Astérisque

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Astérisque, tome 32-33 (1976), p. 113-139

<http://www.numdam.org/item?id=AST_1976__32-33__113_0>

Société Mathématique de France
Astérisque 32-33 (1976)

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by Raoul Bott

1. Introduction.

It is a great privilege to address this colloquium in honour of H. Cartan. Every mathematician of my generation, be he a topologist or geometer or for that matter an algebraist or analyst, has been profoundly influenced by H. Cartan's research exposition and general point of view. In my own case, my first introduction to topology in 1949, were the lectures of Steenrod on fiberbundles on the one hand, and the Cartan Harvard Lectures of the year before on which K. Reidemeister lectured at the I. A. S., to a small private seminar (I believe only E. Specker and I were the audience), on the other. Of course in Professor Reidemeister's rather rambling and philosophical presentation some of the famed French lucidity was lost and that may account for the fact that, in spite of my great admiration for the French school, I have never been quite able to imitate it in my own work.

Nevertheless, the ideas I learned then (largely thanks to much remedial tutoring by my friend Specker), and later, through the years, from the famed Cartan Seminar, are still the foundation of my knowledge; and I would therefore like, on this occasion, above all else to thank Henri Cartan in the name of my whole generation for the selfless way in which he has always worked to make mathematics, on all levels, accessible and intelligible to all of us. He has truly been our teacher.

For my topic here I have chosen to speak on a subject which has fascinated me for the past three years or so and which is especially appropriate here also, because it complements the work of Chern-Weil and H. Cartan on characteristic classes in a manner which I think would have pleased E. Cartan. I am referring to the work of Gelfand-Fuks on the cohomology of the Lie-algebra LM of vector fields on a C^∞ manifold M.

Recall that these "infinitesimal motions" X on M correspond to homogeneous linear first order C^∞ differential operators on M. In view of the fact that quite generally, the commutator of an operator of order n and one of order m is of order $n + m - 1$, it follows that under the law

$$[X,Y] = X \circ Y - Y \circ X$$

these vector fields form a Lie algebra over $\mathbb{R}$; that is, a vector space over $\mathbb{R}$ endowed with a skew symmetric bilinear multiplication $X, Y \to [X,Y]$, subject to the Jacobi identity

$$\Sigma [X,[Y,Z]] = 0$$

where the sum extends over cyclic permutations of the variables.

Now this Lie algebra LM is of course the basic tool in all aspects of modern differential geometry. However, traditionally the emphasis has been on the structure of LM as a $C^\infty(M)$ - module. Thus, for instance, the famous DeRham complex

$$\Omega(M) = \Sigma \, \Omega^q(M)$$

consists of the complex of $C^\infty(M)$ - alternating $C^\infty(M)$ - multilinear maps form

$$LM \times \cdots \times LM \quad \text{to} \quad C^\infty(M) \;,$$

and the exterior derivative

$$d : \Omega^q(M) \longrightarrow \Omega^{q+1}(M)$$

is miraculous and unique in so far as it preserves the $C^\infty(M)$ linearity.

Finally, I think most of us also attributed the finite dimensionality of the DeRham cohomology

$$H_{DR}(M) = \text{Ker } d/\text{Im } d$$

to this same $C^\infty(M)$ - linearity.

Now Gelfand and Fuks had the simple but inspired idea to replace this linearity condition by continuity in the C^∞ topology: Precisely then, they define a complex

$$A(M) = \oplus A^q(M) \tag{1.1}$$

where $A^q(M)$ consists of the multi - $\mathbb{R}$ - linear maps

$$\omega : \underbrace{LM \times \cdots \times LM}_{q} \longrightarrow \mathbb{R}$$

which are continuous in the C^∞ topology on LM, endow $A(M)$ with a differential operator d, given by

$$\begin{aligned} &d\omega(x_1, \cdots, x_{q+1}) \\ &= \sum_{i<j} (-1)^{i+j+1} \omega([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{q+1}), \end{aligned} \tag{1.2}$$

and ask for the cohomology of the resulting differential complex.

I will write $GF(M)$ for its cohomology

$$GF(M) = HA(M) = \text{Ker } d/\text{Im } d \qquad \text{in } (1.1),$$

but before discussing it any further, a word about (1.1) is in order. This formula is best first understood in the context of Lie groups. Given such an object, that is, a group in the category of C^∞ Manifolds, we may naturally single out the subcomplex

$$\Omega\mathfrak{g} \equiv \text{Inv}_G \cdot \Omega(G)$$

of forms on G invariant under left translation. As the DeRham d commutes with translations, $\Omega\mathfrak{g}$ comes equipped with a differential operator, so that $H(\Omega\mathfrak{g})$ is a well defined object, and we further have a natural map

$$H(\Omega\mathfrak{g}) \longrightarrow H(\Omega G) = H_{DR}(G) .$$

Now here $\mathfrak{g}$ really stands for the Lie algebra of Left invariant vector-fields on G, and the d of $\Omega\mathfrak{g}$ is then easily seen to be given by the formula (1.1) if the $X_i \in \mathfrak{g}$, $\omega \in \Omega^q(\mathfrak{g})$. Of course finally (1.1) defines cohomology for any abstract Lie algebra.

In short then, the Gelfand-Fuks complex deals with "the continuous subcomplex of the cochain complex of the abstract Lie algebra LM".

Now LM is clearly in some sense the Lie algebra of the group Diff M of Diffeomorphisms of M, hence in some sense $A(M)$ bears the same relation to Diff(M) as $\Omega(\mathfrak{g})$ bears to G. If one thinks along these lines the continuity assumption also turns out to be essential and natural. Indeed for a Lie group G, the complex $\Omega(\mathfrak{g})$ can easily be characterized by a universal property: To wit: the element $\omega \in \Omega^a(\mathfrak{g})$

assigns to every C^∞ map

$$f : M \longrightarrow G ,$$

the element $\omega(f) = f^*\omega$ in $\Omega^q(M)$, and this function clearly has the following two properties:

a.) <u>Functoriality.</u> For a commuting diagram of C^∞ maps as below,

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "h"] & G \\ M \arrow[ur, "f"'] & \end{tikzcd}$$

we have $\omega(f') = h^*\omega(f)$.

b.) <u>Invariance.</u> Given the diagram

$$M \xrightarrow{f} G \xrightarrow{\ell_g} G$$

where ℓ_g denotes left translation by g, then

$$\omega(\ell_g \circ f) = \omega(f) \quad .$$

Conversely given a function $f \mapsto \omega(f)$ subject to a.) and b.) there is a unique $\omega \in \Omega(\mathfrak{g})$ such that

$$\omega(f) = f^*\underline{\omega}$$

(Indeed just choose $1 : G \longrightarrow G$ to be the identity and set $\underline{\omega} = \omega(1)$.)

Now if we apply this procedure to Diff(M), where we declare a map

$$W \longrightarrow \mathrm{Diff}(M)$$

to be C^∞ if and only if the transpose map

$$W \times M \xrightarrow{f^t} M$$

given by

$$f^t(w, m) = f(w)(m)$$

is C^∞, then one sees without trouble that every such map,

$$f : W \longrightarrow \mathrm{Diff}(M)$$

induces a homomorphism

$$f^! : A(M) \longrightarrow \Omega(W)$$

which is natural and invariant, and as I said before, for this $f^!$ just to exist, the continuity assumption is already essential.

But all this is really a little beside the point - I remark on it as much to show off some familiarity with arrows as to justify the concept - once asked, the problem to compute $H^*\{A(M)\}$ is clearly interesting and in a series of papers Gelfand and Fuks proceeded to investigate it. Among the results they proved the following:

THEOREM (Gelfand-Fuks). The ring $GF^q(M)$ is of finite dimension for each q if M is compact, and in particular, for the circle S^1,

$$GF(S^1) = E(\omega) \otimes \mathbb{R}[y]$$

is the tensor product of an exterior algebra with a 3-dimensional generator ω and a polynomial ring $\mathbb{R}[y]$ with a 2-dimensional generator.

Furthermore, they gave the following explicit representatives for ω and μ: Let S^1 be represented as $\mathbb{R}^1/\mathbb{Z}$ so that vector-fields $f\frac{\partial}{\partial x}$ can be identified with functions f of period 1. With this understood

$$\mu(f,g) = \int_0^1 \det \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} dx$$

$$\omega(f,g,h) = \det \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}_0$$

where the 0 in the second formula denotes evaluation at 0. Of course, any other point would do just as well, and in fact one could also integrate this expression for ω. On the other hand μ can never be represented by a formula that does not involve the behavior of f and g on all of S^1.

2. The local complex.

I do not have time here to describe how Gelfand and Fuks arrived at their results. Rather let me explore only the first step in their computation, for as it turned out later, it furnishes the addendum to the theory of characteristic classes I alluded to earlier.

The first step in question is based on the remark that $\theta \in A(M)$ has a natural support in M, so that it makes sense to speak of the subcomplex $A_K(M)$ of elements whose support lies in K, and therefore in particular of the minimal subcomplexes $A_p(M)$ with support at a point $p \in M$. To study $A_p(M)$ choose a coordinate system $x_1, \cdots, x_n$ centered at p, so that every $X \in L(M)$ is locally given by

$$X = \sum_i a^i \frac{\partial}{\partial x^i}$$

where the a^i are C^∞ functions near p.

From the most elementary properties of distributions it then follows that all the 1-forms of $A_p(M)$ are generated by the finite partial derivatives of the δ - function applied to the components of X. Hence $A_p(M)$ is really independent of M, and therefore a purely local and universal complex which we will often simply denote by A_p:

$$A_p = A_p(M) \qquad \text{for any } M \ni p .$$

Gelfand and Fuks take the position that A_p is again the continuous analogue of $\Omega\mathfrak{g}$. Indeed if $\mathfrak{a}_n$ denotes the Lie algebra of <u>formal</u> vector-fields

$$X = \Sigma\, a_i \frac{\partial}{\partial x_i}$$

where the a_i are formal power series in the x's, and if we topologise $\mathfrak{a}_n$ in the usual power series topology, - x^α is small if $|\alpha|$ is large - then it is obvious that

$$A_p = \Omega_c(\mathfrak{a}_n) \;,$$

is the complex of continuous alternating forms on $\mathfrak{a}_n$.

Well, in any case, to get on with an explicit computation of $H^* A_p = H_c^*(\mathfrak{a}_n)$ we proceed as follows:

In terms of our local coordinates x^i at p, define 1-forms

$$\theta^i_\alpha \in A^1_p = \Omega^1_c(\mathfrak{a}_n) \,,\; i = 1, \cdots, n \,,$$
$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$$

by:

$$\theta^i_\alpha(X) = (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} a^i \Big|_p \,.$$

Here the α's are k-tuples of non negative integers and $|\alpha| = \Sigma\alpha_i$.

Our earlier statement then translates to the assertion that

(2.1) <u>The</u> θ^i_α <u>span</u> A^1_p <u>and hence generate all of</u> A_p.

Note that of course the θ's are symmetric in their lower indices so that $\theta^i_{12} \equiv \theta^i_{21}$, etc.

The sign has been chosen so that the lower indices correspond to Lie - derivatives of θ^i in the direction $\frac{\partial}{\partial x^{\alpha_1}}, \cdots, \frac{\partial}{\partial x^{\alpha_k}}$. That is, we have

$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\theta^i = \theta^i_j$$

$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\cdot\mathcal{L}\left(\frac{\partial}{\partial x^k}\right)\theta^i = \theta^i_{jk}\ , \quad \text{etc,}$$

where $\mathcal{L}$ is still given by ( ) and so commutes with d.

Now because the $\left(\frac{\partial}{\partial x^i}\right)$ commute with one another, the $\mathcal{L}\left(\frac{\partial}{\partial x^i}\right)$ also commute. Hence the map

$$\frac{\partial}{\partial x^{\alpha_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\alpha_n}} \longrightarrow \mathcal{L}\left(\frac{\partial}{\partial x^{\alpha}}\right)\cdots\mathcal{L}\left(\frac{\partial}{\partial x^{\alpha_n}}\right)$$

factors through the symmetric, or polynomial, ring $\mathbb{R}\left[\frac{\partial}{\partial x^i},\cdots,\frac{\partial}{\partial x^n}\right]$ generated by the $\frac{\partial}{\partial x^i}$ over $\mathbb{R}$, and it follows that under $\mathcal{L}$, the complex A^1_p is naturally an $\mathbb{R}\left[\frac{\partial}{\partial x^i}\right]$ module.

From this point of view (2.1) can therefore be stated in the form:

Under $\mathcal{L}$ the 1-forms A^1_p constitute a free module with n generators $\theta^1,\cdots,\theta^n$ over the polynomial ring $\mathbb{R}\left[\frac{\partial}{\partial x^i}\right]$:

$$A^1_p \simeq \mathbb{R}\left[\frac{\partial}{\partial x_i},\cdots,\frac{\partial}{\partial x_n}\right]\ (\theta^1,\cdots,\theta^2)$$

It follows now that structure equations of A_p are completely determined once they are written down for θ^i, and hence by the following, easily verified,

PROPOSITION 2.1.

$$d\theta^i + \theta^i_j \wedge \theta^j = 0\ ^{\dagger}, \tag{2.2}$$

† We use the usual convention that repeated indices are summed.

To see how the formula (2.2) determines all other structure equations, one need only apply $\mathcal{L}\left(\frac{\partial}{\partial x^k}\right)$ etc., to both sides of (2.2) and recall that $\mathcal{L}$ acts by derivations. Explicitly one obtains:

$$d\theta^i_k + \theta^i_{jk} \wedge \theta^j + \theta^i_j \wedge \theta^j_k = 0 \tag{2.3}$$

$$d\theta^i_{k\ell} + \theta^i_{jk\ell} \wedge \theta^j + \theta^i_{jk} \wedge \theta^j_\ell + \theta^i_{j\ell} \wedge \theta^j_k + \theta^i_j \wedge \theta^j_{k\ell} = 0 \tag{2.4}$$

in the first two cases.

Clearly now (2.3) bears a considerable similarity to the structure equations of $GL(n, \mathbb{R})$ and to bring this out further let us define the 2-forms $R^i_j \in A^2_p$ by

$$R^i_j \equiv d\theta^i_j + \theta^i_k \wedge \theta^k_j \ . \tag{2.5}$$

Then (2.3) takes the form

$$R^i_j \equiv - \theta^i_{jk} \wedge \theta^k \tag{2.6}$$

while the structure equations of $GL(n, \mathbb{R})$ of course were $R^i_j \equiv 0$. In fact these forms behave just as the curvature tensor of a "torsion free connection" does, and as we will see later the R^i_j do play a role of the <u>universal</u> or <u>formal</u> curvature forms of a manifold. Here let us just as an exercise in the calculus of $A_p(M)$ derive the classical curvature identities:

PROPOSITION 2.2. <u>The forms $R^i_j \in A^2_p$ have the following properties:</u>

$$R^i_j \wedge \theta^j = 0 \ , \quad dR^i_j = R^i_k \wedge \theta^k_j - \theta^i_k \wedge R^k_j \tag{2.7}$$

<u>which correspond to the two Bianchi Identities in Torsion free Geometry.</u>

Proof: By (2.6)

$$R^i_j \wedge \theta^j = -\theta^i_{jk} \wedge \theta^k \wedge \theta^j \equiv 0$$

because θ^i_{jk} is symmetric in its lower indices.

To obtain the second identity consider the equation (2.4) and multiply it with θ^ℓ summing over the repeated index ℓ. On obtains

$$d\theta^i_{k\ell} \wedge \theta^\ell + \theta^i_{jk} \wedge \theta^j_\ell \wedge \theta^\ell + \theta^i_{j\ell} \wedge \theta^j_k \wedge \theta^\ell + \theta^i_j \wedge \theta^j_{k\ell} \wedge \theta^\ell = 0 \tag{2.8}$$

where the second form of (2.4) has cancelled out by the symmetry of $\theta_{jk\ell}$ in j, k, and ℓ. But (2.8) simply restates (2.7) in the equivalent form

$$-dR^i_k + \theta^i_j \wedge R^i_k - R^i_j \wedge \theta^j_k = 0 .$$

On the cohomological side (2.6) and (2.7) imply that the subalgebra

$$\underline{W} = \left\{\theta^i_j, R^i_j\right\}$$

generated by the indicated forms in A_p is, first of all, finite dimensional and secondly closed under d. Indeed by (2.6) any monomial in the R's of length $> n$ vanishes, while (2.7) together with (2.5) expresses d of the θ's and R's in terms of the θ's and R's. Actually this subcomplex already carries all the cohomology of A_p.

THEOREM: (Gelfand-Fuks) The inclusion

$$\underline{W} \subset A_p(U)$$

induces an isomorphism in cohomology.

The proof of this theorem is nontrivial, and involves some nice elementary invariance theory, to eliminate the forms θ^i_α with $|\alpha| > 2$; but here I have time only to describe the final outcome.

For this purpose let us define the i^{th} Chern polynomial as a function on $GL(n, \mathbb{R})$ by the formula:

$$\Sigma\, t^i c_i(A) = \det(1 + tA) \ . \tag{2.9}$$

It is then clear that each c_i is a polynomial in the entries A^i_j of A, for example

$$\begin{cases} c_1(A) = A^i_i \\ c_2(A) = (-A^i_j A^i_j + (A^i_i)^2)/2 \\ c_n(A) = \det A \end{cases} \tag{2.10}$$

Now as the R^i_j are two-forms and so commute we can evaluate these polynomials c_i on the R^i_j's to obtain forms

$$c_i(R) \in A^{2i}_p \ . \tag{2.11}$$

We call these the Chern classes in A_p and in terms of them the Pontrjagin forms $p_i(R) \in A^{4i}_p$ are defined simply by

$$p_i(R) = c_{2i}(R) \ . \tag{2.12}$$

Of course in the literature one often encounters these forms with factors of $(-1)^i \left(\frac{\sqrt{-1}}{2\pi}\right)^i$, but this is not to the point here, where they are, for the moment, only elements of $\underline{W} \subset A^{4i}_p$. In any case in terms of these $c_i(R)$ one can reduce $\underline{W}$ further to a complex WU_n defined as follows.

Consider the polynomial ring $\mathbb{R}[c_1, \cdots, c_n]$ with (weight of c_i) = $2i$, and let $\underline{\mathbb{R}}[c_1, \cdots, c_n]$ be the quotient of $\mathbb{R}[c_1, \cdots, c_n]$ by the ideal of elements with weight $> 2n$. Here of course the weight w of a monomial $c_1^{\alpha_1} \cdots c_n^{\alpha_n}$ in $\alpha_1 \cdot 1 + \alpha_2 \cdot 2 + \cdots + \alpha_n \cdot n$.

With this understood one sets

$$WU_n = \underline{\mathbb{R}}[c_1, \cdots, c_n] \otimes E(h_1, \cdots, h_n)$$

where $E(h_1, \cdots, h_n)$ denotes an exterior algebra with generators h_i in dimensions $2i - 1$. Finally a differential operator d is introduced in WU_n by setting

$$\begin{aligned} d(c_i \otimes 1) &= 0 \\ d(h_i \otimes 1) &= 1 \otimes c_i \ . \end{aligned} \tag{2.13}$$

The cohomology of A_p is now made rather explicit by:

PROPOSITION 2.3. <u>The map</u> $1 \otimes c_i \to c_i(R)$ <u>has an extension to a map of complexes</u>

$$WU_n \longrightarrow \underline{W} \subset A_p$$

<u>and any such extension induces an isomorphism</u>

$$H^*(WU_n) \xrightarrow{\simeq} H^*(A_p) \simeq H^*(\mathfrak{a}_n) \tag{2.14}$$

As an example, consider the case of $n = 1$. Then $WU_1 \subset \underline{W}$ can be taken to be

$$c_1 = R_1^1 \ , \ h_1 = \theta_1^1 \ ,$$

for we clearly have $dh_1 = c_1$. In this case $H(WU_1)$ is generated by 1, and

$c_1 h_1$ in dim 3. Thus:

$$H^*A_p(\mathbb{R}^1) = \begin{cases} \mathbb{R} \text{ in dim } 0 \\ \mathbb{R} \text{ in dim } 3 \end{cases} . \tag{2.15}$$

The computations of course become more complicated in higher dimensional cases but were carried out by Vey [11] who produced an explicit base for $H^*A_p(\mathbb{R}^n)$ in terms of the monomials in the h's and c's of WU_n.

So much then for our brief excursion into the A_p. We have learned first of all, that this "subcomplex of minimal support" has structure equations which are easy to write down, and which are very analogous to the equations one meets in differential geometry. Secondly we have seen that its cohomology is finite dimensional and computable.

Now in the computation of $H^* A(M)$, the cohomology of $A_p(M)$ plays a fundamental role, which I will have not time to explore here. Let me just give those of you familiar with some topology the barest outlines of the final outcome.

Already Gelfand-Fuks observed that every manifold M carries a bundle EM, which is associated to its principal frame-bundle and whose fibers E_pM, have cohomolog isomorphic to that of $A_p(M)$:

$$H^*(E_pM) \simeq H^* A_p(M).$$

Thus E is in a sense a geometric realization of the family of complexes $A_p(M)$, $p \in M$, and in terms of E, the final result on $H^* AM$ is given by the following theorem which was conjectured independently two years ago by Fuks and myself and for which two independent proofs have recently been given by Haefliger and Segal.

THEOREM (Haefliger, Segal). Let ΓE denote the space of continuous sections of E. Then

$$H^*(AM) \simeq H^*(\Gamma E) \; . \tag{2.16}$$

This beautiful result is, of course, nevertheless a negative one, as it reduces the mysterious $H^*(AM)$ to the complicated but still rather well understood $H^*(\Gamma E)$.

Observe that (2.16) implies, in particular, that

$$H^*(A\mathbb{R}^n) = H^*(\Gamma E) = H^*(E_p) = H^*(A_p\mathbb{R}^n)$$

the second equality being true because E is contractible to E_p over p. Finally let us check (2.16) for the circle, where Gelfand and Fuks made their computations. In this case E_p is the three sphere S^3, which conforms with (2.15). Further as the tangent bundle of S^1 is trivial, we have

$$E = S^3 \times S^1$$

whence $\Gamma E = \mathrm{Map}\,(S^1, S^3)$, the space of maps from S^1 to S^3. Now this function space has well known homology, to wit:

$$H^*\,\mathrm{Map}(S^1, S^3) = \mathbb{R}[y] \otimes E(\omega) \tag{2.17}$$

is a polynomial ring with generator μ in dimension 2, tensored with an exterior algebra with generator ω in dimension 3.

3. On the Relations with Characteristic classes.

I have already remarked on the fact that the basic structure equations of $A_p(M)$ bear a great similarity to the structure equations of torsion free geometry, and I would now like to explain this déjà vu phenomenon.

For this purpose let us think of M as the homogeneous space

$$M = \mathrm{Diff}(M)/\mathrm{Diff}(M;p)$$

where $\mathrm{Diff}(M;p)$ denotes the subgroup keeping p fixed. Now the group $\mathrm{Diff}(M;p)$ has a natural filtration by the subgroups

$$\mathrm{Diff}(M;p) \supset \mathrm{Diff}(M;p^2) \supset \mathrm{Diff}(M;p^3) \supset \cdots$$

keeping p fixed to higher and higher order. Thus

$$\mathrm{Diff}(M;p^k)$$

here denotes those diffeomorphisms whose Taylor expansion at p differ from the identity by terms of order k or higher. Dividing successively by these subgroups and setting

$$J_k(M) = \mathrm{Diff}(M)/\mathrm{Diff}(M;p^{k+1})$$

one obtains a tower of finite dimensional manifolds over M

$$M \longleftarrow J_1(M) \longleftarrow J_2(M) \longleftarrow . \qquad (3.1)$$

on all of which $\mathrm{Diff}(M)$ acts naturally, and it therefore makes sense to speak of the $\mathrm{Diff}(M)$ invariant forms on $J_k(M)$.

The manifolds $J_k(M)$ are of course the higher frame bundles or jet bundles over M, and well known to all geometers, and I claim on these spaces the complex $A_p(M)$ can be naturally identified with the complex of $\mathrm{Diff}(M)$ invariant forms.

Precisely, one has a natural isomorphism :

$$A_p(M) \simeq \lim_{n\to\infty} \mathrm{Inv}_{\mathrm{Diff}(M)} \quad \Omega J_n(M) \,. \tag{3.2}$$

From this point of view, our forms θ^i, θ^i_j, θ^i_{jk}, etc., now appear successively in $J_1(M)$, $J_2(M)$, etc., and although the $d\theta^i$ are of course invariant already in $J_1(M)$, they are not decomposable there. Thus the formula

$$d\theta^i + \theta^i_j \wedge \theta^j = 0 \tag{3.3}$$

only becomes valid in $J_2(M)$, etc.

In short then, the complex $A_p(M)$ provides one with a complete description of the "Tautological" forms on the $J_k(M)$. These of course were known to the geometers all along since E. Cartan I expect, and now in retrospect I recall having learned about them from Kobayashi years ago at the Tucson seminar on Diff. Geometry [8]. However he never pursued the cohomology of this complex, and its significance in the framework of characteristic classes.

Before I explain that significance note that (3.2) corresponds to the well known formula for the invariant forms on a homogeneous space G/H of a Lie group

$$\mathrm{Inv}_G\, \Omega(G/H) \simeq \mathrm{Inv}_H\, \Omega(\mathfrak{g}/\mathfrak{h}) \,, \tag{3.4}$$

in our infinite dimensional case. Indeed in this situation, the Lie algebra of $\mathrm{Diff}(M;p^k)$

clearly corresponds to the vector fields vanishing to order k at p, so that finally as $k \to \infty$,

$$\mathrm{Inv}_G\, \Omega J_\infty M = \Omega \mathfrak{a}_n \quad . \tag{3.5}$$

In any case the isomorphism (3.2) is natural and therefore immediately furnishes us with some sort of characteristic ring in $\lim_k H^*\Omega(J_k M)$ which I simply denote by $H^*(J_\infty M)$.

Now observe that in our tower $J_1(M)$, $J_2(M)$, etc., are all of the same homotopy type. Indeed the fibers of all these projections are all contractible groups. Hence our natural map

$$\begin{array}{ccc} H^*(\mathfrak{a}_n) & \longrightarrow & H^*(J_\infty M) \\ & & \wr\wr \\ & & H^*(J_1 M) \end{array} \tag{3.6}$$

is not quite as far removed from where we want it - that is in $H^*(M)$ - as might at first appear.

In fact at this stage only the following small modification is needed to come up with genuine characteristic classes.

Let us choose coordinates $x^1, x^2, \cdots, x^n$ centered at p, and in terms of these define the groups

$$\mathrm{Diff}_0(M; p^k)$$

as the subgroups of those $f \in \mathrm{Diff}(M)$ whose jet at p takes the form

$$j_p(f)\, x^i = a^i_j x^j + \text{ terms of order} \geq (k+1)$$

with $\| a^i_j \|$ in the group $O(n)$.

Then for $k \geq 2$, we have the exact sequence

$$0 \longrightarrow \mathrm{Diff}(M; p^k) \longrightarrow \mathrm{Diff}_0(M; p^k) \longrightarrow O(n) \longrightarrow 0$$

which gives rise to a corresponding fibering

$$J_k(M) \xrightarrow{\pi} J_k(M)/O(n)$$

where the right hand term is $\mathrm{Diff}(M)/\mathrm{Diff}_0(M; p^k)$.

This construction now induces a natural arrow

$$H^*_c(\mathfrak{a}_n; O_n) \longrightarrow H^*\{J_\infty M/O(n)\} \tag{3.7}$$

where the left hand side denotes the usual relative Lie algebra cohomology with O_n the subalgebra of formal orthogonal vector fields, and the R. H. S. is short for

$$\lim_{k \to \infty} H^*\Omega\{J_k M/O(n)\} .$$

On the other hand $J_k(M)/O(n)$ has the homotopy type of M for every k, so that (3.7) real is really an arrow

$$H^*_c(\mathfrak{a}_n; O_n) \longrightarrow H^*(M) . \tag{3.8}$$

Thus from this point of view the natural characteristic classes of an n - manifold, are given by $H^*_c(\mathfrak{a}_n; O_n)$.

The computations of Gelfand-Fuks can now be modified without too much trouble and one then finds the following result (see [3]).

<u>Proposition.</u> $H^*(\mathfrak{a}_n, O_n) = H^*(WO_n)$ <u>where</u> WO_n <u>is the complex</u>

$$E(h_1, h_3, \cdots) \otimes \mathbb{R}[c_1, \cdots, c_n]$$

with

$$d(1 \otimes h_1) = 1 \otimes c_1$$

$$d(1 \otimes h_3) = 1 \otimes c_3 \text{, etc.}$$

In particular in $\dim \leq n$, $H^*(\mathfrak{a}_n, O_n)$ is isomorphic to the ring generated by the c_{2i}, $4i \leq n$, and, up to a constant factor, these of course correspond to the well known Pontrjagin classes of M

$$c_{2i}(M) = p_i(M) \quad .$$

On the other hand, just as in WU_n there are other classes in $H(\mathfrak{a}_n, O_n)$ of much higher dimension, which therefore seem spurious in the present context.

Nevertheless, as I will explain in a moment, these phantom classes come into their own once we extend our point of view from manifolds to Foliations.

But first a final remark on how and why the formulas (3.3) were reminiscent of curvature formulas; which will also explain why the c_{2i} represent the usual Pontrjagin classes - say of a Riemann structure on M. The point is, of course, that a Riemann structure on M corresponds to a section s of $J_1(M)/O_n$ over M, and the corresponding Levi-Civita connection then defines an O_n - equivariant section

$$J_1(M) \xrightarrow{\nabla} J_2(M) \quad ,$$

under which the R^i_j pull back to the curvature of the Riemann structure which the

$c_{2i}(R^i_j)$, being $O(n)$ basic, correspond to the Pontrjagin classes on M. Once this relation is made, one can link up the computation of $H^*(\mathfrak{a}_n; O_n)$ with the usual machinery of connections and curvature, as well as the Weil algebra of $g\ell(n)$. In fact, the $\underline{W}$ of the previous section is clearly a truncated form of that Weil algebra. All in all the last proposition therefore is a refined version of the classical results.

4. The Characteristic classes of Foliations.

Let me briefly expalin now how the higher classes of WO_n serve as potential characteristic classes of foliations. Roughly speaking the idea is that one may think of a manifold M, as a special case of a foliation - i.e., we think of M as foliated by points.

Thus extended the constructions of the last section are seen to define a natural arrow

$$H^*(\mathfrak{a}_n, O(n)) \xrightarrow{\varphi(\mathfrak{F})} H^*\{M(\mathfrak{F})\} \tag{4.1}$$

to the cohomology of the manifold on which the foliation $\mathfrak{F}$ of codimension n is defined. As now $\dim(M)$ is unrestricted, the total cohomology of the L.H.S. comes into play.

To see how such an arrow comes about, recall that a foliation $\mathfrak{F}$ on M of codimension n can be defined as a subbundle $E_{\mathfrak{F}}$ of the tangent bundle of M which is locally the kernel of submersions of M to $\mathbb{R}^n$:

Thus near each $p \in M$,

$$E \simeq \ker df \tag{4.2}$$

where $f : V \longrightarrow \mathbb{R}^n$, $p \in V$, is a smooth map whose differential is onto at each point. With this understood define $J_k(\mathfrak{F})$ to be the space of k-jets of submersions f, from M to $\mathbb{R}^n$, with target $0 \in \mathbb{R}^n$, and which <u>are compatible with</u> $\mathfrak{F}$ in the sense that $E_{\mathfrak{F}}$ is $\operatorname{Ker} df$.

In the case of the trivial foliation, $(E = 0)$, $J_k(\mathfrak{F})$ is then easily seen to be a local description of our old $J_k(M)$ and in this framework it is not difficult to generalize (3.6) to (4.1). Furthermore under $\varphi(\mathfrak{F})$ the Pontrjagin classes of $H^*(\mathfrak{a}_n, O_n)$ go over into the Pontrjagin classes of the normal bundle TM/E of $\mathfrak{F}$. Thus in particular the truncation of $\underline{W}$ therefore implies the vanishing condition for the characteristic classes of such a normal bundle which I noted some years ago. (At the same time (4.1) also brings us the secondary classes, occasioned by this vanishing phenomenon.)

Thus for $n = 1$, the class $h_1 c_1 \in H^3(\mathfrak{a}_n ; O_n)$ corresponds to the Godbillon, Vey class, which of course started this whole development. The general setting (4.1) for its generalization was noted later by many of us: Bernstein,Rosenfeld in the U.S.S.R., and by Malgrange, and Bott-Haefliger in the West. Of course there are also other points of view which explain these phenomena. In particular the work of Chern-Simons, Cheeger deals with secondary classes quite generally, and Kamber and Tondeur have their own approach to all manner of foliated bundles and their characteristic classes.

Unfortunately however, the most difficult question concerning these classes remains unsolved. We do not know whether all the classes of $H^*(\mathfrak{a}_n ; O_n)$ can be distinguished by actual foliations.

Now the master par excellence for constructing foliations is Thurston, but even he has only been able to detect a few of them. On the other hand of course he has detected these with a vengeance. Indeed, he can forge a foliation $\mathfrak{F}$ on which the class $h_1 c_1^n \in H^{2n+1}(\mathfrak{a}_n, O_n)$ takes on any preassigned real value!

But this really leads us to another topic, beyond the scope of these remarks; all I wanted to do here, apart from saluting Henri Cartan, is to show how a natural question, such as Gelfand-Fuks asked themselves, has borne fruit in many interesting directions and has taught us to rethink old phenomena in new ways.

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