots + T_s)$$

is a canonical divisor on X (in fact, $dz_1 \wedge dz_2$ is a differential form on $(\underline{H} \times \mathbb{C})/G(M)$ which extends meromorphically to X and gives the above canonical divisor). So for U^+ (i.e. "near" the resolution cycle of ∞^+) we have

$$K^2 = (S_1 + \dots + S_r)^2 = \Sigma\,(S_i \circ S_i) + 2 \sum_{i<j} S_i \circ S_j$$
$$= -\,(b_1 + \dots + b_r) + 2r$$

$$e(U^+) = \underset{\substack{\cup S_i \text{ is} \\ \text{connected}}}{1} - \underset{\substack{\cup S_i \text{ contains} \\ \text{one 1-cycle}}}{1} + \underset{\substack{\text{the curves} \\ S_1,\dots,S_r \\ \text{yield } r \\ \text{2-cycles}}}{r} = r$$

$\mathrm{Sign}(U^+) = -r$ (because the cycle has negative definite intersection matrix)

so

$$\frac{1}{3}\,(K^2 - 2e) - \mathrm{Sign} = \frac{1}{3}\,(-\Sigma\, b_i + 2r - 2r) + r = -\frac{1}{3} \sum_{i=1}^{r} (b_i - 3) \qquad \text{for } U^+ .$$

Similarly the other cycle $\sum_{j=1}^{s} T_j$ gives $-\frac{1}{3} \sum_{j=1}^{s} (c_j - 3)$. Therefore we deduce from the signature theorem that

$$\left(-\frac{1}{3} \sum_{1}^{r} (b_i - 3)\right) + \left(-\frac{1}{3} \sum_{1}^{s} (c_j - 3)\right) = 0 .$$

That is, we are naturally led to associate to M a numerical invariant

$$M \mapsto \delta(M) = -\frac{1}{3} \sum_{i=1}^{r} (b_i - 3) ,$$

where $((b_1,\dots,b_r))$ is the cycle associated to M, and the signature theorem then implies that this invariant changes sign when you replace M by $\sqrt{D}\,M$ (this can also be checked directly).

The invariant $\delta(M)$ is the same one as you get from the Atiyah-Patodi-Singer theory on spectral asymmetry.

4. Hilbert modular surfaces

Let $\mathfrak{o} \subset K$ be the ring of integers of K. The group $SL_2(\mathfrak{o})$ acts on $\underline{H} \times \underline{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right)$$

$(a, b, c, d \in \mathfrak{o}, \quad ad - bc = 1, \quad z_1, z_2 \in \underline{H})$.

The group $G = SL_2(\mathfrak{o})/\{\pm 1\}$ acts effectively. The quotient $(\underline{H} \times \underline{H})/SL_2(\mathfrak{o})$ is a (non-compact) complex surface with some quotient singularities coming from those points in $\underline{H} \times \underline{H}$ with non-trivial isotropy groups (the isotropy group of G will always be finite cyclic, and indeed always of order ≤ 6). We can also replace $SL_2(\mathfrak{o})$ by any congruence subgroup $\Gamma \subset G$; if we choose a which acts freely, then the quotient $\underline{H}^2/\Gamma$ will have no singularities. The surface $\underline{H}^2/\Gamma$ can be compactified by adding finitely many points which are all normal singularities ; the number of such points ("cusps") is the class number of K if $\Gamma = G = SL_2(\mathfrak{o})/\{\pm 1\}$. Each of the singularities can be resolved by a cycle $((b_1, \ldots, b_r))$ as above (but now possibly non-primitive, i.e. finite covers of the primitive cycles can occur). Resolving the quotient singularities and the cusps, you get a unique desingularisation

$$Y \rightarrow \overline{\underline{H}^2/\Gamma}$$

($\overline{\underline{H}^2/\Gamma}$ denotes the compactification of $\underline{H}^2/\Gamma$: $\overline{\underline{H}^2/\Gamma} = \underline{H}^2/\Gamma \cup \{\text{cusps}\}$).

The surface Y is non-singular algebraic. It is simply-connected (O. V. Švarčman).

The non-compact "manifold" $\underline{H}^2/\Gamma$ has a well-defined signature (the quotient singularities do not matter since $\underline{H}^2/\Gamma$ is still a rational homology manifold ; the effect of the non-compactness is to make the intersection form degenerate, so that there are zeros in its diagonalized version, but one can still define the signature as the number of positive entries minus the number of negative ones). If Γ operates freely, then

$$(*) \qquad \operatorname{Sign} \underline{H}^2/\Gamma = \sum_\alpha \delta(\gamma_\alpha) \quad ,$$

where γ_α denote the various cusps and $\delta(\gamma_\alpha)$ the invariant of the cusp defined in 3, i.e. $\delta(\gamma_\alpha) = -\frac{1}{3} \sum_{i=1}^{r} (b_i - 3)$ if $((b_1, \ldots, b_r))$ represents the resolution of the cusp. Indeed, for the closed surface Y, the expression

$\frac{1}{3}(K^2 - 2e) - \mathrm{Sign}$ is 0 by the signature theorem, and we can calculate this expression by calculating the contribution from neighbourhoods of the cusps and from the "interior" of Y. Each cusp γ_α gives $\delta(\gamma_\alpha)$, and the interior piece gives $-\mathrm{Sign}(\underline{H}^2/\Gamma)$, because the integral representing $\frac{1}{3}(K^2 - 2e)$ has an integrand which vanishes identically. The sum of these contributions adds up to 0 and gives the above formula $(*)$ for $\mathrm{Sign}\ \underline{H}^2/\Gamma$. This formula holds also if $\Gamma = G = SL_2(\sigma)/\{\pm 1\}$ and the discriminant D is not divisible by 3. In this case there are contributions from the quotient singularities, but they cancel out.

Example. $K = \mathbb{Q}(\sqrt{p})$, $p \equiv 3 (\mathrm{mod}\ 4)$, $p > 3$. Then

$$\mathrm{Sign}\ \underline{H}^2/SL_2(\sigma) = -2h(-p) \qquad (\ h(-p) = \text{class number of } \mathbb{Q}(\sqrt{-p})\ ).$$

Why is this ? The δ-invariants are related to the values of certain L-series at $s = 1$, and their sum is a class number.

The whole story of the cycles associated to modules is related to continued fractions (the process of writing down the cycle associated to M is related to the classical procedure for calculating the fundamental unit by continued fractions). If you analyze this connection and combine it with the above, you find interesting number-theoretical relations between class numbers and continued fractions, e.g. :

THEOREM.- <u>Suppose</u> $3 < p \equiv 3 \ (\mathrm{mod}\ 4)$ <u>prime</u>, $h(4p) =$ class number of $\mathbb{Q}(\sqrt{p}) = 1$, <u>and</u>

$$\sqrt{p} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_{2t} + \cfrac{1}{a_1 + \ddots}}}}$$

primitive period

(the period always starts at a_1 and has even length). <u>Then</u>

$$3h(-p) = \sum_{i=1}^{2t} (-1)^i a_i .$$

Example.

$$\sqrt{23} = 4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{1 + \ddots}}}}}$$

period

$$-1 + 3 - 1 + 8 = 9 = 3h(-23) \ .$$

5. Example (Modular surfaces for $\sqrt{5}$)

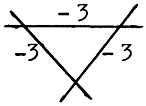
Take $K = \mathbb{Q}(\sqrt{5})$

$$\Gamma = \{A \in SL_2(\mathfrak{o}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2\}/\{\pm 1\}$$

(principal congruence subgroup associated to the ideal $(2) \subset \mathfrak{o}$).

Γ acts freely on $\underline{H}^2$. There are five cusps, so we have to compactify $\underline{H}^2/\Gamma$ by 5 points.

Each one has a resolution



(i.e. given by the non-primitive cycle $((3,3,3))$).

What is the Euler number of $\underline{H}^2/\Gamma$? This is the same as the Euler number of Γ in the sense of cohomology of groups (since $\underline{H}^2$ is contractible and Γ acts freely). We put again $G = SL_2(\mathfrak{o})/\{\pm 1\}$ and first calculate

$e(G)$ (Euler number in the sense of Wall = normalized volume of $\underline{H}^2/G$)

$= 2\zeta_K(-1)$ ($\zeta_K(s)$ = zeta-function of K)

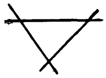
$= \frac{1}{15}$.

Since $G/\Gamma = SL_2(\mathbb{F}_4) \simeq \mathfrak{A}_5$ (alternating group in five objects) we have $|G:\Gamma| = 60$ and

$$e(\Gamma) = \frac{1}{15} \cdot 60 = 4 \ .$$

Now for the resolved compactification Y of $\underline{H}^2/\Gamma$ we have

$$Y = \underline{H}^2/\Gamma \cup 15 \text{ curves} \ ,$$

with the 15 curves lying in 5 configurations  of Euler number 3 , so

$$e(Y) = e(\underline{H}^2/\Gamma) + 15 = 4 + 15 = 19 \ .$$

The group $\mathfrak{A}_5$ operates on Y permuting these 5 "triangles" . Also

$$\mathrm{Sign}(\underline{H}^2/\Gamma) = \Sigma\, \delta(\gamma_\alpha) \ = \ 0$$

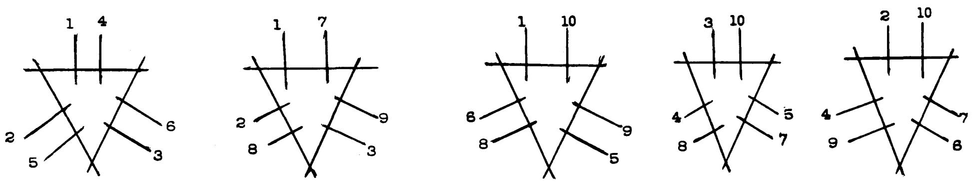
since $\Sigma(b_i - 3) = 0$ for each cusp (indeed, all $b_i = 3$) . Therefore

$$\mathrm{Sign}(Y) = -15$$

($Y = \underline{H}^2/\Gamma \cup 15$ curves with negative definite intersection matrix), so

$$\chi(Y) \ = \ \frac{e(Y) + \mathrm{Sign}(Y)}{4} \ = \ \frac{19 - 15}{4} \ = \ 1 \ .$$

Consider the <u>diagonal</u> $z_1 = z_2$ in $\underline{H} \times \underline{H}$. This gives rise to a curve in $\underline{H}^2/\Gamma$. This curve is isomorphic to $\underline{H}/\Gamma'$, where Γ' is the subgroup of Γ carrying the diagonal to itself and is isomorphic to the ordinary congruence subgroup $\Gamma(2) \subset SL_2(\mathbf{Z})$ divided by $\{\pm 1\}$. The index of $\Gamma(2)$ in $SL_2(\mathbf{Z})$ is 6 . Thus there are $\frac{60}{6} = 10$ "diagonals" in Y (i.e. the transforms of $z_1 = z_2$ in $\underline{H} \times \underline{H}$ under $SL_2(\mathfrak{o})$ fall into 10 equivalence classes modulo Γ , and the images of these are 10 irreducible curves in $\underline{H}^2/\Gamma$, each isomorphic to $\underline{H}/\Gamma(2)$). The curve $\underline{H}/\Gamma(2)$ is known to be compactified by 3 points to a curve of genus 0 . The 10 "diagonals" in Y are all exceptional curves (non-singular, rational, selfintersection number -1). How do they go through the cusps ? We number the diagonals from 1 to 10 ; then the picture of the cusps looks like this :



(each curve has 3 cusps and so must occur 3 times in the diagram).

We now blow down each of the ten diagonals to a point. Since they are exceptional curves and do not intersect one another, this transformation to a new non-singular surface is possible.

On it lie 15 curves (the images of the cusp resolutions), intersecting in a rather complex way. They are exceptional curves on the new surface. Since they intersect each other, the surface is rational. The new surface has Euler number

$$e = 19 - 10 = 9 .$$

This is the same as for a cubic surface. The surface is in fact a cubic surface, namely one introduced by Clebsch in 1871 and given by

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0$$

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

(x_i = homogeneous coordinates in P^4). Of the 27 lines on this cubic surface, the 15 lines we have drawn are those given by $x_i = 0$ (each hyperplane $x_i = 0$ in P^4 cuts the cubic surface in a degenerate cubic curve consisting of 3 lines). Under this identification, the functions $y_i = x_i^{-1}$ $(i = 0,\dots,4)$ correspond to modular forms of weight 2 for Γ (because for a cubic surface the hyperplane section is minus the canonical divisor). These modular forms in fact generate the graded algebra of modular forms for Γ of even weight. This algebra is isomorphic to

$$\mathbf{C}[y_0, y_1, y_2, y_3, y_4] / (\sigma_2, \sigma_4)$$

where σ_i is the i^{th}-elementary symmetric function of the y_j. The graded algebra of modular forms of even weight for $G = SL_2(\mathfrak{o})/\{\pm 1\}$ is the subalgebra of elements invariant under the alternating group $\mathfrak{A}_5$.

6. Curves, class numbers, conjectures

We turn now to something new. Suppose the field has prime discriminant :

$$K = \mathbb{Q}(\sqrt{p}) , \qquad p \equiv 1 \pmod 4 \ \text{ prime.}$$

On $\underline{H}^2/SL_2(\mathfrak{o})$ we shall study a certain curve T_N ($N \geq 1$ an integer). Consider all matrices

$$\begin{pmatrix} a\sqrt{p} & \lambda \\ -\lambda' & b\sqrt{p} \end{pmatrix} \qquad (a , b \in \mathbb{Z} , \ \lambda \in \mathfrak{o})$$

with determinant N. For each such matrix consider the curve

$$a\sqrt{p}\cdot z_1 z_2 + \lambda z_1 - \lambda' z_2 + b\sqrt{p} = 0$$

in $\underline{H}\times\underline{H}$. (This curve is just the graph of a certain fractional linear transformation $\underline{H}\to\underline{H}$.) These various curves are mapped to one another by $SL_2(\mathfrak{o})$. Let T_N be their image in $\underline{H}^2/SL_2(\mathfrak{o})$. Then T_N has only finitely many components (i.e. there are only finitely many $SL_2(\mathfrak{o})$-equivalence classes of matrices as above). If $(\frac{N}{p}) = -1$, then $T_N = \emptyset$ since $N = abp + \lambda\lambda'$ has no solutions.

We want to study the intersection behaviour. Consider the curve

$$T_1 = \text{diagonal} \qquad (\text{image of } z_1 = z_2).$$

If N is not a square, then T_N and T_1 meet transversally in finitely many points (for N a square, T_N contains T_1 as a component). If T_N and T_1 happen to meet in a quotient singularity of order 2 or 3 (no other case can arise since $T_1 \cong \underline{H}/SL_2(\mathbb{Z})$ only has points of these orders), then we count the intersection as $\frac{1}{2}$ or $\frac{1}{3}$ (this is compatible with the homological definition of intersection on the rational homology manifold $\underline{H}^2/SL_2(\mathfrak{o})$).

THEOREM. $$T_N \circ T_1 = \sum_{\substack{x\in\mathbb{Z} \\ 0< 4N - x^2 \equiv 0 \ (\mathrm{mod}\ p)}} H\left(\frac{4N-x^2}{p}\right) \quad \underline{\text{if}}\ N\ \underline{\text{is not a square}},$$

<u>where</u>

$H(M)$ = number of $SL_2(\mathbb{Z})$-equivalence classes of points in $\underline{H}$ which satisfy a quadratic equation over $\mathbb{Z}$ with discriminant $-M$, $\alpha z^2 + \beta z + \gamma = 0$, $\beta^2 - 4\alpha\gamma = -M$ (as usual, a point equivalent to i or $e^{\pi i/3}$ is counted $\frac{1}{2}$ or $\frac{1}{3}$, respectively).

$H(M)$ is essentially a class number ; thus the intersection number of T_N and T_1 on $\underline{H}^2/SL_2(\mathfrak{o})$ is given by a sum of class numbers.

What can one do with these numbers ?

In general $T_N \subset \underline{H}^2/SL_2(\mathfrak{o})$ is not compact. But in Y_o = desingularisation of $\underline{H}^2/SL_2(\mathfrak{o})$ in the cusps (quotient singularities not resolved), the curve T_N will meet the curves S_j of the resolution with some multiplicities. Since

$\det(S_j \circ S_k) \neq 0$ (the matrix $S_j \circ S_k$ is negative definite !), we can find

$$T_N^c = T_N + \text{linear combination of the } S_j$$

such that $T_N^c \circ S_j = 0$ for all j ; that is, we can modify T_N by a linear combination of the S_j so that it is homologous to a "compact cycle" (i.e. one in the image of $H_2(\underline{H}^2/SL_2(\mathfrak{o})) \to H_2(Y_o)$). Here we use complex coefficients for homology. Let $\underline{T}$ be the subspace of $H_2(Y_o)$ spanned by the cycles T_N^c . The volume $\mathrm{vol}(K)$ for $K \in \underline{T}$ equals $\frac{1}{2}\int_K \omega$ where

$$\omega = -\frac{1}{2\pi}\left(\frac{dx_1 \wedge dy_1}{y_1^2} + \frac{dx_2 \wedge dy_2}{y_2^2}\right) .$$

By $K \circ T_N = K \circ T_N^c$ we denote the intersection number in $\underline{H}^2/SL_2(\mathfrak{o})$.

Conjectures

(1.) $\dim \underline{T} = \left[\frac{p-5}{24}\right] + 1$.

(2.) For $K \in \underline{T}$ a homology class, the function

$$\varphi_K(z) = \tfrac{1}{2}\,\mathrm{vol}(K) + \sum_{N=1}^{\infty} (K \circ T_N)\, e^{2\pi i N z} \qquad (z \in \underline{H})$$

is a modular form for $\Gamma_o(p)$ of weight 2 and "Nebentypus", i.e.

$$\varphi_K\left(\frac{az+b}{cz+d}\right) = \left(\frac{a}{p}\right)(cz+d)^2\, \varphi_K(z)$$

for $z \in \underline{H}$, $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$, $c \equiv 0 \pmod p$.

(3.) $K \to \varphi_K$ is an isomorphism between $\underline{T}$ and the space of all modular forms for $\Gamma_o(p)$ of weight 2 and "Nebentypus" whose N^{th} Fourier coefficient $= 0$ whenever $\left(\frac{N}{p}\right) = -1$. (This would imply (1.) since the space of modular forms with these properties has dimension $\left[\frac{p-5}{24}\right] + 1$.)

Of these, (1.) is proved for some p (all $p < 200$) , with $\dim \le \left[\frac{p-5}{24}\right] + 1$ proved for all primes. Conjecture (2.) has been almost completely proved by Zagier (proved for $K = T_1^c$).

The curves T_N were introduced and their intersection behaviour studied for quite other reasons, namely to classify the Hilbert modular surfaces in the sense of the "rough classification scheme" of Kodaira (this has been completely done, for $K = \mathbb{Q}(\sqrt{p})$ with $p \equiv 1 \pmod 4$ prime by Hirzebruch - van de Ven, for arbitrary K by Hirzebruch-Zagier).

One needs the intersection behaviour to find configurations of curves of special forms which imply that the surface is of some particular type.

If we consider the Hilbert modular surfaces again for prime discriminant ($p \equiv 1 \bmod 4$) but divide also by the involution $(z_1, z_2) \longmapsto (z_2, z_1)$, then quite a few of the T_N become exceptional curves when regarded as curves in these modular surfaces for the symmetric Hilbert modular group. If two such exceptional T_N meet, then the surface is rational. For the symmetric modular group rationality happens for exactly 24 prime discriminants.

If a surface is of some particular type, this in turn implies something about the field of modular forms - e.g. if the surface is rational, the function field is a purely transcendental extension of $\mathbb{C}$ of degree 2 . Such facts would be very hard to establish by direct analytic means.

7. Final remarks

The preceding pages represent the complete text of my lecture at the Cartan Colloque. The notes were taken by Don Zagier during the lecture. For references to the literature on the Hilbert modular group and on algebraic surfaces see the bibliographies in

F. Hirzebruch, Hilbert modular surfaces, (L'Enseignement Mathématique, 19 (1973), 183-281).

F. Hirzebruch and D. Zagier, Classification of Hilbert modular surfaces (to appear; preprints available at the Bonn Mathematical Institute).

The study of the example in section 5 was begun by Rosselli, a student in Bonn, and continued in discussions with several mathematicians. I shall write a paper "The Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$ and the icosahedron" which will also give references to the many papers concerning $\mathbb{Q}(\sqrt{5})$ which exist already in the literature.

More information concerning the conjectures in section 6 can be found in my lecture at the Mannheim conference (Kurven auf den Hilbertschen Modulflächen und Klassenzahlrelationen, in Lecture Notes in Mathematics 412, Springer Verlag, 1974). A joint paper with Don Zagier is in preparation.

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