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1. Introduction

Let $M = M^n$, $n \geq 1$, be a compact differentiable manifold without boundary and $\mathfrak{X} = \mathfrak{X}(M)$ be the Banach space of all C^r-flows (i.e., vector fields or ordinary differential equations) on M, endowed with the C^r-topology, $r \geq 2$.

This paper is about the bifurcation theory of flows and in order to clarify its meaning and scope we begin with some general considerations.

Bifurcation theory is the study of maps of a finite dimensional manifold Λ (parameter space) into $\mathfrak{X}$. Of very special interest is the study of the intersection of the image of Λ with the set, $\Sigma \subset \mathfrak{X}$, of all structurally stable flows on M. The first to adopt this point of view was J. Sotomayor [19] who considered the case where $n = 2$ and Λ is an interval. He gives a fairly good description of how, generically, an arc in $\mathfrak{X}$ is situated with respect to Σ. Although many of the concepts of this work can be extended to dimension $n > 2$ [21], it is clear that there one has to settle for much less. For one thing Σ is no longer dense in $\mathfrak{X}$ and won't be so simply described. It is then natural to substitute for Σ the best known open subset of Σ, namely the subset, Δ, of all Morse-Smale flows on M. In [11] this point of view was adopted and a delicate analysis is

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given of how, generically, an arc starting at a point $X \in \Delta$ meets the boundary $\partial\Delta$ for the first time; again the situation is far from simple.

The main goal of the present paper is (Theorem B) to show that given any two flows $X, Y \in \Delta$ there is a <u>simple arc</u> in $\mathfrak{X}$ connecting them. Simple means that the arc contains at most finitely many points outside of Δ and, roughly speaking, for these the singularities and corresponding intersections of stable and unstable manifolds deviate in the least possible way from the structurally stable situation. These exceptional (bifurcation) points will belong to a codimension one sub-manifold $\Sigma_1 \subset \mathfrak{X}$, a natural extension of a set introduced in [19, 20]. The simple arc we get is stable in the sense that any nearby arc exhibits the same type of behavior.

Our theorem then means that one can always go from X to Y with a minimum of topological degeneracy, and this in a stable way. Of course the broken line segment XOY, where O is the zero flow, has only the point O outside Δ but this arc is not stable and O is the epitome of topological degeneracy.

On a somewhat philosophical vein our theorem fits well within Thom's framework of the theory of morphogenesis [24], whose fundamental problem is to study how a "form" is transformed into another.

Simple arcs give a nice, differentiable way of describing some operations with flows which have been used in the literature. For instance, the arguments of [14] give a constructive proof of the existence of a simple arc connecting any two gradient like flows on S^2, the exceptional points exhibiting no saddle connections. The recent work of G. Fleitas [4] implies that our theorem for gradient like flows on M^3 is essentially equivalent to an old theorem by Singer [15] about "moves" on Heegard diagrams. Also, as G. Fleitas pointed out to us, the proof of the h-cobordism theorem in [9] contains as a special case the construction of a simple arc between any two gradient flows on S^n, $n \geq 6$.

We now say a few words about the proof of the theorem. A naive argument to disprove it would run as follows. Let M be the two torus, and $X, Y \in \Delta$, be non vanishing flows such that the rotation number of X is 1 and that of Y is 2. Then along any arc from X to Y the rotation number varies continuously and it is easily verified that every flow with irrational rotation number corresponds to a bifurcation point. This argument is correct as long as the arc stays on a region of $\mathfrak{X}$ where the rotation number is defined; i.e. on a region of non-vanishing flows. What it actually shows is that there is no such arc theorem for diffeomorphisms of S^1. Thus one sees that the bifurcation theory of diffeomorphisms is much more rigid than that of flows. To prove our theorem we start getting rid of the closed orbits through the successive introduction of a saddle node on each closed orbit. One then enters the realm of gradient-like flows, passes to a flow which is locally a gradient, and then to a gradient flow using the methods of [17]. The proof ends by showing, via transversality, that most arcs within the gradient flows are simple. Incidentally, the above mentioned introduction of saddle nodes gives an immediate way to get the Morse inequalitites for Morse-Smale flows from the classical Morse inequalities [16].

2. Definitions and statements of results

Let $x \in M$ and f be a local C^2 diffeomorphism of a neighborhood U of x into M such that $f(x) = x$. One says that x is a <u>hyperbolic</u> fixed point of f if each eigenvalue of the derivative $T_x f$ has absolute value different from one; x is a <u>quasi-hyperbolic</u> fixed point of f if (1) 1 is an eigenvalue of $T_x f$ with multiplicity one, and (2) if $v \neq 0$ is an eigenvector of $T_x f$ corresponding to the eigenvalue 1, then the second derivative of f at x on v, $T_x^2 f(v,v)$, is not in the sum of the eigenspaces of $T_x f$ corresponding to the eigenvalues of $T_x f$ different from 1. Here $T_x^2 f(v,v)$ is taken relative to some coordinate system near x. The non-degeneracy condition (2) does not depend on the coordinate system chosen (see [21]).

Let $X \in \mathfrak{X}^r(M)$, and let ϕ^t be the 1-parameter group corresponding to X (i.e., $\left.\frac{d\phi^t(x)}{dt}\right|_{t=0} = X(x)$). Thus ϕ^t is a diffeomorphism of M for each real t and ϕ^0 is the identity. A critical point x of X is called <u>hyperbolic</u> (<u>quasi-hyperbolic</u>) if it is a hyperbolic (quasi-hyperbolic) fixed point of ϕ^t for $t \neq 0$. A quasi-hyperbolic critical point is also called a <u>saddle node</u>. Let γ be a closed orbit (periodic solution) of X and let H be a small piece of hypersurface through $x \in \gamma$ transverse to γ. There is an induced diffeomorphism θ (called the Poincaré transformation) from a neighborhood of x in H into H defined by $\theta(z) = \phi^{t(z)}(z)$ where $t(z) = \inf\{t: \phi^t(z) \in H, t > 0\}$. The orbit is called <u>hyperbolic</u> (<u>quasi-hyperbolic</u>) if x is a hyperbolic (quasi-hyperbolic) fixed point of θ. These qualities for γ do not depend on the choice of x in γ or H. The orbit of a point $x \in M$ will be denoted $o(x)$.

It is known [5], [6] that any hyperbolic (quasi-hyperbolic) critical point p has a smooth stable manifold $W^s(p) = \{y \in M: \phi^t(y) \to p$ as $t \to \infty\}$ and a smooth unstable manifold $W^u(p) = \{y \in M: \phi^t(y) \to p$ as $t \to -\infty\}$. In the quasi-hyperbolic case, $W^s(p)$ has a smooth boundary $W^{ss}(p)$ consisting of those points y for which $\phi^t(y)$ approaches p exponentially as $t \to \infty$, and $W^u(p)$ has a smooth boundary $W^{uu}(p)$ consisting of the points y for which $\phi^t(y)$ approaches p exponentially as $t \to -\infty$. Similarly, a hyperbolic (quasi-hyperbolic) closed orbit has stable and unstable manifolds obtained by iteration through ϕ^t of the corresponding manifolds for the fixed point of the Poincaré transformation.

A point $x \in M$ is an ω-limit point of X if there are $y \in M$ and a sequence $t_i \to \infty$ such that $\phi^{t_i}(y) \to x$. The point x is an α-limit point of X if it is an ω-limit point of $-X$. The limit set of X, denoted $L(X)$, is the closure of the union of the set of α-limit points and the set of ω-limit points of X. An element $X \in \mathfrak{X}^r(M)$ is called a <u>Morse-Smale</u> flow if

(1) $L(X)$ consists of a finite number of hyperbolic critical points and a finite number of hyperbolic closed orbits.

(2) the stable and unstable manifolds of the orbits in $L(X)$ meet transversely.

These flows have been studied in [16] and [12] where more references may be found. A Morse-Smale flow without closed orbits (i.e. whose limit set consists only of hyperbolic critical points) is called <u>gradient-like</u>.

For a positive integer $j > 0$, let $\mathbb{R}^j$ be the Euclidean space of dimension j. Take coordinates $(x_1,\ldots,x_n, y_1,\ldots, y_s)$ for $\mathbb{R}^n \times \mathbb{R}^{s-1} \times \mathbb{R}^1$, and let $\pi\colon \mathbb{R}^n \times \mathbb{R}^{s-1} \times \mathbb{R}^1 \to \mathbb{R}^1$ be the natural projection onto the last factor; i.e., $\pi(x_1,\ldots, x_n, y_1,\ldots, y_s) = y_s$. Let H and P be submanifolds of a manifold M with $\dim N = n$, $\dim P = p$, $\dim M = m = n + s$, and suppose that $y \in N \cap P$. Let $p_1 = \dim T_yN \cap T_yP$.

We say y is a quasi-transversal intersection of N and P if

(1) $\dim (T_yN + T_yP) = m - 1$ and

(2) if $n + p \geq m$, then there are coordinates $\phi\colon U \subset M \to \mathbb{R}^n \times \mathbb{R}^{s-1} \times \mathbb{R}^1$ near y in M and $\psi\colon V \subset P \to \mathbb{R}^{p_1} \times \mathbb{R}^{p-p_1}$ near y in P such that

(a) $\phi(y) = (0, 0, 0)$, $\phi(N \cap U) \subset \mathbb{R}^n \times \{0\} \times \{0\}$, $\psi(y) = (0, 0)$

(b) $T_y\psi^{-1}(\mathbb{R}^{p_1} \times \{0\}) = T_yN \cap T_yP$

(c) $(0, 0)$ in $\mathbb{R}^{p_1} \times \{0\}$ is a non-degenerate critical point of the map $\pi\psi^{-1}|\mathbb{R}^{p_1} \times \{0\}$.

In other words, the submanifold $\psi^{-1}(\mathbb{R}^{p_1} \times \{0\})$ of P tangent to $T_yN \cap T_yP$ at y has first order contact with N at y. The motivation for this definition is as follows. Suppose $N = N_0$ and $P = P_0$ occur in generic 1-parameter families of manifolds N_t and P_t with N_0 and P_0 meeting quasi-transversely at y. If N_t and P_t are perturbed to families $\tilde{N}_t$ and $\tilde{P}_t$, then there is a t_0 near 0 such that $\tilde{N}_{t_0}$ and $\tilde{P}_{t_0}$ meet quasi-transversely near y, and the qualitative behavior of $\tilde{N}_t \cap \tilde{P}_t$ for t near t_0 is the same as that of $N_t \cap P_t$ for t near 0.

For example, a sphere and a plane in $\mathbb{R}^3$ which are tangent at a point meet quasi-transversely there and can be imbedded in one parameter families as indicated. Similarly, two curves in $\mathbb{R}^3$ which intersect non-tangentially at a point are quasi-transversal there.

Suppose p and q are different hyperbolic critical points of $X \in \mathfrak{X}^r(M)$, and let $\lambda_1,\ldots,\lambda_\alpha$ be the eigenvalues of $T_q\phi^1\ W^u(q)$ with $1 < |\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_\alpha|$, and $\mu_1,\ldots,\mu_\beta$ be the eigenvalues of $T_p\phi^1|W^s(p)$ with $1 > |\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_\beta|$. Let J_q and K_q denote the eigenspaces of $T_q\phi^1|W^u(q)$ corresponding to λ_1 and $\{\lambda_2,\ldots,\lambda_\alpha\}$, and let J_p and K_p denote the eigenspaces of $T_p\phi^1|W^s(p)$ corresponding to μ_1 and $\{\mu_2,\ldots,\mu_\beta\}$. Write $u = \dim W^u(q)$, $s = \dim W^s(p)$, and assume $0 < u < \dim M$, $0 < s < \dim M$. We say that $W^u(p)$ and $W^s(q)$ have an orbit $o(x)$ of <u>type one</u> intersections if the following conditions are satisfied:

(1) λ_1 is real, $1 < \lambda_1 < |\lambda_2| \leq \ldots \leq |\lambda_\alpha|$, and $\dim J_q = 1$

(2) μ_1 is real, $1 > \mu_1 > |\mu_2| \geq \ldots \geq |\mu_\beta|$, and $\dim J_p = 1$

(3) there is a smooth hypersurface H transverse to $o(x)$ at x such that

(a) $W^u(p) \cap H$ and $W^s(q) \cap H$ have a quasi-transversal intersection at x.

(b) $\lim_{t\to\infty} (T_x\phi^t)(T_x(W^u(p) \cap H)) = K_q$ and

$\lim_{t\to-\infty} (T_x\phi^t)(T_x(W^s(q) \cap H)) = K_p$

where the limits are taken with respect to metrics in the appropriate Grassmann bundles.

(4) if $W_1^u(q)$ is the unique ϕ^t invariant $(u-1)$-dimensional submanifold of $W^u(q)$ tangent to K_q, and $W_1^s(p)$ is the unique ϕ^t -invariant $(s-1)$-dimensional submanifold of

$W^s(p)$ tangent to K_p [6], then $W_1^u(q)$ is transverse to all the stable manifolds of critical points different from q, and $W_1^s(p)$ is transverse to all the unstable manifolds of critical points different from p.

Type one intersections were considered in [11]. We next define three subsets Q_1, Q_2, Q_3 of $\mathfrak{X}^r(M)$, which will include the bifurcation points on simple arcs. Q_1 is the set of X such that

(1) $L(X)$ consists of a finite number of critical points and closed orbits, all hyperbolic except one critical point, p, which is quasi-hyperbolic.

(2) the stable and unstable manifolds of orbits in $L(X)$ meet transversely and $(W^u(p)-\{p\}) \cap (W^s(p)-\{p\})$ is a single orbit.

Q_2 is the set of X such that

(1) $L(x)$ consists of a finite number of hyperbolic critical points and one quasi-hyperbolic critical point p,

(2) the stable and unstable manifolds of $L(X)$ meet transversely, and $W^u(p) \cap W^s(p) = \{p\}$.

Q_3 is the set of X such that

(1) $L(X)$ is a finite set of hyperbolic critical points

(2) there are points p and q such that $W^u(p)$ and $W^s(q)$ have an orbit $o(x)$ of type one intersections, and the intersections of stable and unstable manifolds not in $o(x)$ are transverse.

Let $\Sigma_1 = Q_1 \cup Q_2 \cup Q_3$. An element X in Σ_1 will be called a <u>flow of type one</u>.

PROPOSITION 1 - For $r \geq 2$, the set $\Sigma_1 \subset \mathfrak{X}^r(M)$ of flows of type one is an embedded submanifold of $\mathfrak{X}^r(M)$ of codimension one. Moreover, each $X \in \Sigma_1$ has a neighborhood U in $\mathfrak{X}^r(M)$ such that $U - \Sigma_1 \in \Delta$.

This proposition is a generalization of a result due to Sotomayor for $\dim M = 2$ [20]. While more general results of this kind are true, the proposition will suffice for our considerations here.

Let $\Phi^{k,r} = C^k(I, \mathfrak{X}^r(M))$ be the set of C^k mappings of the unit interval $I = [0, 1]$ into the Banach space $\mathfrak{X}^r(M)$ with the uniform C^k topology, $k \geq 1$, $r \geq 2$. For $\xi \in \Phi^{k,r}$, we write $\xi_t = \xi(t)$.

$\Phi^{k,r}$ may be simply described as follows. Let π: $TM \to M$ denote the natural projection, and let $\Phi = \{\eta: I \times M \to TM | \pi\eta(t, x) = x$ for all $t \in I$, $x \in M\}$. If ϕ: $U \times V \to I \times M$ is a local coordinate chart in $I \times M$ with $U \subset I$, $V \subset \mathbb{R}^m$, then relative to ϕ, any $\eta \in \Phi$ is expressed as a mapping η_1: $U \times V \to \mathbb{R}^m$. We call such an η_1 a local expression for η. Let $(t, x_1, \ldots, x_m)$ be the local coordinates in $U \times V$, let $\alpha = (\alpha_1, \ldots, \alpha_j)$, $\beta = (\beta_1 \ldots, \beta_{m-j})$ be j and $(m - j)$ - tuples of non-negative integers, and let γ be a non-negative integer. Set $|\alpha| = \alpha_1 + \cdot + \alpha_j$, $|\beta| = \beta_1 + \cdot + \beta_{m-j}$. Let $\Phi_1^{k,r}$ be the set of all η in Φ such that for any local expression η_1 of η, each partial derivative $\dfrac{\partial^{|\alpha|+|\beta|+\gamma}\eta_1}{\partial X^\alpha \partial t^\gamma \partial X^\beta}$ exists and is continuous for $\gamma \leq k$ and $|\alpha| + |\beta| \leq r$. Here $\partial X^\alpha = \partial X_1^{\alpha_1} \ldots \partial X_j^{\alpha_j}$, $\partial X^\beta = \partial X_{j+1}^{\beta_1} \ldots \partial X_m^{\beta_{m-j}}$ and $0 \leq j \leq m$.

Give $\Phi_1^{k,r}$ the topology of uniform convergence of the indicated partial derivatives. One may check by induction on $k + r$ that the evaluation map ev: $\Phi^{k,r} \to \Phi_1^{k,r}$ defined by $ev(\eta)(t, x) = \eta(t)(x)$ is a homeomorphism. As a consequence, we see that $\Phi^{\infty,\infty}$ is dense in $\Phi^{k,r}$ for all $0 \le k, r$.

A curve $\xi \in \Phi^{k,r}$ will be called <u>simple</u> if it has the following properties

(1) ξ_0 and ξ_1 are in Δ

(2) for $t \in I$, ξ_t fails to be in Δ if and only if it lies in Σ_1

(3) ξ is transverse to Σ_1.

REMARKS: 1) Since Σ_1 is embedded in $\mathfrak{X}^r(M)$, if ξ is a simple curve, then $\xi_t \notin \Delta$ for at most finitely many t's.

2) Using Proposition 1 and the fact that Δ is open in $\mathfrak{X}^r(M)$, one easily sees that the set of simple curves is open in $\Phi^{k,r}$.

For $r \ge 3$, let $\mathfrak{F}^r = C^r(M, \mathbb{R})$ denote the space of C^r real-valued functions on M with the uniform C^r topology, and let G^{r-1} denote the space of C^{r-1} riemannian metrics with its uniform C^{r-1} topology as an open subset of the space of covariant symmetric 2-tensor fields on M.

Recall that given $g \in G^{r-1}$ and $f \in \mathfrak{F}^r$, one defines the gradient flow $\text{grad}_g f$ determined by g and f by the equation

$$g_x(\text{grad}_g f(x), v) = -df_x(v) \quad \text{for} \quad x \in M, \quad v \in T_x M.$$

A function $f \in \mathfrak{F}^r$ is called a Morse function if it has only non-degenerate critical points.

Given a Riemannian metric g and a smooth one parameter family of functions F_t, there is an induced arc of flows given by

$$\xi_t = \text{grad}_g F_t.$$

Similarly, for a fixed Morse function f and a smooth one parameter family of metrics G_t, there is an induced curve of flows given by

$$\xi_t = \mathrm{grad}_{G_t} f.$$

Our main results are the following.

Give $C^k(I,\mathfrak{F}^r)$ and $C^k(I,G^{r-1})$ their uniform C^k-topologies.

THEOREM A - Fix $r \geq 3$, $k \geq 1$.

(1) Given any C^∞-Riemannian metric g on TM, there is an open dense set $U_1 \subset C^k(I,\mathfrak{F}^r)$ of one parameter families of functions on M such that for $F \in U_1$, the induced arc of flows $\xi_t = \mathrm{grad}_g F_t$ is simple.

(2) Given any C^∞ Morse function f on M, there is an open dense set $U_2 \subset C^k(I,G^{r-1})$ of one parameter families of Riemannian metrics on TM such that for $G \in U_2$, the induced arc $\xi_t = \mathrm{grad}_{G_t} f$ is simple.

THEOREM B - Suppose $k \geq 1$ and $r \geq 2$. Then for any two Morse-Smale flows X and Y on M, there is a simple curve $\xi \in \Phi^{k,r}$ such that $\xi_0 = X$ and $\xi_1 = Y$.

Remarks: 1. It is very likely that a simple are ξ is structurally stable in the sense defined by Sotomayor; i.e., if η is near ξ, there is a homeomorphism h: $I \to I$ such that η_t is topologically equivalent to $\xi_{h(t)}$ for all t.

2. Some of the motivation for theorem A(1) came from a theorem due to J. Mather which says that any two stable real-valued functions on M may be joined by a curve of functions containing at most finitely many non-stable elements, [3]. We caution the reader, however,

that, due to the presence of saddle connections, stable functions often give rise to non-structurally stable gradient flows. Furthermore, non-stable functions may yield structurally stable gradient flows as is the case when several non-degenerate critical points have the same image.

3. Proof of Theorem B

In this section we give the proof of Theorem B assuming Theorem A and Proposition 1 have been proved. A discussion of the proofs of these last two results will be deferred to §4.

LEMMA 1 - Let X be a C^r Morse-Smale flow on M. $r \geq 2$. Then there is a simple curve $\xi \in \Phi^{k,r}$ such that $\xi_0 = X$ and $\xi_1 = X_1$ is gradient-like.

Proof: We will show how to break each closed orbit by the introduction of a saddle node which splits into two hyperbolic critical points.

Taking a preliminary approximation, we may assume X is of class C^∞. Let $I_1 = [-1,1]$, D^σ denote the closed unit ball in R^σ, $\sigma = s,u$ with $s+u = \dim M-1$, and let (x,y,v) be coordinates on $D^s \times D^u \times I_1$. Choose a closed orbit γ of X, a point $p \in \gamma$, and a flow box neighborhood U of p so that there is a diffeomorphism ϕ: $U \to D^s \times D^u \times I_1$ such that

(1) $\phi(p) = (0,0,0)$

(2) $\phi^{-1}(D^s \times \{0\} \times I_1) \subset W^s(\gamma)$ and $\phi^{-1}(\{0\} \times D^u \times I_1) \subset W^u(\gamma)$

(3) $X' = \phi_* X = \frac{\partial}{\partial v}$

(4) $U \cap L(X) \subset \gamma$.

Here, of course, $\phi_* X = T\phi \circ X \circ \phi^{-1}$, $\dim W^u(\gamma) = u+1$, and $\dim W^s(\gamma) = s+1$.

Let Y_t be the one-parameter family of vector fields on $D^s \times D^u \times I_1$ defined by $Y_t(x,y,v) = (-x,y,v^2-t)$ for t near 0. Thus, Y_t has no critical points for $t < 0$, one saddle node critical point for $t = 0$, and two hyperbolic critical points for $t > 0$. Take a Euclidean subball D_1^s of D^s centered at 0 such that $\phi^{-1}(\text{bd } D_1^s \times \{0\} \times \{-1\})$ is transverse in $\phi^{-1}(D^s \times D^u \times \{-1\})$ to $W^u(\gamma') \cap \phi^{-1}(D^s \times D^u \times \{-1\})$ for every critical element γ' (i.e., closed orbit or critical point) of X. This can be done since if z is near γ and $z \in W^u(\gamma')$, then $W^u(\gamma')$ contains a $(u+1)$-disk through z C^1 near $\phi^{-1}(\{0\} \times D^u \times I_1)$ by the λ-lemma [12]. Similarly, we may take a subball D_1^u of D^u centered at 0 such that $\phi^{-1}(\{0\} \times \text{bd } D_1^u \times \{1\})$ is transverse in $\phi^{-1}(D^s \times D^u \times \{1\})$ to $W^s(\gamma') \cap \phi^{-1}(D^s \times D^u \times \{1\})$ for every critical element γ' of X.

Now one can choose a neighborhood U_1 of 0 in I_1 and a C^∞ real bump function ψ: $I_1 \to [0, 1]$, taking the value one on U_1 and zero off a slightly bigger set, so that the following properties hold.

Let $Z'(x,y,v) = \psi(v)\, Y_0(x,y,v) + (1-\psi(v))\, X'(x,y,v)$.

Then

(1) Z' has a saddle node critical point at $(0,0,0)$.

(2) $\text{bd } W^s(0,Z') \cap D^s \times D^u \times \{-1\}$ is C^1 near $\text{bd } D_1^s \times \{0\} \times \{-1\}$.

(3) $\text{bd } W^u(0,Z') \cap D^s \times D^u \times \{1\}$ is C^1 near $\{0\} \times \text{bd } D_1^u \times \{1\}$.

These properties imply that the vector field Z on M defined by

$Z(m) = \phi_*^{-1} \circ Z' \circ \phi(m)$, $m \in U$, $Z(m) = X(m)$, $m \in M-U$, is an element of the submanifold $Q_1 \subset \Sigma_1$ defined in §2. Now, let

$Z_t'(x,y,v) = \psi(v)\, Y_t(x,y,v) + (1-\psi(v))\, X'(x,y,v)$, and $Z_t(m) = \phi_*^{-1} \circ Z_t' \circ \phi(m)$, $m \in U$, while $Z_t(m) = X(m)$ for $m \in M-U$.

Then, using arguments similar to those in [11], one can verify that for $\varepsilon > 0$ small, Z_t, $|t| \le \varepsilon$ represents a simple arc with $Z_0 \in Q_1$ and $Z_t \in \Delta$ for $0 < |t| \le \varepsilon$. Employing filtrations, one first checks that the bifurcations of the limit set $L(Z_t)$ are as required. Then the uniform estimates of the λ-lemma [12] insure that the appropriate transversality conditions are fulfilled. §4 contains a few more details on these points.

Now, it is easily shown that, for $\varepsilon > 0$ small, there is a smooth arc ξ_t, $0 \le t \le 1+\varepsilon$ with $\xi_0 = X$, $\xi_t = Z_{t-1}$ for $1-\varepsilon < t < 1+\varepsilon$, and $\xi_t \in \Delta$ for $t \ne 1$. Thus, after reprarmetrizing the arc, we obtain a simple curve ξ_t, $0 \le t \le 1$, such that $\xi_t|_{M-U} = X|_{M-U}$ for all t, $\xi^{-1}(\Sigma_1)$ is a single point, $L(\xi_1) \cap U$ consists of two hyperbolic critical points, and $L(\xi_1) - U = L(X) - \gamma$. The lemma follows by repeating the construction for each closed orbit.

LEMMA 2 - Let X_1 be the gradient-like flow of Lemma 1. There is a curve $\xi \in \Phi^{k,r}$ with $\xi_0 = X_1$, $\xi_t \in \Delta$ for $0 \le t \le 1$, and $\xi_1 = \operatorname{grad}_g f$ for some Morse function f on M and some Riemannian metric g on M.

<u>Proof:</u> We observe that it is sufficient to find a curve ξ_t such that $\xi_0 = X_1$, $\xi_t \in \Delta$ for $0 \le t \le 1$, and $X_2 = \xi_1$ is a gradient in a neighborhood of each critical point. That is, there are a small neighborhood U of $L(X_2)$, a Riemannian metric g on U, and a real-valued function f on U such that $X_2|_U = \operatorname{grad}_g f$. Once this is done, we may

proceed as follows [17]. Changing f by constant values near each critical point, we may assume that $f(p) = \dim W^u(p)$ for each $p \in L(X)$. Then we may extend f to M so that $X_2|_{M-L(X_2)}$ is never tangent to a level surface of f, and the set of critical points of f is $L(X_2)$. Finally, using a partition of unity, one may extend g to M so that $X_2 = \text{grad}_g f$.

We now show how to construct the curve ξ_t alluded to in the proceding paragraph. Fix $p \in L(X)$. As before, we may assume that X_1 is of class C^∞. Making a further preliminary modification, we may arrange for X_1 to be equal to its linear part near p. This can be done with a C^1 small approximation within the C^∞ vector fields on M.

Thus, if $\dim W^u(p) = u$ and $\dim W^s(p) = s$, $0 \le u \le \dim M$ there is a neighborhood U of p in M and a C^∞ diffeomorphism ϕ: $U \to D^s \times D^u$ such that

(1) $\phi(p) = (0,0)$

(2) $\phi_* X_1(x,y) = (Ax,By)$
where A: $\mathbb{R}^s \to \mathbb{R}^s$ is a linear automorphism whose eigenvalues have negative real parts, and B: $\mathbb{R}^u \to \mathbb{R}^u$ is a linear automorphism whose eigenvalues have positive real parts.

(3) $U \cap L(X_1) = \{p\}$.

Here again $D^s(D^u)$ is the closed unit disk in $\mathbb{R}^s(\mathbb{R}^u)$, and (x,y) are coordinates on $D^s \times D^u$.

Let A_t, $0 \le t \le 1$, be a smooth curve of linear isomorphisms of $\mathbb{R}^s$ so that $A_0 = A$, $A_1 x = -2x$ for all x, and for each t, the eigenvalues of A_t have negative real parts. Similarly, let B_t, $0 \le t \le 1$, be a smooth curve of linear isomorphisms of $\mathbb{R}^u$ so that $B_0 = B$, $B_1 y = 2y$ for all y, and the eigenvalues of B_t have positive real parts for each t. Now choose small neighborhoods $U_1 \subset U_2$ of $(0,0)$ in $D^s \times D^u$ and a C^∞ real bump function $\psi: D^s \times D^u \to [0,1]$ such that $\psi(x,y) = 1$, $(x,y) \in U_1$, and $\psi(x,y) = 0$, $(x,y) \in D^s \times D^u - U_2$. Let $Y_t(x,y) = (A_t x, B_t y)$, $0 \le t \le 1$, and define

$\xi_t(m) = \phi_*^{-1}(\psi\phi(m)\ Y_t(\phi(m)) + (1-\psi\phi(m))\phi_* X_1(\phi(m)))$, $m \in U$, and $\xi_t(m) = X_1(m)$, $m \in M-U$.

Then $\xi_t \in \Delta$ for all $0 \le t \le 1$ by the construction of A_t, B_t and the λ-lemma. Also, $\xi_1\Big|_{\phi^{-1}(U_1)}$ is a gradient. Now, repeating this process for each critical point of X_1 gives the lemma.

REMARKS: 1) Let X be as in Lemma 1, and let $Y = \xi_1$ be the gradient vector field obtained in Lemma 2. For each integer $0 \le \lambda \le \dim M$, let C_λ be the number of critical points of Y whose unstable manifolds have dimension λ. Then $C_\lambda = M_\lambda + \overline{M}_{\lambda+1}$ where M_λ is the number of critical points of X whose unstable manifolds have dimension λ, and $\overline{M}_\lambda$ is the number of closed orbits of X whose unstable manifolds have dimension λ. Thus the usual Morse inequalities for the C_λ's give rise to the Morse inequalities for X obtained by Smale in [16].

2) In [7], K. Meyer constructs Lyapunov functions for a Morse-Smale flow. With methods similar to the proof of Lemma 2, one may show that if f is any Lyapunov function for a gradient like Morse-Smale flow X_1, then there is a curve ξ_t, $0 \le t \le 1$, with $\xi_0 = X_1$,

$\xi_t \in \Delta$ for all t, and $\xi_1 = \mathrm{grad}_g f$ for some metric g on M.

We now prove Theorem B. Let $X_0, X_1 \in \Delta$. Say that $X_0 \equiv X_1$ if there is a simple curve $\xi \in \Phi^{k,r}$ such that $\xi_0 = X_0$ and $\xi_1 = X_1$. Clearly this is an equivalence relation on Δ. Our goal is to prove that $X \equiv Y$ for any X and Y in Δ.

Using Lemmas 1 and 2, we may find Morse functions f_0, f_1 and Riemannian metrics g_0, g_1 such that $\mathrm{grad}_{g_0} f_0$ and $\mathrm{grad}_{g_1} f_1$ are in Δ, $X \equiv \mathrm{grad}_{g_0} f_0$, and $Y \equiv \mathrm{grad}_{g_1} f_1$. Changing f_0 and g_1 slightly as in [17], we may assume $\mathrm{grad}_{g_1} f_0 \in \Delta$. Now choose arcs G_t in $C^k(I, \mathcal{G}^{r-1})$ and F_t in $C^k(I, \mathfrak{F}^r)$, $0 \leq t \leq 1$, such that $G_0 = g_0$, $G_1 = g_1$, $F_0 = f_0$, and $F_1 = f_1$. Applying Theorem A, we may perturb G and F to conclude that $\mathrm{grad}_{g_0} f_0 \equiv \mathrm{grad}_{g_1} f_0 \equiv \mathrm{grad}_{g_1} f_1$. Hence $X \equiv Y$.

4. Proof of Proposition 1 and Theorem A

Proof of Proposition 1: The proof combines arguments in [11] and [21]. Similar methods were use to prove related results for diffeomorphisms in [11].

A critical element of X is either a critical point or a closed orbit. The period of a critical point is 0, whereas the period of a closed orbit γ is $\inf\{t > 0: \phi^t(x)=x, x\in\gamma\}$. Fix a riemannian metric g on TM. If γ is a hyperbolic or quasi-hyperbolic critical element of X, then the metric g induces metrics g_s on $W^s(\gamma)$ and g_u on $W^u(\gamma)$. Let d_u and d_s denote the distance functions induced on $W^u(\gamma)$ and $W^s(\gamma)$, and, for a positive integer n, set $B^u_\gamma(n) = \{y \in W^u(\gamma): d_u(y,\gamma) < n\}$ and $B^s_\gamma(n) = \{y \in W^s(\gamma): d_s(y,\gamma) < n\}$. Let $\Gamma_X(n)$ denote the set of critical elements of X of period less than n which we think of as a set of subsets of M or (taking its union) a subset of M.

Let $Q_1(n)$ be the set of all $X \in \mathfrak{X}^r(M)$ such that

(1) X has one quasi-hyperbolic critical point p, and the remaining critical elements of period less than n are hyperbolic

(2) for $\gamma_1, \gamma_2 \in \Gamma_X(n)$, $B^u_{\gamma_1}(n)$ is transverse to $B^s_{\gamma_2}(n)$. It

follows from [21] that each $Q_1(n)$ is an embedded submanifold of $\mathfrak{X}^r(M)$ of codimension one [1]. Also, if $Y \in Q_1(n)$, with quasi-hyperbolic critical point p, there are neighborhoods U of Y in $\mathfrak{X}^r(M)$, V of p, V' of $\Gamma_Y(n) - \{p\}$ in M, and a C^{r-1} function $f: U \to R$ such that $f^{-1}(0) = Q_1(n) \cap U$, and the following holds. If $Y_1 \in U - Q_1(n) = f^{-1}(R - \{0\})$, then

(1) $\Gamma_{Y_1}(n) \cap V'$ has only hyperbolic critical elements

(2) $\Gamma_{Y_1}(n) \cap V$ is either empty or has two hyperbolic critical points

(3) for any $\gamma, \nu \in \Gamma_{Y_1}(n)$, $B^u_\gamma(n)$ is transverse to $B^s_\nu(n)$.

Now, let $X \in Q_1$ have the quasi-hyperbolic critical point p. Clearly, $X \in Q_1(n)$ for each n. We will show that if n is large enough and U is a small neighborhood of X as above, then $Q_1(n) \cap U \subset Q_1$

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[1] Actually, $Q_1(n)$ is an open subset of a submanifold considered in [21].

and $U - Q_1(n) \subset \Lambda$. This will prove Proposition 1 for Q_1.

For Y near X, let ϕ_Y^t be the 1-parameter group associated to Y. In the notation of [11], choose a filtration for X separating its cycles. This amounts to a sequence $M = M_n \supset M_{n-1} \supset \dots \supset M_i \supset \emptyset$ of compact submanifolds with boundary of M (except M and $\emptyset$ of course) such that for $i \leq n$,

(4) $M_{i-1} \subset \text{int } M_i$

(5) $\phi_X^t(M_i) \subset \text{int } M_i$ for $t > 0$

(6) $\bigcup_i (\bigcap_{t \in \mathbb{R}} \phi_X^t(M_i - M_{i-1})) = L(X) \cup (W^u(p) \cap W^s(p))$.

Now for n larger than all of the periods of the critical elements of X, choose neighborhoods V of p, V' of $\Gamma_X(n) - \{p\}$, U of X as above. From (4), (5), and (6), one sees that if these neighborhoods are small and $Y \in U \cap Q_1(n)$, then $L(Y) \subset V \cup V'$, $L(Y) \cap V'$ is hyperbolic, $L(Y) \cap V$ is a quasi-hyperbolic critical point, and $W^u(p) \cap W^s(p) - \{p\}$ is a single orbit. Now the λ-lemma gives that for n large, U small, and $Y \in U \cap Q_1(n)$, $W^u(\gamma,Y)$ is transverse to $W^s(\nu,Y)$ for any critical elements γ,ν of Y. Thus $U \cap Q_1(n) \subset Q_1$. Also, if $Y \in U - Q_1(n)$, by (2) and the fact that $W^u(p) \cap W^s(p) - \{p\}$ is a single orbit, $L(Y)$ has either a hyperbolic closed orbit near $W^u(p) \cap W^s(p)$ or two critical points in V. Thus $L(Y)$ has only finitely many critical elements. Moreover, shrinking U again, the λ-lemma yields the transversality conditions to give $U - Q_1(n) \subset \Delta$.

The proof of Proposition 1 for Q_2 is similar. For Q_3, proceed as follows. Let $Q_3(n)$ be the set of X in $\mathfrak{X}^r(M)$ such that $L(X)$ is a finite number of hyperbolic critical points, and, for some fixed

$p,q \in L(X)$, $B_p^u(n) \cap B_p^s(n)$ has exactly one orbit of type one intersections, the other orbits being transversal intersections. Then, by [21], $O_3(n)$ is an embedded codimension one submanifold of $\mathfrak{X}^r(M)$. Again one shows (by means of a filtration and the λ-lemma) that for $X \in O_3$ there are an integer $n > 0$ and a neighborhood U of X in $\mathfrak{X}^r(M)$ such that $U \cap O_3(n) \subset U \cap O_3$ and $U - O_3(n) \subset \Delta$. Note that X is Ω-stable [18] this time so the control on $L(X)$ is easy.

Proof of Theorem A: (1) The proof is a refined application of transversality methods (see e.g. [1], [23]) where one has to verify that the local approximations may be made without modifying the metric.

Fix a C^∞ metric g, and $f \in C^k(I,\mathfrak{F}^r)$, $k \geq 1$, $r \geq 3$. We show that f may be approximated by a curve ξ such that $\mathrm{grad}_g \xi$ is simple. This is enough since the set of simple curves is open.

For a map h: $N \to P$ between manifolds and $W \subset P$ a submanifold, we write $h \pitchfork W$ to mean that h is transverse to W.

We may assume with a preliminary change in f that $f \in C^r(I,\mathfrak{F}^r)$, so that the evaluation map $ev(f)$ is in $C^r(I\times M,\mathbb{R})$. Let T^*M_0 denote the zero-section of the cotangent bundle T^*M, π: $I\times M \to I$ be the projection, and let $F = F(f)$: $I\times M \to T^*M$ be given by $F(t,m) = df_t(m)$. By standard transversality arguments, there is a dense open set $\mathcal{B}_1 \subset C^r(I,\mathfrak{F}^r)$ such that if $f \in \mathcal{B}_1$, then $F \pitchfork T^*M_0$ and $\pi\big|_{F^{-1}(T^*M_0)}$ has only non-degenerate critical points with distinct values. Thus for $f \in \mathcal{B}_1$, $F^{-1}(T^*M_0)$ is a one-dimensional manifold, and each f_t has only finitely many critical points. Hence $L(\mathrm{grad}_g f_t)$ is finite for each t, and $\mathrm{grad}_g f_t \in \Lambda$ if and only if it is

Kupka-Smale (i.e., it has only hyperbolic critical points and their stable and unstable manifolds meet transversely). For $f \in \mathcal{B}_1$, let $C(f)$ be the set of critical points of $\pi | F^{-1}(T^*M_0)$, and let $C_1(f) = \pi C(f)$.

One may verify with local coordinates that if $(x,t) \in C(f)$, x is a quasi-hyperbolic critical point of $\text{grad}_g f_t$, and if $(x,t) \in F^{-1}(R^*M_0) - C(f)$, then x is a hyperbolic critical point of $\text{grad}_g f_t$.* With a refinement of the proof of Theorem A in [17] (so that g is unchanged), one may find a residual subset $\mathcal{B}_2 \subset \mathcal{B}_1$ such that for $f \in \mathcal{B}_2$ and $t \in C_1(f)$, the stable and unstable manifolds of $L(\text{grad}_g f_t)$ are transverse - which means that $\text{grad}_g f_t \in Q_2$. Moreover, since $F \pitchfork T^*M_0$, the curve $\text{grad}_g f$ is transverse to Q_2 at t.

By Proposition 1, for $f \in \mathcal{B}_2$, there is a neighborhood U of $C_1(f)$ such that $\text{grad}_g f_t \in \Delta$ for $t \in U - C_1(f)$. Fixing such an f, we show that it may be modified on a neighborhood of $I-U$ so that $\text{grad}_g f$ becomes simple.

Let $J \subset I$ be a finite union of closed intervals with $\text{bd}\, J \subset \text{int}\, U$ and $J \cap C_1(f) = \emptyset$. There is an open neighborhood $\mathcal{B}_4$ of f in $C^r(I,\mathfrak{F}^r)$ so that for $h \in \mathcal{B}_4$ and $t \in J$, $L(\text{grad}_g h_t)$ is a finite set of hyperbolic critical points. Clearly, it suffices to show that a residual set $\mathcal{B}_5 \subset \mathcal{B}_4$ has the property that if $f \in \mathcal{B}_5$, $t \in J$, then $\text{grad}_g f_t \notin \Delta$ if and only if $\text{grad}_g f_t \in Q_3$ and $\text{grad}_g f \pitchfork Q_3$. We first show that for f in a residual set $\mathcal{B}_5 \subset \mathcal{B}_4$, $\text{grad}_g f_t \notin \Delta$, $t \in J$, if and only if there is one orbit of quasi-transversal intersections of stable and unstable manifolds. Then we show how to make the exceptional orbits into type one intersections.

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* This follows by taking suitable local coordinate expressions for f_t near x (see [2] and [3]).

For $t \in J$, $f \in \mathcal{B}_4$, let $x_{1t}, \ldots, x_{nt}$ denote the critical points of $\text{grad}_g f_t$, and suppose that $W^u(x_{it})$, $W^s(x_{it})$ are their respective unstable and stable manifolds. Let $u_i = \dim W^u(x_{it})$, $s_i = \dim W^s(x_{it})$, and let D^σ denote the closed unit ball in the Euclidean space R^σ, $0 < \sigma < \dim M$. For each i,t, there are embeddings $\psi_t^{u_i}\colon D^{u_i} \to M$, $\psi_t^{s_i}\colon D^{s_i} \to M$ so that

(1) $\psi_t^{u_i}(0) = \psi_t^{s_i}(0) = x_{it}$

(2) $\psi_t^{u_i}(D^{u_i}) \subset W^u(x_{it})$, $\psi_t^{s_i}(D^{s_i}) \subset W^s(x_{it})$.

Let ϕ_t^ν denote the one parameter group determined by $\text{grad}_g f_t$ (i.e. $\left.\frac{d}{d\nu}\phi_t^\nu(x)\right|_{\nu=0} = \text{grad}_g f_t(x)$). For each i,t, there is an induced embedding $\bar{\psi}_t^{u_i}\colon R^{u_i} \to M$ defined by

$$\bar{\psi}_t^{u_i}(y) = \phi_t^{r(y)}\psi_t^{u_i}(r(y)^{-1}y)$$

where $r(y) = 1 + |y|$. Since $f \in C^r(I,\mathfrak{F}^r) \approx C^r(I\times M,\mathbb{R})$, the mapping $\rho_{ij} = \rho_{ij}(f)\colon (t,x,y) \to (\psi_t^{s_i}(x), \bar{\psi}_t^{u_j}(y))$ of $J \times \text{bd}\, D^{s_i} \times \mathbb{R}^{u_j}$ into $M\times M$ is C^{r-1} (for $|y| \neq 0$) by stable manifold theory (this can be gotten most easily from the proof in [7]).

Let $\text{diag} = \{(x,y) \in M\times M \mid x = y\}$ denote the diagonal in $M\times M$, and let $\pi\colon J \times \text{bd}\, D^{s_i} \times \mathbb{R}^{u_j} \to J$ be the projection. Since $r \geq 3$, there is a residual set $\mathcal{B}_5' \subset \mathcal{B}_4$ such that $f \in \mathcal{B}_5'$ implies that $\rho_{ij} \pitchfork \text{diag}$ and $\pi|\rho_{ij}^{-1}(\text{diag})$ has non-degenerate critical points with distinct images. If (t,x,y) is a critical point of $\pi|_{ij}^{-1}(\text{diag})$, conditions (1) and (2)(a) in the definition of a quasi-transversal intersection are satisfied at $\psi_t^{s_i}(x) = \bar{\psi}_t^{u_j}(y)$. With a further approximation, we may fulfill the remaining condition and make $\psi_t^{s_i}(x) = \bar{\psi}_t^{u_j}(y)$ a quasi-transversal intersection of $W^u(x)$ and $W^s(y)$. Also, all other points of $\rho_{ij}^{-1}(\text{diag})$ correspond to transverse intersections. Restricting $\mathcal{B}_5'$ further we may assume that if $f \in \mathcal{B}_5'$, then for each $t \in J$, $\text{grad}_g f_t$ has at most one orbit of quasi-transversal intersections, the other intersections being transverse.

Now we restrict $\mathcal{B}_5'$ to $\mathcal{B}_5$ so that the quasi-transversal intersections become of type one. Suppose $x \in W^u(p) \cap W^s(q)$ and $o(x)$ is and orbit of quasi-transversal intersections for $\text{grad}_g f_t$ with $t \in J$, $f \in \mathcal{B}_5$. Let $\text{grad}_g f_t = \xi_t$, and let ϕ_t^1 be the time-one map of ξ_t. With a first restriction, we may assume the eigenvalues of $T_q\phi_t^1 W^u(q)$ and $T_p\phi_t^1| W^s(p)$ are distinct, say these are $1 < \lambda_1 < \dots < \lambda_\alpha$ and $1 > \mu_1 > \dots > \mu_\beta$, respectively. Let $E^s \oplus E_1^u \oplus \dots \oplus E_\alpha^u$ be the decomposition of T_qM where $E^s = T_q W^s(q)$ and E_i^u is the eigenspace of $T_q \phi_t^1$ corresponding to λ_i. From invariant manifold theory [6], there is a C^1 diffeomorphism $\zeta: \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^{\alpha-1} \to M$ such that

(a) $\zeta(0,0,0) = q$

(b) $\zeta(\mathbb{R}^s \times \{0\} \times \{0\}) \subset W^s(q)$

(c) $x \in \zeta(\mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^{\alpha-1})$

(d) $\zeta(\mathbb{R}^s \times \mathbb{R} \times \{0\})$ is locally invariant under the one parameter group ϕ_t^ν of ξ_t (i.e., $\phi_t^\nu(\zeta(\mathbb{R}^s \times \mathbb{R} \times \{0\}) \cap \zeta(\mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^{\alpha-1}) \subset \zeta(\mathbb{R}^s \times \mathbb{R} \times \{0\})$ for $\nu > 0$).

With a second restriction of $\mathcal{B}_5$, we may assume that $W^u(p)$ is transverse to $\zeta(\mathbb{R}^s \times \mathbb{R} \times \{0\})$ which will insure that the first Grassmann condition in part (3) of the definition of type one intersection holds. Proceeding similarly in a coordinate system near p, we may arrange for the second part of part (3) to hold. Finally, we obtain part (4) by means of approximations as in [17], but without changing the metric g. Then we have that $\mathrm{grad}_g f_t \not\in \Delta$ if and only if $\mathrm{grad}_g f_t \in \mathcal{Q}_3$.

The condition that $\rho_{ij}(f) \pitchfork \mathrm{diag}$ insures that $\mathrm{grad}_g f|J \pitchfork \mathcal{Q}_3$. This completes the proof of part (1) of Theorem A.

The proof of part (2) is similar, and in fact, easier. This is because for a fixed Morse function f, any curve of riemannian metrics $g \in C^k(I,\mathcal{G}^{r-1})$ gives rise to a curve $\mathrm{grad}_g f$ so that $L(\mathrm{grad}_{g_t} f)$ is finite and hyperbolic for each t. Also, all of the approximations in [17] away from the critical points of a given gradient flow of a Morse function may be realized by perturbations of the metric.

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