THOMAS W. KÖRNER

A Pseudofunction on a Helson Set. I

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This paper has been written in such a way that the reader may extract from it various sub-papers, each complete in itself.

(1) Section 1. Ideas behind the proof. Simple example of their use.

(2) Sections 2, 3. Complete proof of the main theorem with no deviations.

(3) Sections 1, 2, 3.

(4) Section 1, section 9 up to the end of the proof of Theorem 9.3 (1), sections 2 and 3, Theorem 1.1' in section 6, section 8 up to but not including the proof of Theorem 8.1. This reading gives all the results of general interest.

(5) Sections 1, 2, 3, 4 glance at the results of section 5 but omit the messy proofs, run rapidly through the rather trivial results of section 6, read the proof of theorem 1.1', read and consider carefully the statements of the results in section 7, read the proof of Theorem 7.1 and a selection of the proofs in the remainder of the section, section 8, section 9 omitting details of the proofs from the end of the proof of Theorem 9.3 (1). This reading gives most of the results of specialist interest without involving the reader in the messiest proofs.

Unless the reader has a special reason for choosing otherwise, selection (4) offers the best value (possibly reading the proof of Theorem 8.1 or perhaps omitting the details of section 3). In any case, an attempt has been made (at the occasional cost of slight repetition) to allow the reader to browse through the paper at will.

Two apologies are necessary. As we mention at the end of section 1 many of the proofs can be simplified by using the group D^{∞} rather than $\underline{\underline{T}}$. It is the author's opinion that the ideas of the paper are more natural in $\underline{\underline{T}}$ (even if easier in D^{∞}). However, it is instructive to rewrite the contents of sections 2 and 3 in terms of D^{∞}. Certain of the results in section 9 were first obtained in special groups and then reproved for $\underline{\underline{T}}$. Here again the reader may find it pleasanter to work in D^{∞}.

The second apology concerns the inequalities used. Many are much cruder than they could be, often being in the form "$10^{10}x \geqslant 1$ for $x \geqslant 1/2$". In justification for this we note that our methods either produce the best quantitative result without any fine estimates or give a qualitative result with quantitative estimates so bad as to be useless even when all the immediately obvious refinements have been made. To give an example, the quantity $N(\varepsilon, K, \lambda)$ in Lemma 2.1 may readily be bounded by $A(K, \lambda)\varepsilon^{-1}$. Improvements in the bounds of $A(K, \lambda)$ will not yield better qualitative or quantitative results. Only by replacing ε^{-1} by $(-\log \varepsilon)^{-1}$ or some similar improvement could we obtain genuinely superior results and this I do not know how to do (if indeed it is possible).

§ 1. INTRODUCTION. We follow the notation of [4]. We shall assume a knowledge of the definitions given there and their simplest consequences. We write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\chi_n(t) = \exp int$ for $t \in \mathbb{T}$ and $S(\mathbb{T}) = \{f \in C(\mathbb{T}) : |f(t)| = 1 \text{ for all } t \in \mathbb{T}\}$. We call a closed set E weak Kronecker (respectively weak Dirichlet) if for any $\mu \in M^+(E)$, $\varepsilon > 0$, $f \in S(E)$ we have $\inf_{m\in\mathbb{Z}} \mu\{t : |\chi_m(t) - f(t)| \geqslant \varepsilon\} = 0$ (respectively if for any $\mu \in M^+(E)$, $\varepsilon > 0$ we have $\liminf_{m\to\infty} \mu\{t : |\chi_m(t) - 1| \geqslant \varepsilon\} = 0$. It is clear that a weak Kronecker set is an independent Helson-1 set and Björk and Kaufman noticed that by the theorem of Hahn-Banach any weak Dirichlet set is automatically an N set (thus clearing up a long standing question). The converse results are also true. If the reader is unfamiliar with these concepts he should read Lemma 4.1 before proceeding further.

We have written this paper in such a way that Sections § 2 to § 3 form a self contained proof of

THEOREM 1.1. There exists a weak Kronecker set carrying a non zero pseudo-function.

Since a Helson set of synthesis cannot carry a true pseudomeasure and since no measure on a Helson set is a pseudofunction [4] we have

COROLLARY 1.2. (i) (Malliavin) Sets of non synthesis exist ;

(ii) (Piatečki Shapiro) Sets of multiplicity but not of strong multiplicity exist ;

(iii) Independent N sets need not be of uniqueness.

One way of expressing the theorem is to say that sets which are thin in the sense that they are of interpolation may be thick in the sense that they are of multiplicity. We contrast the beautiful positive result of Varopoulos [18].

LEMMA 1.3. No Kronecker set can support a true pseudomeasure.

We defer the discussion of the other results to § 4. The remainder of the present Section is devoted to heuristic considerations intended to show why we attack the problem as we do. Except in the case of Lemma 1.11, the proofs we shall offer will be either non-existent or inadequate. This is because we shall only need Lemma 1.11 in what follows (and then only for Section 9). I hope nevertheless that the discussion, though formally unnecessary, will be found helpful if not before then at least after the main proof has been read.

In a fascinating note [1] Drury introduces the following idea. Suppose E is a Dirichlet set. Then we can find $m(r) \to \infty$ with $\|1 - \chi_{m(r)}\|_{C(E)} \leqslant \varepsilon_r \to 0$. Set $E^* = \{x : |1 - \chi_{m(r)}(x)| \leqslant \varepsilon_r\}$. Then $E^* \supseteq E$ and E^* is a closed Dirichlet set. Moreover, because E^* has a very simple form (E^* is "as big as it can be"), E^* has certain nice properties which Drury exploits. In some sense E^* is a shadow of E, but the notion is difficult to formalize, since the shadow is not unique (it depends on the choice of ε_r and $m(r)$).

In studying questions on the union of AA^+ sets (themselves related to the unsolved question whether a closed countable set is a ZA^+ set) it is natural to ask whether we can use the shadow technique to discuss the union of 2 Dirichlet sets E_1, E_2 (a Dirichlet set is AA^+ with constant 1 (Lemma 4.2)). If a shadow E_2^* of E_2 lies entirely inside or (apart from an arbitrarily small neighbourhood of 0) outside a shadow E_1^* of E_1, the situation is simple to analyze. What happens if the shadows must intersect on a set E_{12} say ?

In spite of the vagueness of the description, it turns out to be quite easy to construct a situation which (in my opinion) corresponds in its essential features with that described in the question.

LEMMA 1.4. Given $1/10 > \varepsilon > 0$, we can find closed sets $E_1, E_2, E_{12} \subseteq \mathbb{T}$, measures $\mu_1 \in M^+(E_1)$, $\mu_2 \in M^+(E_2)$, $\mu_{12} \in M^+(E_{12})$ with $\|\mu_1\| = \|\mu_2\| = \|\mu_{12}\| = 1$, sequences of integers $0 < N < M_1(1) < M_1(2) < \ldots$ and $0 < N < M_2(1) < M_2(2) < \ldots$ such that

(i) $\|\chi_{M_i(j)} - 1\|_{C(E_i \cup E_{12})} \to 0$ as $j \to \infty$

(ii) $|\hat{\mu}_i(m)| \geqslant \varepsilon$ implies $|\hat{\mu}_{12}(m) - \hat{\mu}_i(m)| \leqslant \varepsilon$

(iii) $\min(|\hat{\mu}_1(m)|, |\hat{\mu}_2(m)|) \leqslant \varepsilon$ for $|m| \geqslant N$.

This is the first and simplest of a series of similar results, and it is important to realise what it does and what it does not mean. We have two sets E_1 and E_2 which are Dirichlet (condition (i)) but such that if we can get close to 1 on E_1 with a character χ_m then we cannot get close to 1 on E_2 with the same character (condition (iii)) at least if $|m| \geqslant N$. We say nothing and can say nothing about the case $|m| \leqslant N$, for example χ_0 clearly is 1 both on E_1 and E_2, and if we demand E_1, E_2 in a small neighbourhood of 0, χ_m will be close to 1 on both E_1, E_2 for m small.

On the other hand E_{12} tries to imitate both E_1 and E_2. Slightly more precisely, E_{12} imitates E_1 when it is possible to get close to 1 with a character

χ_m on E_1 and imitates E_2 (with respect to the character $\chi_{m'}$) when $\chi_{m'}$ is close to 1 on E_2 (condition (ii)). We note that although $|\hat{\mu}_i(m)|$ large implies $\hat{\mu}_{12}(m)$ close to $\hat{\mu}_i(m)$ and so large, it is not true that $|\hat{\mu}_{12}(m)|$ large implies $\max(|\hat{\mu}_1(m)|, |\hat{\mu}_2(m)|)$ large (consider for example $m = M_1(r) + M_2(r)$).

Let us fix our ideas by considering

LEMMA 1.5. If E_1, E_2, E_{12} and N are as in Lemma 1.4, then $\|\sum_{r=N}^{\infty} a_r\chi_r - 1\|_{C(E_1 \cup E_2 \cup E_{12})} \leqslant \varepsilon$ implies $\sum |a_r| \geqslant 3 - 23\varepsilon$.

Proof. If $\sum_{r=N}^{\infty} |a_r| \geqslant 3$ then we are home, so we may suppose $\sum_{r=N}^{\infty} |a_r| \leqslant 3$. Write $\Lambda(i) = \{r \geqslant N : |\hat{\mu}_i(r)| \geqslant \varepsilon\}$. Then

$$\begin{aligned}
|\sum_{r\in\Lambda(i)} a_r \hat{\mu}_i(r) - 1| &\leqslant |\sum_{r\geqslant N} a_r \hat{\mu}_i(r) - 1| + \sum_{r\geqslant N, r\notin\Lambda(i)} |a_r||\hat{\mu}_i(r)| \\
&\leqslant |\sum_{r\geqslant N} a_r \hat{\mu}_i(r) - 1| + 3\varepsilon \\
&= |\int (\sum_{r\geqslant N} a_r\chi_r - 1)d\mu_i| + 3\varepsilon \\
&\leqslant \|\sum_{r\geqslant N} a_r\chi_r - 1\|_{C(E_i)} + 3\varepsilon \\
&\leqslant 4\varepsilon \qquad [i = 1,2].
\end{aligned}$$

Also, noting that $\Lambda(1)\cap\Lambda(2) = \emptyset$ we have

$$\begin{aligned}
&|\sum_{r\notin\Lambda(1)\cup\Lambda(2), r\geqslant N} a_r \hat{\mu}_{12}(r) + 1| \leqslant \\
&\leqslant |\sum_{r\geqslant N} a_r \hat{\mu}_{12}(r) - 1| + |\sum_{r\in\Lambda(1)} a_r \hat{\mu}_{12}(r) - 1| + |\sum_{r\in\Lambda(2)} a_r \hat{\mu}_{12}(r) - 1| \\
&\leqslant |\sum_{r\geqslant N} a_r \hat{\mu}_{12}(r) - 1| + |\sum_{r\in\Lambda(1)} a_r \hat{\mu}_1(r) - 1| + |\sum_{r\in\Lambda(2)} a_r \hat{\mu}_2(r) - 1|
\end{aligned}$$

$$+\sum_{r\in\Lambda(1)}|a_r||\hat{\mu}_{12}(r)-\hat{\mu}_1(r)|+\sum_{r\in\Lambda(2)}|a_r||\hat{\mu}_{12}(r)-\hat{\mu}_2(r)|$$

$$\leqslant |\sum_{r\geqslant N} a_r\,\hat{\mu}_{12}(r)-1|+4\varepsilon+4\varepsilon+3\varepsilon+3\varepsilon$$

$$=\ \left|\int(\sum_{r\geqslant N} a_r\chi_r-1)d\mu_{12}\right|+14\,\varepsilon$$

$$\leqslant \|\sum_{r\geqslant N} a_r\chi_r-1\|_{C(E_{12})}+14\,\varepsilon$$

$$\leqslant 15\varepsilon\ .$$

Since $|\hat{\mu}_i(r)|\leqslant\|\mu_i\|=1$, $|\hat{\mu}_{12}(r)|\leqslant\|\mu_{12}\|=1$, it follows that $\sum_{r\ \Lambda(1)}|a_r|\geqslant 1-4\varepsilon$, $\sum_{r\in\Lambda(2)}|a_r|\geqslant 1-4\varepsilon$, $\sum_{r\in\Lambda(1)\cup\Lambda(2),r\geqslant N}|a_r|\geqslant 1-15\,\varepsilon$, so $\sum_{r\geqslant N}|a_r|\geqslant 3-23\varepsilon$ as stated.

This result is best possible. If E_1, E_2 are Dirichlet, then, given any $N\in\mathbb{Z}$ and $\delta>0$, we can find $\sum_{r\geqslant N}|a_r|\leqslant 3+\delta$ such that $\sum_{r\geqslant N}a_r\chi_r(e)=1$ for all $e\in E_1\cup E_2$ (see [12] or Lemma 4.1).

Crudely speaking, approximating 1 on E_1 forces us to approximate 1 on E_{12}. Similarly, approximating 1 on E_2 forces us to approximate 1 on E_{12}. But, since approximating 1 on E_1 is no help in approximating 1 on E_2 and vice versa, by approximating 1 on E_1 and E_2 $(\sum_{\Lambda(1)}a_r\chi_r \underset{E_1}{\simeq} 1$, $\sum_{\Lambda(2)}a_r\chi_r \underset{E_1}{\simeq} 1)$ we approximate 2 on E_{12} $(\sum_{\Lambda(1)\cup\Lambda(2)}a_r\chi_r \underset{E_{12}}{\simeq} 2)$ and we must approximate -1 on E_{12} $(\sum_{r\notin\Lambda(1)\cup\Lambda(2)}a_r\chi_r \underset{E_{12}}{\simeq} -1)$ to get what we want $(\sum a_r\chi_r \underset{E_1\cup E_2\cup E_{12}}{\simeq} 1)$.

The correct generalization is clear.

LEMMA 1.6 (i) Given $\varepsilon>0$, $n\geqslant 1$, we can find an N and for each $S\subseteq\{1,2,\dots,n\}$ a closed set E_S and a measure $\mu_S\in M^+(E_S)$ with $\mu_S\ =1$, and for each $1\leqslant i\leqslant n$ a sequence of integers $0<n<M_i(1)<M_i(2)<\dots$ such that

(i) $\|\chi_{M_i(j)} - 1\|_{C(E_S)} \to 0$ as $j \to \infty$ for each $i \in S$ $[\emptyset \neq S \subseteq \{1,2,..,n\}]$;

(ii) $|\hat{\mu}_S(m)| \geqslant \varepsilon$ implies $|\hat{\mu}_S(m) - \hat{\mu}_T(m)| \leqslant \varepsilon$ for all $S \subseteq T \subseteq \{1,2,..,n\}$, $|m| \geqslant N$;

(iii) $|\hat{\mu}_{S \cap T}(m)| \geqslant \max(|\hat{\mu}_S(m)|, |\hat{\mu}_T(m)|) - \varepsilon/8n$ for all $S,T \subseteq \{1,2,..,n\}$, $|m| \geqslant N$;

(iv) $|\hat{\mu}_\emptyset(m)| \leqslant \varepsilon/2$ for all $|m| \geqslant N$.

LEMMA 1.6 (ii) Under the conditions on the first part we have, writing $E = \bigcup_{\emptyset \neq S \subseteq \{1,2,...,n\}} E_S$, that

$$\|\sum_{r=N}^{\infty} a_r \chi_r - 1\| \leqslant \varepsilon \quad \text{implies} \quad \sum_{r=N}^{\infty} |a_r| \geqslant 2^n - 1 - B(n)\varepsilon$$

(where $B(n)$ depends only on n).

I strongly recommend the deduction of Lemma 1.6 (ii) from Lemma 1.6 (i) as an exercise. The result is again best possible ([12] or Lemma 4.1). A more sophisticated version of Lemma 1.6 (ii) is proved in full as Theorem 7.1 of this paper. However, the finite version of Lemma 1.6 (i) is even more useful.

LEMMA 1.7 (The Linked Set Lemma). Given $1 > \varepsilon$, $\eta > 0$, we can find an $N(\varepsilon, \eta) \geqslant 1$ with the following property : -

Given $1 > \delta > 0$ and $m \geqslant 1$ we can find a *monotonic increasing* function $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ (such that $h(r) > r$) with the following properties : -

Given $N(\varepsilon, \eta) = \frac{1}{2} M(0) < h(M(0)) < M(1) < h(m(1)) < \ldots < h(M(m)) < M(m+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$, we can find finite sets $E_S \subseteq [-\varepsilon, \varepsilon]$, and measures $\mu_S \in M^+(E_S)$ with $\|\mu_S\| = 1$ $[S \subseteq \{1,2,...,m\}]$ such that

(i) $M(m+1)E_S = 0$ (i.e. $E_S \subseteq \{2\pi r/M(m+1) : r \in \mathbb{Z}\}$) $[S \subseteq \{1,2,\ldots,m\}]$

(ii) $\|\chi_{M(i)} - 1\|_{C(E_S)} \leqslant \delta$ for each $i \in S$

(iii) $|\hat{\mu}_S(r)| \geqslant \eta$ implies $|\hat{\mu}_S(r) - \hat{\mu}_T(r)| \leqslant \eta$ for all $S \subseteq T \subseteq \{1,2,..,m\}$, $M(m+1) - N \geqslant |r| \geqslant N$

(iv) $|\hat{\mu}_S(r)|, |\hat{\mu}_T(r)| \geqslant \eta$ implies $|\hat{\mu}_{S\cap T}(r)| \geqslant \min(|\hat{\mu}_S(r)|, |\hat{\mu}_T(r)|) - \eta$ $\left[S,T \subseteq \{1,2,\ldots,m\},\ M(m+1) - N \geqslant |r| \geqslant N\right]$

(v) $|\hat{\mu}_\emptyset(r)| \leqslant \eta$ for all $M(m+1) - N \geqslant |r| \geqslant N$.

Remark. Together (iii) and (iv) show that if $|\mu_S(r), \mu_T(r)| \geqslant 2\eta$ then $|\hat{\mu}_T(r) - \hat{\mu}_S(r)| \leqslant 2\eta$.

REMARKS. (1) We shall exhibit the main idea in the construction of these sets at the end of this section by proving an analogous result for D^∞ (Lemma 1.12).

(2) In constructing measures or pseudomeasures as limits of finitely supported measures μ_n, it is sometimes necessary and usually convenient to choose supp μ_n with M supp $\mu_n = 0$ for some M. In this case $\hat{\mu}_n$ is periodic with period M and knowledge of $\hat{\mu}_n(m)$ for $k < |m| \leqslant k + M$ gives us $\hat{\mu}_n(m)$ for all m. We frequently make use of this fact without drawing attention to it (e. g. knowing $|\hat{\mu}_n(m)| \leqslant 1$ for $|m| \leqslant M/2$ we then deduce $\|\mu_n\|_{PM} \leqslant 1$).

As we noted in the first paragraph of this section, Björk and Kaufman remarked independently that if E is a closed set with $\limsup_{p\to\infty} \mu\{x : |\chi_p(x) - 1| \leqslant \delta\} = \|\mu\|$ for all $\mu \in M^+(E)$, $\delta > 0$ (i. e. if E is weak Dirichlet), then $1|E$ lies in the uniform closure of $B_1 = \left\{\sum_{r=1}^{\infty} a_r\chi_r|E : \sum_{r=1}^{\infty}|a_r| \leqslant 1\right\}$ (by Hahn Banach). This leads us to ask whether 1 lies in $\{\sum a_r\chi_r|E : \sum_{r=1} |a_r| < \infty\}$ and this in turn prompts the question : - if $\limsup_{p\to\infty} \mu\{x : |\chi_p(x) - 1| \leqslant \delta\} \geqslant \lambda\|\mu\|$ for all $\mu \in M^+(E)$, $\delta > 0$, can we say that $1|E$ lies in the uniform closure of $B_{\lambda^{-1}} = \{\sum_{r=1} a_r\chi_r|E : \sum_{r=1} |a_r| \leqslant \lambda^{-1}\}$? The answer is no.

Consider the set E of Lemma 1.6 (ii). We have

$$\begin{aligned}\limsup_{p\to\infty} \mu\{x : |\chi_p(x) - 1| \leqslant \delta\} &\geqslant \frac{1}{m} \sum_{i=1}^{m} \limsup_{p\to\infty} \mu\{x : |\chi_{M_i(p)}(x) - 1| \leqslant \delta\} \\ &\geqslant \frac{1}{m} \sum_{i=1}^{m} \mu(\bigcup_{i\in S} E_S) \\ &\geqslant \frac{1}{m}\mu(E) = \frac{1}{m}\|\mu\|\end{aligned}$$

for $\mu \in M^+(E)$, $\delta > 0$ whilst $1|E \notin B_m$, indeed if $\zeta > 0$ then (for ε sufficiently small) $1|E \notin B_{1/(2^m-1-\zeta)}$.

Can this counter example be improved ? We note at once that writing

$E' = \bigcup_{\text{card } S \geqslant q} E_S$ we have

$$\limsup_{p\to\infty} \mu\{x : |\chi_p(x) - 1| \leqslant \delta\} \geqslant \frac{1}{m} \sum_{i=1}^{m} \mu(\bigcup_{i\in S, \text{card } S\geqslant q} E_S) \geqslant \frac{q}{m}\|\mu\|$$

for all $\mu \in M^+(E')$, $\delta > 0$. But what can we say about $k = \inf\{s : 1|E'$ belongs to the uniform closure of $B'_s = \{\sum_{r=1} a_r\chi_r|E' : \sum_{r=1}|a_r| \leqslant s\}\}$. It is not terribly difficult to see that $k = K(m,q) + o(1)$ as $\varepsilon \to 0$ for some $K(q , m)$ depending only on m and q. Direct calculation shows $K(2,1) = 3, K(4,2) = 5, K(6,3) = 7$. (Thus for example, taking $m = 6$, $q = 3$, we have $\limsup_{p\to\infty} \mu\{x : |\chi_p(x) - 1| \leq \delta\} \geqslant \frac{1}{2}\|\mu\|$ for all $\mu \in M^+(E')$, $\delta > 0$, but, given $\zeta > 0$, we know, provided ε is small enough, that $1 = \sum_{r=1} a_r\chi_r(e)$ for all $e \in E'$ implies $\sum_{r=1}|a_r| \geqslant 7 - \zeta$).

However, the calculations then become complicated (for example $K(8,4) = 11+1/7$), and I am indebted to Conway for the key step in the proof that $\lim_{m\to\infty} \sup\{K(m,q) : q \geqslant \lambda m\} = \infty$ for $1 > \lambda > 0$. (The second half of Section § 3 is devoted to this proof.) Knowing this fact we have directly from Lemma 1.6 (i)

LEMMA 1.8. Given $K>0$, $1 > \lambda > 0$, we can find a closed set E such that $\lim_{p\to\infty} \sup\{x : |\chi_p(x) - 1| \leqslant \delta\} \lambda\|\mu\|$ for all $\mu \in M^+(E)$, $\delta > 0$, yet $\sum_{r=1}^{\infty} a_r\chi_r(e) = 1$ for all $e \in E$ implies $\sum |a_r| \geqslant K$.

(This point will be examined more closely in Section § 7).

More importantly we have, by similar arguments, from Lemma 1.7

LEMMA 1.9. Given $1 \geqslant \varepsilon > 0$, $1 > \lambda > 0$, $K > 1$, we can find an $N = N(\varepsilon, K, \lambda) \geqslant 1$ and an $m(K, \lambda) \geqslant 1$ such that, given $\delta > 0$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ (such that $h(r) > r$) with the following property : -

Given $N(\varepsilon, K, \lambda) = \frac{1}{2}M(0) < h(M(0)) < M(1) < h(M(1)) < \ldots < h(M(m)) < M(m+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$, we can find a finite set $E \subseteq [-\varepsilon, \varepsilon]$ such that

(i) $M(m+1)E = 0$

(ii) $\sup_{1\leqslant j\leqslant m} \mu\{e : |\chi_{M(j)}(e) - 1| \leqslant \delta\} \geqslant \lambda\|\mu\|$ for all $\mu \in M^+(E)$

(iii) $\sum_{r=M(0)}^{M(m+1)-M(0)/2} a_r\chi_r(e) = 1$ for all $e \in E$ implies $\sum_{r=M(0)/2}^{M(m+1)-M(0)/2} |a_r| \geqslant K$.

In practice it is often more convenient to use the equivalent

LEMMA 1.9'. As for Lemma 1.9 with condition (ii) replaced by

(ii') There exist $b_{M(j)} \geqslant 0$ with $\sum_{j=1}^{m} b_{M(j)} = 1$ and $\sum_{j=1}^{m} b_{M(j)}\chi_{M(j)} - 1 \leqslant \delta + 2(1 - \lambda)$.

(It is not claimed that the smallest possible value of the $N(\varepsilon, K, \lambda)$ and so on is the same in the two versions of the Lemma).

Using either version of the lemma we can, for example, improve Lemma 1.8 to give

LEMMA 1.10 (i). Given $K > 0$, we can find a weak Dirichlet set E such that $\sum_{r=N}^{\infty} a_r \chi_r(e) = 1$ for all $e \in E$ implies $\sum_{r=N}^{\infty} |a_r| \geqslant K$.

In fact with a little more work we can show that

LEMMA 1.10 (ii). Given $K > 0$, we can find a weak Dirichlet set $E \subseteq [-\varepsilon, \varepsilon]$ such that $\sum_{r=1}^{\infty} a_r \chi_r(e) = 1$ for all $e \in E$ implies $\sum_{r=1}^{\infty} |a_r| \geqslant K$.

This gives in turn, after a few modifications,

LEMMA 1.10 (iii). There exists a weak Dirichlet set E such that $\| \sum_{r=1}^{\infty} a_r \chi_r - 1 \|_{C(E)} > 0$ for all $\sum |a_r| < \infty$ (i. e. E is not a ZA^+ set).

An improved version of this result is proved in full as Theorem 8.1.

It is clear that some sort of connection exists between the question of the existence of a weak Dirichlet set which is not ZA^+ and the question of the existence of a weak Dirichlet set which is not of uniqueness : we therefore turn our attention to this second question.

We know ([5] p. 53) that a closed set E is of multiplicity if and only if it supports a non zero pseudofunction. If we want to construct a weak Dirichlet set E and a non zero pseudofunction T with E as support, it is natural to attempt to re-interpret Lemma

1.9' in terms of pseudomeasures.

LEMMA 1.9". If E is as in Lemma 1.9', then we can find a $T \in M(E) = PM(E)$ such that

(iii") $\hat{T}(0) = 1 = \|T\|_{PM(E)}$, $|\hat{T}(r)| \leqslant K^{-1}$ for all $M(0) \leqslant r \leqslant M(m+1) - M(0)$.

Proof. Since $\sum = \{\sum_{r=M(0)/2}^{M(m+1)-M(0)/2} a_r \chi_r | E : \sum a_r < K\}$ is a convex circled set and $1 \in \sum$ it follows by the theorem of Hahn-Banach that there exists an S $S \in PM(E)$ with $|\langle S,f\rangle| < 1 = |\langle S,1\rangle|$ for all $f \in \sum$. In particular, taking $f = K\chi_r$ $[M(0)/2 \leqslant r \leqslant M(m+1) - M(0)/2]$, we have $|\hat{S}(r)| \leqslant K^{-1}$ for $M(0)/2 \leqslant r \leqslant M(m+1) - M(0)/2$, $\hat{S}(0) = 1$. We remind the reader that $\hat{S}$ is periodic with period $M(m+1)$. Since $\hat{S}$ takes only a finite number of values, we can find an $|s| \leqslant M(0)/2$ such that $|\hat{S}(s)| = \|S\|_{PM}$. Set $T = \chi_s S/\|S\|_{PM}$. Then $\hat{T}(0) = 1 = \|T\|_{PM}$ and $|\hat{T}(r)| = |\hat{S}(r - s)|/\|S\|_{PM} \leqslant |\hat{S}(r - s)| \leqslant K^{-1}$ for all $M(0) \leqslant r \leqslant M(m+1) - M(0)$ as required.

We obtain immediately

LEMMA 1.11. There exists a weak Dirichlet set E which is of multiplicity.

Proof. Set $T_0 = \delta_0$, $\varepsilon_1 = 1$, $m_0 = 0$, $M_0(1) = 4N(1,2,3/4)$ (with the notation of Lemma 1.9'). We construct $T_r \in M^+$, $\varepsilon_{r+1} > 0$, $m_r = 0$, $M_r(m_r+1)$ inductively as follows. Suppose $M_r(m_r+1) \geqslant 4N(\varepsilon_{r+1}, 2^{r+1}, 1-2^{-r-2})$.

By Lemmas 1.9' and 1.9" we can find successively $M_{r+1}(0) \leqslant M_r(m_r+1)/2$, $m_{r+1} \geqslant 1$, $M_{r+1}(1)$ a (positive) integral multiple of both $8M_{r+1}(0)$ and $M_r(m_r+1)$, $M_{r+1}(2)$, $M_{r+1}(3)$, ..., $M_{r+1}(m_{r+1})$ such that $M_{r+1}(s + 1)$ is a (positive) integral multiple of $M_{r+1}(s)$ $[1 \leqslant s \leqslant m_{r+1}-1]$, $\varepsilon_{r+1}/8 > \varepsilon_{r+2} > 0$ such that $M_{r+1}(m_{r+1})\varepsilon_{r+2} \leqslant 2^{-r-4}$, $M_{r+1}(m_{r+1}+1)$ a (positive) integral multiple of $M_{r+1}(m_{r+1})$ such that $M_{r+1}(m_{r+1}+1) \geqslant 4N(\varepsilon_{r+2}, 2^{r+2}, 1-2^{-r-3})$, $T_{r+1} \in M$ with supp $T_{r+1} = E_{r+1} \subseteq [-\varepsilon_{r+1}, \varepsilon_{r+1}]$ such that

(i) $M_{r+1}(m_{r+1}+1)E_{r+1} = 0$

(ii) There exist $b_{M_{r+1}(j)} \geqslant 0$ with $\sum_{j=1}^{m_{r+1}} b_{M_{r+1}(j)} = 1$ and

$$\left\| \sum_{j=1}^{m_{r+1}} b_{M_{r+1}(j)} \chi_{M_{r+1}(j)} - 1 \right\| \leqslant 2^{-r-1} + 2^{-r-1} = 2^{-r}$$

(iii)' $\hat{T}_{r+1}(0) = \|T_{r+1}\|_{PM} = 1$, $|\hat{T}_{r+1}(s)| \leqslant 2^{-r-1}$ for all $M_{r+1}(0) \leqslant |s| \leqslant M_{r+1}(m_{r+1}+1) - M_{r+1}(0)$.

(Note that we could demand $M_{r+1}(s) \geqslant K(r,s)M_r(s)$ for some $K(r,s)$ chosen in advance $[0 \leqslant s \leqslant m_{r+1},\ r \geqslant 1]$).

Set $S_{r+1} = S_r * T_{r+1}$, $S_o = T_o$. By (iii)' $\hat{S}_r(0) = \|S_r\|_{PM} = 1$ so by the weak $*$ compactness of the unit ball in $PM(\underline{\underline{T}})$, S_r has a weak $*$ limit point S with $\hat{S}(0) = 1$ (so $S \neq 0$). Again by (iii)' $|\hat{S}_k(s)| \leqslant |\hat{T}_r(s)| \leqslant 2^{-r-1}$ for all $M_{r+1}(0) \leqslant |s| \leqslant M_{r+1}(m_{r+1}+1) - M_{r+1}(0)$ and $k \geqslant r+1$. Since $M_{r+1}(m_{r+1}) - M_{r+1}(0) \geqslant M_{r+1}(m_{r+1}+1)/2$, we have $|\hat{S}_k(s)| \leqslant 2^{-r-1}$ for all $M_{r+1}(0) \leqslant |s| \leqslant M_{r+2}(0)$, $k \geqslant r+1$ and so $|\hat{S}(s)| \leqslant 2^{-r-1}$ for all $M_{r+1}(0) \leqslant |s| \leqslant M_{r+2}(0)$. Thus S is a non zero pseudofunction.

On the other hand, setting $F_{n+1} = F_n + E_{n+1} = \{x + y : x \quad F_n,\ y \quad E_{n+1}\}$, we have that F_{n+1} converges topologically to a closed set E with $\sup\{|x-y| : x \in F_n,\ y \in E\} \leqslant 2\varepsilon_{n+1}$. Since supp $S_n \subseteq F_n$, we have Supp $S \subseteq E$ and so E supports a non zero pseudofunction.

On the other hand given $y \in E$ we can certainly find an $x \in F_{n+1}$ with $|x-y| < 2M_{n+1}(m_{n+1})_{n+1} \leqslant 2^{-n}$ and so with $|\chi_{M_n(j)}(x) - \chi_{M_n(j)}(y)| \leqslant 2^{-n}$. It follows that $|\sum b_{M_n(j)}\chi_{M_n(j)}(y) - 1| \leqslant 2^{-n} + |\sum b_{M_n(j)}\chi_{M_n(j)}(x) - 1| \leqslant 2^{-n+1} \to 0$ as $n \to \infty$. Thus E is weak Dirichlet.

Remark. E is a translational set (un ensemble de translation [5] Ch. 1). We remind the reader of the first example, due to Piatecki Shapiro, of a set of multiplicity which is not of strict multiplicity. This was the set $p = \{\sum_{r=0}^{\infty} \varepsilon_r 2^{-r} : \sum_{r=0}^{n} \varepsilon_r \leqslant n/k$ for all $n \geqslant 0$, $\varepsilon_r = 0$ or $\varepsilon_r = 1\}$ $[k > 2]$ (see [15]). As constructed $E \not\subseteq P$ and $P \not\subseteq E$ but E and P have certain resemblances.

Having got Lemma 1.11 it is natural to try a similar attack on Theorem 1.1. Surprisingly only one further difficulty arises. It turns out that we must be able to bound $\|T\|_M$ in Lemma 1.9'' independently of $M(m+1)$. (We shall draw attention to the point in the argument where this fact is needed). But (as the reader may have suspected) our derivation of Lemma 1.9'' threw away more of our knowledge of E than necessary. In fact (introducing more notations than we have specifically defined) we can find a_S with $\sum_{\emptyset \neq S \subseteq \{1,2,..,n\}} |a_S| \leqslant C(K\ \lambda)$ such that $\sum a_S \mu_S = T$ has the property (iii)'''. In particular

LEMMA 1.9'''. We can ensure that T in Lemma 1.9'' satisfies

(iv) $\|T\|_{PM} \leqslant C(K\ \lambda)$

where C only depends on K and λ (and, in particular, C does not depend on $M(m+1)$).

We shall restate Lemma 1.9'' as Lemma 2.1 (The central Lemma).

As we stated above some of the work simplifies if we work in D^{∞}. We conclude this section by showing how this happens. The results proved will not be required again and the reader who is unhappy working in D^{∞} may skip the rest of the section (except for the last paragraph). However the calculations of the first part of Section 3 may be easier to understand in the light of what follows.

I should like to thank Dr. Drury for convincing me that the results of this paper generalize to, and simplify in, D^{∞}. In practice any reader who can make sense out of the paragraph that follows knows enough to follow the remainder of the section.

Recall first that D^{∞} is the product ΠD_2 of countably many copies of the topological group $D_2 = \{0, 1\}$. The elements of D^{∞} are thus sequences (ε_j) with $\varepsilon_j = 0, 1$. The dual of D^{∞} consists of sequences (χ_j) with $\chi_j = 0, 1$ and only a finite number of χ_j non zero. We have $\langle(\chi_j), (\varepsilon_j)\rangle = \prod_{j=1}^{\infty}(-1)^{\varepsilon_j \chi_j}$.
We write in an obvious (but non standard) manner $(\varepsilon_j) = \varepsilon(\sum_{k=1}^{\infty} 2\varepsilon_j 3^{-k})$, $(\chi_j) = \chi(\sum_{k=1}^{\infty} \chi_k)$. Thus for example $\varepsilon(2/3 + 2/27)$ is the sequence $(1, 0, 1, 0, 0, \ldots) \in D^{\infty}$, $\chi(7)$ is the sequence $(1, 1, 1, 0, 0, \ldots) \in D^{\infty}$ and $\langle \chi(7), \varepsilon(2/3 + 2/27)\rangle = (-1)^1 (-1)^0 (-1)^1 = 1$.

Let us prove an analogue of Lemma 1.7 (The Linked Set Lemma) for D^{∞}.

Lemma 1.7'. Let $0 \leqslant r_1 < r_2 < \ldots < r_{m+1}$ be integers. Set $\sigma_k = \mathop{*}\limits_{s=r_k}^{r_{k+1}-1} ((\delta_{\varepsilon(0)} + \delta_{\varepsilon(2/3^s)})/2)$ (when $*$ represents a convolution product and δ_x is the Dirac measure at x). Writing $\mu_S = \mathop{*}\limits_{i \notin S} \sigma_k$, $E_S = \operatorname{supp} \mu_S$ $[S \subseteq \{1, 2, \ldots, m\}]$ we have $\mu_S \in M^+$, $\|\mu_S\| = 1$ and

(i) $\|\chi(2^{r_{m+1}}) - 1\|_{C(E_S)} = 0$

(ii) $\|\chi(j) - 1\|_{C(E_S)} = 0$ for $2^{r_i} \leqslant j < 2^{r_{i+1}-1}$, $i \notin S$

(iii) Let $0 \leqslant j < 2^{r_{m+1}}$ so that $j = \sum_{t=0}^{r_{m+1}-1} \gamma_t 2^t$

$\gamma_t = 0, 1$. Let $R = \{k : \gamma_t = 0$ for all $r_k \leqslant t < \gamma_{k+1}\}$. Then

$\hat{\mu}_S(\chi(j)) = 1$ if $R \subseteq S$

$\hat{\mu}_S(\chi(j)) = 0$ otherwise.

Proof. Direct calculation (which the reader should do).

Remark. If in Lemma 1.7 we could take $\delta = \eta = 0$ and restrain $\hat{\mu}_S(r)$ to take the values 0 and 1 the conditions (iii), (iv) and (v) would reduce to something very like (iii) in this lemma.

The fact that condition (iii) of Lemma 1.7' is so much arithmetically simpler than conditions (iii), (iv) and (v) of Lemma 1.7 (though in fact they represent the same phenomenon) enables us to obtain a version of Lemma (2.1) (The Central Lemma) very quickly. Consider the finite space Ψ of non empty subsets of $\{1, 2, \ldots, m\}$ and the collection of functions $f_S : X \to \mathbb{R}$ given by $f_S(R) = 1$ if $S \subseteq R \in \Psi$ $f_S(R) = 0$ otherwise $[S \subseteq \{1, 2, \ldots, m\}]$. Let $K(m,q) = \inf\{\sum_{S \neq \emptyset} A_S : \sum A_S f_S(R) = 1$ for all $R \in \Psi$, card $R \geqslant q\}$.

LEMMA 1.12 (a). With the notation above there exist $a_R \in \mathbb{C}$ $|R \in \Psi$, card $R \geqslant q|$ such that

$$\sum a_R = 1, \quad \Big|\sum_{\text{card } R \geqslant q} a_R f_S(R)\Big| \leqslant K(m,q)^{-1} \quad \text{for} \quad S \neq \emptyset.$$

(b). With the notation of (1) and of Lemma 1.7' we have, writing $T = \sum_{\text{card } R \geqslant q} a_R \mu_R$ and $E = \text{supp } T$, that

(i) $\| \chi(2^{r_{m+1}}) - 1 \|_{C(E)} = 0$

(ii) $\text{card}\{1 \leqslant k \leqslant m : \langle \chi(2^{r_k}), e\rangle \neq 1\} \geqslant q$

(iii) $\|T\|_{PM} = \hat{T}(\chi(0)) = 1 \geqslant K(m,q)|\hat{T}(\chi(j))|$ for $2^{r_1} \leqslant j < 2^{r_{m+1}}$

(iv) $\|T\|_M \leqslant \sum_{\text{card } R \geqslant q} |a_R| = C(m,q)$, say, where $C(m,q)$

depends only on m and q.

Proof. (a) Hahn-Banach (we shall reprove this in Section 3).

(b) Since $\text{supp } T \subseteq \bigcup_{\text{card } R \geqslant q} \text{supp } \mu_k = \bigcup_{\text{card } R \geqslant q} E_k$, conditions (i) and (ii) follow directly from conditions (i) and (ii) of Lemma 1.7'.

To prove (iii) observe first that if $0 \leqslant j < 2^{r_{m+1}}$ so that $j = \sum_{t=0}^{r_{m+1}-1} \gamma_t 2^t$ $[\gamma_t = 0 , 1]$ we have by Lemma 1.7' (iii)

$$\hat{T}(\chi(j)) = \sum_{\text{card } R \geqslant q} a_R \hat{\mu}_k(\chi(j)) = \sum_{\text{card } R \geqslant q} a_R f_S(R)$$

where $S = \{k : \gamma_t = 0 \text{ for all } r_k \leqslant t < \gamma_{k+1}\}$. If $0 \leqslant j < 2^{r_1}$ then $S = \emptyset$ and we have

$$\hat{T}(\chi(j)) = \sum_{\text{card } R \geqslant q} a_k = 1 .$$

If $2^{r_1} \leqslant j < 2^{r_{m+1}}$ then $S \neq \emptyset$ and by (a) we have

$$|\hat{T}(\chi(j))| = |\sum_{\text{card } R \geqslant q} a_R f_S(R)| \leqslant K(q,m)^{-1} .$$

Since $\hat{T}(\chi(j))$ is periodic in j with period 2^{r_m+1} (by (i)) this gives (iii).

Finally (iv) follows from the fact that $\mu_R \; _M = 1$ and so $\|T\|_M \leqslant \sum |a_R| \, \|\mu_R\|_M = \sum |a_R|$.

Why do we claim that Lemma 1.12 is a version of Lemma 1.9''' ? Choose a $1 > \lambda > 0$. If we set $q = [\lambda m] + 1$ then Lemma 1.12 (ii) has the form of Lemma 1.9'(ii).

If we knew that $K([\lambda m], m) \to \infty$ as $m \to \infty$ then Lemma 1.12 (iii) would have the form of Lemma 1.9"(iii) and we would be home. It is this combinatorial fact that we shall establish in the second part of Section 3.

I must conclude this introduction by expressing my thanks to several people. To Dr. N. Th. Varopoulos and Dr. S. Drury for their continual advice and encouragement and for suggesting the problems on AA^+ which led to this paper ; to professors J.-P. Kahane and C. McGehee and Dr. J. Stegeman for reading the first draft and suggesting several improvements in presentation (and to other people whose suggestions have simply been incorporated without acknowledgement) ; and finally to Dr. J. H. Conway (though there is no possibility that the reader could overlook his contribution).

§ 2. THE CONSTRUCTION FOR THE MAIN THEOREM.

The object of this section is to construct a non zero pseudofunction S supported by a weak Kronecker set. We use the following central lemma which we discussed in § 1 and shall prove in § 3.

LEMMA 2.1 (The Central Lemma). Given $K > 1$, $1 > \lambda > 0$, we can find a $C(K, \lambda) \geqslant 1$ and an $m(K, \lambda) \in \underline{\underline{Z}}^+$ with the following property : -

Given $1 > \varepsilon > 0$ we can find an $N(\varepsilon, K, \lambda) \geqslant 1$ with the following property :-

Given $\delta > 0$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ (such that $h(r) > r$) with the following property : -

Given $N(\varepsilon, K, \lambda) = \frac{1}{2} M(0) < h(M(0)) < M(1) < h(M(1)) < M(2) < \ldots < h(M(m)) < M(m+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$ we can find a finite set $E \subseteq [-\varepsilon, \varepsilon]$ and $T \in M(E) = PM(E)$ such that

(i) $M(m+1)E = 0$

(ii) There exist $b_{M(j)} \geqslant 0$ with $\sum_{j=1}^{m} b_{M(j)} = 1$ and $\left\| \sum_{j=1}^{m} b_{M(j)} \chi_{M(j)} - 1 \right\|_{C(E)} \leqslant \delta + 2(1-\lambda)$

(ii)' $\operatorname{card}\{1 \leqslant r \leqslant m : |\chi_{M(r)}(x) - 1| \leqslant \delta\} \geqslant \lambda m$ for all $x \in E$

(iii) $\|T\|_{PM} = \hat{T}(0) = 1 \geqslant K \sup_{M(m+1)-M(0) \geqslant r \geqslant M(0)} |\hat{T}(r)|$

(iv) $\|T\|_M \leqslant C$.

Remark 1. Conditions (ii) and (ii)' are, of course, essentially equivalent. We shall use which ever version is more convenient.

Remark 2. The condition $M(r+1)$ an integral multiple of $M(r)$ is artificial.

We shall prove a stronger version of the Central Lemma in which the condition is dropped as Lemma 5.2' (the proof relies on the same ideas but is messier).

Remark 3. Our argument will depend crucially on the order in which we are allowed to choose our constants and the manner in which they depend on one another. In particular it is extremely important that, though C depends on K and λ, it does not depend on $M(m+1)$ which can be taken as large as we like without allowing $\|T\|_{PM} > C$.

We shall need to choose a sequence f_n of functions in $S(\mathbb{T})$ such that $\sup_{|t-s| \leqslant 2^{-2n+2}} |f_n(t) - f_n(s)| \leqslant 2^{-n-10}$ and the f_n are uniformly dense in $S(\mathbb{T})$.

We shall obtain E and S by constructing inductively a sequence of measure μ_M supported on finite sets E_n and taking E to be the topological limit of the E_n, S to be a weak $\sigma(A, PM)$ limit point of the μ_n.

We construct E_n, μ_n subject to the following inductive condition (here $N(\varepsilon , K , \lambda)$ is defined as in Lemma 2.1, but for convenience we suppose, as we may, that $N(\varepsilon , K , \lambda)$ increases as K increases.)

INDUCTIVE CONDITION $L(n)$. At the conclusion of the n^{th} step we have a finite set E_n, a measure $\mu_n \in M(E_n)$, $2^{-4n-8} > \varepsilon(n) > 0$ and an integer $P(n) \geq 1$ such that

(i) $\hat{\mu}_n(0) = 1$

(ii) $\|\mu_n\|_{PM} \leqslant 2 - 2^{-n}$

(iii) $N(\varepsilon(n)/2 , 2^{n+4} \|\mu_n\|_M, 1 - 2^{-n-4}) \leqslant P(n)$.

Remark 1. The initial steps of our construction have no particular importance.

We could for example take $E_0 = \{0\}$, μ_0 the unit mass δ_0 at the origin, $2^{-8} > \varepsilon(0) > 0$ arbitrary, $P(0) = N(\varepsilon(0)/2\ ,\ 2^4\ ,\ 1-2^{-4})$.

Remark 2. Conditions (i) and (ii) have no great importance (they ensure that 0 cannot be a weak $*$ limit point of the μ_n and that $\|\mu_n\|_{PM}$ is bounded). On the other hand (iii) is connected with the most delicate part of our proof. The reader should thus take especial note of the points where we use (iii) and the points where we ensure that (iii) will be satisfied at $n+1$.

LEMMA 2.2. Given E_n, μ_n, $\varepsilon(n)$, $P(n)$ satisfying the inductive condition $L(n)$, we can find E_{n+1}, μ_{n+1}, $\varepsilon(n+1)$, $P(n+1)$ satisfying condition $L(n+1)$ such that in addition

(iv) $\varepsilon(n)/8 \geqslant \varepsilon(n+1)$

(v) $P(n+1) \geqslant 4P(n)$

(vi) $|\hat{\mu}_{n+1}(r)| \leqslant |\hat{\mu}_n(r)| + 2^{-n-4}$ for all $|r| \leqslant P(n)$

(vii) $|\hat{\mu}_{n+1}(r)| \leqslant 2^{-n-2}$ for all $P(n) \leqslant |r| \leqslant P(n+1)$

(viii) $\sup_{y \in E_{n+1}} \inf_{x \in E_n} |x - y| \leqslant \varepsilon(n)$.

Further,

(ix) There exists a $g_{n+1} \in A(T)$, $\|g_{n+1}\|_{A(\mathbf{T})} = 1$ such that g_{n+1} has real Fourier coefficients and writing $F_{n+1} = E_{n+1} + [-2\varepsilon(n+1)\ ,\ 2\varepsilon(n+1)]$ we have

$$\sup_{x \in F_{n+1}} |g_{n+1}(x) - f_{n+1}(x)| \leqslant 2^{-n-1}.$$

Proof. Using the result and the notation of the Central Lemma (2.1) together with part (iii) of Inductive Condition $L(n)$ we proceed as follows :

Write $m = m(2^{n+4} \|\mu_n\|_M, \ 1 - 2^{-n-4})$. We can select $P(n) < M(1) < M(2) < \ldots$

$\ldots < M(n) < M(n+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$ and

(1) $$M(1)\varepsilon(n) \geqslant 2^{n+16} P(n) \|\mu_n\|_M$$

(2) $$M(r+1) \geqslant 2^{n+16} M(r) \qquad [1 \leqslant r \leqslant m]$$

whilst choosing $0 < \varepsilon(n+1) < \min(\varepsilon(n)/32, 2^{-n-16}/M(m))$ we have

(3) $$M(m+1) \geqslant 16M(m) + N(\varepsilon(n+1)/2, 2^{n+5} \|\mu_n\|_M C(2^{n+4} \|\mu_n\|_M, 1-2^{-n-4}), 1-2^{-n-5})$$

in such a way that we can find a $T_{n+1} \in M(\underline{\underline{T}})$ with the following properties. If we write $E_{n+1} = \operatorname{supp} T_{n+1}$ then

(4) $$E^*_{n+1} \subseteq [-\varepsilon(n)/2 \ , \ \varepsilon(n)/2]$$

(5) $$M(m+1)E^*_{n+1} = 0.$$

(6) There exist $b_{M(j)} \geqslant 0$ with $\sum_{j=1}^{m} b_{M(j)} = 1$ and $\| \sum_{j=1}^{m} b_{M(j)} \chi_{M(j)} - 1 \| \leqslant 2^{-n-3}$

(7) $$\|T_{n+1}\|_{PM} = \hat{T}_{n+1}(0) = 1 \geqslant 2^{n+4} \|\mu_n\|_M \sup_{M(m+1)-P(n) \geqslant r \geqslant P(n)} |\hat{T}_{n+1}(r)|$$

(8) $$\|T_{n+1}\|_M \leqslant C(2^{n+4} \|\mu_n\|_M, \ 1-2^{-n-4}).$$

We put $P(n+1) = M(m+1) - P(n)$.

Let $E_n = \{e_1, e_2, \ldots, e_\ell\}$ (with $e_1, e_2, \ldots, e_\ell$ distinct).

By (1) and (2) we can find distinct $e'_u = 2\pi \sum_{j=1}^{m} \frac{V_{u,j}}{M(j)}$ with $V_{u,j} \in \underline{\underline{Z}}$ $[1 \leqslant u \leqslant \ell]$ such that

(9) $$|e_u - e'_u| \leqslant \varepsilon(n)/4 \qquad [1 \leqslant u \leqslant \ell]$$

(10) $$|e_u - e'_u| \leqslant 2^{-n-12}/P(n) \|\mu_n\|_M \qquad [1 \leqslant u \leqslant \ell]$$

(11) $$|\chi_{M(j)}(e'_u) - f_{n+1}(e_u)| \leqslant 2^{-n-5} \qquad [1 \leqslant u \leqslant \ell, \ 1 \leqslant j \leqslant m].$$

We note that (10) yields

(10') $\quad |\chi_r(e_u) - \chi_r(e'_u)| \leqslant 2^{-n-8}/\|\mu_n\|_M$ for all $|r| \leqslant P(n)$, $1 \leqslant u \leqslant \ell$.

Thus, if we write μ'_n for that measure with support

$$E'_n = \{e'_1, e'_2, \ldots, e'_\ell\} \quad \text{and} \quad \mu'_n(\{e'_u\}) = \mu_n(\{e_u\}) \quad [1 \leqslant u \leqslant \ell]$$

we have

(10") $$|\hat{\mu}'_n - \hat{\mu}_n(r)| \leqslant 2^{-n-8} \quad \text{for all} \quad |r| \leqslant P(n).$$

We remark that

(12) $$M(m+1)e'_u = 0.$$

Finally we tidy up by noting that (4), the inductive condition $2^{-4n-8} > \varepsilon(n) > 0$ and the definition of f_n give $|f_{n+1}(e_u) - f_{n+1}(e'_u)| \leqslant 2^{-n-1}$ so that (11) gives

(11') $$|\chi_{M(j)}(e'_u) - f_{n+1}(e'_u)| \leqslant 2^{-n-4}.$$

We set $\mu_{n+1} = \mu'_n * T_{n+1}$, $E_{n+1} = E'_n + E^*_{n+1}$ and proceed to demonstrate that the E_{n+1}, μ_{n+1}, $\varepsilon(n+1)$, $P(n+1)$ which we have constructed satisfy the conclusions of the lemma.

Since E'_n, E^*_{n+1} are finite, E_{n+1} is, and since $\mu'_n \in M(E'_n)$, $T_{n+1} \in M(E^*_{n+1})$, we have $\mu_{n+1} \in E_{n+1}$. Further, by (5) and (12) we have $M(m+1)E_{m+1} = 0$. By the choice of $\varepsilon(n+1)$ we have at once $\varepsilon(n+1) < 2^{-4n-12}$. By (7) we have $\hat{\mu}_{n+1}(0) = \hat{\mu}'_n(0)\,\hat{T}_{n+1}(0) = 1.1 = 1$ and so condition $L(n+1)(\underline{\underline{i}})$ is satisfied. By (7) and (10")

$$|\hat{\mu}_{n+1}(r)| \leqslant |\hat{\mu}'_n(r)|\,|\hat{T}_{n+1}(r)| \leqslant (|\mu_{n+1}(r)| + 2^{-n-8}\,\|T\|_{PM}$$
$$\leqslant 2 - 2^{-n-1} \quad \text{for} \quad |r| \leqslant P(n).$$

By (7) and the definition of μ'_n

$$|\hat{\mu}_{n+1}(r)| \leqslant |\hat{\mu}_n'(r)|\ |\hat{T}_{n+1}(r)| \leqslant \|\mu_n'\|_M\, 2^{-n-4}/\|\mu_n\|_M = 2^{-n-4}$$

for $P(n) \leqslant r \leqslant M(m+1) - P(n)$.

Thus $\sup_{-P(n) \leqslant r \leqslant M(m+1)-P(n)} |\hat{\mu}_{n+1}(r)| \leqslant 2 - 2^{-n-1}$. But $M(m+1)E_{n+1} = 0$ and so $\hat{\mu}_{n+1}$ has period $M(m+1)$. It follows that $\|\mu_{n+1}\|_{PM} \leqslant 2 - 2^{-n-1}$, i.e. condition $L(n+1)(\underline{\underline{ii}})$ is satisfied.

To show that condition $L(n+1)(\underline{\underline{iii}})$ is satisfied, we remark that by (8)

$$\|\mu_{n+1}\|_M \leqslant \|\mu_n'\|_M \|T_{n+1}\|_M = \|\mu_n\|_M \|T_{n+1}\|_M \leqslant \|\mu_n\|_M\, C(2^{n+4}\|\mu_n\|_M,\ 1-2^{-n-4})$$

whilst by (3) (since $M(m) > P(n)$)

$$P(n+1) \geqslant M(m+1) - M(m) \geqslant N(\varepsilon(n+1)/2, 2^{n+5}\|\mu_n\|_M\, C(2^{n+4}\|\mu_n\|_M, 1 - 2^{-n-4}), 1-2^{-n-5}).$$

Thus $P(n+1) \geqslant N(\varepsilon(N+1)/2\ ,\ 2^{n+5}\|\mu_{n+1}\|_M\ ,\ 1-2^{-n-4})$ as required, and we have succeeded in re-establishing the induction.

We turn to the remaining conditions of the lemma. Condition $(\underline{\underline{iv}})$ follows directly from the definition of $\varepsilon(n+1)$, $(\underline{\underline{v}})$ follows from (10") and (7) which together give

$$\begin{aligned} |\hat{\mu}_{n+1}(r)| &\leqslant |\hat{\mu}_{n+1}'(r)|\ |\hat{T}_{n+1}(r)| \\ &\leqslant (|\hat{\mu}_n(r)| + 2^{-n-8})\|T_{n+1}\|_{PM} \\ &\leqslant |\hat{\mu}_n(r)| + 2^{-n-8} \qquad \text{for all } |r| \leqslant P(n) \end{aligned}$$

and $(\underline{\underline{vii}})$ was proved in the paragraph immediately following the definition of T_{n+1}.

The proof of condition $(\underline{\underline{ix}})$ is slightly more complicated.

Set $g_{n+1} = \sum_{j=1}^{m} b_{M(j)}\chi_{M(j)}$. Automatically $g_{n+1} \in A(T)$, $\|g_{n+1}\|_{A(T)} = 1$ and g_{n+1} has real Fourier coefficients. If $x \in F_{n+1} = E_{n+1} + [-2\varepsilon(n+1)\ ,\ 2\varepsilon(n+1)]$ $= E_n' + E_{n+1}^* + [-2\varepsilon(n+1)\ ,\ 2\varepsilon(n+1)]$ then by definition we can write $x = e' + e + y$ where $e' \in E_n'$, $e \in E_{n+1}$ and $|y| \leqslant 2\varepsilon(n+1)$. Thus

$$
\begin{aligned}
|g_{n+1}(x) - f_{n+1}(x)| \leqslant\ & \Big|\sum_{j=1}^{m} b_{M(j)} f_{n+1}(e') - f_{n+1}(e')\Big| \\
& + |f_{n+1}(e') - f_{n+1}(x)| \\
& + \Big|\sum_{j=1}^{m} b_{M(j)} \chi_{M(j)}(e^*) f_{n+1}(e') - f_{n+1}(e')\Big| \\
& + \Big|\sum_{j=1}^{m} b_{M(j)} \chi_{M(j)}(e^*)(\chi_{M(j)}(e') - f_{n+1}(e'))\Big| \\
& + \Big|\sum_{j=1}^{m} b_{M(j)} \chi_{M(j)}(e^*) \chi_{M(j)}(e') - g_{n+1}(x)\Big| \\
= 0 +\ & \sup_{t-s \leqslant 4\varepsilon(n+1)} |f_{n+1}(t) - f_{n+1}(s)| \\
& + \Big|\sum_{j=1}^{m} b_{M(j)} \chi_{M(j)}(e^*) - 1\Big| + \sup_{u \in E_n'} \ \sup_{1 \leqslant j \leqslant m} |\chi_{M(j)}(e') - f_{n+1}(e')| \\
& + \Big|\sum_{j=1}^{m} b_{M(j)} \chi_{M(j)}(e' + e)(1 - \chi_{M(j)}(y))\Big| \\
\leqslant\ & 0 + 2^{-n-4} + 2^{-n-3} + 2^{-n-4} + 2\varepsilon(n+1) \sup_{1 \leqslant j \leqslant m} M(j) \\
\leqslant\ & 2^{-n-1}
\end{aligned}
$$

using the fact that $\sum_{j=1}^{m} b_{M(j)} = 1$ and that f_{n+1} varies slowly (this is the point at which the condition $\sup_{t-s \leqslant 2^{-2n-2}} |f_n(t) - f_n(s)| \leqslant 2^{-n-10}$ which we imposed earlier becomes important ; for any $f \in S(\mathbf{T})$ our construction will eventually give good approximations but only when, as must happen if $\varepsilon(n) \to 0$, we know that f is almost constant on every interval of length 4 (n+1), the results numbered (b) and (11') and the fact that $1 - \chi_{M(j)}(y)$ is small (since we chose $\varepsilon(n+1) \leqslant 2^{-n-16}/M(m)$ y represents a very small perturbation relative to the wavelengths considered).

Remark 1. Let us see, once again, how we bounded $|\hat{T}_{n+1}(r)| = |\hat{\mu}'_{n+1}(r)|\ |\hat{T}_{n+1}(r)|$. For $|r| \leqslant P(n)$ we used the fact that, since μ'_{n+1} is only a slight perturbation of μ_{n+1}, $\hat{\mu}'_n(r)$ is close to $\hat{\mu}_n(r)$, together with the fact that $|\hat{T}_{n+1}(r)| \leq \|T_{n+1}\|_{PM} = 1$. On the other hand if r is large (for example comparable with $M(1)$) μ'_n is not longer a small perturbation of μ_n compared with the wave-length of χ_r. We can (at least at first sight) only use the crudest possible estimate $|\hat{\mu}'_n(r)| \leq \|\mu'_n\|_M = \|\mu_n\|_M$. Since we want $|\hat{\mu}_{n+1}(r)| \leq 2^{-n-2}$, this means that we must ensure $|\hat{T}_{n+1}(r)| \leq K^{-1}$, where $K = 2^{n+2}\|\mu_n\|_M$. What about the gap $|r| > P(n)$, but r not very large compared with $P(n)$? It is possible (and in our construction actually happens) that $|\hat{\mu}_n(r)|$ is comparable with 1 for some r strictly comparable with $P(n)$. We cannot, therefore, use a perturbation argument to show $|\hat{\mu}'_n(r)|$ small (and in our construction $|\hat{\mu}'_n(r)|$ is, in fact, close to 1 for some $|r| \leq 2P(n)$). To get $|\hat{\mu}_{n+1}(r)|$ small, we must therefore take $|\hat{T}_{n+1}(r)|$ small, and with the apparatus at our disposal, the simplest thing to do is to take $|\hat{T}_{n+1}(r)| \leq K^{-1}$. (It is possible to proceed otherwise, but the most obvious ways simply transfer part of the difficulty from one section of the proof to another).

We have thus decided to have $|\hat{T}_{n+1}(r)| \leq K^{-1}$ for $P(n) \leq r \leq M(m+1) - P(n)$. In order to have E_{n+1} close to E_n, we also want $\operatorname{supp} T_{n+1} \subseteq [-\varepsilon(n)/2, \varepsilon(n)/2]$. Thus if we are to use Lemma 2.1, we must have $P(n) \geq N(\varepsilon(n)/2\ ,\ 2^{n+2}\|\mu_n\|_M, 1-2^{-n-4})$, i.e. we must have a condition of type (iii). How are we to re-establish such a condition for n+1 ? We know that $\|\mu_{n+1}\|_M \leq \|\mu'_n\|_M \|T_{n+1}\|_M = \|\mu_n\| \|T_{n+1}\|_M$. But (and this is the important point) $\|T_{n+1}\|_M$ is independent of $M(m+1)$ and so of $P(n+1) = M(m+1) - P(n)$. By choosing $M(m+1)$ sufficiently large (iii) is automatically satisfied.

Remark 2. The chain of ideas that leads to the proof of (ix) runs as follows. We know that if $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2$ is Kronecker, then, if E_1, E_2 are closed and uncountable, $E_1 + E_2$ is not Helson. On the other hand if $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2$ is Dirichlet, then, if E_1, E_2 are closed, $E_1 + E_2$ is Dirichlet. Thinking about these 2 results, we are led to consider the following obvious fact. If $f \in S(\underline{\underline{T}})$, E_1 and E_2 are closed, χ_M is close to f on E_1 and χ_M is close to 1 on E_2, then, provided supp E_2 is a small neighborhood close to 0 and f varies slowly, χ_M is close to f on $E_1 + E_2$. Similarly if χ_M is close to f on most of E_1, then χ_M will be close to f on most of $E_1 + E_2$ and this is the content of (ix).

We are now in a position to give the

Proof of Theorem 1.1. Construct a sequence $(E_{n+1}, \mu_{n+1}, \varepsilon(n+1), P(n+1))$ satisfying the conclusions of Lemma 2.2 $[n = 0, 1, 2, \ldots]$. Let E be the topological limit of the E_n, i. e. let

$$E = \{x : \text{we can find } x_n \in E_n \text{ with } x_n \to x \text{ as } n \to \infty\},$$

(note that E is automatically closed). Conditions (iv) and (vii) of Lemma 2.2 show that, given $y \in E$ we can find an $x \in E_n$ with $|x - y| \leqslant 2\varepsilon(n)$. Condition (ix) now gives $\sup_{x\in E} |g_n(x) - f_n(x)| \leqslant 2^{-n}$ since the f_n are uniformly dense in $S(\underline{\underline{T}})$ this means that $\inf_{n\in\mathbb{Z}} \sup_{x\in E} |g_n(x) - f(x)| = 0$ for all $f \in S(T)$ and so E is weak Kronecker.

On the other hand, since (by the Inductive Condition $L(n)$ (ii)) $\|\mu_n\|_{PM}$ is bounded, the weak $\sigma(A, PM)$ compactness of the unit ball shows that μ_n has a weak limit point S say (see also Remark 1 following this proof). We investigate the properties of S.

If $f \in A(\mathbb{T})$, $\operatorname{supp} f \cap E = \emptyset$, then $\operatorname{supp} f \cap E_n = \emptyset$ and so $\langle f, \mu_n \rangle = 0$ for all sufficiently large n (recall that $\mu_n \in M(E_n)$). Thus $\langle f, S \rangle = 0$ and $\operatorname{supp} S \subseteq E$. Since (by the Inductive Condition $L(n)$ (i)) $\hat{\mu}_n(0) = 1$ for all n, it follows that $\hat{S}(0) = 1$ and in particular $S \neq 0$. Using conditions (vi) and (vii) of Lemma 2.2, we see that $|\hat{\mu}_q(r)| \leq 2^{-n-1}$ for all $P(n) \leq |r| \leq P(n+1)$, $q \geq n+1$ and so $|\hat{S}(r)| \leq 2^{-n-1}$ for all $P(n) \leq |r| \leq P(n+1)$. Since (by (v)) $P(n) \to \infty$ as $n \to \infty$, this shows that $\hat{S}(r) \to 0$ as $|r| \to \infty$. Thus S is a non zero pseudofunction and the proof is complete.

Remark 1. A useful remark of Salem (proved in Lemma 4.1) tells us that $\|1 - \chi_r\|_{C(F)}$ small implies $\|1 - \chi_r\|_{A(F)}$ small. In particular, if T is a pseudomeasure with $\|T\|_{PM} \leq 2$ say and support in $[-\varepsilon, \varepsilon]$ with $\varepsilon > 0$ small, it follows from the fact that $\|1 - \chi_r\|_{C([-\varepsilon, \varepsilon])}$ is small for r near to 0 that $\hat{T}(r)$ is close to $\hat{T}(0)$. Thus, if μ_{n+1} is constructed as in the proof of Lemma 2.2, we have $\hat{T}_{n+1}(r)$ close to 1, $\hat{\mu}'_n(r)$ close to $\hat{\mu}_n(r)$ and so $\hat{\mu}_{n+1}(r)$ close to $\hat{\mu}_n(r)$ for $|r|$ not too large. Doing the calculations explicitly (they are simple and can be simplified still further by taking $\varepsilon(n) \to 0$ very rapidly) we see that $\hat{\mu}_n(r)$ converges as $n \to \infty$ for each r. In this way we see that $\mu_n \to S$ weakly for some $S \in PM$. We thus have S as the limit (not a limit point) of the μ_n and avoid the use of the axiom of choice (nor do we need this axiom anywhere else in this paper.

Remark 2. We note that S is synthesised boundedly by the measures μ_n (all that the theorem says is that S is not synthesisable by measures with support in E).

There are other criteria of thinness besides interpolation properties. In Section § 6 we shall prove (using very minor modifications of the ideas above)

THEOREM 1.1'. Given $\delta(n) \to 0$ $n \geqslant 1$, we can find a weak Kronecker set E which is not of synthesis together with a sequence of integers $Q(n) \to \infty$ such that

$$E \subseteq \{x : x - 2\pi r/Q(n) \leqslant \delta(Q(n)) \quad \text{for some} \quad 1 \leqslant r \leqslant Q(n)\}.$$

By picking $\delta(n) \to 0$ fast enough we can ensure that E is Dirichlet and indeed satisfies the Salem covering condition (see [4]), and that, given H a continuous increasing function with $H(0) = 0$, E has Hausdorff H-measure 0 (cf. also the Hertz arithmetic condition [5] p. 124, [3]).

We remark that a result of Kahane ([4] p. 97) shows that a Dirichlet set cannot support a non zero pseudofunction.

We shall also indicate a proof of

THEOREM 1.1". Given $H : \underline{\underline{R}} \to \underline{\underline{R}}^+$ continuous increasing with $H(0) = 0$, we can find a weak Kronecker set E with Hausdorff H measure 0 which supports a non zero pseudofunction.

The proof of Theorem 1.1" will involve results proved in Section § 5 but the proof of Theorem 1.1' can, if the reader so wishes, be read now.

§ 3. PROOF OF THE CENTRAL LEMMA. Let $m \geqslant q \geqslant 1$ Consider the finite space ψ of non empty subsets of $\{1, 2, \ldots, m\}$ and the collection of functions $f_S : X \to \mathbb{R}$ given by

$$f_S(R) = 1 \qquad \text{if} \quad S \subseteq R \in \Psi$$

$$f_S(R) = 0 \qquad \text{otherwise} \qquad [S \subseteq \{1, 2, \ldots, m\}].$$

Let $\Phi(m,q) = \{S \in \Psi ,\ \text{card}\, S \geqslant q\}$

$$K(m,q) = \inf\Big\{\sum_{S\neq\emptyset} |A_S| \ :\ \sum A_S\, f_S(R) = 1 \quad \text{for all} \quad R \in \Phi(m,q)\Big\}.$$

If the reader has glanced at the last part of Section 1 he will recall that we reduced the problem of proving a version of Lemma 2.1 for D^∞ to that of showing $K([\lambda m], m) \to \infty$ as $m \to \infty$ for all $1 > \lambda > 0$. In the first part of this section we shall show that the proof of Lemma 2.1 for $\mathbf{T}$ can also be reduced to the same problem. The reasoning is parallel to the reasoning for D^∞ but as we promised to readers unhappy with D^∞ we shall make no use of the proof given for D^∞. In the second part we show that indeed $K([\lambda m]\ , m) \to \infty$ and so complete the proof of the result stated in the title of this paper.

First, therefore, we must prove Lemma 1.7 (the Linked Set Lemma). To do this we require various elementary results on the measure

$$\mu_{\varepsilon,M} = \sum_{|r| \leqslant M\varepsilon/2\pi} \delta_{2\pi r/M} / \text{card}\{r : |r| \leqslant M\varepsilon/2\pi\}$$

where M is a positive integer and $1 > \varepsilon > 0$.

LEMMA 3.1. (i) If $1 > \varepsilon > 0$, $M \geqslant 1$ then $\mu_{\varepsilon,M} \in M^+([-\varepsilon, \varepsilon])$, $\|\mu_{\varepsilon,M}\| = 1$ and $M \,\text{supp}\, \mu_{\varepsilon,M} = 0$.

(ii) If $1 > \varepsilon$, $\eta > 0$ and $M \geqslant 80\varepsilon^{-1}\eta^{-1}$ then $|\hat{\mu}_{\varepsilon,M}(s)| \leqslant \eta$ for all $40\varepsilon^{-1}\eta^{-1} \leqslant s \leqslant M - 40\varepsilon^{-1}\eta^{-1}$.

(iii) If $1 > \varepsilon$, $\eta > 0$ and we write μ_ε for the restriction of Lebesgue measure to $[-\varepsilon, \varepsilon]$ normalized to give $\|\mu_\varepsilon\| = 1$, then

$|\hat{\mu}_{\varepsilon,M}(s) - \hat{\mu}(s)| \leqslant \eta$ for all $|s| \leqslant \eta M/40$.

(iv) If $1 > \varepsilon, \eta > 0$ and $M(1)$, $M(2)$ are positive integers with $M(2) \geqslant M(1)$, then

$$|\hat{\mu}_{\varepsilon,M(1)}(s) - \hat{\mu}_{\varepsilon,M(2)}(s)| \leqslant \eta \quad \text{for} \quad |s| \leqslant \eta M/80 .$$

(v) If $1 > \varepsilon, \eta > 0$ and $M(1)$, $M(2)$ are positive integers with $M(2)$ a positive integral multiple of $M(1)$ and $M(1) \geqslant 3200\,\varepsilon^{-1}\eta^{-2}$ then

$$|\hat{\mu}_{\varepsilon,M(2)}(s)| \geqslant \eta \quad \text{implies} \quad |\hat{\mu}_{\varepsilon,M(1)}(s) - \hat{\mu}_{\varepsilon,M(2)}(s)| \leqslant \eta .$$

Proof. (i) Obvious.

(ii) Write $2N+1 = \operatorname{Card}\{r : |r| \leq M\varepsilon/2\pi\}$, $P = [40\varepsilon^{-1}\eta^{-1}]$. We have

$$\begin{aligned}(2N+1)|\hat{\mu}_{\varepsilon,M}(s)| &= \Big|\sum_{r=-N}^{N} \hat{\delta}_{2\pi r/M}(s)\Big| \\ &= \Big|\sum_{r=-N}^{N} \exp(2\pi irs/M)\Big| \\ &= \left|\frac{\sin(N+\frac{1}{2})2\pi s/M}{\sin \pi s/M}\right| \\ &\leqslant \left|\frac{1}{\sin \pi s/M}\right| \\ &\leqslant \frac{M}{P} \leqslant (2N+1)\eta \qquad \text{for all} \quad P \leqslant s \leqslant M-P\end{aligned}$$

as required.

(iii)
$$\begin{aligned}|\hat{\mu}_{\varepsilon,M}(s) - \hat{\mu}_{\varepsilon}(s)| &= \left|\int \chi_s \, d\mu_{\varepsilon,M} - \int \chi_s \, d\mu_{\varepsilon}\right| \\ &\leqslant \sup_{|x-y| \leqslant 2\pi/M} |\chi_s(x) - \chi_s(y)| \\ &\leqslant \sup_{|x-y| \leqslant 2\pi/M} |s(x-y)| \\ &\leqslant \eta \qquad \text{for all} \quad |s| \leqslant \eta M/40\end{aligned}$$

(iv) This follows at once from (iii).

(v) By (ii) (and the fact that, by (i), $\hat{\mu}_{\varepsilon,M}(s)$ is periodic in s with period M) we know that $|\hat{\mu}_{\varepsilon,M(2)}(s)| \geqslant \eta$ implies $s = k\,M(2) + r$ with $k \in \mathbb{Z}$, $|r| \leqslant 40\,\varepsilon^{-1}\eta^{-1}$. In particular, $s = \ell M(1) + r$ with $k \in \mathbb{Z}$, so that

$$\hat{\mu}_{\varepsilon,M(2)}(s) = \hat{\mu}_{\varepsilon,M(2)}(r)\ ,\ \hat{\mu}_{\varepsilon,M(1)}(s) = \hat{\mu}_{\varepsilon,M(2)}(r) \quad \text{and by (iv)}$$

$$|\hat{\mu}_{\varepsilon,M(2)}(s) - \hat{\mu}_{\varepsilon,M(1)}(s)| = |\hat{\mu}_{\varepsilon,M(2)}(r) - \hat{\mu}_{\varepsilon,M(1)}(r)| \leqslant \eta$$

(since $|r| \leqslant 40\,\varepsilon^{-1}\eta^{-1} \leqslant \eta M(1)/80$).

We can now give the

Proof of Lemma 1.7. Let $N(\varepsilon,\eta) = 160{,}000\,\varepsilon^{-1}\eta^{-1}$, $h(r) = 128.10^6([\eta^{-3}]+1)m^2 r$. We put $\sigma_i = \mu_{\delta/10M(i-1),M(i)}$, $\sigma_i' = \mu_{\delta/10M(i-1),M(i+1)}$ $[1 \leqslant i \leqslant m]$ and $\sigma_0' = \mu_{\varepsilon/2,M(1)}$. Let us write $*$ for the convolution product. If $S \subseteq \{1, 2, \ldots, m\}$, we write $\mu_S = \mathop{*}_{i\in S} \sigma_i * \mathop{*}_{i\in S,0\leqslant i\leqslant m} \sigma_i'$ and take $E_S = \operatorname{supp} \mu_S$.

The following facts are obvious

(a) $E_S = \sum_{i\in S} \operatorname{supp} \sigma_i + \sum_{i\notin S,0\leqslant i\leqslant m} \operatorname{supp} \sigma_i'$

(b) $M(i) \operatorname{supp} \sigma_r, M(i+1) \operatorname{supp} \sigma_i' = 0$ $\quad [1 \leqslant i \leqslant m]$

$M(1) \operatorname{supp} \sigma_i' = 0$

(c) $M(i-1) \operatorname{supp} \sigma_i, M(i-1) \operatorname{supp} \sigma_i' \subseteq [-\delta/10, \delta/10]$ $\quad [1 \leqslant i \leqslant m]$.

Further, we have ensured that

(d) $M(i+1)$ is a multiple of $M(i)$, $M(i+1) \geqslant 128.10^6([\eta^{-3}] + 1)m^4 i$ $\quad [1 \leqslant i \leqslant m]$, $M(0) = 3200\,\varepsilon^{-1}\eta^{-1}$.

From (a), (b) and (d) we have at once $E_S \subseteq [-\varepsilon, \varepsilon]$ and

(i) $M(n+1)E_S = 0$.

Suppose now $x \in E_S$. Then by (a) $x = \sum_{j=0}^{n} x_j$ where $x_j \in \operatorname{supp} \sigma_j$ if $j \in S$,

$x_j \in \operatorname{Supp} \sigma'_j$ if $j \notin S$ $[0 \le j \le n]$. If $i \in S$ we have by (b) $M(i)x_j = 0$ for $0 \le j \le i$, whilst by (c) and (d) $M(i)x_j \in [-10^{i-j}\delta, 10^{i-j}\delta]$ for $i+1 \le j \le n$. Thus

$$|\chi_{M(i)}(x) - 1| = |\chi_{M(i)}(\sum_{j=i+1}^{n} x_j) - 1| \le |M(i)\sum_{j=i+1}^{n} x_j| \le \delta$$

and we have

(ii) $\|\chi_{M(i)} - 1\|_{C(E_S)} \le \delta$ whenever $i \in S$.

Since the convolution of positive measures is positive $\mu_S \in M^+(E_S)$. Further

$\|\mu_S\| = \prod_{i \in S} \|\sigma_i\| \prod_{i \notin S, 0 \leqslant i \leqslant m} \|\sigma'_i\| = 1$. We know that

$\hat{\mu}_S(r) = \prod_{i \in S} \hat{\sigma}_i(r) \prod_{i \notin S, 0 \leqslant i \leqslant m} \hat{\sigma}'_i(r)$ and that $|\hat{\sigma}_i(r)| \leqslant \|\sigma_i\| = 1$, $|\hat{\sigma}'_i(r)| \leqslant \|\sigma'_i\| = 1$. In particular, therefore, $|\hat{\mu}_S(r)| \geqslant \eta$ implies $|\hat{\sigma}'_j(r)| \geqslant \eta/m$ for all $j \in S$. Lemma 3.1 (v) now shows that $|\hat{\sigma}'_j(r) - \hat{\sigma}_j(r)| \leqslant \eta/m$ for all $j \in S$. Thus if $S \subseteq T \subseteq \{1, 2, \ldots, m\}$ we have

$$|\hat{\mu}_S(r) - \hat{\mu}_T(r)| = |\prod_{i \in S} \hat{\sigma}_i(r) \prod_{i \in T, 0 \leqslant i \leqslant n} \hat{\sigma}'_i(r)(\prod_{i \in T \setminus S} \hat{\sigma}'_i(r) - \prod_{i \in T \setminus S} \hat{\sigma}_i(r))|$$

$$\leqslant |\prod_{i \in T \setminus S} \hat{\sigma}'_i(r) - \prod_{i \in T \setminus S} \hat{\sigma}_i(r)|$$

$$\leqslant \operatorname{card}(T \setminus S) \sup_{i \in T \setminus S} |\hat{\sigma}'_i(r) - \hat{\sigma}_i(r)|$$

$$\leqslant \eta$$

(using the fact that $|ab - cd| \leqslant |a|\,|b-c| + |c|\,|a-d|$).

Thus in particular

(iii) $|\hat{\mu}_S(r)| \geqslant \eta$ implies $|\hat{\mu}_S(r) - \hat{\mu}_T(r)| \leqslant \eta$

whenever $S \subseteq T \subseteq \{1, 2, \ldots, m\}$ and (since $|a| \leqslant \eta$ implies $|b| \geqslant |a| - \eta$ automatically)

(iv) $|\hat{\mu}_{S\cap T}(r)| \geqslant \min(|\hat{\mu}_S(r)|, |\hat{\mu}_T(r)|) - \eta$

whenever $S, T \subseteq \{1, 2, \ldots, n\}$.

Finally, since by Lemma 3.1 (i) $|\hat{\sigma}_i(r)| \leqslant \eta$ for $M(i)/2 \leqslant |r| \leqslant M(i)/2$ we have $|\hat{\mu}_\emptyset(r)| = \prod_{0\leqslant i\leqslant n} |\hat{\sigma}_i'(r)| \leqslant \eta$ for $M(0)/2 \leqslant |r| \leqslant M(n+1) - M(n)/2$, so noting that $\hat{\mu}_\emptyset$ is periodic with period $M(n+1)$ we have

(v) $|\hat{\mu}_\emptyset(r)| \leqslant \eta$ for $M(0)/2 \leqslant |r| \leqslant M(n+1) - M(0)/2$.

So much for the construction of our linked sets. Now let us see how to use them. First we note that Lemma 1.7 can be restated in a more attractive way.

LEMMA 1.7'. Given $1 > \varepsilon, \eta > 0$ we can find an $N(\varepsilon, \eta) \geqslant 1$ with the following property : -

Given $1 > \delta > 0$ and $m \geqslant 1$ we can find a monotonic increasing function $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ (such that $h(r) > r$) with the following property : -

Given $N(\varepsilon, \eta) < \frac{1}{2} M(0) < h(M(0)) < M(1) < h(M(1)) < \ldots < h(M(m)) < M(m+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$ we can find finite sets $E_S \subseteq [-\varepsilon, \varepsilon]$ and measures $\mu_S \in M^+(E_S)$ with $\|\mu_S\| = 1$ $[S \subseteq \{1, 2, \ldots, m\}]$ such that

(i) $M(m+1)E_S = 0$

(ii) $\|\chi_{M(i)} - 1\|_{C(E_S)} \leqslant \delta$ for each $i \in S$

(iii) For each $N \leqslant |r| \leqslant M(m+1) - N$ there exists a $|\lambda_r| \leqslant 1$ and an $\emptyset \neq R(r) \subseteq \{1, 2, \ldots, m\}$ such that $|\hat{\mu}_S(r) - \lambda_r f_{R(r)}(S)| \leqslant 6(m+1)\eta$.

Proof. Define $N(\varepsilon, \eta)$, h and so on as in Lemma 1.7. We wish to show that conditions (iii), (iv) and (v) of that lemma imply condition (iii) of this one. Suppose therefore that $N \leqslant |r| \leqslant M(m+1) - N$.

Let $\Gamma = \{T \subseteq \{1, 2, \ldots, m\} : |\hat{\mu}_T(r)| \geqslant 2\eta(\text{card } T + 1)\}$. If Γ is empty set $\lambda = 0$. By Lemma 1.7 if $S, T \in \Gamma$ then $S \cap T \in \Gamma$. Thus if Γ is non empty there exists a unique member $R(r)$ of $\sum$ with the smallest number of elements. Set $\lambda_r = \hat{\mu}_{R(r)}(r)$. Using (iii) and (iv) of Lemma 1.7 we have the required result.

Once we have reformulated Lemma 1.7 as Lemma 1.7' we are in a position to use

LEMMA 3.2. There exist $a_T \in \mathbb{C}$ $[T \in \Phi(m,q)]$ such that

(i) $\sum_{T \in \Phi(m,q)} a_T = 1$

(ii) $|\sum_{T \in \Phi(m,q)} a_T f_S(T)| \leqslant (K(m,q))^{-1}$ for all $S \in \Psi$.

Proof. Write $\tilde{f}_T = \tilde{f}_T | \Psi$. Then

$$\Pi = \{\sum_{T \in \Phi(m,q)} A_T f_T : \sum |A_T| < K(q,m)\}$$

is balanced convex subset of $C(\Psi)$ such that $1 \notin \Pi$. By Hahn-Banach, therefore, there exist a $\tau \in M(\Psi)$ with $\int 1 \, d\tau = 1$ and $\left|\int g \, d\tau\right| < 1$ for all $g \in \Pi$. In particular $\left|\int \tilde{f}_\tau d\tau\right| < (K(m,q))^{-1}$. But this is precisely the desired result.

Remark 1. We use a finite dimensional version of Hahn-Banach (which in particular does not depend on the Axiom of Choice).

Remark 2. Note that simple considerations of symmetry show that we can take $a_T \in \mathbb{R}$ and $a_T = a_S$ whenever card T = card S.

Remark 3. In order to estimate $K(m,q)$ we have dualised our problem but there is no reason why the a_T could not be found directly. A suitably happy guess would remove the need for the combinatorial lemmas which conclude this section.

LEMMA 3.3. Let μ_T be chosen as in the conclusion of Lemma 1.7' and a_T as in the conclusion of Lemma 3.2. Then writing $C(m,q) = \sum_{U \in \Phi(m,q)} |a_U|$ and $S = \sum_{U \in \Phi(m,q)} a_U \mu_U$, $E = \operatorname{supp} S$ we have

(i) $M(m+1)E = 0$

(ii) $\| m^{-1} \sum_{r=1}^{m} \chi_{M(r)} - 1 \|_{C(E)} \leqslant 2(m-q)/m + \delta$

(ii)' $\operatorname{card} \{1 \leqslant r \leqslant m : |\chi_{M(r)}(x) - 1| \leqslant \delta\} \leqslant (m-q)/m$ for all $x \in E$

(iii)'(a) $\hat{S}(0) = 1$

(b) $|\hat{S}(r)| \leqslant (K(m,q))^{-1} + 6(m+1)C(m,q)\eta$ for all $M(0)/2 \leqslant |r| \leqslant N-M(0)/2$

(iv) $\| S \|_M \leqslant C(m,q)$.

Proof. If $x \in \operatorname{supp} S$ then $x \in E_R$ for some R with $\operatorname{card} R \geqslant q$. Thus by Lemma 1.7 (i) and (ii) $M(m+1)x = 0$ and $|\chi_{M(i)}(x) - 1| \leqslant \delta$ whenever $i \in R$, so that $|\chi_{M(i)}(x) - 1| \leqslant \delta$ for at least q values of $1 \leqslant i \leqslant m$. Thus (i), (ii) are true and it only remains to verify (iii) and (iv). This is a matter of simple calculation : -

$$\hat{S}(0) = \sum_{U \in \Phi(m,q)} a_U \hat{\mu}_U(0) = \sum a_r = 1$$

$$|\hat{S}(r)| = |\sum_{U \in \Phi(m,q)} a_U \hat{\mu}_U(r)|$$

$$\leqslant |\sum_{U \in \Phi(m,q)} a_U \lambda_r f_{R(r)}(U)| + 6(m+1) \sum_{U \in \Phi(m,q)} a_U \eta$$

$$\leqslant |\lambda_r| (K(m,q))^{-1} + 6(m+1)\, C(m,q)\, \eta$$

$$\leqslant (K(m,q))^{-1} + 6(m+1)C(m,q)\eta$$

$$\|S\|_M \leqslant \sum_{U\in\Phi(m,q)} |a_U| \, \|\mu_U\| = C(m,q).$$

We now make some slight modifications in the form of Lemma 3.3 to bring it into line with Lemma 2.1 (the Central Lemma).

LEMMA 3.4. Suppose m,q positive integers with $1 > q/m > \lambda > 0$. Then given $1 > \varepsilon > 0$ we can find an $N(\varepsilon , m , q) \geqslant 1$ with the following property : -

Given $\delta > 0$ we can find a monotonic increasing function $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ (such that $h(r) > r$) with the following property : -

Given $N(\varepsilon , m , q) = \frac{1}{2} M(0) < h(M(0)) < M(1) < h(M(1)) < M(2) < \ldots < h(M(m)) < M(m+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$ we can find a finite set $E \subseteq [-\varepsilon, \varepsilon]$ and $T \in M(E) = PM(E)$ such that

(i) $M(m+1)E = 0$

(ii) $\|m^{-1} \sum_{r=1}^{m} \chi_{M(r)} - 1\|_{C(E)} \leqslant 2(1 - \lambda) + \delta$

(ii)' $\operatorname{card}\{1 \leqslant r \leqslant m : |\chi_{M(r)}(x) - 1| \leqslant \delta\} \geqslant \lambda m$ for all $x \in E$

(iii) $\|T\|_{PM} = \hat{T}(0) = 1 \geqslant (K(m,q) + 1)^{-1} \sup_{M(m+1)-M(0) \geqslant r \geqslant M(0)} |\hat{T}(r)|$

(iv) $\|T\|_M \leqslant C(m,q)$.

Proof. Take $\eta = (24(m+1)C(m,q)K(m,q)+1)^{-2}$ in Lemma 3.3. then gives

(a) $\hat{S}(0) = 1$

(b) $|\hat{S}(r)| \leqslant (K(m,q)+1)^{-1}$ for all $M(0)/2 \leqslant |r| \leqslant N-M(0)/2$.

Let v be that integer for which $|\hat{S}(r)|$ takes its maximum value in the range $-M(0)/2 \leqslant r \leqslant M(0)/2$. By (a) $|\hat{S}(v)| \geqslant 1$. Since $\hat{S}(r)$ is periodic with period N we have $|\hat{S}(v)| \geqslant |\hat{S}(r)|$ for all $r \in \mathbb{Z}$. Set $T = \chi_{-v}S/(S(v))^{-1}$. We have at once

$\hat{T}(O) = 1 = \|T\|_{PM}$ and $|\hat{T}(r)| \leqslant (K(m,q) + 1)^{-1}$ for all $M(0) \leqslant |r| \leqslant N-M(0)$.

Further $\operatorname{supp} T = \operatorname{supp} S = E$, $\|T\|_M \leqslant \|S\|_M$ and the lemma follows.

We have now reduced our problem to that of finding a good lower bound on $K(m\ ,\ q)$ and this we procede to do

LEMMA 3.5. The general solution of $\sum A_T f_T(S) = 1$ for all $S \in \Phi(m,q)$ is $A_T = (-1)^{\text{card } T-1}(1 + \sum_{\emptyset \neq U \subseteq T} B_U)$ $[T \in \Psi]$, where $B_U = 0$ if $\text{card } U > q$ but otherwise may be chosen freely.

Proof. Observing that

$f_T(S) = 0$ for $\text{card } S < \text{card } T$

$f_T(S) = 0$ for $\text{card } S = \text{card } T$, $T \neq S$

$f_S(S) = 1$

we see that the matrix $(f_T(S))$ $[S \in \Phi(m,q)\ ,\ T \in \Psi]$ is "triangular" and so has rank $\text{card } \Phi(m,q)$ (the largest possible rank ; the reader may find it instructive to write out $(f_T(S))$ in full for $m = 4$, $q = 1, 2, 3, 4$). Thus the system of equations

$$(*) \qquad \sum A_T f_T(S) = 1 \qquad [S \in \Phi(m,q)]$$

has exactly

$$\text{card}\left\{T : T \in \Psi\right\} - \text{card } \Phi(m,q) = \sum_{r=1}^{q} \left\{T : T \in \Psi\ ,\ \text{card } T = r\right\}$$

linearly independent solutions.

On the other hand

$$\sum_{T \in \Psi} f_T(S)(-1)^{\text{card } T-1}(1 + \sum_{\emptyset \neq U \subseteq T} B_U) = \sum_{S \supseteq T \neq \emptyset} (-1)^{\text{card } T-1}(1 + \sum_{\emptyset \neq U \subseteq T} B_U)$$

$$= \left(\sum_{r=1}^{\text{card } S} \binom{\text{card } S}{r}(-1)^{r-1}\right) + \sum_{1 \leqslant \text{card } U \leqslant q} B_U\left(\sum_{U \leqslant T \leqslant S} (-1)^{\text{card } T-1}\right)$$

$$= 1 + \sum_{1 \leqslant \text{card}\, U \leqslant q} (-1)^{\text{card}U} B_U \Big(\sum_{r=0}^{\text{card}S - \text{card}\,U} (-1)^r \binom{\text{card}S - \text{card}U}{r}\Big)$$

$$=1 \quad \text{for all} \quad S \in \Phi(m,q).$$

Thus $A_T = (-1)^{\text{card}T-1}(1 + \sum_{\emptyset \neq U \subseteq T} B_U)$ $[\emptyset \neq T \subseteq \{1, 2, \ldots, m\}]$, where $B_U = 0$ if $\text{Card}\, U > q$, but otherwise may be chosen freely, is a solution and, by the paragraph above, is the most general solution.

LEMMA 3.6 (i). $K(m,q) \geqslant \inf_{D_j \in \mathbb{R},\, D_0=1} \sum_{r=0}^{m} \Big|\sum_{t=0}^{q} \binom{m-q}{r-t} D_t\Big| - 1$.

Proof. Let A_T be given in the form used in the statement of Lemma 3.5. We have

$$\sum_{\{1,2,\ldots,m\} \supseteq T \neq \emptyset} |A_T| = \sum_{\{1,2,\ldots,m\} \supseteq T \neq \emptyset} \Big|1 + \sum_{\emptyset \neq U \subseteq T} B_U\Big|$$

$$= \sum_{r=1}^{m} \sum_{\text{card}T=r} \Big|1 + \sum_{\emptyset \neq U \subseteq T} B_U\Big|$$

$$\geqslant \sum_{r=1}^{m} \Big|\sum_{\text{card}T=r} 1 + \sum_{\text{card}T=r} \sum_{\emptyset \neq U \subseteq T} B_U\Big|$$

$$= \sum_{r=1}^{m} \Big|\binom{m}{r} + \sum_{s=1}^{r} \binom{m-s}{r-s} C_s\Big|$$

where $C_s = \sum_{\text{card}U=s,\, U \subseteq \{1,2,\ldots,m\}} B_U$. Simplifying by writing $C_0 = 1$, and noting that $C_s = 0$ for $s > q$, we have

$$\sum_{\{1,2,\ldots,m\} \supseteq T \neq \emptyset} |A_T| \geqslant \sum_{r=0}^{m} \Big|\sum_{s=0}^{q} \binom{m-s}{r-s} C_s\Big| - 1.$$

Now using the identity

$$\binom{m-s}{r-s} = \sum_{j=0}^{q-s} \binom{q-s}{q-r+j}\binom{m-q}{j}$$

obtained by equating coefficients of x^{r-s} in $(1+x)^{m-s} = (1+x)^{q-s}(1+x)^{m-q}$, we have

$$\sum_{s=0}^{q} \binom{m-s}{r-s} C_s = \sum_{s=0}^{q} \sum_{j=0}^{q-s} \binom{m-q}{j}\binom{q-s}{q-r+j} C_s$$

$$= \sum_{t=0}^{q} \binom{m-q}{r-t} D_t$$

where $D_t = \sum_{s=0}^{t} \binom{q-s}{q-t} C_s$ so that $D_o = 1$, $D_t = 0$ for $t > q$. Thus

$$\sum_{\{1,2,..,m\} \supseteq T \neq \emptyset} |A_T| \geqslant \sum_{r=o}^{m} \left|\sum_{t=o}^{q} \binom{m-q}{r-t} D_t\right| - 1$$ where $D_o = 1$. It follows that

$$K(m,q) \geqslant \inf_{D_o=1, D_t \in \underline{\underline{C}}} \sum_{r=o}^{m} \left|\sum_{t=o}^{q} \binom{m-q}{r-t} D_t\right| - 1$$

$$= \inf_{D_o=1, D_t \in \underline{\underline{R}}} \sum_{r=o}^{m} \left|\sum_{t=o}^{q} \binom{m-q}{r-t} D_t\right| - 1$$

as required.

Remark. It may be helpful to write out the formulae in full for a special case such as $m = 7$, $q = 3$.

At this point the author stuck completely. What follows is due to Dr. J. H. Conway to whom I should like to offer my most grateful thanks.

Observe first as a trivial consequence of Lemma 3.6 (i)

LEMMA 3.6 (ii). $K(m,q) \geqslant \sqrt{\left(\inf_{D_o=1, D_t \in \underline{\underline{R}}} \sum_{r=o}^{m} \left(\sum_{t=o}^{q} \binom{m-q}{r-t} D_t\right)^2\right)} - 1.$

Proof. Use Lemma 3.6 (i) and the obvious inequality

$$\left(\sum_{i=1}^{n} |x_i|\right)^2 \geqslant \sum_{i=1}^{n} |x_i|^2 .$$

We are now in a position to use

LEMMA 3.7 (Conway) :

$$\inf_{x_1, x_2, \ldots, x_r \in \underline{\underline{R}}, x_o=1} \sum_{k=o}^{n+r} \left(\sum_{j=o}^{r} \binom{n}{k-j} x_j\right)^2 = \frac{(n+r+1)(n+r+2)\ldots(2n+r)}{(r+1)(r+2)\ldots(n+r)} .$$

Proof. Set $y_j = (-1)^j x_j$ for $0 \leqslant j \leqslant r$, $y_j = 0$ if $-n \leqslant j \leqslant -1$ or $r+1 \leqslant j \leqslant n+r$.

Then, setting $\nabla z_r = z_r - z_{r-1}$, we have

$$\sum_{k=o}^{n+r} \left(\sum_{j=o}^{r} \binom{n}{k-j} x_j\right)^2 = \sum_{k=o}^{n+r} (\nabla^n y_k)^2.$$

We wish to minimize

$$f(x_1, x_2, \ldots, x_r) = \sum_{k=o}^{n+r} \left(\sum_{j=o}^{r} \binom{n}{k-j} x_j\right)^2.$$

Observe first that $f(\underset{\sim}{x}) \to 0$ as $|\underset{\sim}{x}| \to \infty$ so that f has a global minimum. If we can show that f has a unique stationary point x^* then this must be that global minimum.

Suppose therefore that $\underset{\sim}{w} = (w_1, w_2, \ldots, w_r)$ is a stationary point for f. Taking partial derivatives with respect to $x_1, x_2, \ldots, x_r$ we have (setting $w_o = 1$) that

$$\sum_{k=o}^{n+r} \binom{n}{k-p} \left(\sum_{j=o}^{r} \binom{n}{k-j} w_j\right) = 0 \qquad [1 \leqslant p \leqslant r].$$

Recalling the formula $\sum_{r=o}^{s} \binom{n}{r}\binom{n}{s-r} = \binom{2n}{s}$ obtained by equating coefficients of x^s in $(1+x)^n(1+x)^n = (1+x)^{2n}$ we see that

$$\sum_{j=o}^{r} \binom{2n}{n+p-j} w_j = 0 \qquad [1 \leqslant p \leqslant r].$$

But setting $y_j = (-1)^j w_j$ for $0 \leqslant j \leqslant r$, $y_j = 0$ if $1-n \leqslant j \leqslant -1$ or $r+1 \leqslant j \leqslant n+r$ this may be rewritten as

$$\nabla^{2n} y_k = 0 \qquad [1-n \leqslant k \leqslant 1+r-n].$$

Combining the conclusions of the last three paragraphs we see that some solution of

$$\nabla^{2n} y_k = 0 \qquad [1-n \leqslant k \leqslant 1+r-n],$$

subject to $y_o = 1$, $y_j = 0$ for $1-n \leqslant j \leqslant -1$ or $r+1 \leqslant j \leqslant n+r$, satisfies

$$\inf f(\underset{\sim}{x}) = \sum_{k=o}^{n+r} (\nabla^n y_k)^2 .$$

Now the general solution of

$$\nabla^{2n} z_k = 0 \qquad \text{for all} \quad k$$

is a $2n-1$ th degree polynomial. Thus the system of equations with boundary conditions set up for y_k above has the unique solution

$$y_k = y(k) \qquad\qquad [0 \leqslant k \leqslant r]$$

where $y(j) = (-1)^n \frac{(j+n-1)(j+n-2)\ldots(j+1)}{(n-1)(n-2)\ldots 1} \frac{(j-r-1)\ldots(j-r-n)}{(r+1)\ldots(r+n)}$.

We thus have

$$\inf f(\underset{\sim}{x}) = \sum_{k=o}^{n+r} (\nabla^n y)(k)$$

$$= \frac{(n+r+1)(n+r+2)\ldots(2n+r)}{(r+1)(r+2)\ldots(n+r)}$$

(the last formula being obtained directly for $r = 0$ and extended by induction on r) and the lemma is proved.

Remark. We only need this lemma to show (as Lemma 3.8 below) that $K(ap, bp) \to \infty$ as $p \to \infty$. But to show this a much weaker and less detailed result would suffice. It would be very pleasant if such a simpler result could be obtained by more transparent combinatorial means. However a direct analytic rather than combinatorial proof of the result may be difficult to find, precisely because we use the result to show the failure of common analytic averaging descriptions(e.g. $M(E) = A(E)'$) to characterize certain situations (e.g. $M(E) = PM(E)$).

LEMMA 3.8. If $a > b \geqslant 1$ are fixed integers, then $K(ap\ ,\ bp) \to \infty$ as $p \to \infty$.

Proof. By Lemmas 3.6(ii) and 3.7

$$K(ap\ ,\ bp) \geqslant \sqrt{\left(\frac{(ap+1)(ap+2)\ldots(1ap-bp)}{(bp+1)(bp+2)\ldots ap}\right)}$$

$$\geqslant \sqrt{\left(\frac{(ap+1)(ap+2)\ldots(ap+k)}{(bp+1)(bp+2)\ldots(bp+k)}\right)} \qquad \text{(for } (b-a)p \geqslant k)$$

$$\to \left(\frac{a}{b}\right)^{k/2} \qquad \text{as } p \to \infty .$$

Since k was arbitrary $K(ap, bp) \to \infty$ as $p \to \infty$.

This is the last link in the chain of results we needed. We now have a

Proof of Lemma 2.1. Suppose $1 > \lambda > 0$, $K > 0$ given. Choosing $a > b \geqslant 1$ integers with $\lambda > b/a$, we know by Lemma 3.8 that we can find $p \geqslant 1$ an integer with $K(ap, bp) \geqslant K+1$. Set $m = ap$, $q = bp$, in Lemma 3.4. The conclusions of Lemma 2.1 can then be read off directly from the conclusions of Lemma 3.4 and we are done.

§ 4. FURTHER RESULTS.

In this section we discuss the results to be obtained in the remainder of the paper by modifications of methods used in the first part. Before doing so we assemble some background results in Lemma 4.1. The reader is warned that we shall feel free to use these results or their proofs without reference. He should also note that although the results and proofs are easy they are not trivial. Lemma 4.1(i) goes back at least as far as Salem and has been used to great effect by Varopoulos. Lemma 4.1(ii) is a simple example of its use, due to Kahane. Lemma 4.1(iii) answers a question which remained open for over 30 years (under only a slightly heavier disguise). The credit for ending this remarkable situation is due to Björk and Kaufman (separately).

LEMMA 4.1.

(i) If $\varepsilon > 0$, $n \in \mathbb{Z}$ and E is a closed set with $\| \chi_n - 1 \|_{C(E)} \leqslant \varepsilon$, then $\| \chi_n - 1 \|_{A(E)} \leqslant 200\, \varepsilon^{1/2}$.

(ii) A Dirichlet set cannot support a non zero pseudo function.

(iii) A closed set E is weak Dirichlet if and only if given $\varepsilon > 0$, $N \in \mathbb{Z}$, we can find $a_{N+1}, a_{N+2}, \ldots \geqslant 0$ with $\sum_{r=N+1}^{\infty} a_r = 1$ and $\| \sum_{r=N+1}^{\infty} a_r \chi_r - 1 \|_{C(E)} \leqslant \varepsilon$.

(iii)' Suppose X is a compact Hausdorff space and $h_n \in C(X)$ with $\| h_n \|_{C(X)} \leqslant K$ are given such that $\liminf_{n \to \infty} \int | h_n - 1 | \, d\mu = 0$ for all $\mu \in M^+(X)$. Then given $\varepsilon > 0$ we can find $a_1, a_2, \ldots \geqslant 0$ such that $\sum_{n=1}^{\infty} a_n = 1$, $\| \sum_{n=1}^{\infty} a_n h_n - 1 \|_{C(X)} \leqslant \varepsilon$.

(iv) A closed set E is weak Kronecker if and only if given $f \in S(\mathbb{T})$ and $\varepsilon > 0$ we can find $a_r \geqslant 0$ with $\sum_{n=-\infty}^{\infty} a_n = 1$ and $\| \sum_{n=-\infty}^{\infty} a_n \chi_n - f \| \leqslant \varepsilon$.

Proof. (i) We may suppose $0 < \varepsilon < \frac{1}{4}$. Let $I_\varepsilon : \mathbb{R} \to \mathbb{R}$ be given by

$$I_\varepsilon(x) = x \qquad \text{for} \quad -\varepsilon \leqslant x \leqslant \varepsilon$$

$$I_\varepsilon(x) = 2\varepsilon - x \qquad \text{for} \quad \varepsilon \leqslant x \leqslant 2\varepsilon$$

$$I_\varepsilon(x) = -2\varepsilon - x \qquad \text{for} \quad -2\varepsilon \leqslant x \leqslant -\varepsilon$$

$$I_\varepsilon(x) = 0 \qquad \text{otherwise.}$$

Consider $f_\varepsilon(t) = I_\varepsilon(\sin t/2)$ $[t \in (-\pi, \pi]]$. Clearly f_ε is continuously differentiable almost everywhere, so that $m \hat{f}_\varepsilon(m) = \hat{f}'_\varepsilon(m)$ and by Parsefal's formula $\sum_{n=-\infty}^{\infty} | \hat{f}'_\varepsilon(n) |^2 = \| \hat{f}' \|_{L^2}^2 \leqslant 18.8\, \varepsilon$. Thus

$$\sum_{n=-\infty}^{\infty} | \hat{f}_\varepsilon(n) | = | \hat{f}_\varepsilon(0) | + \sum_{n \neq 0} n^{-1} | n \hat{f}_\varepsilon(n) |$$

$$\leqslant \Big(\sum_{n\neq 0} n^{-2} \sum_{n\neq 0} |n\,\hat{f}_{\varepsilon}(n)|^2\Big)^{1/2}$$

$$\leqslant 100\,\varepsilon^{1/2}.$$

Writing $f_{\varepsilon,n}(x) = I_{\varepsilon}\sin(nx/2)$ $[x \in (-\pi,\pi]]$ we have (making the correct identifications) $f_{\varepsilon,n} = f\circ\chi_n \in A(\mathbf{T})$ and so $\|f_{\varepsilon,n}\|_{A(\mathbf{T})} = \|f_{\varepsilon}\|_{A(\mathbf{T})}$. In particular therefore, if E is a closed set such that $\|1-\chi_n\|_{C(E)} \leqslant \varepsilon$ we have

$$\|1-\chi_n\|_{A(E)} = \|\chi_{n/2} - \chi_{-n/2}\|_{A(E)} = 2\|\sin nx/2\|_{A(E)}$$

$$= 2\|f_{\varepsilon,n}\|_{A(E)} \leqslant 200\,\varepsilon^{1/2}.$$

(ii) Let T be a pseudo measure on E a Dirichlet set. We can find $r(n) \to \infty$, $\varepsilon(n) \to 0$ with $\|\chi_{r(n)} - 1\|_{C(E)} \leqslant \varepsilon(n)$. With the notation of (i)

$$|\hat{T}(m) - \hat{T}(r(n)+m)| = 2|\langle T, e^{ir(n)t} f_{\varepsilon(n),r(n)}\rangle|$$

$$\leqslant 2\|T\|_{PM}\|f_{\varepsilon(n),r(n)}\|_{A(T)}$$

$$\to 0 \qquad\qquad \text{as } n\to\infty.$$

Thus $\limsup_{k\to\infty}|\hat{T}(k)| = \sup_{k\in\mathbf{Z}}|\hat{T}(k)|$.

(iii) The necessity follows from (iii)' below. To prove sufficiency we note that if $a_r \geqslant 0$, $\sum_{r=N+1}^{\infty} a_r = 1$ then $\|\sum_{r=N+1}^{\infty} a_r\chi_r - 1\|_{C(E)} \leqslant \varepsilon$ implies $\|\sum_{r=N+1}^{\infty} a_r(\chi_r - 1)\|_{C(E)} \leqslant \varepsilon$ which in turn implies $\|\sum_{r=N+1}^{\infty} a_r|\mathrm{Re}(\chi_r - 1)|\|_{C(E)} \leqslant 4\varepsilon$ (use the fact that $|\chi_r| = 1$). Thus if $\mu \in M^+(E)$ we obtain $\sum_{r=N+1}^{\infty} a_r\int|\mathrm{Re}(\chi_r - 1)|\,d\mu \leqslant 4\varepsilon\|\mu\|$, and so, for at least one $q \geqslant N+1$, we have $\int|\mathrm{Re}(\chi_q - 1)|\,d\mu \leqslant 4\varepsilon\|\mu\|$ which in turn yields $\mu\{x : |\chi_q(x) - 1| \geqslant 8\varepsilon^{1/2}\} \leqslant 8\varepsilon^{1/2}\|\mu\|$ (our estimates, as usual, being on the safe side). Allowing $\varepsilon \to 0$, $N\to\infty$ we have the result required.

(iii)' Let $\Gamma = \{\sum_{r=1}^{\infty} a_r h_r - 1 : \sum_{r=1}^{\infty} a_r = 1, \quad a_r \geqslant 0\}$. Then Γ is a convex subset of $C(E)$ and so its uniform closure $\bar{\Gamma}$ is also convex. Hence if $0 \notin \bar{\Gamma}$ the theorem of Hahn Banach shows that there exists a $\mu \in M(E)$ with $\left|\int h \, d\mu\right| \geqslant \delta > 0$ for all $h \in \bar{\Gamma}$. In particular $\left|\int (h_n - 1) d\mu\right| \geq \delta > 0$ and so $\int |h_n - 1| \, d|\mu| \geq \delta$ for all n. The contradiction proves the result.

(iv) Proof as for (iii).

Remarks.

(i) Lemma 4.1 (i) illuminates the privileged position of Dirichlet sets as against weak Dirichlet sets. Dirichlet sets are not simply sets with good uniform approximation properties but also, owing to the structure of $A(\mathbb{T})$, with good $A(\mathbb{T})$ norm approximation properties. The same kind of remark applies to Kronecker sets. Since the moral of this paper is that Kronecker sets are not typical Helson sets and Dirichlet sets are not typical N sets the point is worth thinking about.

(ii) Our proof of Lemma 4.1 (iii)' uses the Axiom of Choice. One simple consequence of Lemma 4.1 (iii)' is the following (well known in the theory of sup norm algebras). If $h_n \in C(X)$, $\|h_n\|_{C(X)} \leqslant K$ and $h_n(X) \to 0$ as $n \to \infty$ for each $x \in X$ then given $\varepsilon > 0$ we can find $a_r \geqslant 0$ such that $\sum_{n=1}^{\infty} a_n = 1$ and $\|\sum_{n=1}^{\infty} a_n h_n - 1\|_{C(X)} \leqslant \varepsilon$. Can we avoid appealing to non classical theorems in the proof of this ?

(iii) For further results on weak Kronecker and weak Dirichlet sets the reader is refered to [10] and [11]. But his time might be better spent reading [1] and [18] which form an elegant commentary on Lemma 4.1 (i).

Having got these results out of the way we turn to discuss the remainder of this paper.

In Section § 5 we prove certain technical variations of Lemma 1.7 (the Linked Set Lemma) of which the following are typical and the most important.

LEMMA 4.2 (The Central Lemma Second Version). Given $K > 1$, $1 > \lambda > 0$, we can find a $C(K, \lambda) \geq 1$, an $m(K, \lambda) \in \underline{\underline{Z}}^+$ and an $N_o(K, \lambda) \geq 1$ with the following property : -

Given $\varepsilon > 0$, $\delta, \gamma > 0$, $\rho_1, \rho_2 > 0$ and $H : \underline{\underline{R}} \to \underline{\underline{R}}$ a continuous monotonic increasing function with $H(0) = 0$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ (such that $h(r) > r$) with the following property : -

Given $\varepsilon^{-1} N_o(K, \lambda) \leq \frac{1}{2} M(0) < h(M(0)) < M(1) < h(M(1)) < M(2) < \ldots < h(M(m)) < M(m+1)$ with $\frac{1}{2} M(0)$ and $M(r)$ integral $[1 \leq r \leq m+1]$, we can find a finite set $E \subseteq [-\varepsilon, \varepsilon]$ such that

(i) $M(m+1)E = 0$

(ii) There exist $b_{M(j)} \geq 0$ with $\sum_{j=1}^{m} b_{M(j)} = 1$ and $\| \sum b_{M(j)} \chi_{M(j)} - 1 \|_{C(E)} \leq \delta + (1-\lambda)$.

(ii)' $\text{card}\{1 \leq r \leq m : |\chi_{M(r)}(x) - 1| \leq \delta\} \geq \lambda m$ for all $x \in E$

(iii) $\|T\|_{PM} = T(0) = 1 \geq K \sup_{M(m+1)-M(0) \geq r \geq M(0)} |\hat{T}(r)|$

(iv) $\|T\|_M \leq C$

(v) $E + [-\rho_1/(M(m+1)-M(0)), \rho_1/(M(m+1)-M(0))]$ can be covered by intervals of length $\ell_i \leq 2\rho_2$ such that $\sum H(\ell_i) \leq \gamma$.

LEMMA 1.7' (The Linked Set Lemma Second Version). The condition $M(r+1)$ a multiple of $M(r)$ may be dropped in Lemma 1.7 and we can demand taking $1/10 > \delta > 0$ that if f_δ is the trapezoidal function with $f_\delta(-\delta) = f_\delta(\delta) = 0$, $f_\delta(-\delta^2) = f_\delta(\delta^2) = 1$, f_δ linear on $[-\delta, -\delta^2][-\delta^2, \delta^2]$, $[\delta^2, \delta]$, $\mathbb{T} \setminus (-\delta, \delta)$ we have

$$\text{(vi)}' \quad \begin{aligned} & f_\delta(\chi_{M(r)}(e)) = 1 \quad \text{if} \quad e \in E_S \quad r \in S \\ & f_\delta(\chi_{M(r)}(e)) = 0 \quad \text{if} \quad e \in E_S \quad r \notin S \qquad [1 \leqslant r \leqslant m,\ S \subseteq \{1, 2, .., m\}] \end{aligned}$$

(In particular the E_S are disjoint.)

Remark. Condition (vi)' is simply a picturesque way of saying $|\chi_{M(r)}(e) - 1| \leqslant \delta^2$ for all $e \in E_S$, $r \in S$, $|\chi_{M(r)}(e) - 1| \geqslant \delta$ for all $e \in E_S$, $r \notin S$.

LEMMA 1.6'. We can ensure that the E_S of Lemma 1.6 are so constructed that there exist $1/10 \geqslant \delta(j) > 0$, $\delta(j) \to 0$ as $j \to \infty$ such that (using the notation of Lemma 1.6)

$$\text{(i)}' \quad \begin{aligned} & f_{\delta(j)}(\chi_{M_i(j)}(e)) = 1 \quad \text{if} \quad e \in E_S, \quad i \in S \\ & f_{\delta(j)}(\chi_{M_i(j)}(e)) = 0 \quad \text{if} \quad e \in E_S, \quad i \in S \end{aligned}$$

$$[1 \leqslant i \leqslant m, \quad \emptyset \neq S \subseteq \{1, 2, \ldots, m\}] .$$

(In particular the E_S are disjoint.)

The proofs (except in the case of Lemma 1.6' where we have not bothered to prove the slightly easier Lemma 1.6) are simply more complicated versions of the proofs already given, the basic ideas being the same. In order to simplify the overall argument of the proof of the existence of a Helson set of non synthesis, we proved the weakest result necessary ; in what follows we shall be less economical.

I should advise against spending too much time on Section § 5 which is only included for the following two reasons. Firstly it is sometimes convenient when trying to work out a construction to have results like the Central Lemma in their most general form ; though usually it turns out later that a weaker form will do equally well. Secondly we start each of the remaining sections by exhibiting a simple technical variation on the methods used in the first three sections and then pushing it as far as it will go. For some of the more complex and less interesting refinements we use general results obtained in Section § 5 rather than redeveloping the necessary machinery. If the reader finds himself interested in a result or proof which uses a lemma from Section § 5 he should either take the lemma on trust or work out some substitute for himself.

With these exceptions the remaining sections are independent of each other (and to a large extent even of Sections § 1 and § 2).

Section § 6 completes our discussion of the relation between synthesis and various thinness conditions by proving THEOREMS 1.1' and 1.1" which we quoted without proof at the end of Section § 2. The proof of THEOREM 1.1" uses Lemma 4.2 (v). We also discuss (without reaching any profound conclusions) the relation between our results and those of Varopoulos and Kaijser on sets $E \subseteq \mathbb{T}$ with $A(E) = V(D^{\infty})$. Sections § 6 and § 9 leave open a number of interesting questions. (But the interest of Section § 9 lies in what we can answer whilst unfortunately the main interest of Section § 6 lies in what we fail to answer.)

The remainder of the paper deals with the question with which we started in

Section § 1. "If we can approximate 1 in measure, how well can we approximate it in A ?"

Consider the Banach sub-algebra $A^+(\underline{\underline{T}}) = \{\sum_{r=0}^{\infty} a_r \chi_r : \sum_{r=0}^{\infty} |a_r| < \infty\} \subseteq A(\mathbb{T})$. We call a closed set $E \subseteq \underline{\underline{T}}$ a zero set (ZA^+ set) if there exists a $0 \neq f \in A^+$ with $f(e) = 0$ for all $e \in E$. If $A^+(E) = A(E)$ we say that E is an AA^+ set with constant $C = \sup_{0 \neq f \in A(E)} \frac{\|f\|_{A^+(E)}}{\|f\|_{A(E)}}$ (by a well known theorem of Banach $C < \infty$).

Clearly $C \geq \sup_{r \in \underline{\underline{Z}}} \|\chi_r | E\|_{A^+(E)}$. On the other hand, given $\varepsilon > 0$, we can find for each $r \in \underline{\underline{Z}}$ an $X_r \in A$ with $\|X_r\|_{A(E)} \leq C + \varepsilon$, $X_r|E = \chi_r|E$. Thus if $f = \sum_{r=-\infty}^{\infty} a_r \chi_r \in A$ we have $f|E = (\sum_{r=-\infty}^{\infty} a_r X_r)|E$ and so

$$\|f\|_{A^+(E)} \leq \|\sum_{r=-\infty}^{\infty} a_r X_r\|_{A^+(E)} = \|\sum_{r=-\infty}^{\infty} a_r X_r\|_{A(E)} \leq \sum |a_r| \, \|X_r\|_{A(E)}$$
$$\leq (C + \varepsilon)\|f\|_{A(E)}.$$

Hence

$$\begin{aligned} C &= \sup_{n \in \underline{\underline{Z}}} \|\chi_n|E\|_{A^+(E)} \\ &= \sup_{n \in \underline{\underline{Z}}} \inf\left\{\sum_{r=0}^{\infty} |a_r| : \chi_n(e) = \sum_{r=0}^{\infty} a_r \chi_r(e) \quad \text{for all} \quad e \in E\right\} \\ &= \sup_{n \in \underline{\underline{Z}}} \inf\left\{\sum_{r=-n}^{\infty} |a_r| : 1 = \sum_{r=-n}^{\infty} a_r \chi_r(e) \quad \text{for all} \quad e \in E\right\} \\ &= \limsup_{n \to \infty} \inf\left\{\sum_{r=n}^{\infty} |a_r| : 1 = \sum_{r=n}^{\infty} a_r \chi_r(e) \quad \text{for all} \quad e \in E\right\}. \end{aligned}$$

Note that an AA^+ set is automatically a ZA^+ set.

Varopoulos has noted

LEMMA 4.1 (v). Every Dirichlet set is AA^+ with constant 1.

Proof. Suppose $\varepsilon > 0$, $f \in A(T)$ given. Then we can find an N such that $\sum_{n=-N}^{N} |\hat{f}(n)| \geq \|f\|_{A(T)} - \varepsilon/2$. Further by Lemma 4.1(i) we can find a $P > N$ such that $\|\chi_P - 1\|_{A(E)} < \varepsilon/2(\|f\|_{A(T)} + 1)$. Setting $g = \sum_{n=-N}^{N} \hat{f}(n)\chi_{n+P}$ we have $\|g - f\|_{A(E)} < \varepsilon$ and $g \in A^+(T)$.

The remainder of the proof is standard but we give it in full, since in Section § 7 we shall repeatedly use results obtained by this kind of argument without going into the proof in detail. Given $\varepsilon > 0$, $f \in A(E)$ it follows by the results of the first paragraph that we can find $g_k \in A^+(T)$ such that

$$\|f - g_1|E\|_{A(E)} \leq \varepsilon/2 \quad , \quad \|g_1\|_{A^+(T)} \leq \|f\|_{A(E)}$$

$$\|f - (g_1+g_2+\ldots+g_n)|E\|_{A(E)} \leq \varepsilon/2^n \quad , \quad \|g_n\|_{A^+(T)} \leq \varepsilon/2^{n-1}.$$

Now $\sum_{i=1} g_i$ converges in $A(T)$ norm to g, say, where $g \in A^+(T)$, $\|g\|_{A(T)} \leq \|f\|_{A(E)} + \varepsilon$ and $\|f - g|E\|_{A(E)} = 0$. Thus $f = g|E$ and $f \in A^+(E)$, $\|f\|_{A^+(E)} \leq \|f\|_{A(E)} + \varepsilon$. Since ε was arbitrary it follows that $\|f\|_{A^+(E)} = \|f\|_{A(E)}$ and the lemma is proved.

We also have the following results due to Drury (vi) and myself (vii).

LEMMA 4.1 .

(vi) If E is Dirichlet then we can find $f \in A^+(T)$ with $f(e) = 1$ if $e \in E$, $|f(x)| < 1$ if $x \notin E$.

(vii) If E_1, E_2 are AA^+ sets with constants C_1, C_2 then $E_1 \cup E_2$ is AA^+ with constant at most $C_1 + C_2 + C_1C_2$. In particular, by (v), the union of n Dirichlet sets is AA^+ with constant at most $2^n - 1$.

(vii)' Suppose X is a compact Hausdorff space and Y_1, Y_2 are closed subsets of X with $Y_1 \cup Y_2 = X$. Then, if $g_1, g_2 \in C(X)$, $g_j|Y_j = 1|Y_j$ $[j = 1,2]$ we have $g_1 + g_2 - g_1g_2 = 1$.

Proof. (vi) See [1]. This result is a considerable refinement of (v).

(vii) Observing that if $g_j \in A(T)$, $\hat{g}_j(r) = 0$ for $r \leqslant w$ then

$$G = g_1 + g_2 - g_1g_2 \in A(\underline{\underline{T}}), \quad \|G\|_{A(\underline{\underline{T}})} \leqslant \|g_1\|_{A(\underline{\underline{T}})} + \|g_2\|_{A(\underline{\underline{T}})} + \|g_1\|_{A(\underline{\underline{T}})}\|g_2\|_{A(\underline{\underline{T}})}$$

and $\hat{G}(r) = 0$ for $r \leqslant w$ we see that (vii)' implies (vii).

(vii)' Observe that $G = 1 - (1-g_1)(1-g_2)$.

In Sections § 7 and § 8 we show that the results of Lemma 4.1 are (in some sense) best possible. The first part of Section § 7 is devoted to the proof of

THEOREM 7.1.

(i) Given $C_1, C_2 \geqslant 1$ we can find closed sets E_1, E_2 such that $E_1 \cap E_2$ consists of 1 point, E_i is AA^+ with constant C_i $[i = 1,2]$ but $E_1 \cup E_2$ is independent with AA^+ constant $C_1+C_2+C_1C_2$.

(ii) Given $C_1, C_2 \geqslant 1$, $\varepsilon > 0$ we can find disjoint closed sets E_1, E_2 such that E_i is AA^+ with constant C_i $[i = 1,2]$ but $E_1 \cup E_2$ is independent with AA^+ constant at least $C_1 + C_2 + C_1C_2 - \varepsilon$.

(iii) Given $\varepsilon > 0$ we can construct n Dirichlet sets $E_1, E_2, \dots, E_n$ which are disjoint (respectively have $E_i \cap E_j = \{x\}$ $[i \neq j]$ for some $x \in \underline{\underline{T}}$) such that $\bigcup_{i=1}^{n} E_i$ is independent with AA^+ constant $2^n - 1 - \varepsilon$ (respectively AA^+ constant $2^n - 1$).

We use the existence of the "strongly separating" function $f_{\delta(j)}$ in the new version of Lemma 1.6 given in Section § 5 and one of the ideas of Drury in his paper [1] (refered to in Section § 1) to construct sets E which will have AA^+ constant exactly C.

We shall also prove the following result.

THEOREM 7.2. We can find $\Lambda(1), \Lambda(2) \subseteq \underline{\underline{Z}}^+$ and closed sets E_1, E_2 such that $E_1 \cap E_2 = \{x\}$ for some $x \in \underline{\underline{T}}$, $E_1 \cup E_2$ is independent, E_j is $AA_{\Lambda(j)}$ with constant 1 $[j = 1,2]$ but

$$\sum_{r \in \Lambda(1) \cup \Lambda(2) \cup -\Lambda(1) \cup -\Lambda(2) \cup \{0\}} |a_r| < \infty \quad , \quad \sum_{r \in \Lambda(1) \cup \Lambda(2) \cup -\Lambda(1) \cup -\Lambda(2) \cup \{0\}} a_r \chi_r(e) = 0$$

for all $e \in E_1 \cup E_2$ has no non-trivial solutions (so in particular $E_1 \cup E_2$ is not $AA_{\Lambda(1) \cup \Lambda(2)}$).

Section § 8 (which, as we have stated before, does not use Section § 7) is devoted to a proof of

THEOREM 8.1. There exists an independent weak Dirichlet set which is not ZA^+.

The second part of Section § 7 deals with a question suggested by the discussion in the two paragraphs preceeding Lemma 1.9. (The question is thus "internally generated" and the reader may prefer to ignore it.) As a temporary notation call a closed set E <u>s-Weak Dirichlet</u> if, given any $\sigma \in M^+(E)$, $n \in \underline{\underline{Z}}$, $\delta > 0$, we can find an $m \geqslant n$ with

$$\sigma\{x \in E : |\chi_m(x) - 1| \leqslant \delta\} \geqslant s\|\sigma\| \qquad [0 \leqslant s \leqslant 1].$$

Just as Dirichlet sets differ from weak Dirichlet sets so weak Dirichlet sets differ from s-Weak Dirichlet sets $[0 \le s < 1]$. The full result we wish to contrast with Lemma 4.1 (iii) is

THEOREM 7.3. Given any $1 > s \geq 0$ we can find an independent s-Weak Dirichlet set E such that, given any $R > 0$, we can find an $\varepsilon(R) > 0$ such that for any $\sum_{m \geq 1} |a_m| \leq R$ we have $\|\sum_{m \geq 1} a_m \chi_m - 1\|_{C(E)} \geq \varepsilon(R)$.

Moreover Theorem 7.3 is (at least qualitatively) best possible.

LEMMA 7.4. If $1 \geq s > 0$ and E is an s-Weak Dirichlet set, then, given any $\varepsilon > 0$, we can find an $R(\varepsilon) > 0$ such that $\limsup_{n \to \infty} \inf\{\|\sum_{m \geq n} a_m \chi_m - 1\|_{C(E)} : \sum_{m \geq n} |a_m| \leq R(\varepsilon)\} \leq \varepsilon$.

In the final section, Section § 9, we consider the tilde algebra

$$\tilde{A}(E) = \{f \in C(E) : \exists f_n \in A(E) \text{ with } \sup_{n \geq 1} \|f_n\|_{A(E)} < \infty, \quad \|f_n - f\|_{C(E)} \to 0\}$$

where, if $f \in \tilde{A}(E)$,

$$\|f\|_{\tilde{A}(E)} = \inf\{\sup_{n \geq 1} \|f_n\|_{\tilde{A}(E)} : \|f_n - f\|_{C(E)} \to 0 \quad \text{as} \quad n \to \infty\}.$$

It is easily verified that $(\tilde{A}(E), \| \|_{\tilde{A}(E)})$ is a Banach algebra. Beurling raised the question whether $\tilde{A}(E) = A(E)$. McGehee and Katznelson showed that in fact there exists a countable E with $\tilde{A}(E) \neq A(E)$ (they proved rather stronger results which the reader will find in [7], [8]), but left open the question whether the embedding of $(A(E), \| \|_{A(E)})$ in $(\tilde{A}(E), \| \|_{\tilde{A}(E)})$ is always isometric. Varopoulos [19] (or [13]) showed that this is not so and that there exist closed sets $E \subseteq \underline{\underline{T}}$ with $A(E)$ not even closed in $(\tilde{A}(E), \| \|_{\tilde{A}(E)})$. Further E may be chosen to be of synthesis.

The interest of this last sentence lies in the following well known equivalence

LEMMA 4.3. The following 3 statements are equivalent for a closed set $E \subseteq \mathbb{T}$ of synthesis

(i) $A(E)$ is closed in $(\tilde{A}(E), \| \ \|_{\tilde{A}(E)})$ with $\sup_{f \in A(E)} \frac{\|f\|_{A(E)}}{\|f\|_{\tilde{A}(E)}} \leqslant K$.

(ii) If $T \in PM(E)$, then we can find $\mu_\alpha \in M(E)$ with $\|\mu_\alpha\|_{PM} \leqslant K \|T\|_{PM}$, $\mu_\alpha \to T$ weakly.

(iii) If $T \in PM(E)$, then we can find $\mu_n \in M(E)$ with $\|\mu_n\|_{PM} \leqslant K \|T\|_{PM}$, $\mu_n \to T$ weakly.

Proof. The equivalence of (i) and (ii) follows from Hahn-Banach. The equivalence of (ii) and (iii) follows from the Banach-Steinhaus theorem.

We say that a set E satisfying the conditions of Lemma 4.2 is of <u>bounded</u> or <u>sequential</u> synthesis with constant K. Varopoulos thus proved

LEMMA 4.4 (Varopoulos). There exist sets which are of synthesis but not of bounded synthesis.

Two other proofs of this result have since been found, the first by Varopoulos ([20]) and the second by Katznelson and McGehee ([8]). The reader will find further information well presented in the works cited ([7], [8], [19], [13]).

The definition of $\tilde{A}(E)$ by itself suggests that the method of this paper might be applicable to the study of tilde algebras. We shall prove the following new results.

THEOREM 9.2. There exists a translational set (ensemble de translation) E with $A(E)$ not closed in $\tilde{A}(E)$.

THEOREM 9.3.

(i) There exist closed sets E_1, E_2 intersecting at a single point such that E_1, E_2 are of bounded synthesis with constant 1 yet $E_1 \cup E_2$ is not of bounded synthesis.

(ii) There exist closed sets E_1 , E_2 intersecting at a single point such that $(A(E_i), \| \|_{A(E_i)})$ is isometrically embedded in $(\tilde{A}(E_i), \| \|_{A(E_i)})$ $[i = 1, 2]$, and $E_1 \cup E_2$ is independent yet $A(E_1 \cup E_2)$ is not closed in $(\tilde{A}(E_1 \cup E_2), \| \|_{\tilde{A}(E_1 \cup E_2)})$.

In proving his theorem Varopoulos characteristically uses methods which require great ingenuity in invention but little computation in execution. It would not surprise me if by increasing the amount of detail a proof of Theorem 9.3 could be extracted from his first proof of Lemma 4.3 given in [19]. He has pointed out in conversation that a weaker result can easily be recovered from this proof.

LEMMA 9.1. There exist disjoint closed sets E_i of bounded synthesis with constant 1 such that $\bigcup_{i=0}^{\infty} E_i$ is closed and of synthesis but not of bounded synthesis.

Indeed I strongly suspect that a result considerably finer than those so far obtained is true and could be obtained by his methods. I conjecture that there exists a closed set E with $A(E)$ dense in $(\tilde{A}(E), \| \|_{\tilde{A}(E)})$ but $A(E) \neq \tilde{A}(E)$.

§ 5. TECHNICAL IMPROVEMENTS.

The first part of this section is devoted to the proof and use of

LEMMA 5.1. Let $1/8 > \varepsilon$, $\eta, \chi > 0$ be such that $\eta > 128\chi$ and

$Q \geqslant 10^{12}\varepsilon^{-1}\eta^{-1}\varkappa^{-2}$. Set $Q_1 = [10^{-8}Q\eta]$, $Q_o = [10^{-20}Q\eta^3]$. Suppose further we are given $1 \geqslant \gamma > 0$ with $H : \mathbb{R} \to \mathbb{R}$ a continuous monotonic function with $H(0) = 0$, together with $1/8 > \rho > 0$ such that $8Q \leqslant \varkappa^2$, $Qh(4\rho) \leqslant \gamma$.

Given $Q' \geqslant 10^{12}\ -1\eta^{-2}$ an integer and closed sets $E_1, E_2 \subseteq \mathbb{T}$ (not necessarily disjoint), we can find $E_1', E_2' \subseteq T$ and linear maps $L_i : M(E_i) \to M(E_i')$ $[i = 1,2]$ such that

(1) $Q'E_i' = 0$

(2) $\sup\limits_{e_i' \in \operatorname{supp} L_i\sigma} \inf\limits_{e_i \in \operatorname{supp}\sigma} |e_i - e_i'| \leqslant \varepsilon$ for all $\sigma \in M(E_i)$

(3) $|(L_i\sigma - \sigma)^\wedge(r)| \leqslant \eta\ \sigma$ for all $|r| \leqslant Q_o$, $\sigma \in M(E_i)$

(4) $L_i\sigma \in M^+(E_i')$, $\|L_i\sigma\|_M = \|\sigma\|_M$ for all $\sigma \in M^+(E_i)$

(and so in particular $(L_i\sigma)^\wedge(0) = \hat{\sigma}(0)$ for all $\sigma \in M(E_i)$) $[i = 1,2]$

(5) $|\chi_Q(x_1) - 1| \leqslant \varkappa^2$, $|\chi_Q(x_2) - 1| \geqslant \varkappa$ for all x_i with $|x_i - e_i'| \leqslant$ for some $e_i' \in E_i'$.

(6a) $|(L_2\sigma)^\wedge(r)| \leqslant \eta\|\sigma\|_M$ for all $Q' - Q_1 \geqslant r \geqslant Q_1$, $\sigma \in M(E_2)$

(6b) $|(L_2\sigma)^\wedge(r)| \leqslant |\hat{\sigma}(r)| + \eta\|\sigma\|_M$ for all $|r| \leqslant Q$

(6c) $|(L_1\sigma)^\wedge(r)| \leqslant \eta\|\sigma\|_M + |\hat{\sigma}(r)|$ for all $r \in \mathbb{Z}$, $\sigma \in M(E_1 \cap \{2\pi r/Q : 1 \leqslant r \leqslant Q\})$

(7) $|(L_2\sigma_2 - L_1\sigma_1)^\wedge(r)| \leqslant |(\sigma_2 - \sigma_1)^\wedge(r)| + \eta$ for all $|r| \leqslant Q_1$, $\sigma_i \in M(E_i)$, $\|\sigma_i\|_M \leqslant 1$ $[i = 1,2]$.

(8) $\{x_1 : |x_1 - e_1'| \leqslant \rho$ for some $e_1' \in E_1'\}$ can be covered by intervals of length $\ell_i \leqslant 2\rho$ with $\sum H(\ell_i) \leqslant \gamma$.

Remark. The introduction of the maps L_i may obscure a fairly simple situation.

If the reader is unhappy about their meaning, he should skim through the proof that follows and the use of Lemma 5.1 in proving Lemma 5.2 considering the effect of L_1, L_2 on 2 fixed measures σ_1, σ_2. In any case the reader will surely find it easier to consider the proof that follows as a collection of hints towards constructing his own proof. (However, the proof is given in full detail.)

Proof. This is a more complicated version of the proof of Lemma 1.7 from which we borrow the results of Lemma 3.1 together with the notation $\mu_{\varepsilon,M}$.

Define $L'' : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ by

$$L''\sigma = \sum_{s=1}^{Q} \sigma\left(\left[(2s-1)\pi/Q\ ,\ (2s+1)\pi/Q\right)\right)\delta_{2\pi s/Q} \qquad [i = 1,2].$$

If K is a compact set in $\underline{\underline{T}}$, write $K'' = \bigcup_{\sigma \in M(K)} \operatorname{supp} L''\sigma$. It is clear that L'' is linear and

(2)'' $\quad \sup_{k \in \operatorname{supp}\sigma}\ \inf_{k'' \in \operatorname{supp} L''\sigma} |k - k''| \leqslant \pi/Q$ for all $\sigma \in M(K)$

(3)'' $\quad |(L''\sigma - \sigma)^\wedge(r)| \leqslant \|\sigma\|_M \sup_{x \leqslant \pi/Q} |\chi_r(x) - 1| \leqslant \|\sigma\| \pi Q_1/Q$ for $|r| \leqslant Q_1$

(4)'' $\quad L''\sigma \in M^+(K'')$, $\|L''\sigma\|_M = \|\sigma\|_M$ for all $\sigma \in M^+(K)$

(5)'' $\quad QK'' = 0$.

For each integral $1 \leqslant s \leqslant Q$ we can find an integral $1 \leqslant t(s) \leqslant Q'$ such that $|2\pi s/Q - 2\pi t(s)/Q'| \leqslant \pi/Q'$. Writing $U = \{2\pi s/Q : 1 \leqslant s \leqslant Q\}$ and define $L''' : M(U) \to M(\underline{\underline{T}})$ by

$$L'''\sigma = \sum_{s=1}^{Q} \sigma(\{2\pi s/Q\})\delta_{2\pi t(s)/Q'} \ .$$

If $K \subseteq U$ write $K''' = \bigcup_{\sigma \in M(K)} \operatorname{supp} L'''\sigma$. It is clear that L''' is linear and

(1)''' $\quad Q'L''' = 0$

(2)''' $\sup_{K''' \in \operatorname{supp} L'''\sigma} \ \inf_{k \in \operatorname{supp}\sigma} |k - k'''| \leqslant \pi/Q'$ for all $\sigma \in M(K)$

(3)''' $|(L'''\sigma - \sigma)^\wedge(r)| \leqslant \|\sigma\| \pi |r|/Q'$ for all $\sigma \in M^+(K)$

(4)''' $L'''\sigma \in M^+(K''')$, $\|L'''\sigma\|_M = \|\sigma\|_M$ for all $\sigma \in M^+(K)$

(5)''' $\|\chi_Q - 1\|_{C(K''')} \leqslant \pi Q/Q'$.

We have now defined 2 "shifting" operations ; we now define 4 "convolution" operations. Write $\varepsilon' = 10^4/Q_1\eta$ so that in particular $0 \leqslant \varepsilon' \leqslant \varepsilon/10$, $Q_1\varepsilon' \geqslant 10^4\eta^{-1}$, $Q_o\varepsilon' \leqslant 10^{-10}\eta$. Write $\varkappa' = 10^{-4}\eta/Q$ so that in particular $0 \leqslant \varkappa' = \varkappa^2/10Q$, $0 \leqslant \varkappa' \leqslant \varepsilon/10$, $Q_1\varkappa' \geqslant 10^4\eta^{-2}$, $Q\varepsilon \leqslant 10^{-4}\eta$. We now define maps $L_i' : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ $[i = 0, 1]$, $L_2' : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ by

$$L_o'\sigma = \sigma * \mu_{\varkappa', Q'}$$

$$L_1'\sigma = \sigma * \mu_{\varepsilon', Q}$$

$$L_2'\sigma = \sigma * \mu_{\varepsilon', Q'} .$$

It is immediate that L_o', L_1', L_2' are linear.

The definition of L_2' (a modification of L_2') is more complex. For every $a \in \underline{\underline{T}}$ write

$$G_a = \{x = a + 2\pi r/Q' : |x - a| \leqslant \varepsilon' \ , \ \inf_{1 \leqslant t \leqslant Q} |x - 2\pi t/Q| > 4\pi\varkappa/Q$$

and if $\sigma \in M(\underline{\underline{T}})$, write L_2' for that function $C(\underline{\underline{T}}) \to \underline{\underline{C}}$ given by

$$L_2'\sigma(f) = \int \langle f \ , \ \gamma(x)\rangle \, d\sigma(x)$$

with $\gamma(a) = \frac{1}{\operatorname{card} G_a} \sum_{x \in G_a} \delta_a$. It is clear that $L_2'\sigma \in M(\underline{\underline{T}})$ and that L_2' is a linear map $M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$. The most important fact about L_2' is that since $|\gamma(a) - \mu_{\varepsilon'/2, Q'} * \delta_a| \leqslant 16\varkappa$ we have

(9) $\|L_2'^{*}\sigma - L_2'\sigma\| \leqslant 16\varkappa\|\sigma\|$

so, in particular,

$$(9)' \qquad |(L_2'^{*}\sigma)^{\wedge}(r) - (L_2'\sigma)^{\wedge}(r)| \leqslant 16\,\varkappa\|\sigma\| \quad \text{for all} \quad r \in \underline{\underline{Z}},$$

and this enables us to use estimates for $L_2'^{*}\sigma^{\wedge}(r)$, such as those obtained in Lemma 1.8, to obtain estimates for $L_2'\sigma^{\wedge}(r)$.

This we proceed to do. Suppose K is a compact set in $\underline{\underline{T}}$. Write $K_i' = \bigcup_{\sigma \in M(K)} \operatorname{supp} L_i'\sigma$ $[i = 0,1,2]$. It is clear that

$$(2)_0' \qquad \sup_{k_0' \in \operatorname{supp} L_0'\sigma} \ \inf_{k_0 \in \operatorname{supp}\sigma} |k_0 - k_0'| \leqslant \varkappa' \quad \text{for all} \quad \sigma \in M(K)$$

$$(2)_j' \qquad \sup_{k_j' \in \operatorname{supp} L_j'\sigma} \ \inf_{k_j \in \operatorname{supp}\sigma} |k_j - k_j'| \leqslant \varepsilon' \quad \text{for all} \quad \sigma \in M(K), \quad j = 1,2$$

$$(3)_{0(a)}' \qquad |(L_0'\sigma)^{\wedge}(r)| \leqslant |\hat{\sigma}(r)||\hat{\mu}_{\varkappa',Q'}(r)| \leqslant |\hat{\sigma}(r)| \quad \text{for all} \quad r$$

$$(3)_{0(b)}' \qquad |(L_0'\sigma - \sigma)^{\wedge}(r)| \leqslant \|\sigma\| \sup_{|x| \leqslant \varkappa'} |\chi_r(x) - 1| \leqslant \|\sigma\| \; |r|\varkappa'$$

$$(3)_{0(c)}' \qquad |(L'\sigma)^{\wedge}(r)| \leqslant \|\sigma\|\eta \quad \text{for all} \quad Q' - 40\varkappa'^{-1}\eta^{-1} \geqslant r \geqslant 40\varkappa'^{-1}\eta^{-1}$$

$$(3)_{1(a)}' \qquad |(L_1'\sigma)^{\wedge}(r)| \leqslant |\hat{\sigma}(r)| \quad \text{for all} \quad r$$

$$(3)_{1(b)}' \qquad |(L_1'\sigma - \sigma)^{\wedge}(r)| \leqslant \|\sigma\|_M |r|\,\varepsilon'$$

$$(3)_{2(b)}' \qquad |(L_2'\sigma - \sigma)^{\wedge}(r)| \leqslant \|\sigma\|_M \sup_{|x| \leqslant \varepsilon'} |\chi_r(x) - 1| \leqslant \|\sigma\|_M |r|\,\varepsilon'$$

and

$$(4)_i' \qquad L_i\sigma \in M^{+}(K_i'), \quad \|L_i'\sigma\|_M = \|\sigma\|_M \quad \text{for all} \quad \sigma \in M^{+}(K).$$

By Lemma 1.8(ii) (which we also used above in proving $(3)_{0(c)}'$) we have

$$(6)_{2(a)}' \qquad |(L_2'^{*}\sigma)^{\wedge}(r)| = |\hat{\mu}_{\varepsilon',Q}(r)||\hat{\sigma}(r)| \leqslant 80\,\varepsilon'^{-1}Q'^{-1}\|\sigma\|$$

for all $Q_1 \leqslant r \leqslant Q - Q_1$, whilst trivially

$$(6)_{2(b)}' \qquad |(L_2\sigma)^{\wedge}(r)| \leqslant |\hat{\sigma}(r)| \quad \text{for all} \quad r,$$

so that, using $(9)'$, we obtain

$(6)'_{2(a)}$ $\qquad |(L'_2\sigma)^\wedge(r)| \leqslant (40\varepsilon'^{-1}Q'^{-1} + 16\mathcal{K})\|\sigma\|$

for all $Q_1 \leqslant r \leqslant Q-Q_1$ and

$(6)'_{2(b)}$ $\qquad |(L'_2\sigma)^\wedge(r)| \leqslant |\hat{\sigma}(r)|$ for all r.

Next, using Lemma 1.8(iii), we have

$$(7)'^{*}_{12} \quad |(L'_1\sigma)^\wedge(r) - (L'^{*}_2\sigma)^\wedge(r)| = |((\mu_{\varepsilon',Q} - \mu_{\varepsilon',Q'})*\sigma)^\wedge(r)|$$

$$\leqslant |(\mu_{\varepsilon',Q} - \mu_{\varepsilon',Q'})^\wedge(r)|\,\|\sigma\|$$

$$\leqslant 400(Q_1/Q + Q_1/Q')\,\|\sigma\|$$

so that by (9)'

$$(7)'_{12} \quad |(L'_1\sigma)^\wedge(r) - (L'_2\sigma)^\wedge(r)| \leqslant (400(Q_1/Q + Q_1/Q') + 16\mathcal{K})\,\|\sigma\|$$

for all $|r| \leqslant Q_1$.

We are now in a position to define L_i, E'_i and obtain their properties. Write

$$L_1 = L'_0 \circ L''' \circ L'_1 \circ L''|E_1$$

$$L_2 = L'_2 \circ L''' \circ L''|E_2$$

$$E'_1 = \bigcup_{\sigma\in M(E_1)} \text{supp } L_1\sigma$$

$$E'_2 = \bigcup_{\sigma\in M(E_2)} \text{supp } L_2\sigma.$$

Observe first that $L_1 : M(E_1) \to M(E'_1)$, $L_2 : M(E_2) \to M(E'_2)$ being the compositions of linear mappings are themselves linear. If $\sigma \in M(E_1)$ then by (5)'' Q supp $L''\sigma = 0$ so, by the definition of L'_1, $Q\text{ supp}(L'_1 \circ L'')\sigma \subseteq Q(\text{supp } L''\sigma + \text{supp } \mu_{\varepsilon',Q}) = 0$ (the notation is slightly abusive). (We used this fact implicitly to give $L''' \circ L'_1 \circ L''$ well defined.) By (1)''' $Q'\text{ supp}(L''' \circ L'_1 \circ L'')\sigma = 0$ and so $Q'\text{ supp } L_1\sigma \subseteq Q'(\text{supp}(L''' \circ L'_1 \circ L'') + \text{supp } \mu_{\mathcal{K}',Q'}) = 0$. Again $L'''(L''\sigma)$ is indeed well defined and $Q'\text{ supp } L'''(L''\sigma) = 0$ and so, examining the definition of L'_2, we have

$Q' \operatorname{supp}(L_2' \circ L''' \circ L'')\sigma = 0$. We have thus proved (1). By (2)'', (2)''', $(2)_o'$ and $(2)_i'$ we have

$$\sup_{e_i' \in \operatorname{supp} L_i\sigma} \inf_{e_i \in \operatorname{supp}\sigma} |e_i - e_i'| \leqslant \pi/Q + \pi/Q' + \varkappa' + \varepsilon' \leqslant \varepsilon$$

for all $\sigma \in M(E_i)$ $[i = 1,2]$ so that (2) is proved. Next we note that by (4)'', (4)''', $(4)_o'$

$$\|L''\sigma\| \leqslant \|\sigma\|, \quad \|L'''\sigma\| \leqslant \|\sigma\|, \quad \|L_i'\sigma\| \leqslant \|\sigma\| \quad |i = 0,1,2| \ ;$$

we shall use these facts repeatedly. For example (3)'', (3)''' and $(3)_i'$ now give

$$|(L_i\sigma - \sigma)^\wedge(r)| \leqslant \|\sigma\| \, (\pi Q_1/Q' + \pi Q_1/Q' + Q_o\varepsilon' + Q_o\varkappa') \leqslant \eta\|\sigma\|$$

for $|r| \leqslant Q_o$, so (3) is proved.

As we remarked above $Q \operatorname{supp}(L_1' \circ L'')\sigma = 0$ for all $\sigma \in M(E_1)$ so by (2)''' and $(2)_o'$

$$\begin{aligned}|\chi_Q(x_1) - 1| &\leqslant |\chi_Q(x_1) - \chi_Q(e_1)| + |\chi_Q(e_1) - 1| \\ &\leqslant Q|x_1 - e_1| + Q\pi/Q' + Q\varkappa' = Q\rho + 10^{-3}\pi Q + \varkappa^2/10 \leqslant \varkappa^2\end{aligned}$$

whenever $|x_1 - e_1'| \leqslant \rho$ for some $e_1' \in E_1'$. Again $Q' \operatorname{supp}(L''' \circ L'')\sigma = 0$ for all $\sigma \in M(E_2)$ so examining the definition of L_2' and in particular the form of the measure $\gamma(2\pi s/Q')$ we see that $e_2' \in \operatorname{supp} L_2'((L''' \circ L'')\sigma)$ implies $\inf_{1 \leqslant t \leqslant Q} |e_2' - 2\pi t/Q)| \geqslant 4\pi\varkappa/Q$. Thus

$$\begin{aligned}\inf_{1 \leqslant t \leqslant Q} |x_2 - 2\pi t/Q| &\geqslant \inf_{1 \leqslant t \leqslant Q} |e_2' - 2\pi t/Q| - \\ &\geqslant 4\pi\varkappa/Q - \varkappa^2/8Q \geqslant 2\pi\varkappa/Q\end{aligned}$$

and so $|\chi_Q(x_2) - 1| \geqslant \varkappa$ for all x_2 with $|x_2 - e_2'| \leqslant \rho$ for some $e_2' \in E_2'$. Thus (5) is proved.

From $(6)_1'$ we obtain

$$|(L_2\sigma)^\wedge(r)| \leqslant (80\varepsilon'^{-1} Q_o^{-1} + 16\varkappa)\|(L''' \circ L'')\sigma\|$$

$$\leqslant (80\,\varepsilon'^{-1} Q_o^{-1} + 16\varkappa)\|\sigma\|$$

$$\leqslant \eta\|\sigma\| \qquad \text{for} \quad Q' - Q_1 \geqslant r \geqslant Q_1, \qquad \sigma \in M(E_2)$$

so that (6a) is verified. Again

$$|(L_2\sigma)^\wedge(r)| \leqslant |(L''' \circ L''\sigma)^\wedge(r)|$$

$$\leqslant |\hat{\sigma}(r)| + \|\sigma\| \, (\pi Q_1/Q + \pi Q_1/Q')$$

$$\leqslant |\hat{\sigma}(r)| + \|\sigma\| \, \eta \qquad\qquad \text{for all} \quad |r| \leqslant Q_1$$

and (6b) follows. The remaining assertions are also easily verified. We now obtain a version of the linked set Lemma which is easier to handle than our original one.

LEMMA 5.2. If $1 > \varepsilon$, $\eta > 0$ and $N(\varepsilon, \eta) = 1600([\varepsilon^{-1}\eta^{-1}] + 1)$, then, given $1 > \delta > 0$ and $m \geqslant 1$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}$ such that $10^{100m}\eta^{-3m}\delta^{-2m}r \geqslant h(r) \geqslant r$ with the following properties : -

Given $N(\varepsilon, \eta) = \frac{1}{2}M(0) < h(M(0)) < M(1) < \dots < h(M(m)) < M(m+1)$ we can find finite sets $E_S \subseteq [-\varepsilon, \varepsilon]$ and measures $\mu_S \in M^+(E_S)$ with $\|\mu_S\| = 1$ $[S \subseteq \{1,2,\dots,m\}]$ such that

(i) $M(m+1)E_S = 0$

(ii) $f_\delta(\chi_{M(p)}(e)) = 1$ if $e \in E_S$, $p \in S$

$f_\delta(\chi_{M(p)}(e)) = 0$ if $e \in E_S$, $p \in S$

(iii) $|\hat{\mu}_S(r)| \geqslant \eta$ implies $|\hat{\mu}_S(r) - \hat{\mu}_T(r)| \leqslant \eta$

for all $S \subseteq T \subseteq \{1,2,\dots,m\}$, $M(m+1) - N \geqslant |r| \geqslant N$

(iv) $|\hat{\mu}_S(r)|$, $|\hat{\mu}_T(r)| \geqslant \eta$ implies $|\hat{\mu}_{S\cap T}(r)| \geqslant \min(|\hat{\mu}_S(r)|, |\hat{\mu}_T(r)|) - \eta$

$[S, T \subseteq \{1,2,\dots,m\},\ M(m+1) - N \geqslant |r| \geqslant N]$

(v) $|\hat{\mu}_\emptyset(r)| \leqslant \eta$ for all $M(m+1) - N \geqslant |r| \geqslant N$

(Recall that f_δ is the trapezoidal function with $f_\delta(-\delta) = f_\delta(\delta) = 0$, $f_\delta(-\delta^2) = 1$, f_δ linear on $[-\delta, -\delta^2]$, $[-\delta^2, \delta^2]$, $[\delta^2, \delta]$, $\mathbb{T} \setminus [-\delta, \delta]$).

Proof. In order to prove the result inductively, we replace $\delta, \eta, \varepsilon$ in condition $E_S \subseteq [-\varepsilon, \varepsilon]$ and in the condition (ii), (iii), (iv), (v) by $\delta(1 - 2^{-m-4})$, $\eta(1 - 2^{-m-4})$, $\varepsilon(1 - 2^{-m-4})$ calling the new conditions so obtained $(ii)_m$, $(iii)_m$, $(iv)_m$, $(v)_m$; thus, for example, (v) becomes

$(v)_m$ $\quad |\hat{\mu}_\emptyset(r)| \leqslant \eta(1 - 2^{-m-4})$.

If we take $h(r) = 4r$, $\mu_\emptyset = \mu_{\varepsilon/2, M(1)}$, it is at once clear (using Lemma 3.1) that the (modified) lemma is true for $m = 0$.

Now suppose that the (modified) Lemma is true for $m = n$.

Then in particular we can find $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $10^{100n}\eta^{-3n}\delta^{-2n} r \geqslant h(r) \geqslant 10^{50(n+1)}\eta^{-3n}\delta^{-2n} r$ such that, given $M(\varepsilon, \eta) = \frac{1}{2} M(0) < h(M(0)) < M(1) < \ldots$ $\ldots < h(M(n)) < M(n+1) < h(M(n+1)) < M(n+2)$, we can find finite sets $E_S^{(n)} \subseteq [-\varepsilon(1 - 2^{-m}), \varepsilon(1 - 2^{-m})]$ and measures $\mu_S^{(n)} \subseteq M^+(E_S^{(n)})$ with $\|\mu_S^{(n)}\| = 1$ $[S \subseteq \{1, 2, \ldots, m\}]$ such that

$(i)_n$ $\quad M(n+1)E_S^{(n)} = 0$

$(ii)_n$ $\quad f_{\delta(1-2^{-n})}(\chi_{M(r)}(e)) = 1$ if $e \in E_S^{(n)}$, $r \in S$

$\quad f_{\delta(1-2^{-n})}(\chi_{M(r)}(e)) = 0$ if $e \in E_S^{(n)}$, $r \notin S$

$(iii)_n$ $\quad |\hat{\mu}_S^{(n)}(r)| \geqslant \eta(1-2^{-n})$ implies $|\hat{\mu}_S^{(n)}(r) - \hat{\mu}_T^{(n)}(r)| \leqslant \eta(1-2^{-n})$

for all $S \subseteq T \subseteq \{1, 2, \ldots, n\}$

$(iv)_n$ $\quad |\hat{\mu}_{S \cap T}^{(n)}(r)| \geqslant \min(|\hat{\mu}_S^{(n)}(r)|, |\hat{\mu}_T^{(n)}(r)|) - \eta(1-2^{-n})$

$(v)_n$ $\quad |\hat{\mu}_\emptyset^{(n)}(r)| \leqslant \eta(1-2^{-n})$ for all $M(n+1) - N \geqslant r \geqslant N$.

We now apply Lemma 5.1.

Since $M(n+1) \geqslant 10^{12}[\delta/(M(n)2^{-n-12})](2^{-n-7}\eta)^{-2}((1-2^{-n-5})\delta)^{-2}$, $M(n+2) \geqslant 10^{12}\, 8(2^{-n-7}\delta)^2\, M(n+1)$ and $M(n) \leqslant 10^{-24} M(n+1)(2^{-n-7}\eta)^{-3}$. Lemma 5.1 shows (setting $Q' = M(n+2)$, $Q = M(n+1)$, $Q_o = M(n)$ and writing $Q_1 = M^*(n) = 10^{-8} M(n+1)(2^{-n-7}\eta))$ that, putting $E^{(n)} = \bigcup_{S\subseteq\{1,2,\dots,n\}} E_S^{(n)}$, we can find $E_1^{(n+1)}, E_2^{(n+1)} \subseteq \mathbb{T}$ and linear maps $L_i : M(E^{(n)}) \to M(E_i^{(n+1)})$ $[i = 1,2]$ such that

(1) $M(n+2)E_i^{(n+1)} = 0$

(2) $\sup_{e_i'\in \operatorname{supp} L_i\sigma}\ \inf_{e_i\in\operatorname{supp}\sigma} |e_i - e_i'| \leqslant \delta\, 2^{-n-7}/M(n)$ for all $\sigma \in M(E_i^{(n)})$

(3) $|(L_i\sigma - \sigma)\hat{}(r)| \leqslant 2^{-n-7}\eta\,\|\sigma\|$ for all $|r| \leqslant M(n)$, $\sigma \in M(E_i^{(n)})$

(4) $L_i\sigma \in M^+(E_i^{(n+1)})$, $\|L_i\sigma\|_M = \|\sigma\|_M$ for all $\sigma \in M(E_i^{(n)})$

(5) $|\chi_Q(x_1) - 1| \leqslant ((1-2^{-n-5})\delta)^2$, $|\chi_Q(x_2) - 1| \geqslant (1 - 2^{-n-5})\delta$

(6a)' $|(L_2\sigma)\hat{}(r)| \leqslant 2^{-n-7}\eta\,\|\sigma\|_M$ for all $M(n+2)-M^*(n) \geqslant |r| \geqslant M^*(n)$

(6b)' $|(L_2\sigma)\hat{}(r)| \leqslant |\hat{\sigma}(r)| + 2^{-n-7}\eta\,\|\sigma\|_M$ for all $|r| \leqslant M^*(n)$

(6c)' $|(L_1\sigma)\hat{}(r)| \leqslant |\hat{\sigma}(r)| + 2^{-n-7}\eta\,\|\sigma\|_M$ for all $r \in \mathbb{Z}$, $\sigma \in M(E^{(n)})$

(7)' $|(L_2\sigma_2 - L_1\sigma_1)\hat{}(r)| \leqslant |(\sigma_2 - \sigma_1)\hat{}(r)| + 2^{-n-6}\eta$ for all $|r| \leqslant M^*(n)$, $\sigma_i \in M(E^{(n)})$, $\|\sigma_i\|_M \leqslant 1$ $[i = 1,2]$.

For each $S \subseteq \{1,2,\dots,n\}$ set $\mu_S^{(n+1)} = L_2\mu_S^{(n)}$, $\mu_{S\cup\{n+1\}}^{(n+1)} = L_1\mu_S^{(n)}$ and write $E_U^{(n+1)} = \operatorname{supp}\mu_U^{(n+1)}$ for all $U \subseteq \{1,2,\dots, n+1\}$. The condition $E_U \subseteq [-\varepsilon(1-2^{-n-1}), \varepsilon(1-2^{-n-1})]$ follows from (2) (observing that $2^{-n-7}/M(n) \leqslant \varepsilon\, 2^{-n-2}$). Condition $(i)_{n+1}$ follows from (1), $(ii)_{n+1}$ follows from $(ii)_n$ and (2) for $1 \leqslant r \leqslant n$ and from (5) for $r = n+1$. Condition $(v)_{n+1}$ follows from (6a) and $(v)_n$ (note that $M(n+1) - N \geqslant M^*(n)$ and that $\hat{\mu}_U^{(n+1)}(r)$ is periodic in r with period $M(n+1)$.

There remain the proofs of $(iii)_{n+1}$ and $(iv)_{n+1}$ which we shall do by splitting up into cases. If $S \subseteq \{1,2,\ldots,n+1\}$, we shall write $S^* = S \setminus \{n+1\}$. To prove $(iii)_{n+1}$ suppose $S \subseteq T \subseteq \{1,2,\ldots,n+1\}$ and $|\hat{\mu}_S^{(n+1)}(r)| \geqslant \eta(1-2^{-(n+1)})$. If $|r| \leqslant M^*(n)$ then (6b) and (6c) show that $|\hat{\mu}_{S^*}^{(n)}(r)| \geqslant |\hat{\mu}_S^{(n+1)}(r)| - 2^{-(n+7)}\eta$ and so $|\hat{\mu}_{S^*}^{(n)}(r)| \geqslant \eta(1-2^{-n})$. Since $T^* \supseteq S^*$, (iii)' now shows $|\hat{\mu}_{S^*}^{(n)}(r) - \hat{\mu}_{T^*}^{(n)}(r)| \leqslant \eta(1-2^{-n})$. Using (6b) if $n+1 \in S$, (6c) if $n+1 \notin T$ and (7) if $n+1 \in T$, $n+1 \notin S$, we obtain $|\hat{\mu}_S^{(n+1)}(r) - \hat{\mu}_T^{(n+1)}(r)| \leqslant \eta(1-2^{-n}) + \eta 2^{-n-7} \leqslant \eta(1-2^{-n-1})$ as required. If $M(n+2) - M^*(n) \geqslant r \geqslant M^*(n)$, then (6a) shows that $n+1 \in S$ (and so $n+1 \in T$). We now use (6c) to show that

$|\hat{\mu}_{S^*}^{(n)}(r)| \geqslant |\hat{\mu}_S^{(n+1)}(r)| - 2^{-n-7}\eta \geqslant (1-2^{-n})\eta$ so that by $(iii)_n$ $|\hat{\mu}_{T^*}^{(n)}(r) - \hat{\mu}_{S^*}^{(n)}(r)| \leqslant (1-2^{-n})\eta$ and by (6c) again $|\hat{\mu}_T^{(n+1)}(r) - \hat{\mu}_S^{(n+1)}(r)| = |L_1(\mu_{T^*}^{(n)} - \mu_{S^*}^{(n)})\hat{}\,(r)|$
$\leqslant |\hat{\mu}_{T^*}^{(n)}(r) - \hat{\mu}_{S^*}^{(n)}(r)| + \|\mu_{T^*} - \mu_{S^*}\| 2^{-n-7} \leqslant (1-2^{-n})\eta + 2.2^{-n-7}\eta \leqslant (1-2^{-n-1})\eta$. Condition $(iii)_{n+1}$ follows on observing that $\hat{\mu}_U^{(n+1)}$ has period $M(n+2)$.

If $\sigma_1, \sigma_2 \in M(E^{(n)})$, $|\hat{\sigma}_1(r)| \geqslant |\hat{\sigma}_2(r)|$ then $\lambda\hat{\sigma}_1(r) = \hat{\sigma}_2(r)$ for some $|\lambda| \leqslant 1$, and so, if $|r| \leqslant M^*(n)$, we have $|L_i(\lambda\sigma_1 - \sigma_2)\hat{}\,(r)| \leqslant 2^{-n-7}\eta(|\lambda| \|\sigma_1\| + \|\sigma_2\|)\eta$ (by (6b) and (6c)). In particular $|(L_i\, \sigma_i)\hat{}\,(r)| \geqslant |(L_i\, \sigma_2)\hat{}\,(r)| + 2^{-n-7}(\|\sigma_1\| + \|\sigma_2\|)$. Thus, if $S, T \subseteq \{1,2,\ldots,n+1\}$ and $|r| \leqslant M^*(n)$, $(iv)_n$ and (7) imply

$|\hat{\mu}_{S\cap T}^{(n+1)}(r)| \geqslant \min(|\hat{\mu}_S^{(n+1)}(r)|, \ |\hat{\mu}_T^{(n+1)}(r)|) - 2^{-n-6}\eta - 2^{-n-6}\eta - (1-2^{-n})\eta$

$\leqslant \min(|\hat{\mu}_S(r)|, \ |\hat{\mu}_T(r)|) + (1-2^{-n-1})\eta$. On the other hand, if $M(n+1) - M^*(n) \geqslant r \geqslant M^*(n)$, then, by (6a), the relation $|\hat{\mu}_{S\cap T}^{(n+1)}(r)| \geqslant \min(|\hat{\mu}_S^{(n+1)}(r)|, \ |\hat{\mu}_T^{(n+1)}(r)|) - (1-2^{-n-1})\eta$ is trivially satisfied unless $n+1 \in S \cap T$. But if $n+1 \in S\cap T$, then the relation stated is satisfied because of (6b) and $(iv)_n$. Condition $(iv)_{n+1}$ is thus proved.

Let us state and prove another easy consequence of Lemma 5.1.

LEMMA 5.3. Suppose $1 \geq \varepsilon, \eta, \delta > 0$ given together with positive integers N and k, then there exists an $M_0(\varepsilon, \eta, \delta, k, N) \geq 4N$ such that, given $K \geq 1 > \rho, \gamma > 0$ and a monotonic increasing function $H : \underline{\underline{R}}^+ \to \underline{\underline{R}}^+$ with $H(0) = 0$ and $M'' \geq M_0$, $M \geq M_0 M'' K^{-1}$ such that $MH(4\rho) \leq \gamma$, $M' \geq M_0 M\rho^{-2}$ together with finite sets $E_S \subseteq [-\varepsilon(1-2^{-k}), \varepsilon(1-2^{-k})]$ and measure $\mu_S \in M^+(E_S)$ with $\|\mu_S\| = 1$ $[S \subseteq \{1,2,\ldots,m\}]$ such that

$(i)_k$ $\quad ME_S = 0$

$(iii)_k$ $\quad |\hat{\mu}_S(r)| \geq \eta(1-2^{-k})$ implies $|\hat{\mu}_S(r) - \hat{\mu}_T(r)| \leq \eta(1-2^{-k})$

for all $S \subseteq T \subseteq \{1,2,\ldots,m\}$, $M - N \geq |r| \geq N$

$(iv)_k$ $\quad |\hat{\mu}_S(r)|, \ |\hat{\mu}_T(r)| \geq \eta(1-2^{-k})$ implies

$|\hat{\mu}_{S\cap T}(r)| \geq \min(|\hat{\mu}_S(r)|, \ |\hat{\mu}_T(r)|) - \eta(1-2^{-k})$ $[S, T \subseteq \{1,2,\ldots,m\}$, $M-N \geq |r| \geq N]$

$(v)_k$ $\quad |\hat{\mu}_\emptyset(r)| \leq \eta(1-2^{-k})$ for all $M-n \geq |r| \geq N$,

then, for $p \in \{1,2,\ldots,m\}$ fixed we can find finite sets $E'_S \subseteq [-\varepsilon(1-2^{-k-1}), \varepsilon(1-2^{-k-1})]$ and measures $\mu'_S \in M^+(E'_S)$ with $\|\mu'_S\| = 1$ $[S \subseteq \{1,2,\ldots,m\}]$ such that

$(i)_{k+1}$ $\quad M'E'_S = 0$

$(ii)_{k+1}$ $\quad f_{\delta(1-2^{-k-1})}(\chi_M(e')) = 1 \quad$ if $\quad e' \in E'_S, \quad r \in S$

$\quad f_{\delta(1-2^{-k-1})}(\chi_M(e')) = 0 \quad$ if $\quad e' \in E'_S, \quad r \notin S$

$(iii)_{k+1}$ $\quad |\hat{\mu}'_S(r)| \geq \eta(1-2^{-k-1})$ implies $|\hat{\mu}'_S(r) - \hat{\mu}'_T(r)| \leq \eta(1-2^{-k})$ for all $S \subseteq T \subseteq \{1,2,\ldots,m\}$, $M' - N \geq |r| \geq N$

$(iv)_{k+1}$ $\quad |\hat{\mu}'_S(r)|, \ |\hat{\mu}'_T(r)| \geq \eta(1-2^{-k-1})$ implies

$|\hat{\mu}'_{S\cap T}(r)| \geq \min(|\hat{\mu}'_S(r)|, \ |\hat{\mu}'_T(r)|) - \eta(1-2^{-k-1})$ for all $S, T \subseteq \{1,2,\ldots,m\}$, $M' - N \geq |r| \geq N$

$(v)_{k+1}$ $\quad |\hat{\mu}_\emptyset(r)| \leq \eta(1-2^{-k-1})$ for all $M' - N \geq |r| \geq N$,

$(vi)_{k+1}$ $\quad |\hat{\mu}_S(r) - \hat{\mu}_{S'}(r)| \leqslant \eta 2^{-k-1}$ for all $|r| \leqslant M''$

$(vii)_{k+1}$ $\quad \sup_{e' \in E'_S} \inf_{e \in E_S} |e - e'| \leqslant \delta^2 2^{-k-1} M'^{-1}$

$(viii)_{k+1}$ $\quad \bigcup_{r \in S} \{x : |x - e| \leqslant K/M'$ for some $e \in E'_S\}$ can be covered by intervals of length $\ell_j \leqslant \rho$ with $\sum H(\ell_j) \leqslant \gamma$.

Proof. By Lemma 5.1 (and the remark that $H(4/\rho Q) \to 0$ as $Q \to \infty$) there exists an M_o depending only on $\varepsilon, \eta, \delta, \gamma, \rho > 0$ such that if M'', M', M, E_S are as given in the hypotheses, the following is true. Write $F_1 = \bigcup_{p \in S} E_S$, $F_2 = \bigcup_{p \notin S} E_S$. Then we can find $F'_i \subseteq \mathbb{T}$, $M - N \geqslant M_1 \geqslant M''$, and linear maps $L_i : M(F_i) \to M(F'_i)$ $[i = 1,2]$ such that

(1) $\quad M' F'_i = 0$

(2) $\quad \sup_{e'_i \in \operatorname{supp} L_i \sigma} \inf_{e_i \in \operatorname{supp} \sigma} |e - e'_i| \leqslant \chi \delta \eta 2^{-k-4} M''^{-1}$

(3) $\quad |(L_i\sigma - \sigma)\hat{}\,(r)| \leqslant 2^{-k-4}\eta \|\sigma\|$ for all $|r| \leqslant M''$, $\sigma \in M(F_i)$

(4) $\quad L_i\sigma \in M^+(F'_i)$, $\|L_i\sigma\| = \|\sigma\|$ for all $\sigma \in M^+(F_i)$

(5) $\quad f_{\delta(1-2^{-k-1})}(\chi_M(e')) = 1$ if $e' \in F'_1$

$\quad f_{\delta(1-2^{-k-1})}(\chi_M(e')) = 0$ if $e' \in F'_2$

(6a) $\quad |(L_2\sigma)\hat{}\,(r)| \leqslant 2^{-k-4}\eta \|\sigma\|$ for all $M' - M_1 \geqslant r \geqslant M_1$, $\sigma \in M(F_2)$

(6b) $\quad |(L_2\sigma)\hat{}\,(r)| \leqslant |\hat{\sigma}(r)| + 2^{-k-4}\eta \|\sigma\|$ for all r

(6c) $\quad |(L_1\sigma)\hat{}\,(r)| \leqslant |\hat{\sigma}(r)| + 2^{-k-4}\eta \|\sigma\|$ for all r

(7) $\quad |(L_2\sigma_2 - L_1\sigma_1)\hat{}\,(r)| \leqslant |(\sigma_2 - \sigma_1)\hat{}\,(r)| + 2^{-k-4}\eta$ for all $|r| \leqslant M_1$, $\sigma_i \in M(F_i)$, $\|\sigma_i\|_M \leqslant 1$ $\quad [i = 1,2]$

(8) $\quad \{x_1 : |x_1 - e'_1| \leqslant K/M'$ for some $e'_1 \in F'_1\}$ can be covered by intervals

of length $\ell_j \leqslant \rho$ with $\sum H(\ell_j) \leqslant \gamma$.

Set $\mu'_S = L_1\mu_S$ if $p \in S$, $\mu'_S = L_2\mu_S$ if $p \notin S$, $E'_S = \operatorname{supp}\mu'_S$. Conditions $(i)_{k+1}$, $(ii)_{k+1}$, $(vi)_{k+1}$, $(vii)_{k+1}$ and $(viii)_{k+1}$ follow from (1), (5), (3), (2) and (8) respectively. Since $p \notin \emptyset$, condition $(v)_{k+1}$ follows from $(v)_k$ and (6b) for $M-N \geqslant r \geqslant N$ (so by periodicity $M'-N \geqslant r \geqslant M'-M+N$) and from (6a) for $M'-M+N \geqslant r \geqslant M-N$ (note that $M_1 \leqslant M-N$). To prove $(ii)_{k+1}$, $(iv)_{k+1}$ we argue by cases. By periodicity it suffices to consider $M' - M_1 \geqslant r \geqslant N$.

If $N \leqslant r \leqslant M_1$, then by (6b) or (6c) $|\hat{\mu}'_S(r)| \geqslant \eta(1-2^{-k-1})$ implies $|\hat{\mu}_S(r)| \geqslant \eta(1-2^{-k})$ and so, if $S \subseteq T$, $|\hat{\mu}_T(r) - \hat{\mu}_S(r)| \leqslant \eta(1-2^{-k})$, so that by (6b) (if $p \in S,T$), (6c) (if $p \notin S,T$) or (7) (if $p \in T$, $p \notin S$) $|\hat{\mu}'_T(r) - \hat{\mu}'_S(r)| \leqslant \eta(1-2^{-k-1})$, so $(iii)_{k+1}$ is true. If $M_1 \leqslant r \leqslant M-M_1$, then if $p \notin S$, $(iii)_{k+1}$ is vacuously true by (6a). If $p \in S \subseteq T$, then $p \in T$ and $(iii)_{k+1}$ follows from $(iii)_k$ and (6c).

Similarly suppose $S,T \subseteq \{1,2,\ldots,m\}$. If $p \notin S \cap T$, then $(iv)_{k+1}$ is vacuously true for $M_1 \leqslant r \leqslant M-M_1$ and follows from $(iv)_k$ and (6b) or (7) for $N \leqslant r \leqslant M_1$. If $p \in S \cap T$, then $(iv)_{k+1}$ follows from (6c) and $(iv)_k$.

As an immediate corollary of Lemmas 5.2 and 5.3 we have

LEMMA 5.4. If $1 > \varepsilon$, $\eta > 0$ and $N(\varepsilon, \eta) = 1600([\varepsilon^{-1}\eta^{-1}] + 1)$, then given $1 > \delta$, γ, $\rho > 0$, $K, m \geqslant 1$ and $H : \mathbb{R}^+ \to \mathbb{R}^+$ a monotonic increasing function with $H(0) = 0$, we can find a monotonic increasing function $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $h(r) > r$ with the following properties : -

Given $2N(\varepsilon, \eta) = \frac{1}{2}M(0) < h(M(0)) < M(1) < \ldots < h(M(m)) < M(m+1)$, we can find μ_S, E_S satisfying the conclusions of Lemma 5.2 such that additionally

(vi) $\bigcup_{\emptyset \neq S \subseteq \{1,2,\dots,m\}} \{x : |x - e| \leqslant K/M(m+1)\}$ can be covered by intervals length $\ell_i \leqslant \rho$ with $\sum H(\ell_i) \leqslant \gamma$.

Proof. Fix $\varepsilon, \eta, \delta, \gamma, k, m$ once for all. Let $W(n)$ be the statement. Given $K_o \geqslant K$, there exists a monotonic increasing function $h_n : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $h_n(r)$ r with the following properties : -

Given $2N(\varepsilon, \eta) = \frac{1}{2} M(0) < h(M(0)) < M(1) < \dots < h(M(m)) < M_n(m+1)$, we can find finite sets $E_{S,n} \subseteq [-\varepsilon(1-2^{-n-4}), \varepsilon(1-2^{-n-4})]$ and measures $\mu_{S,n} \in M^+(E_{S,n})$ with $\|\mu_{S,n}\| = 1$ such that

$(i)_n$ $M_n(m+1)E_{S,n} = 0$

$(ii)_n$ $f_{\delta(1-2^{-n-4})}(\chi_{M(p)}(e)) = 1$ if $e \in E_{S,n}$, $p \in S$

$f_{\delta(1-2^{-n-4})}(\chi_{M(p)}(e)) = 0$ if $e \in E_{S,n}$, $p \notin S$

$(iii)_n$ $|\hat{\mu}_{S,n}(r)| \geqslant \eta(1-2^{-n-4})$ implies $|\hat{\mu}_{S,n}(r) - \hat{\mu}_{T,n}(r)| \leqslant \eta(1-2^{-n-4})$ for all $S \subseteq T \subseteq \{1,2,\dots,m\}$, $M_n(m+1) - N \geqslant |r| \geqslant N$

$(iv)_n$ $|\hat{\mu}_{S,n}(r)|$, $|\hat{\mu}_{T,n}(r)| \geqslant \eta(1-2^{-n-4})$ implies $|\hat{\mu}_{S\cap T,n}(r)| \geqslant \min(|\hat{\mu}_{S,n}(r)|, |\hat{\mu}_{T,n}(r)|) - \eta(1-2^{-n-4})$ $[S,T \subseteq \{1,2,\dots,m\}$, $M_n(m+1) - N \geqslant |r| \geqslant N]$

$(v)_n$ $|\hat{\mu}_{\emptyset,n}(r)| \leqslant (1-2^{-n-4})\eta$ for all $M_n(m+1) - N \geqslant |r| \geqslant N$

$(vi)_n$ If $1 \leqslant p \leqslant n$, then $\bigcup_{p \in S} \{x : |x - e| \leqslant 4K_o/M_n(m+1)$ for some $e \in E_S\}$ can be covered by intervals of length $\ell_i \leqslant \rho$ such that $\sum H(\ell_i) = \gamma/m$.

For $n = 0$ this is a re-statement (with coarser inequalities) of Lemma 5.2. On the other hand if $W(n)$ is true, we can apply Lemma 5.3, taking $k = n + 4$, $0 < K < 1/(8K_o)$,

$M_n(m+1) \geqslant M_o(\varepsilon, \eta, \delta, m, N)$ and $M_{n+1}(m+1) \geqslant M_n(m+1)M_o(\varepsilon, \eta, \delta, m, N)K^{-1}$, choosing a $\rho_{n+1} = \rho$ such that $M_{n+1}(m+1)H(4\rho_{n+1}) \leqslant \gamma/m$ and $K \geqslant 8K_o$. Lemma 5.3 now gives us $E_{S,n+1}$, $\mu_{S,n+1}$ satisfying $W(n+1)$.

This last assertion is checked as follows. Conditions $(i)_{n+1}$, $(iii)_{n+1}$, $(vi)_{n+1}$, $(v)_{n+1}$ and that part of $(iv)_{n+1}$ dealing with $p = n+1$ follow from $(i)_{k+1}$ (with $M' = M_{n+1}(m+1))$, $(iii)_{k+1}$ and $(viii)_{k+1}$ of Lemma 5.3. Conditions $(ii)_{n+1}$, $(iv)_{n+1}$ for $1 \leqslant p \leqslant n$ follow from $(ii)_n$ and $(iv)_n$ together with the observation that, since $(vii)_{k+1}$ gives $\sup_{e' \in E_{S,n+1}} \sup_{e \in E_{S,n}} |e - e'| \leqslant \delta^2 \eta 2^{-n-5}/M_{n+1}(m+1)$, we have $\sup_{e' \in E_{S,n}} \inf_{e \in E_{S,n}} |\chi_{M(p)}(e) - \chi_{M(p)}(e')| \leqslant \delta^2 2^{-n-5} M(p)/M_{n+1}(m+1) \leqslant \delta^2 2^{-n-5}$ and $\bigcup_{p \in S} \{x : |x - e| \leqslant 4K_o/M_n(m+1)$ for some $e \in E_S\} \subseteq \bigcup_{p \in S} \{x : |x - e| \leqslant 4K_o/M_{n+1}(m+1)$ for some $e \in E_S\}$ for all $1 \leqslant p \leqslant n$.

Thus by induction $W(m)$ is true. But $W(m)$ is a statement of the conclusion of the lemma, and so we are through.

The arguments of Section § 3 show us that, using Lemma 5.2 and Lemma 5.4 respectively in place of Lemma 1.7, we get the following 2 improvements on Lemma 2.1 (the Central Lemma).

LEMMA 5.2'. Given $K > 1$, $1 > \lambda > 0$ we can find a $C(K, \lambda) \geqslant 1$, a $B(K, \lambda) > 0$, an $N_o(K, \lambda) \geqslant 1$ and an $m(K, \lambda) \in \underline{\underline{Z}}^+$ with the following property : -

Given $\varepsilon, \delta > 0$ and $N \geqslant \varepsilon^{-1} N_o(K, \lambda)$ an integer, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ such that $\delta^{2m} r \geqslant h(r) \geqslant r$ with the following property:-

Given $N = \frac{1}{2} M(0) < h(M(0)) < M(1) < \ldots < h(M(m)) < M(m+1)$ we can find a finite set $E \subseteq [-\varepsilon , \varepsilon]$ and a measure $T \in M(E)$ such that

(i) $M(m+1)E = 0$

(ii) $f_\delta(\chi_{M(r)}(e))$ takes the value 0 or 1 if $e \in E$, $1 \leqslant r \leqslant m$

card $\{1 \leqslant r \leqslant m : f_\delta(\chi_{M(r)}(x)) = 1\} \geqslant \lambda m$ for all $x \in E$

(iii) $\|T\|_{PM} = \hat{T}(0) = 1 \geqslant K \sup_{M(m+1)-M(0) \geqslant r \geqslant M(0)} |\hat{T}(r)|$

(iv) $\|T\|_M \leqslant C$.

LEMMA 5.4'. Given $K > 1$, $1 > \lambda > 0$ we can find a $C(K , \lambda) \geqslant 1$, an $N_o(K , \lambda) \geqslant 1$ and an $m(K , \lambda) \in \underline{\underline{Z}}^+$ with the following property : -

Given $\varepsilon , \delta > 0$ and $N \geqslant \varepsilon^{-1} N_o(K , \lambda)$ together with $K \geqslant 1$, $1 \geqslant \rho$, $\gamma > 0$ and H a continuous monotonic increasing function $H : \underline{\underline{R}} \to \underline{\underline{R}}$ with $H(0) = 0$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ such that $h(r) > r$ with the following property : -

Given $N = \frac{1}{2} M(0) < h(M(0)) < M(1) < \ldots < h(M(m)) < M(m+1)$ we can find a finite set $E \subseteq [-\varepsilon , \varepsilon]$ and a measure $T \in M(E)$ such that

(i) $M(m+1)E = 0$

(ii) $f_\delta(\chi_{M(r)}(e))$ takes the value 0 or 1 if $e \in E$, $1 \leqslant r \leqslant m$

card $\{1 \leqslant r \leqslant m : f_\delta(\chi_{M(r)}(x)) = 1\} \geqslant \lambda m$ for all $x \in E$

(iii) $\|T\|_{PM} = \hat{T}(0) = 1 \geqslant K \sup_{M(m+1)-M(0) \geqslant r \geqslant M(0)} |\hat{T}(r)|$

(iv) $\|T\|_M \leqslant C$

(v) $\{x : |e - x| \leqslant K/M(m+1)$ for some $e \in E\}$ may be covered by intervals of length $\ell_i \leqslant \rho$ such that $\sum H(\ell_i) \leqslant \gamma$.

The essential improvement on Lemma 2.1 in Lemma 5.2' lies in the fact that we can now choose $M(r+1)$ without the restriction $M(r)$ divides $M(r+1)$. This is frequently very helpful in constructing proofs (it would have simplified the construction of the e'_u in the proof of Lemma 2.2 for example). The fact that we can take h linear in Lemma 5.2' like the condition (v) of Lemma 5.4' shows that supp T is in some sense rather thin, and we shall use these 2 results in Section § 6 ("How Thin Can a Set of Non Synthesis Be ?").

However, the main purpose of this section is to prove the following result.

LEMMA 5.5. Given $\varepsilon, \eta > 0$, we can find an $N(\varepsilon, \eta)$ such that, given $m \in \underline{\underline{Z}}^+$, $x \in \underline{\underline{T}}$, and F a closed set in $\underline{\underline{T}}$ such that $\mathrm{Gp}F \neq \underline{\underline{T}}$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ with $h(r) > r$ having the following property : -

Given $M_p(j)$ $[1 \leqslant p \leqslant m,\ 0 \leqslant j]$ with $N = M_1(0)$, $h(M_p(j)) \leqslant M_{p'}(j')$ whenever $0 \leqslant j < j'$, $1 \leqslant p$, $p' \leqslant m$ or $0 \leqslant j = j'$, $1 \leqslant p < p' \leqslant m$, we can find closed sets $E_S \subseteq [x - \varepsilon, x + \varepsilon]$ and measures $\mu_S \in M^+(E_S)$ with $\|\mu_S\| = 1$ $[S \subseteq \{1,2,\ldots,m\}]$ such that

(i) $f_{2^{-j}}(\chi_{M_p(j)}(e)) = 1$ if $e \in E_S$, $p \in S$

$f_{2^{-j}}(\chi_{M_p(j)}(e)) = 0$ if $e \in E_S$, $p \notin S$

(ii) $|\hat{\mu}_S(r)| \geqslant \eta$ implies $|\hat{\mu}_S(r) - \hat{\mu}_T(r)| \leqslant \eta$ for all $S \subseteq T \subseteq \{1,2,\ldots,m\}$, $|r| \geqslant N$

(iii) $|\hat{\mu}_{S\cap T}(r)| \geqslant \max(|\hat{\mu}_S(r)|, |\hat{\mu}_T(r)|) - \eta$ for all $S, T \subseteq \{1,2,\ldots,m\}$, $|r| \geqslant N$

(iv) $|\hat{\mu}_{\emptyset}(r)| = \eta$ for all $|r| \geqslant N$

(v) Writing $E = \bigcup_{S \subseteq \{1,2,\ldots,m\}} E_S$ we have E independent and $\mathrm{Gp}E \cap \mathrm{Gp}F = \{0\}$.

Remark. In the proof of one or two of the less important results in Section § 7 we shall need the following further condition (whose (easy) proof is left to the reader).

(vi) $|\hat{\mu}_S(M_p(j) + k)| \leqslant 2^{-j}$ for all $|k| \leqslant j$, $p \notin S$.

This is in fact a consequence of the method we used to construct the E_S but the reader may find it easier to incorporate an explicit extra step in the construction.

It is clear that, apart from condition (v), Lemma 5.5 is a direct consequence of Lemma 5.2 and repeated inductive use of Lemma 5.3 (note that if μ' is a translate of $\mu \in M(\underline{\underline{T}})$ then $|\hat{\mu}'(r)| = |\hat{\mu}(r)|$). To get (v) we shall use

LEMMA 5.6. Given $1 > \delta$, $\eta > 0$ and N_1, N_2 positive integers with $N_1 \delta \leqslant 10^{-2}\eta$, $N_2 \geqslant 12800\,[\eta^{-1}\delta^{-1}]$ together with $\gamma > 0$, $k \in \underline{\underline{Z}}^+$ and F a closed set with $\mathrm{Gp}F \neq \underline{\underline{T}}$, we can find an $N_3(\delta, \eta, N_2, \gamma, k, F) \geqslant 4N_2$ such that, given $N_4 \geqslant N_3$, there exists a linear map $L : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ such that

(1) $N_4 \operatorname{supp} L\sigma = 0$ for all $\sigma \in M(\underline{\underline{T}})$

(2) $\inf_{e' \in \operatorname{supp} L\sigma} \sup_{e \in \operatorname{supp} \sigma} |e - e'| \leqslant \delta$ for all $\sigma \in M(\underline{\underline{T}})$

(3) $|(L\sigma - \sigma)^{\wedge}(r)| \leqslant \eta \|\sigma\|$ for all $|r| \leqslant N_1$, $\sigma \in M(\underline{\underline{T}})$

(4) $|(L\sigma)^{\wedge}(r)| \leqslant |\hat{\sigma}(r)| + \eta\|\sigma\|$ for all $r, \sigma \in M(\underline{\underline{T}})$

(5) $|(L\sigma)^{\wedge}(r)| \leqslant \eta\|\sigma\|$ for all $N_4 - N_1 \geqslant |r| \geqslant N_1$, $\sigma \in M(\underline{\underline{T}})$

(6) $L\sigma \in M^+(\underline{\underline{T}})$, $\|L\sigma\| = \|\sigma\|$ for all $\sigma \in M^+(\underline{\underline{T}})$

(7) Suppose $\sigma \in M(\underline{\underline{T}})$. Then, given $x_1, x_2, \ldots, x_k \in F$, $y_1, y_2, \ldots, y_k \in \text{Supp}\,\sigma$ with $\inf\limits_{1 \leqslant i < j \leqslant k} |y_i - y_j| \geqslant \gamma$ we have automatically $\sum\limits_{i=1}^{k} x_i + \sum\limits_{i=1}^{k} m_i y_i \neq 0$ whenever $1 \leqslant \sum\limits_{i=1}^{k} |m_i| \leqslant k$.

This result follows at once by combining the three parts of the following lemma.

LEMMA 5.7(i). Given $1 > \delta$, $\eta > 0$ and N_1, N_2 positive integers with $N_1\delta \leqslant 10^{-2}\eta$, $N_2 \geqslant 12800\,[\eta^{-1}\delta^{-1}]$ we can find an $N_3(\delta, \eta, N_2) \geqslant 4N_2$ such that, given $N_4 \geqslant N_3$, there exists a linear map $L : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ satisfying conditions (1) to (6) of Lemma 5.6 ;

(ii) Given $1 > \delta$, $\eta > 0$ and N_1, N_2, k_o positive integers with $N_1\delta \leqslant 10^{-2}\eta$, $N_2 \geqslant 12800\,[\eta^{-1}\delta^{-1}]$ we can find an $N_3(\delta, \eta, N_2, k_o) \geqslant 4N_2$ such that, given $N_4 \geqslant N_3$, N_4 a multiple of N_2, there exists a linear map $L : M(\{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\}) \to M(\underline{\underline{T}})$ such that conditions (1) to (6) of Lemma 5.6 are satisfied and additionally

(8) If $\sigma \in M\{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\}$ we have $|(L\sigma)(\{2\pi s/N_4\})| \leqslant \|\sigma\|\eta k_o/10^2 N_4$ for all $1 \leqslant s \leqslant N_4$;

(iii) Given $1 > \gamma_o, \delta, \eta > 0$ and N_1, N_2, k_o positive integers, $N_1\delta \leqslant 10^{-2}\eta$, $N_2 \geqslant 12800\,[\eta^{-1}\delta^{-1}\gamma_o^{-1}]$ together with $\gamma > \gamma_o > 0$, $k_1, k_o, k \in \underline{\underline{Z}}^+$, F a closed set with $\text{Gp}F \neq \underline{\underline{T}}$, we can find an $N_3(\delta, \eta, N_2, \gamma, \gamma_o, k_1, k_o, k, F) \geqslant 4N_2$ such that, given $N_4 \geqslant N_3$ and k closed intervals $I_1, I_2, \ldots, I_k$ each of length γ, there exists a linear map $L : M(\{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\}) \to M(\{2\pi r/N_4 : 1 \leqslant r \leqslant N_4\})$ satisfying conditions (1), (2), (6) of Lemma 5.6 for all $\sigma \in M(\{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\})$ and conditions (3), (4), (5) of Lemma 5.6 for all $\sigma \in M(\{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\})$ such that

$\sup_{a\in\mathbb{T}} |\sigma|([a-\gamma, a+\gamma]) \leqslant \eta/80k$ together with

(7)' If $y_1' \in I_i \cap \operatorname{supp}\sigma$ for some $\sigma \in M(\{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\})$, $|y_i - y_i'| \leqslant 2\pi k_1/N_4$, $x_1, x_2, \ldots, x_k \in F$ then $\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} m_i y_i \neq 0$ whenever $1 \leqslant \sum_{i=1}^{k} |m_i| \leqslant k$ $[m_i \in \mathbb{Z}]$.

(8)' If $\sigma \in M(\{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\})$ and $\sup_{a\in\mathbb{T}} |\sigma|([a-\gamma_o, a+\gamma_o]) \leqslant 3^{-k_o}\eta/10k$ then $\sup_{a\in\mathbb{T}} |L\sigma|([a-\gamma_o, a+\gamma_o]) \leqslant \eta 3^{-k}/80k$

$$\sup_{a\in\mathbb{T}} |L\sigma|([a-\gamma, a+\gamma]) \leqslant \sup_{a\in\mathbb{T}} |\sigma|([a-\gamma, a+\gamma]) + 3^{-k_o}\eta/10k.$$

Proof.

(i) Consider L_2 in Lemma 5.1.

(ii) Set $N_3 = 10^3(k_o+1)N_2$ and let $L\sigma = \sigma * \mu_{\delta, N_4}$.

Conditions (1) to (6) follow from Lemma 3.1. Condition (8) is evident (look at the case σ a point mass).

(iii) Set $U_1 = \{2\pi r/N_2 : 1 \leqslant r \leqslant N_2\}$, $I = \bigcup_{i=1}^{k} I_i + [\gamma_o, -\gamma_o]$, $U_o = U_1 \cap I$, $F_k = \left\{\sum_{i=1}^{k} x_i : x_i \in F\right\}$. Since F is closed, so is F_k. Since $\mathrm{Gp}F \neq \mathbb{T}$, F_k can contain no intervals. Thus we can find N_3 a multiple of $8N_2$ such that for each $e \in U_o$ there exists an $e'(e)$ with $e'(e) = 2\pi s(e)/N_3$ for some $s(e) \in \mathbb{Z}$ such that $|e'(e) - e| \leqslant \delta\gamma_o/4$, and such that writing $J(e) = [e'(e) - 8\pi k_1/N_3, e'(e) + 8\pi k_1/N_3]$ we have that the $J(e)$ are disjoint, $J(e) \subseteq I_i$ if $e \in I_i$, $J(e) \cap I_i = \emptyset$ if $e_i \in I_i$ and, if $y_1, y_2, \ldots, y_k$ belong to distinct $J(e)$, then $\sum_{i=1}^{k} m_i y_i \notin F_k$ whenever $1 \leqslant \sum_{i=1}^{k} |m_i| \leqslant k$ $[m_i \in \mathbb{Z}]$.

Set $L_1\sigma = (\sigma | (U_1 \setminus U_o)) * \mu_{\delta\gamma_o, N_4}$ and let $L_2\sigma$ be that measure on

$\{2\pi r/N_4 : 1 \leq r \leq N_4\} \cap U_o$ with $L_2\sigma(e'(e)) = \sigma(e)$ for all $\sigma \in M\{2\pi r/N_2 : 1 \leq r \leq N_2\}$. It is clear that $L = L_1 + L_2$ on $M(\{2\pi r/N_2 : 1 \leq r \leq N_2\}$. Since

$|L\sigma - \sigma * \mu_{\delta \gamma, N_4}| \leq 4\ k \sup_{a \in \mathbb{T}} |\sigma|([a-\gamma, a+\gamma])$ so that

$|(L\sigma - \sigma * \mu_{\delta \gamma, N_4})\hat{}(r)| \leq 2\ k \sup_{a \in \mathbb{T}} |\sigma|([a-\gamma_o, a+\gamma_o])$, conditions (3), (4), (5) follow from Lemma 3.1 (for $\sigma \in M(\{2\pi r/N_2 : 1 \leq r \leq N_2\})$ such that

$\sup_{a \in \mathbb{T}} |\sigma|([a-\gamma, a+\gamma]) \leq \eta/80k)$. Conditions (1), (2), (3) and (8)' are true by inspection. Condition (7)' is a direct consequence of the last sentence of the paragraph above.

Proof of Lemma 5.6. By using (i) of Lemma 5.7 followed by (ii) twice, we can find P_1, P_2 (depending on δ, γ, N_2 and k) with $P_1 \geq 10^{-3}\gamma$ (provided simply that M_o is a positive multiple of M', where M' depends only on δ, η, N_2 and γ) and a linear map $L_o : M(\mathbb{T}) \to M(\mathbb{T})$ such that the following conditions are satisfied for $m \geq 0$

$(1)_m$ $\quad M'_m \operatorname{supp} L_m\sigma = 0$ for all $\sigma \in M(\mathbb{T})$

$(2)_m$ $\quad \inf_{e' \in \operatorname{supp} L_m\sigma} \sup_{e \in \operatorname{supp} \sigma} |e - e'| \leq \delta(1 - 2^{-m-1})$

$(3)_m$ $\quad |(L_m\sigma - \sigma)\hat{}(r)| \leq \eta(1-2^{-m-1})\|\sigma\|$ for all $|r| \leq N_1$, $\sigma \in M(\mathbb{T})$

$(4)_m$ $\quad |(L_m\sigma)\hat{}(r)| \leq |\hat{\sigma}(r)| + \eta(1-2^{-m-1})\|\sigma\|$ for all $r, \sigma \in M(\mathbb{T})$

$(5)_m$ $\quad |(L_m\sigma)\hat{}(r)| \leq \eta(1-2^{-m-1})\|\sigma\|$ for all $M_m - N_1 \geq |r| \geq N_1$, $\sigma \in M(\mathbb{T})$

$(6)_m$ $\quad L_m\sigma \in M^+(\mathbb{T})$, $\|L_m\sigma\| = \|\sigma\|$ for all $\sigma \in M^+(\mathbb{T})$

$(7)_m$ $\quad$ If $\sigma \in M(\mathbb{T})$, then $\sup_{a \in \mathbb{T}} |L_m\sigma|\left(\left[a - \frac{\pi}{P_1}, a + \frac{\pi}{P_1}\right]\right) \leq \eta(2-2^{-m})/160k$

$(8)_m$ $\quad \sup_{a \in \mathbb{T}} |L\sigma|([a - \frac{\pi}{P_2}, a + \frac{\pi}{P_2}]) \leq \eta 3^{-k} 4^{-10-m}/160k$ where $k_4 = \binom{p}{k}$

$(9)_m$ $\quad$ We can find $\sum(t) = \{I_i, \ldots, I_{i_k}\}$ $[1 \leq t \leq \binom{p}{k}]$ some dictionary ordering of the sets of k elements of the form $I_r = \left[\frac{2\pi r}{P_2}, \frac{2\pi(r+1)}{P_2}\right]$ such that, if $1 \leq s \leq \min(m, \binom{p}{k})$ and $y'_1, y'_2, \ldots, y'_k$ belong to distinct elements of $\sum(s)$, we have for all $|y_i' - y_i| \leqslant 2\pi q_m/M_n$, $x_i \in F$ that $\sum m_i y_i + \sum x_i \neq 0$ whenever $1 \leqslant \sum_{i=1}^{k} |m_i| \leqslant k$ $|m_i \in Z|$.

By Lemma 5.7(iii) we know that if, for all $q_m \in \underline{\underline{Z}}^+$, we can find an $L_m : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ and an $M_m \geqslant M_o$ such that conditions $(1)_m$ to $(9)_m$ are satisfied then, for all $q_{m+1} \in \underline{\underline{Z}}^+$ provided only M_{m+1} is a multiple of $M_m M'_{m+1}(\delta, \eta, M_m, kq_{m+1})$ for some M'_{m+1} fixed, we can find an $L_{m+1} : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ such that conditions $(1)_{m+1}$ to $(9)_{m+1}$ are satisfied. (We need to be able to choose q_m freely to satisfy the condition written " $N_2 \geqslant 12800\,[\eta^{-1}\delta^{-1}]$ " in Lemma 5.7(iii) whilst at the same time ensuring that the set $L_{m+1}\sigma$ will be sufficiently close to $L_m\sigma$ for $(9)_m$ to imply $(9)_{m+1}$ by reference to the condition (2) of Lemma 5.7(iii).)

Since we know that $(1)_o$ to $(9)_o$ can be satisfied, it follows that $(1)_{\binom{P}{m}}$ to $(9)_{\binom{P}{m}}$ can be satisfied (for all $q_{\binom{P}{m}} \in \underline{\underline{Z}}^+$). An application of Lemma 5.7(i) completes the proof.

Proof of Lemma 5.5. Let $P(n)$ be the statement that we can find a monotonic increasing function $h_q : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ with $h_q(r) > r$ $[1 \leqslant q \leqslant n]$ such that writing $(p(q), j(q))$ for the q^{th} element in the usual dictionary ordering of (p,j) $[1 \leqslant p \leqslant m, 0 \leqslant j]$ $(j(q) \leqslant j(q+1),\ p(q) < P(q+1)$ if $j(q) = j(q+1))$, the following statement is true : -

Given $N = M(0) < h_1(M(0)) < M(1) < h_2(M(1)) < \dots < h_n(M(n-1)) < M(n) < h_n(M(n)) < M'(n+1)$ we can find $2^{-n} \geqslant \delta(n) > 0$ and closed sets $E_S^{(n)} \subseteq [x-\varepsilon(1-2^{-n-4}), x+\varepsilon(1-2^{-n-4})]$ and measures $\mu_S^{(n)} \in M^+(E_S^{(n)})$ with $\|\mu_S^{(n)}\| = 1$ such that

$(i)_n$ $\quad f_{2^{-j(t)}(1-2^{-n-4})}(\chi_{M(2t)}(e)) = 1$ if $e \in E_S^{(n)}$, $p(t) \in S$, $1 \leqslant 2t \leqslant n$

$\quad f_{2^{-j(t)}(1-2^{-n-4})}(\chi_{M(2t)}(e)) = 0$ if $e \in E_S^{(n)}$, $p(t) \notin S$, $1 \leqslant 2t \leqslant n$

$(ii)_n$ $|\hat{\mu}_S^{(n)}(r)| \geqslant \eta(1-2^{-n-4})$ implies $|\hat{\mu}_S^{(n)}(r)| - \hat{\mu}_T^{(n)}(r)| \leqslant \eta(1-2^{-n-4})$

for all $S \subseteq T \subseteq \{1,2,\ldots,m\}$, $M(n+1) - N \geqslant r \geqslant N$

$(iii)_n$ $|\hat{\mu}_{S\cap T}^{(n)}(r)| \geqslant \max(|\hat{\mu}_S^{(n)}(r)|, \ |\hat{\mu}_T^{(n)}(r)|) - \eta(1-2^{-n-4})$

for all $S, T \subseteq \{1,2,\ldots,m\}$ $M(n+1) - N \geqslant r \geqslant N$

$(iv)_n$ $|\hat{\mu}_\emptyset^{(n)}(r)| \leqslant \eta$ for all $M(n+1) - N \geqslant |r| \geqslant N$

$(v)_n$ Given $x_1, x_2, \ldots, x_u \in F$ (where $u = \left[\frac{n}{2}\right]$), $y_1', y_2', \ldots, y_u' \in \bigcup_{\emptyset \neq S} E_S^{(n)}$ such that $|y_i' - y_j'| \geqslant 2^{-u}$ for $i \neq j$, we have $\sum x_i + \sum m_i y_i \neq 0$ for all $|y_i - y_i'| \leqslant \delta(n)$, $1 \leqslant \sum_{i=1}^{k} |m_i| \leqslant k$ $[m_i \in \mathbb{Z}]$.

$(vi)_n$ $M'(n+1)\delta(n)\eta \geqslant 2^{-4n-4}\,12800$, $M(n)\delta(n) \leqslant 2^{-4n-4}$

$(vii)_n$ $M'(n+1)e = 0$ for all $e \in E_S^{(n)}$.

It follows from Lemma 5.6 for n odd and from Lemma 5.5 for n even that $P(n+1)$ is true and that moreover, given $E_S^{(n)}$, $\mu_S^{(n)}$ as in the statement of $P(n)$, we can find $E_S^{(n+1)}$, $\mu_S^{(n+1)}$ as in the statement of $P(n+1)$ with

$(viii)_{n+1}$ $\inf_{e \in E_S^{(n)}} |e - e'| \leqslant \delta(n)/2$ for all $e' \in E_S^{(n+1)}$

$(ix)_{n+1}$ $|(\mu_S^{(n)} - \mu_S^{(n+1)})\hat{\ }(r)| \leqslant 2^{-n}$ for $|r| \leqslant M(n)$.

Further by Lemma 5.2 $P(0)$ is true.

Thus setting $h(r) = h_n(r)$ for $h_{n+1}(h_n(\ldots(h_1(N)))) > r \geqslant h_n(h_{n-1}(\ldots(h_1(N))))$ we see that given $N = M(0) < h(M(0)) < M(1) < h(M(1)) < \ldots$ we can construct $E_S^{(n)}$, $\mu_S^{(n)}$, $\delta(n)$ satisfying conditions $(i)_n$ to $(ix)_n$ $[n \geqslant 2]$. It is clear (by $(x)_n$ and $(xi)_n$) that $E_S^{(n)}$ converges (topologically) to a closed set E_S and $\mu_S^{(n)}$ converges weakly to a $\mu_S \in M^+(E_S)$ with $\|\mu_S\| = 1$. Condition (i) follows from $(i)_n$, $(iv)_n$ and $(vii)_n$; (iii) from $(iii)_n$; (iv) from $(iv)_n$ (note that we have not claimed that $E_\emptyset$ is of measure 0); and (v) from $(v)_n$ and $(viii)_n$.

We conclude the section by obtaining a consequence of Lemma 5.5 which stands in the same relation to it as Lemma 2.1 stands to Lemma 1.7 or Lemmas 5.2' and 5.4' to Lemmas 5.2 and 5.4.

LEMMA 5.8. Given $K > 1$, $1 > \lambda > 0$, we can find an $C(K, \lambda) \geq 1$ and an $m(K, \lambda) \in \underline{\underline{Z}}^+$ with the following property : -

Given $1 > \varepsilon > 0$, we can find an $N(\varepsilon, K, \lambda) \geq 1$ with the following property :-

Given $x \in \underline{\underline{T}}$ and F a closed set in $\underline{\underline{T}}$ such that $\mathrm{Gp}F \neq \underline{\underline{T}}$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ (such that $h(r) > r$) with the following property :-

Given $M_p(j)$ $[1 \leq p \leq m,\ 0 \leq j]$ with $2N(\varepsilon, K, \lambda) = M_1(0)$, $h(M_p(j)) \leq M_{p'}(j')$ whenever $0 \leq j < j'$, $1 \leq p, p' \leq m$ or $1 \leq p < p' \leq m$, we can find a closed set $E \subseteq [x-\varepsilon, x+\varepsilon]$ and a measure $T \in M(E)$ such that

(i) E is the union of a finite collection $\mathcal{E}$ of disjoint closed sets such that for each $E' \in \mathcal{E}$ we can find a subset $\theta(E') \subseteq \{1, 2, \ldots, m\}$ with

$$f_{2^{-j}}(\chi_{M_p(j)}(e)) = 1 \qquad \text{for all } e \in E', \quad p \in \theta(E')$$

$$f_{2^{-j}}(\chi_{M_p(j)}(e)) = 0 \qquad \text{for all } e \in E', \quad p \notin \theta(E')$$

(ii) $\mathrm{card}\, \theta(E') \geq \lambda m$

(iii) $\|T\|_{PM} = |\hat{T}(0)| = 1 \geq K \sup_{|r| \geq M(0)} |\hat{T}(r)|$

(iv) $\|T\|_M \leq C$

(v) E is independent, $\mathrm{Gp}E \cap \mathrm{Gp}F = \{0\}$.

Consequently

(ii)' $\limsup_{r\to\infty} \sigma\{x\in E : |\chi_r(x) - 1| \leqslant \delta\} \geqslant (1-\lambda)\|\sigma\|$ for all $\sigma\in M^+(E)$, indeed

(ii)'' $\lim_{u\to\infty} \inf_{\sigma\in M^+(E), \|\sigma\|=1} \sup_{m\geqslant p\geqslant 1} \inf_{r\geqslant u} \sigma\{x\in E : |\chi_{M_p(r)}(x) - 1| \leqslant \delta\} \geqslant (1-\lambda)$

for all $\delta > 0$.

(iii)' $\inf\{\|\sum_{|r|\geqslant M(0)} a_r\chi_r - 1\|_{C(E)} : \sum_{|r|\geqslant M(0)} |a_r| \leqslant K/2\} \geqslant 1/2C$.

Remark 1. We could have maintained uniformity with e. g. Lemma 2.1 in the conclusions (i) to (ii) by talking about points rather than sets ("For every $e\in E$ we can find a subset $\theta(e) \subseteq \{1,2,\ldots,m\}$ such that card $\theta(e) \geqslant \lambda m$ and $f_{2^{-j}}(\chi_{M_p(j)}(e))$ takes the value 1 or 0 according as j belongs or does not belong to $\theta(e)$"). However, in order to avoid measurability questions the pointwise conditions would have in any case to be translated back into the equivalent closed set conditions.

Remark 2. We will deduce (iii)' from (iii). If the reader examines our proof of (iii) he will find that it was obtained (via a simple version of Hahn Banach) from what is essentially just another form of (iii)'. We leave it to the reader to produce a direct proof. (In the original versions of the proofs in this paper we obtained results like (iii)' directly (e. g. Lemma 1.9(iii)), but this complicated the proof of results like (iii) (in particular Lemma 1.9'' (iv)) in which we are, in general, more interested.)

Proof of Lemma 5.8.

(ii)' We have, by (i) and (ii),

$$\sum_{p=1}^{m} \sigma\{x\in E : |\chi_{M_p(j)}(x) - 1| \leqslant \delta\} = \sum_{p=1}^{m} \sum_{E'\in\mathcal{E}} \sigma\{x\in E' : |\chi_{M_p(j)}(x) - 1| \leqslant \delta\}$$

$$\geqslant \sum_{E'\in\mathcal{E}} (1-\lambda)m\sigma(E')$$

$$= (1-\lambda)m\sigma(E)$$

so that

$$\sup_{1 \leqslant p \leqslant m} \sigma\{x \in E : |\chi_{M_p(j)}(x) - 1| \leqslant \delta\} \geqslant (1 - \lambda)\|\sigma\|$$

provided only $j \geqslant j_o(\delta)$ for some j_o independent of $\|\sigma\|$ $[\delta > 0,\ \sigma \in M^+(E)]$.

(iv)' By (iv) and (iii)

$$\begin{aligned}\Big\|\sum_{|r| \geqslant M(0)} a_r \chi_r - 1\Big\|_{C(E)} &\geqslant \Big|\int\Big(\sum_{|r| \geqslant M(0)} a_r \chi_r - 1\Big)dT\Big| \\ &= \Big|\sum_{|r| \geqslant M(0)} a_r \hat{T}(r) - 1\Big| \\ &\geqslant 1 - \sum_{|r| \geqslant M(0)} |a_r|\,|\hat{T}(r)| \\ &\geqslant 1 - (K/2)K^{-1} = 1/2\end{aligned}$$

for all $\sum_{|r| \geqslant M(0)} |a_r| \leqslant K/2$.

The proof of the main part of Lemma 5.8 follows mutatis mutandis that of Lemma 2.1. In particular using the arguments and notations of the first part of Section § 3, we see that Lemma 5.8 follows from Lemma 5.5 and

LEMMA 3.3'. Given $m \geqslant q \geqslant 1$ we can find $a_U \in \underline{\underline{C}}$ $[U \in \Phi(m,q)]$ such that $\sum_{U \in \Phi(m,q)} a_U = 1$ with the following property :-

Given $1 > \alpha > 0$ with $K(m,q) - \alpha > 1$ we can find an $\eta_o(\alpha)$ with the following property :-

If we take $\eta_o(\alpha) > \eta > 0$ in the hypotheses of Lemma 5.5 then, taking the μ_S as in the conclusion of Lemma 5.5, we have, writing $T_o = \sum_{U \in \Phi(m,q)} a_U \mu_U$ that

(iii) $\quad 1 \geqslant (K(m,q) - \alpha) \sup_{|r| \geqslant M(0)} |\hat{T}_o(r)|$.

Proof. We use the proof of Lemma 3.3 with the condition "$M(0)/2 \leqslant r \leqslant M(m+1) - M(0)/2$" replaced by "$M(0)/2 \leqslant |r|$".

This concludes a section which contains no really new ideas and from which we shall only need the statements of the lemmas. (Further, their use can always be avoided in any specific case by ad hoc arguments incorporating perhaps only one part of their proofs.) By placing all the messy work together, I hope to ensure that the ideas of the remaining proofs can stand out more clearly. However, before resuming the main lines of our argument in Section § 7 onwards, we detour slightly to discuss some easy applications of the results above in constructing Helson sets of non synthesis satisfying various thinness conditions.

§ 6. HOW THIN CAN A SET OF NON SYNTHESIS BE ?

We arrange the results of this section roughly in order of decreasing interest. The arguments are of a considerably lower standard of difficulty than those in the rest of the paper. Lemma 6.2 is independent of Section § 5 and the lemmas which conclude the section are very simple consequences of results of Varopoulos (and a related result of Kaijser given as Lemma 6.4).

LEMMA 6.1 (Varopoulos). If K_1, K_2 are perfect non empty disjoint sets and $K_1 \cup K_2$ is Kronecker, then K_1+K_2 contains a closed set of non synthesis.

We proved our Helson set E of non synthesis by constructing a pseudo function on it. Is the result affected if we demand E Dirichlet (so that by the result of Kahane proved in Lemma 4.1, $\sup_n |\hat{T}(n)| = \limsup_{n\to\infty} |\hat{T}(n)|$ for all $T \in PM(E)$) ? The answer is no.

THEOREM 1.1'. Given $\delta(n) > 0$ $[n \geq 1]$ we can find a weak Kronecker set E which is not of synthesis together with a sequence of integers $Q(n) \to \infty$ such that $E \subseteq \{x: |x-2\pi r/Q(n)| \leq \delta(Q(n))$ for some $1 \leq r \leq Q(n)\}$.

As a corollary we obtain

LEMMA 6.2. Given H_1 a continuous increasing function $H_1 : \mathbb{R}^+ \to \mathbb{R}^+$ with $H_1(0) = 0$ and $H_2 : \mathbb{R}^+ \to \mathbb{R}^+$ continuous with $H_2(x) \to \infty$ as $x \to \infty$, we can find a weak Kronecker set E of non synthesis such that

(1) E has Hausdorff H_1 measure 0 ;

(2) E is Dirichlet ;

(3) We can find $\delta_i \to 0$, $q_i \to \infty$ such that $H_2(\delta_i^{-1}) \geq q_i$ yet E can be covered by at most q_i intervals of length less than or equal to δ_i.

(Note that if $H_2(x) = \log x$ condition (3) is Salem's condition and implies (2) ([4] p. 95).)

Proof of Lemma 6.2. This is trivial. Since $H_1(x), H_2(x), x \to 0+$ as $x \to 0+$ we can find $\delta(n) > 0$ such that

(i) $nH_1(\delta(n)/2) < 2^{-n}$

(ii) $n\delta(n) < 2^{-n}$

(iii) $nH_2((\delta(n)/2)^{-1}) < 2^{-n}$.

Constructing E as in Theorem 1.1' we have

(i) E can be covered by intervals $I_s = \left[\frac{2\pi s}{Q(r)}, \frac{2\pi(s+1)}{Q(r)}\right]$ of length $\ell_s = \frac{2\pi}{Q(r)}$ such that $\sum H_1(\ell_s) \leq 2^{-n}$

(ii) $\|\chi_{Q(r)} - 1\|_{C(E)} \leqslant 2^{-r}$

(iii) E can be covered by at most $Q(r)$ intervals I_s of length ℓ_s such that $H_2(\ell_s^{-1}) \geqslant Q(r)$.

The result follows.

Theorem 1.1' also shows (as we remarked earlier) how delicately Herz's arithmetic condition for synthesis ([5] p. 124 and Section 9 of this paper) depends on strictly arithmetical properties of $\underline{\underline{T}}$.

It would be more interesting to decide whether we could construct a Helson set of non synthesis inside the K_1+K_2 (which is automatically Dirichlet) of Lemma 6.1. I have not succeeded in doing so, but neither have I found a good reason for supposing that the method of this paper is inapplicable. (We shall discuss the problem again in the last part of this section from the statement of Lemma 6.4 onwards.)

Our proof of Theorem 1.1' mimics as far as possible the proof of Theorem 1.1 in Section § 2. We construct E_n, μ_n subject to the inductive condition below. ($N(\varepsilon, K, \lambda)$, f_n are defined as in Section § 2 and we suppose for convenience here and throughout this section that $1 > \delta(0)$ and $(n+1)^{-2} 2^{-n-3} \delta(n) \geqslant \delta(n+1) \geqslant 0$).

INDUCTIVE CONDITION L(n). At the conclusion of the n^{th} step we have a finite set E_n, a measure $\mu_n \in M(E_n)$ and an integer $Q(n) \geqslant 1$ such that

(i) $\hat{\mu}_n(0) = 1$

(ii) $\|\hat{\mu}_n\|_{PM} \leqslant 2 - 2^{-n}$

(iii) $Q(n)E_n = 0$.

LEMMA 6.3. Given E_n, μ_n, $Q(n)$ satisfying the inductive condition $L(n)$, we can find E_{n+1}, μ_{n+1}, $Q(n+1)$, $P(n+1)$ satisfying condition $L(n+1)$ such that, in addition

(iv) $|\hat{\mu}_{n+1}(r)| \leqslant |\hat{\mu}_n(r)| + 2^{-n-4}$ for all $|r| \leqslant Q(n)$

(v) $|\hat{\mu}_{n+1}(r)| \leqslant 2^{-n-4}$ for all $P(n+1) \leqslant r \leqslant Q(n+1) - P(n+1)$

(vi) $\sup_{y \in E_{n+1}} \inf_{x \in E_n} |x-y| \leqslant \frac{\delta(Q(n))}{2}$.

Further

(vii) If F is any closed set with $\sup_{y \in F} \inf_{x \in E_{n+1}} |x-y| \leqslant \delta(Q(n+1))$ then we can find $f = \sum_{r=P(n+1)}^{Q(n+1)-P(n+1)} a_r \chi_r$, $a_r \geqslant 0$, $\sum_{r=P(n+1)}^{Q(n+1)-P(n+1)} a_r = 1$ such that $\|f - f_{n+1}\|_{C(F)} \leqslant 2^{-n}$.

Proof. Write $\varepsilon(n) = \delta(Q(n))(2^{+n+8} \|\mu_n\|_M N(2^{-4}\delta(Q(n)), 2^{n+5}\|\mu_n\|_M, 1-2^{-n-5}))^{-1}$. The work of this paragraph is somewhat simplified by using Lemma 5.2' but we could have copied the first two paragraphs of the proof of Lemma 2.2 directly. Let $E = \{e_1, e_2, \ldots, e_\ell\}$ (with $e_1, e_2, \ldots, e_\ell$ distinct). We can find e_i' linearly independent over $\mathbb{Q}$ with $|e_i'' - e_i| \leqslant \varepsilon(n)/2$. By Kronecker's theorem there exists an M such that $\sup_{1 \leqslant i \leqslant \ell} |\chi_M(e_i) - f_{n+1}(e_i)| \leqslant 2^{-n-6}$. By the continuity of χ_M, f_{n+1} there exists a $Q \in \mathbb{Z}^+$ and e_i' distinct with $|e_i' - e_i''| \leqslant \varepsilon(n)/2$ (and so $|e_i - e_i'| \leqslant \varepsilon(n)$), $Qe_i' = 0$ and $\sup_{1 \leqslant i \leqslant \ell} |\chi_M(e_i) - f_{n+1}(e_i)| \leqslant 2^{-n-b}$ (and so $\sup_{1 \leqslant i \leqslant \ell} |\chi_{M+tQ}(e_i') - f_{n+1}(e_i')| \leqslant 2^{-n-5}$ for all $t \in \mathbb{Z}$). Set $E_n' = \{e_i' : 1 \leqslant i \leqslant \ell\}$ and write μ_n' for that measure with support E_n' and $\mu_n'(e_i') = \mu_n(e_i)$. By Lemma 5.2' we can find $m \geqslant 1$ and $N(2^{-4}\delta(Q(n)), 2^{n+5}\|\mu_n\|_M, 1-2^{-n-5}) = P(n) < M(1) < M(2) < \ldots < M(m) < M(m+1) = Q(n+1)$

and $T_{n+1} \in M\ ([-2^{-n-5}\delta(Q(n)), 2^{-n-5}\delta(Q(n))])$ with support E^*_{n+1} such that $M(j)$ is congruent to P modulo Q and $M(m+1)$ is congruent to zero modulo Q $[1 \leqslant j \leqslant m]$ whilst

(a) $M(m+1)E^*_{n+1} = 0$

(b) $\|m^{-1} \sum_{j=1}^{m} \chi_{M(j)} - 1\|_{C(E^*_{n+1})} \leqslant 2^{-n-3}$

c) $\|T_{n+1}\|_{PM} = \hat{T}_{n+1}(0) = 1 \geqslant 2^{n+4}\|\mu_n\|_M \sup_{M(m+1)-P(n) \geqslant r \geqslant P(n)} |\hat{T}_{n+1}(r)|$.

The arguments of Lemma 2.2 now show that writing $E_{n+1} = E'_n + E^*_{n+1}$, $\mu_{n+1} = \mu'_n * T_{n+1}$ the conclusions of the lemma hold.

Proof of Theorem 1.1'. Construct a sequence $(E_{n+1}, \mu_{n+1}, Q(n+1), P(n+1))$ satisfying the conclusions of Lemma 6.3 $[n = 0, 1, 2, \ldots]$. Let E be the topological limit of the E_{n+1} and S a (in fact the) weak limit point of the μ_{n+1}. As in the proof of Theorem 1.1 $0 \neq S \in PM(E)$, $E \subseteq \{x : |x - 2\pi r/Q(n)| \leqslant \delta(Q(n))$ for some $1 \leqslant r \leqslant Q(n)\}$ and E is weak Kronecker, indeed given $f \in S(T)$ and $\varepsilon > 0$ we can find an $n \geqslant 1$ and a $g \in A(T)$ with $\|g\|_{A(T)} = 1$ and $\operatorname{supp} \hat{g} = \{r : \hat{g}(-r) \neq 0\} \subseteq \{r : P(n+1) - M(n+1) \geqslant r \geqslant M(n+1)\}$, yet $\|g - f\|_{C(E)} \leqslant \varepsilon$. Thus for any measure $\sigma \in M(E)$ we have $\liminf_{n\to\infty} \left\{\left|\int (g-f)d\sigma\right| : g = \sum_{r=M(n+1)}^{P(n+1)-M(n+1)} a_r \chi_r,\ \sum_{r=M(n+1)}^{P(n+1)-M(n+1)} |a_r| \leqslant 1\right\} = 0$.

Taking $g = \chi_{-k}$ this yields

$$\liminf_{n\to\infty} \inf \left\{\left|\hat{\sigma}(k) - \sum_{r=M(n+1)}^{P(n+1)-M(n+1)} a_r\, \hat{\sigma}(r)\right| : \sum_{r=M(N+1)}^{P(n+1)-M(n+1)} |a_r| \leqslant 1\right\} = 0$$

so $\limsup_{n\to\infty} \sup \{|\hat{\sigma}(r)| : P(n+1)-M(n+1) \geqslant r \geqslant M(n+1)\} \geqslant |\hat{\sigma}(k)|$ for all k whence $\limsup_{n\to\infty} \sup \{|\hat{\sigma}(r)| : P(n+1)-M(n+1) \geqslant r \geqslant M(n+1)\} \geqslant \|\sigma\|_{PM}$. But conditions (iv) and (v) show that

$$\limsup_{n\to\infty} \sup \left\{|\hat{S}(r)| : P(n+1) - M(n+1) \geqslant r \geqslant M(n+1)\right\} = 0$$

so S is a true pseudomeasure. Since a Helson set of synthesis cannot support a true pseudomeasure (the dual $(A(E))'$ of $A(E)$ is the set of synthesisable pseudomeasures, if E is Helson $(A(E))' = (C(E))' = M(E)$), we are done.

Next we prove

THEOREM 1.1". Given $H : \underline{\underline{R}}^+ \to \underline{\underline{R}}^+$ continuous increasing with $H(0) = 0$, we can find a weak Kronecker set E with Hausdorff H measure 0 which supports a non zero pseudofunction.

The reader may recall a remark of Salem which says that if $\mu \in M(E)$ where E is independent then if $|\hat{\mu}(r)|$ tends to 0 as $r \to \infty$ it does so slower than any power of r. We remark that even for S, the pseudofunction constructed in Theorem 1.1, although an explicit lower bound can be put on the rate at which $\hat{S}(r)$ tends to 0, the speed of convergence so given is extremely slow (by comparison for example with $(\log\log \dots \log n)^{-1}$). It is doubtful whether the direct methods of this paper can give rapid convergence but similar methods also fail to give rapid convergence in many situations where such convergence is known to exist. Thus nothing in this paper constitutes evidence against such statements as "Given $c_n \geqslant 0$ such that $c_n n^{-\alpha} \to \infty$ as $n \to \infty$ for any α we can find a weak Kronecker set E and an $0 \neq S \in PM(E)$ such that $|\hat{S}(n)| \leqslant c_n$ for all n".

The proof of Theorem 1.1" follows that of Theorem 1.1 (to which the reader is asked to refer) using Lemma 5.2' in place of Lemma 2.1.

We construct E_n, μ_n subject to the Inductive Condition $L(n)$ of Section § 2 with $N(\varepsilon, K, \lambda) = \varepsilon^{-1} N_o(K, \lambda)$ (in the notation of Lemma 5.4).

LEMMA 2.2". As for Lemma 2.2 with the additional condition

(x) If F is as in (ix) then there exist intervals I_i of length $\ell_i \leqslant \varepsilon(n)$ such that $\sum H(\ell_i) \leqslant 2^{-n}$ and $\bigcup I_i \supseteq F$.

Proof. Write $m = m(2^{n+4}||\mu_n||_M, \ 1-2^{-n-4})$. Set $K = 16(N_o(2^{n+5}||\mu_n||_M C(2^{n+4}||\mu_n||_M, \ 1-2^{-n-4}), \ 1-2^{-n-5}))^{-1}$. By Lemma 5.4' we can select $P(n) < M(1) < M(2) < \ldots < M(m) < M(m+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$ and

(1) $M(1)\varepsilon(n) \geqslant 2^{n+16} P(n)||\mu_n||_M$

(2) $M(r+1) \geqslant 2^{n+16} M(r)$ $\qquad [1 \leqslant r \leqslant m]$

whilst, putting $\varepsilon(n+1) = K/M(m+1)$, we have

$$0 < \varepsilon(n+1) < \min(\varepsilon(n)/32 \ , \ 2^{-n-16} M(m))$$

(so that we have statement (3) : -

$M(m+1) \geqslant 16M(m) + (\varepsilon(n+1)/2)^{-1} N_o(2^{n+5}||\mu_n||_M, C(2^{n+4}||\mu_n||_M, 1-2^{-n-4}), (1-2^{-n-5}))$

in such a way that we can find a $T_{n+1} \in M(\underline{\underline{T}})$ with the following properties. If we write $E^*_{n+1} = \text{supp } T_{n+1}$ then statements (4) to (8) of the proof of Lemma 2.2 apply and additionally we have (using Lemma 5.2'(v)) the statement

(8a) $E^*_{n+1} + [-\varepsilon(n+1) \ , \ \varepsilon(n+1)]$ can be covered by intervals of length $\ell_i \leqslant \varepsilon(n)$ such that $\sum H(\ell_i) \leqslant 2^{-n-1}/\text{card } E_n$.

The remainder of the proof follows the proof of Lemma 2.2 (from statement (8) onwards) word for word. Since by (8a) $E^*_{n+1} + [-\varepsilon(n+1) \ , \ \varepsilon(n+1)] + e'_u$ can be covered by intervals of length $\ell_i \leqslant \varepsilon(n)$ such that $\sum H(\ell_i) \leqslant 2^{-n-1}/\text{card } E'_n$, it follows that $E_{n+1} = E^*_{n+1} + [-\varepsilon(n+1), \varepsilon(n+1)] + E'_n$ can be similarly covered with $\sum H(\ell_i) \leqslant 2^{-n-1}$. Condition (ix) of $L(n+1)$ now follows.

Proof of Theorem 1.1". As for Theorem 1.1. Since $E \subseteq E_n + [-\varepsilon(n), \varepsilon(n)]$ it follows from condition (ix) of $L(n)$ that, for all n, E can be covered by intervals of length ℓ_i with $\sum H(\ell_i) \leqslant 2^{-n}$. It follows that E has Hausdorff H measure 0.

There is another thinness condition due to Carleson.

Carleson's Condition. A closed set E is said to satisfy Carleson's condition if it is of (Lebesgue) measure 0 and the complement of E (which is automatically of the form $\bigcup_{i=1}^{\infty} I_i$ where the I_i are disjoint open intervals (called the complementary intervals) of length ℓ_i say) satisfies $\sum_{i=1}^{\infty} \ell_i \log 1/\ell_i < \infty$.

We make the obvious remark :

LEMMA 6.4. Suppose E is the topological limit of sets $E_n = \{x_{1n}, x_{2n}, \ldots, x_{nn}\}$ with $|x_{rn} - x_{rm}| \leqslant \inf_{1 \leqslant t < s \leqslant n} |x_{tn} - x_{sn}|/4 = \gamma_n$ say for all $1 \leqslant r \leqslant n < m$ and $\inf_{1 \leqslant r \leqslant n-1} |x_{rn} - x_{nn}| \leq \delta_n$. Then, if E is of Lebesgue measure 0 and $\sum \delta_n \log(1/\delta_n) < \infty$ it follows that E satisfies Carleson's condition.

Proof. This is trivial. Let $J_1 = \mathbb{T}$, J'_n be an interval (x_{sn}, x_{nn}) with $|x_{sn} - x_{nn}| = \inf_{1 \leqslant r \leqslant n-1} |x_{rn} - x_{nn}|$ and $J_n = J'_n + [-\gamma_n, \gamma_n]$. It is clear that there exists an injective map $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $J'_{h(i)} \supseteq I_i$. Thus

$$\sum \ell_n \log 1/\ell_n \leqslant \sum (\delta_n - 2\gamma_n) \log(1/(\delta_n + 2\gamma_n))$$

$$\leqslant 2 \sum \delta_n(\log \delta_n^{-1} + \log 2) < \infty$$

and the result is proved.

THEOREM 1.1'''. In Theorem 1.1' (and so in Lemma 6.2) we may demand that E satisfies the Carleson Condition.

To prove this we need.

LEMMA 6.3'''. As for Lemma 6.3 with the additional condition

(viii) If $E_{n+1} \cap S(x,\delta(Q(n))) = \{x\} + \{y_1, y_2, \ldots, y_k\}$ for some $x \in E_n$ with $y_1 < y_2 < \ldots < y_k$ and $|y_i - x| \leqslant \delta(Q(n))$, then $\sum_{i=1}^{k}(y_{i+1}-y_i)\log(1/(y_{i+1}-y_i)) \leqslant 2^{-n}/\text{card } E_n$.

Proof. We follow the proof of Lemma 6.3 and 2.2. By the argument of the second paragraph of Lemma 5.2 there exists a constant $D_o \geqslant 1$ such that if $D \geqslant D_o$ and $M_o(1)$, $M_o(2)$, $\ldots$, $M_o(m_o+1)$ is a sequence of integers with $M_o(1) \geqslant D$, $M_o(j+1) \geqslant D_o M_o(j)$, then (writing $E_n = \{e_1, e_2, \ldots, e_\ell\}$ with the e_u distinct) we can find e'_u with $M_o(m_o+1)e'_u = 0$ such that

$(9)_o$ $\quad |e_u - e'_u| \leqslant 16D^{-1} \leqslant \delta(Q(n))/4$

$(10)_o$ $\quad |\chi_{M_o(j)}(e'_u) - f_{n+1}(e'_u)| \leqslant 2^{-n-5} \quad [1 \leqslant u \leqslant \ell, \quad 1 \leqslant j \leqslant m_o]$.

Introducing the notation of Lemma 5.2' we see that, setting

$$B_o = 2^{n+16}\max(D_o, B(2^{n+5}||\mu_n||_M, 1-2^{-n-4}), N_o(2^{n+5}||\mu_n||_M, 1-2^{-n-4}))$$

$$m = m(2^{n+5}||\mu_n||_M, 1-2^{-n-4})$$

and choosing $0 < \varepsilon(n+1) \leqslant \delta(Q(n))/4$ such that

$$4m\,\varepsilon(n+1)\log 2\, B_o \leqslant 2^{-n-2}/\text{card}\, E_n$$

and finally setting

$$P(n) = (\,[\varepsilon(n+1)]^{-1} + 1)N_o(2^{n+5}\|\mu_n\|_M\,,\ 1-2^{-n-4})\,,$$

we can choose $P(n) = M(0) < M(1) < \ldots < M(m+1) = Q(n+1)$ together with $T_{n+1} \in M\ \ ([-\varepsilon(n+1)\,,\ \varepsilon(n+1)])$ and distinct e'_u such that the following is true : -

(9) $\quad |e_u - e'_u| \leqslant 2^{-n-16}/P(n) \leqslant \delta(Q(n))/4$

(10) $\quad |\chi_{M(j)}(e'_u) - f_{n+1}(e'_u)| \leqslant 2^{-n-5} \quad [1 \leqslant u \leqslant \ell,\ \ 1 \leqslant j \leqslant m_o]$

(10a) $\quad BM(j) \leqslant M(j) \leqslant 2\,BM(j)$

(4) $\quad M(m+1)E^*_{n+1} = 0$ where $E^*_{n+1} = \text{supp}\, T_{n+1}$

(6) $\quad \|m^{-1}\sum_{j=1}^{m}\chi_{M(j)} - 1\|_{C(E^*_{n+1})} \leqslant 2^{-n-3}$

(7) $\quad \|T_{n+1}\|_M = \hat{T}_{n+1}(0) = 1 \geqslant 2^{n+4}\|\mu_n\|_M \sup_{M(m+1)-P(n) \geqslant r \geqslant P(n)} |\hat{T}_{n+1}(r)|$

(8) $\quad \|T_{n+1}\|_M \leqslant C(2^{n+4}\|\mu_n\|_M\,,\ 1-2^{-n-4})\,.$

The arguments of Lemma 2.2 now show that, writing $E_{n+1} = E'_n + E^*_{n+1}$, $\mu_{n+1} = \mu'_n * T_{n+1}$ (where μ'_n , E'_n are defined exactly as in Lemma 2.2), the conclusions of the lemma (with the possible exception of (viii)) hold.

To prove (viii) we note that writing $E^*_{n+1} = \{y_1, y_2, \ldots . y_k\}$ with $-\varepsilon(n+1) < y_1 < y_2 < y_3 < \ldots < y_k < \varepsilon(n+1)$ we have

$$\begin{aligned}\sum_{i=1}^{k}(y_{i+1} - y_i)\log(1/(y_{i+1} - y_i)) &\leqslant \sum_{i=1}^{k}(y_{i+1} - y_i)\log(M(m+1)/2\pi)\\ &\leqslant 2\varepsilon(n+1)\log(M(m+1)/2\pi)\\ &\leqslant 2\varepsilon(n+1)\log(2^m B_o^m\, P(n))\\ &\leqslant 2\varepsilon(n+1)\log(2^m B_o^m\,((\varepsilon(n+1)^{-1} + 1)))\end{aligned}$$

$$\leqslant 2\varepsilon(n+1)((m+1)\log 2 B_o - 2 \log \varepsilon(n+1))$$

$$\leqslant 2^{-n}/\text{card } E_n .$$

Condition (viii) now follows by inspection.

Proof of Theorem 1.1'''. As for Theorem 1.1'. The fact that E satisfies the Carleson condition follows from conditions (viii) and (vi) of Lemma 6.3'. These imply that we can write $E_n = \{x_1, x_2, \dots, x_s\}$, $E_{n+1} = \{x'_1, x'_2, \dots, x'_t\}$ $[1 \leqslant s < t]$ in such a way that $\sup_{1 \leqslant i \leqslant s} |x'_i - x_i| \leqslant \delta(Q(n)) \leqslant \sup_{1 \leqslant i < j \leqslant s} |x_i - x_j|/2^n$ and $\sup_{s < i \leqslant t} \inf_{1 \leqslant j < i} |x'_i - x'_j| \leqslant \delta_i$ with $\sum_{i=s+1}^{t} \delta_i \log \delta_i^{-1} \leqslant 2^{-n}$. An application of Lemma 6.4 gives the required result.

I do not know whether we can demand that the E in Theorem 1.1'' satisfy Carleson's condition. I suspect that such a result, even if true, would require a new idea for its proof.

In so far as the results above deal with sets of non synthesis rather than independent Helson sets of non synthesis, they can be obtained in a very much simpler manner using Lemma 6.1 (the result of Varopoulos). Compare Lemma 6.2 with the following result :

LEMMA 6.5. Given H a continuous increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ with $H(0) = 0$ and $\delta(n) > 0$, we can find K a perfect Kronecker set such that writing $K_o = K + K$ we have

(i) $K \subseteq \left\{x : \left|\frac{2\pi r}{Q(n)} - x\right| < \delta(Q(n)) \text{ for some } r\right\}$ for some $Q(n) \to \infty$

(ii) The complementary intervals of I_i of K_o have length ℓ_i with $\sum H(\ell_i) < \infty$.

In particular by Lemma 6.1 there exists a Dirichlet set E of non synthesis with the properties (i) and (ii). If we take $H(x) = x \log x^{-1}$ we obtain E satisfying the Carleson condition.

Proof. We shall construct K directly. But (under the non restrictive condition H concave) it would suffice to construct a perfect set E with properties (i) and (ii) and remark that every perfect set has a perfect Kronecker subset. Choose $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $h(r) \leqslant r$ and h takes every value on $\mathbb{Z}^+$ infinitely often. Choose x_{11} with $Q(1)x_{11} = 0$ for some $Q(1) \in \mathbb{Z}^+$. Set $A = H(|2x_1|) + H(2\pi - |2x_1|)$. By the continuity of H we can find $\varepsilon(1) > 0$ such that if $|y_1 - x_1| < \varepsilon(1)$, then $H(|2y_1|) + H(2\pi - |2y_1|) \leqslant A + (1 - 2^{-1})$. We shall construct inductively a sequence of sets $E_n = \{x_{1n}, x_{2n}, \ldots, x_{nn}\}$ together with $\varepsilon(n) > 0$, $Q(n) \in \mathbb{Z}$ in such a way that

$(i)_n$ $\quad |x_{rn} - x_{sn}| \geqslant 16\varepsilon(n)$ $\quad$ for $\quad 1 \leqslant r < s \leqslant n$

$(ii)_n$ $\quad$ If $\quad |y_r - x_{rn}| \leqslant \varepsilon(n)$ $\quad [1 \leqslant r \leqslant n]$ then, writing $\{y_1, y_2, \ldots, y_n\}^* = \{y_r + y_s : 1 \leqslant r \leqslant s \leqslant n\}$, we know that the complementary intervals of $\{y_1, y_2, \ldots, y_n\}^*$ have length ℓ_{kn} with $\sum H(\ell_{kn}) \leqslant A + (1-2^{-n})$.

Suppose therefore that we have constructed E_n, $\varepsilon(n)$, $Q(n)$ satisfying $(i)_n$ and $(ii)_n$. We can find $x'_{1n}, x'_{2n}, \ldots, x'_{nn}$ (linearly) independent (over $\mathbb{Q}$) with $|x'_{rn} - x_{rn}| \leqslant \varepsilon(n)/16$. We can find $x'_{n+1\,n}$ such that $x'_{1n}, x'_{2n}, \ldots, x'_{nn}, x'_{n+1\,n}$ are independent $|x'_{h(n)\,n} - x'_{n+1\,n}| \leqslant \varepsilon(n)/16$ and $(n+1)H(2|x'_{h(n)\,n} - x'_{n+1\,n}|) \leqslant 2^{-n-5}$. By Kronecker's theorem there exists an integer $N(n+1) \geqslant 8Q(n)$ such that $\sup_{1 \; r \; n+1} |\chi_{N(n+1)}(x'_{rn}) - f_{n+1}(x'_{rn})| < 2^{-n-5}$. By the continuity of the functions involved we can find an $0 < \varepsilon'(n+1) < \varepsilon(n)/2^{16}$, a $Q(n+1) \in \mathbb{Z}$ with $Q(n+1) \geqslant 8(n+1)$ and

$x_{1\,n+1}, x_{2\,n+1}, \ldots, x_{n+1\,n+1}$ with $|x_{r\,n+1} - x'_{r\,n}| < \varepsilon'(n+1)$, $Q(n+1)x_{r\,n+1} = 0$

such that

$(i)_{n+1}$ $\quad |x_{r\,n+1} - x_{s\,n+1}| \geqslant 16\varepsilon'(n+1)$ for $1 \leqslant r < s < n+1$

$(ii)'_{n+1}$ $\quad n\,H(2K) < 2^{-n-4}$ whenever $0 \leqslant K < |x_{h(n)\,n+1} - x_{n+1\,n+1}| + 2\varepsilon'(n+1)$

$(iii)_{n+1}$ $\quad \sup_{1 \leqslant r \leqslant n+1} |\chi_{N(n+1)}(y_r) - f_{n+1}(y_r)| < 2^{-n-4}$

whenever $\sup_{1 \leqslant r \leqslant n+1} |y_r - x_{r\,n+1}| \leqslant \varepsilon'(n+1)$.

We note

$(iv)_{n+1}$ $\quad \sup_{1 \leqslant r \leqslant n} |x_{r\,n} - x_{r\,n+1}| \leqslant \varepsilon'(n)/2$, $\quad |x_{h(n)\,n} - x_{n+1\,n+1}| \leqslant \varepsilon'(n)/2$

and that $(ii)'_{n+1}$ gives

$(ii)_{n+1}$ $\quad$ If $|y_r - x_{r\,n+1}| \leqslant \varepsilon'(n+1)$ $[1 \leqslant r \leqslant n+1]$ then, writing ℓ_{ks} for the lengths of the complementary intervals of $\{y_1, y_2, \ldots, y_s\}^*$ $[1 \leqslant s \leqslant n+1]$, we have

$$\sum H(\ell_{k\,n+1}) \leqslant \sum H(\ell_{k\,n}) + \sum_{r=1}^{n} H(|(y_r + y_{n+1}) - (y_r + y_{h(n)})|) + H(2|y_{n+1} - y_{h(n)}|)$$

$$\leqslant A + (1 - 2^{-n}) + (n+1)H(2K)$$

$$= A + (1 - 2^{-n-1})$$

(using $(ii)_n$).

We complete the inductive step by setting $E_{n+1} = \{x_1, x_2, \ldots, x_{n+1}\}$ and $\varepsilon(n+1) = \min(\varepsilon'(n+1), \delta(Q(n+1)))/8$.

Since $\sup_{x \in E_{n+1}} \inf_{y \in E_n} |x - y| \leqslant \varepsilon(n) < \varepsilon(n-1)/4$ we see that E_n converges topologically to a closed set K. Condition $(iii)_{n+1}$ ensures that K is Kronecker. The choice of $\varepsilon(n+1) \leqslant \delta(Q(n+1))/8$ together with the remark that $\{x : |x - \frac{2\pi r}{Q}| \leqslant \delta/2$ for some $r\} + \{x : |x - \frac{2\pi r}{Q}| \leqslant \delta/2$ for some $r\} \subseteq \{x : |x - \frac{2\pi r}{Q}| \leqslant \delta$ for some $r\}$ ensures that condition (i) of the lemma holds. Finally condition (ii) of the lemma follows from $(ii)_n$ using precesely the same trivial argument as we used for Lemma 6.4. This concludes the proof.

Taking K_1, K_2 perfect non empty disjoint closed subsets of K we have by Lemma 6.1 that K_1+K_2 contains a closed set E of non synthesis such that (since $K_o \supseteq E$)

(i) $E \subseteq \left\{x : \left|\frac{2\pi r}{Q(n)} - x\right| < \delta(Q(n)) \text{ for some } r\right\}$

(ii) The complementary intervals I_i of E have length ℓ_i with $\sum H(\ell_i) < \infty$.

Again we have the two celebrated results of Malliavin and Rudin respectively which say that any closed set which is of strong multiplicity contains a subset of non synthesis [14] and that there exist closed independent sets of strong multiplicity. (Moreover Rudin sets may be obtained, modifying either the original proof [16] or that in [10] with any given Hausdorff H-measure.) Combining the two results we obtain an independent closed set of non synthesis with (for some given $H : \underline{\underline{R}}^+ \to \underline{\underline{R}}^+$, monotonic, $H(0) = 0$) Hausdorff H-measure 0.

At this stage I asked myself whether we can construct, for example, weak Kronecker sets of non synthesis with some fixed Hausdorff H-measure. However, as the reader may already have realized, the question is essentially trivial. It is not difficult to extract (from Sections § 2, § 3 and § 4 for (i), from Section § 5 of [10] for (ii); (iii) is a result of Wik [21] given a very elegant proof by Kaufman in [9].)

LEMMA 6.6 (i). Let $H : \underline{\underline{R}}^+ \to \underline{\underline{R}}^+$ be a continuous increasing function with $H(0) = 0$. Given K a closed set such that $\mathrm{Gp}K \neq \underline{\underline{T}}$ and

$0 < P_1(1) < P_2(1) < P_2(2) < P_3(1) < P_3(2) < P_3(3) < P_4(1) < \dots$ integers, we can find a closed set E with Hausdorff H-measure 0, carrying a true pseudofunction and such that $GpE \cap GpK = \{0\}$, and $\liminf_{n\to\infty} \int |f_r - \chi_{P_n(r)}| d\mu = 0$ for all $\mu \in M^+(E)$.

(ii) Let H and $P_n(r)$ be as in (i), $\alpha \geqslant 0$. Given E a closed set such that $GpE \neq \mathbb{T}$, we can find a closed set K with Hausdorff H-measure α such that $GpE \cap GpK = \{0\}$ and $\liminf_{n\to\infty} \|f_r - \chi_{P_n(r)}\|_{C(K)} = 0$ for all $r \geq 1$.

(iii) (Wik) Kronecker sets of Hausdorff H-measure α exist.

(iv) There exists a weak Kronecker set L with Hausdorff H-measure α carrying a non zero pseudofunction.

Proof. (i) and (ii) are left as exercises to the reader (because the results are not terribly interesting and the proofs introduce no new ideas and not because the proofs are particularly simple).

(iii) is, as stated above, proved elsewhere.

(iv) Using either (i) and (iii) or (ii) and Theorem 1.1, we can find $0 < P_1(1) < P_2(1) < P_2(2) < \dots$ and independent closed sets E and K such that $GpE \cap GpK = \{0\}$, E carries a pseudofunction, E has Hausdorff H-dimension 0, K has Hausdorff H-dimension α, $\liminf_{n\to\infty} \int |f_r - \chi_{P_n(r)}| d\mu = 0$ for all $\mu \in M^+(E)$, $\|f_r - \chi_{P_n(r)}\|_{C(K)} \to 0$. Setting $L = E \cup K$, we see that it has the required properties (for example $\liminf_{n\to\infty} \int |f_r - \chi_{P_n(r)}| d\mu = 0$ for all $\mu \in M^+(L)$, so L is weak Kronecker).

If we recast the problem to avoid such a trivial solution we obtain

LEMMA 6.6 (v). There exists a weak Kronecker set E and a pseudofunction $T \in PM(E)$ such that E has Hausdorff H-measure α and supp $T = E$.

But the proof simply involves a more detailed investigation of the proof of Theorem 1.1" and the results used in obtaining it. (Note, however, that though it is possible to obtain Lemma 6.6 (v) by following Theorem 1.1" step by step, it is simpler to construct pseudofunctions $T_1, T_2, \dots$ with $\|T_i\|_{PM} \leq 1$, Hausdorff H-measure of $E_1 = \text{supp } T_i$ equal to zero, such that $E_i \cap E_j = \emptyset$ for $i \neq j$, and E the topological limit of $\bigcup_{i=1}^{n} E_i$ is weak Kronecker and has Hausdorff H-measure α. Setting $T = \sum 2^{-i} T_i$ we have supp $T = E$ and T a pseudofunction).

It is much more interesting to recall the extraordinary result of Kaijser [6] (the notation is standard ; see e. g. [20]).

LEMMA 6.7 (Kaijser). There exists an $1 > \alpha > 0$ such that if E_1, E_2 are perfect non empty disjoint sets and $E_1 \cup E_2$ is Helson $\beta > \alpha$, then the map $T : A(E_1 + E_2) \to V(E_1, E_2)$ given by

$$T\left(\sum_{r=-\infty}^{\infty} a_r \chi_r \,|\, E_1 + E_2\right) = \sum_{r=-\infty}^{\infty} a_r \chi_r \,|\, E_1 \otimes \chi_r \,|\, E_2$$

is well defined and gives a topological isomorphism.

What is remarkable about this result is that α may be chosen less than 1 yet cannot be taken arbitrarily close to zero for any one of the following 3 reasons (I have placed them in what seems to me increasing order of finality).

LEMMA 6.8. (i) We can find E_1, E_2 non empty disjoint perfect sets such that

E_1 E_2 is Helson with constant $1/\sqrt{2}$ but there exist $x_1, x_3 \in E_1$, $x_2, x_4 \in E_2$ such that $x_1 + x_3 = x_2 + x_4$ (so that T in Lemma 6.7 is not well defined).

(ii) (Varopoulos) We can find E_1, E_2 non empty disjoint perfect sets such that $E_1 \cup E_2$ is Helson but $E_1 + E_2 = \underline{\underline{T}}$ (so $A(E_1+E_2) = A(\underline{\underline{T}}) \neq V(E_1, E_2)$).

(iii) We can find E_1, E_2 non empty disjoint perfect sets such that $E_1 \cup E_2$ is Helson with constant $1/2$, $E_1 \cup E_2$ independent, yet there exist $\sigma_i \in M^+(E_i)$ $[i = 1,2]$ such that $(\sigma_1 * \sigma_2)\hat{}(r) \to 0$ as $|r| \to \infty$ (so by duality considerations $A(E_1+E_2) \neq V(E_1, E_2)$).

Proof. (i) Take $x_1 = \pi(1/6 + \sqrt{3})$, $x_2 = \pi(2/6 + \sqrt{3})$, $x_3 = \pi(3/6 + \sqrt{3})$, $x_4 = \pi(4/6 + \sqrt{3})$. It is easy to construct E_1, E_2 disjoint such that $x_1, x_3 \in E_1$, $x_2, x_4 \in E_2$ such that for each $|\lambda| = 1$, q, r integers with $0 \leq q \leq 3$ and $1 \leq r$ and for each L closed with $\{x_1, x_2, x_3, x_4\} \cap L = \emptyset$, $L \subseteq E_1 \cup E_2$ we have $\liminf_{p \to \infty} (\|\chi_{np+q} - f_r\|_{C(L)} + |\chi_{np+q}(\sqrt{3}) - \lambda|) = 0$. It is clear that $E_1 \cup E_2$ has the same Helson constant as $\{x_1, x_2, x_3, x_4\}$, i. e. $E_1 \cup E_2$ has Helson constant $1/\sqrt{2}$. But $x_1 + x_3 = x_2 + x_4$.

(ii) We shall give a version of this result in Lemma 6.10.

(iii) We may indeed take E_1, E_2 Kronecker. The result is then a version of Theorem 7 of [10].

Suppose E_1, E_2 are non empty disjoint perfect subsets. If $E_1 \cup E_2$ is Kronecker, then Varopoulos has shown that $E_1 + E_2$ is of synthesis and so (under the natural identification) $F \subseteq E_1 + E_2$ is of synthesis for $V(E_1, E_2)$ if and only if it is of synthesis for $A(\underline{\underline{T}})$ ([20]).

On the other hand, if $E_1 \cup E_2$ is Helson-1 and E_1 is of non synthesis, then the equivalence fails (consider $F = E_1 + e_2$ with $e_2 \in E_2$). (Though, of course, it remains true that F is of synthesis for $V(E_1, E_2)$ whenever it is of synthesis for $A(\underline{\underline{T}})$. What has happened is that $E_1 + E_2$ is not of synthesis, so that the dual of $A'(E_1+E_2)$ is (making the correct identification) strictly included in $PM(E_1+E_2)$).

It might be easier to construct a Helson set of non synthesis inside E_1+E_2 (supposed of synthesis) where E_1, E_2 are perfect and disjoint, if we relax the condition $E_1 \cup E_2$ Kronecker and replace it by $E_1 \cup E_2$ Helson β (with β as in Kaijser's theorem. See e. g. Lemma 6.1.)

This question has an equivalent and more fundamental re-statement. Does $V(D^\infty, D^\infty)$ have associated sets of interpolation which are not of non synthesis ? If the answer is not trivial or obtained by some simple variation of the methods above which I have overlooked, then it may be rather deep. Let us report some unpublished work of Varopoulos : - every countable independent set in $D^\infty \times D^\infty$ is Kronecker for $V(D^\infty, D^\infty)$ (a Kronecker set for V being a closed set $E \subseteq D^\infty \times D^\infty$ such that, if $f \in S(E)$, there exist $u_n, v_n \in S(D^\infty)$ with $\|u_n \otimes v_n - f\|_{C(E)} \to 0$). This should be contrasted with the existence of independent countable Dirichlet non Kronecker sets in $\underline{\underline{T}}$, and gives rise to the unanswered question : - does there exist an independent closed set in $D^\infty \times D^\infty$ which is not of interpolation (or indeed merely not of interpolation with constant 1) ? (Recall the existence of an independent Dirichlet non Helson set in $\underline{\underline{T}}$.)

The generalizations to locally compact Abelian groups rather than tensor algebras are however immediate.

LEMMA 6.9.

(i) $D_p = \prod_{i=1}^{\infty} \underline{\underline{Z}}_p$ contains a Helson-1 set which supports a true pseudofunction ;

(ii) If G is a non discrete locally compact Abelian group, then G contains a Helson-1 set which supports a true pseudomeasure ;

(iii) If G is a non discrete locally compact Abelian group, then G contains an independent Dirichlet non Helson set.

Remark 1. It is clear that no reasonable analogue of (iii) exists if G is discrete but on the other hand we have not given (ii) in its strongest possible form for the non discrete case.

Remark 2. In (iii) the definition of independence must be appropriate (see e. g. [13] XIII, 3.6).

Proof. (i) This is proved step by step as in the case of $\underline{\underline{T}}$ (apart from certain simplifications described at the end of Section 1).

(ii) Use the structure theorems. Note that it suffices to prove the result for any closed subgroup F of G. The result is true for $F = \underline{\underline{R}}$ (by the result for $\underline{\underline{T}}$), for $F = D_p$ (by (i)), for F compact and the closure of a subgroup generated by an element of infinite order (proof as for $\underline{\underline{T}}$) or for $F = \prod \underline{\underline{Z}}(p(i))$ where $p(i) \nearrow \infty$ very rapidly (proof as in (i) or as for $\underline{\underline{T}}$). Since G must contain a closed subgroup F with one of these properties, the full result is proved.

(iii) Similar but simpler considerations allow us to extend the proof of Theorem 9 of [10] in the same way.

In order to remind the reader of the kind of thing which can be done (admittedly on a very small scale) when we consider the sum of 2 sets, here is a minor improvement of the result of Varopoulos given as Lemma 6.5 (ii).

LEMMA 6.10. There exist 2 disjoint Kronecker sets E_1, E_2 with $E_1 \subseteq -[\pi/40, \pi/40] \cup [19\pi/40, 21\pi/40] \cup [29\pi/40, 31\pi/40] = L_1$, $E_2 \subseteq [2\pi/40, 18\pi/40] \cup [22\pi/40, 28\pi/40] \cup [32\pi/40, 38\pi/40] = L_2$ so that $E_1 \cup E_2$ has Helson constant at least $\alpha = (1/276)$ and measures $\tau_1 \in M^+(E_1)$, $\tau_2 \in M^+(E_2)$ such that $\tau_1 * \tau_2$ is a strictly positive infinitely differentiable function.

Proof. (For the constant 1/276 see [11] Lemma 1.13; that $\alpha > 0$ is a consequence of the fact that $E_1 \cap E_2 = \emptyset$.) We modify Lemma 3.4 [10] to which the reader is referred for details. Throughout μ is Haar measure and L^1 is $L^1(\mu)$. Since $\operatorname{int} L_1 + \operatorname{int} L_2 = \underline{\underline{T}}$, we can find h_1, g_1 infinitely differentiable positive functions such that $h_1 * g_1(x) \geq \delta$ for all $x \in \underline{\underline{T}}$ for some $\delta > 0$, $\operatorname{supp} h_1 \subseteq L_1$, $\operatorname{supp} g_1 \subseteq L_2$ and $\|h_1\|_{L^1} = \|g_1\|_{L^1} = 1$.

As an inductive hypothesis suppose now we have found h_n, g_n non negative infinitely differentiable functions such that $h_n * g_n(x) \geq \delta(2^{-1} + 2^{-n})$ for all $x \in \underline{\underline{T}}$. Then, taking $\sigma_{nN} = A_{nN} \sum \delta_{2\pi r/N}\, g_n(2\pi r/N)$ where $A_{nN} > 0$ is such that $\|\sigma_{nN}\| = \|g_n\|_{L^1}$ we have $\|\sigma_{nN} * k - g * k\|_{C(\underline{\underline{T}})} \to 0$ as $N \to \infty$ for all $k \in C(\underline{\underline{T}})$ and so $\|(\sigma_{nN} * k)^{(p)} - (g * k)^{(p)}\|_{C(\underline{\underline{T}})} = \|\sigma_{nN} * k^{(p)} - g * k^{(p)}\|_{C(\underline{\underline{T}})} \to 0$ as $N \to \infty$ for all $k \in C^p(\underline{\underline{T}})$. In particular therefore we can find an N such that $\|(\sigma_{nN} * h_n)^{(p)} - (g_n * h_n)^{(p)}\|_{C(\underline{\underline{T}})} \leq \delta 2^{-n-4}$ for all $0 \leq p \leq n$. By the continuity of $h_n, h_n^{(1)}, \ldots, h_n^{(n)}$ we can find $x_1, x_2, \ldots, x_N$ linearly independent with

$x_r \in \text{int supp } g_n$ if $g_n(2\pi r/N) \neq 0$ and $\sup_{1 \leqslant r \leqslant N} |x_r - 2\pi r/N|$ so small that writing $\sigma_n = A_{nN} \sum \delta_{x_r} g_n(2\pi r/N)$ we have $\|(\sigma_{nN} - \sigma_n) * h_n^{(p)}\|_{C(\underline{\underline{T}})} \leqslant \delta 2^{-n-4}$ and so $\|(\sigma_n * h_n)^{(p)} - (g_n * h_n)^{(p)}\|_{C(\underline{\underline{T}})} \leqslant \delta 2^{-n-3}$ for all $0 \leqslant p \leqslant n$. By Kronecker's theorem there exists a $Q(n)$ such that $\sup_{1 \leqslant r \leqslant N} |\chi_{Q(n)}(x_r) - f_n(x_r)| \leqslant 2^{-n-4}$ and by the continuity of $\chi_{Q(n)}, f_n$ and $h_n^{(p)}$ we can find an $\varepsilon_n > 0$ such that

$\sup_{|x-y| \leqslant 2\varepsilon_n} \sup_{0 \leqslant p \leqslant n} |h_n^{(p)}(x) - h_n^{(p)}(y)| \leqslant \delta 2^{-n-4}/\|\sigma_n\|_M$, whilst writing

$I_r = [x_r - \varepsilon_n, x_r + \varepsilon_n]$ we have $\|\chi_{Q(n)} - f_n\|_{C(I_r)} \leqslant 2^{-n-3}$ and, for every r such that $g_n(2\pi r/N) \neq 0$, $I_r \subseteq \text{int supp } g_n$ $[1 \leqslant r \leqslant N]$. Pick K_n an infinitely differentiable positive function with $\text{supp } K_n \subseteq [-\varepsilon_n, \varepsilon_n]$, $\int K_n(x)dx = 1$. Setting $g_{n=1} = \sigma_n * K_n$ we have g_{n+1} non negative infinitely differentiable and

$(A)_{n,1}$ $\quad \text{supp } g_{n+1} \subseteq \text{supp } g_n$

$(B)_{n,1}$ $\quad \inf_{Q \in \underline{\underline{Z}}} \|\chi_Q - f_n\|_{C(\text{supp } g_{n+1})} \leqslant 2^{-n-3}$

$(C)_{n,1}$
$$\begin{aligned} \|(g_{n+1} * h_n)^{(p)} - (g_n * h_n)^{(p)}\|_{C(\underline{\underline{T}})} &\leqslant \|(g_{n+1} * h_n)^{(p)} - (\sigma_n * h_n)^{(p)}\|_{C(\underline{\underline{T}})} \\ &\quad + \delta 2^{-n-3} \\ &\leqslant \|\sigma_n\|_M \|K_n * h_n^{(p)} - h_n^{(p)}\|_{C(\underline{\underline{T}})} + \delta 2^{-n-3} \\ &\leqslant 2^{-n-2} \quad \text{for all } 0 \leqslant p \leqslant n. \end{aligned}$$

$(D)_{n,1}$ $\quad \|g_{n+1}\|_{L^1} = \|g_n\|_{L^1}$

Similarly we can find h_{n+1} non negative infinitely differentiable with

$(A)_{n,2}$ $\quad \text{supp } h_{n+1} \subseteq \text{supp } h_n$

$(B)_{n,2}$ $\quad \inf_{Q \in \underline{\underline{Z}}} \|\chi_Q - f_n\|_{C(\text{supp } h_{n+1})} \leqslant 2^{-n}$

$(C)_{n,2}$ $\quad \|(g_{n+1} * h_{n+1})^{(p)} - (g_{n+1} * h_n)^{(p)}\|_{C(\underline{\underline{T}})} \leqslant \delta 2^{-n-3}$

$(D)_{n,2} \qquad \|h_{n+1}\|_{L^1} = \|h_n\|_{L^1}$,

and so

$(E)_n \qquad \|(g_{n+1} * h_{n+1})^{(p)} - (g_n * h_n)^{(p)}\|_{C(\mathbb{T})} \leqslant \delta 2^{-n-2}$

(whence, in particular, $h_{n+1} * g_{n+1}(x) \geqslant \delta(2^{-1} + 2^{-n-1})$ and the induction may be restarted).

Now $h_n\mu$, $g_n\mu$ are positive measures with $\|h_n\mu\|_M = \|g_n\mu\|_M = 1$. Thus they have weak limit points τ_1, τ_2 say with $\tau_1, \tau_2 \in M^+(\mathbb{T})$, $\|\tau_1\|_M, \|\tau_2\|_M = 1$. By $(A)_{n,i}$, $(B)_{n,i}$ supp $\tau_i \subseteq L_i$, supp τ_i is Kronecker $[i = 1,2]$. By $(D)_n$, $h_n * g_n$ converges in the space of infinitely differentiable functions under the usual topology ($k_n \to 0$ if and only if $\|k_n^{(p)}\|_{C(T)} \to 0$) to F say. By the inductive hypothesis $F(x) \geqslant \delta/2$ for all $x \in \mathbb{T}$. Thus $\tau_1 * \tau_2 = F$ is an infinitely differentiable strictly positive function.

Let us also prove

LEMMA 6.11. There exist disjoint Kronecker sets K_1, K_2 such that $K_1 \cup K_2$ is a perfect weak Kronecker set and $K_1 + K_2$ contains a weak Kronecker set of non synthesis.

Proof. We proceed by means of an inductive construction on the lines of our proof of Theorem 1.1' to which we ask the reader to refer ; for convenience we take $f_{2n} = f_{2n+1}$, $\varepsilon(n)$ as given there, and $\delta(n)$ also satisfying the conditions imposed. Suppose that at the n^{th} step we have disjoint finite sets L_n, K_n, E_n with $\inf_{x \in L_n, y \in K_n} |x-y| \geqslant 2^{-2} + 2^{-n}$ and $L_n + K_n \supseteq E_n$, a measure $\mu_n \in M(E_n)$ and an integer $Q(n) \geqslant 1$ obeying inductive condition $L(n)$ and condition (iii)' $Q(n)(L_n \cup K_n) = 0$.

We claim that we can find L_{n+1}, K_{n+1}, E_{n+1}, μ_{n+1}, $Q(n+1)$ satisfying the inductive hypothesis just stated (but with n replaced by $n+1$), the conditions of Lemma 6.3, and the additional conditions

(vii)' Condition (vii) holds with E_{n+1} replaced by $L_{n+1} \cup K_{n+1}$;

(viii) If n is even, then, given any closed set F with $\sup_{y \in F} \inf_{x \in K_{n+1}} |x-y| \leqslant \delta(Q(n+1))$, we can find an $P(n+1) \leqslant R \leqslant Q(n+1)-P(n+1)$ such that $\|\chi_R - f_n\|_{C(F)} \leqslant 2^{-n}$. If n is odd, the same relation holds with K_{n+1} replaced by L_{n+1}.

The proof is very easy. We take the case n even. Suppose $E_n = \{e_1, \ldots, e_\ell\}$ with the e_ℓ distinct. Each $e_u = x_u + y_u$ with $x_u \in L_n$, $y_u \in K_n$. We can find successively $x''_1, y''_1, x''_2, y''_2, \ldots$ independent such that the $e''_u = x''_u + y''_u$ are independent and $|x''_u - x_u|$, $|y''_u - y_u| \leqslant \varepsilon(n)/4$. Again, if $L_n = \{x_1, \ldots, x_\ell, \ldots, x_v\}$, $K_n = \{y_1, \ldots, y_\ell, \ldots, y_w\}$ we can choose successively $x''_{\ell+1}, x''_{\ell+2}, \ldots, x''_v$ and $y''_{\ell+1}, \ldots, y''_w$ such that the $x''_1, \ldots, x''_v$, $y''_1, \ldots, y''_w$ are independent ; moreover, we can find M_1, M_2, R such that

(α) $\sup_{1 \leqslant i \leqslant \ell} |\chi_{M_1}(e''_i) - f_{n+1}(e''_i)| \leqslant 2^{-n-6}$

(β) $\max\left(\sup_{1 \leqslant i \leqslant v} |\chi_{M_2}(x''_i) - f_{n+1}(x''_i)|, \sup_{1 \leqslant i \leqslant w} |\chi_{M_2}(y''_i) - f_{n+1}(y''_i)|\right) \leqslant 2^{-n-6}$

By the continuity of χ_{M_1}, χ_{M_2}, χ_R, f_{n+1} there exists a $Q'(n+1) \in \mathbb{Z}^+$ and x'_u, y'_u distinct such that

(γ) $|x'_u - x''_u| \leqslant \varepsilon(n)/4$, $|y'_u - y''_u| \leqslant \varepsilon(n)/4$

(δ) $Q'(n+1)x'_u = Q'(n+1)y'_u = 0$

and so, writing $L'_n = \{x'_u : 1 \leqslant u \leqslant v\}$, $K_{n+1} = \{y'_u : 1 \leqslant u \leqslant w\}$, $E'_n = \{x'_u + y'_u : 1 \leqslant u \leqslant \ell\}$

and μ_n' for that measure with support E_n' and $\mu_n'(\{x_u' + y_u'\}) = \mu_n(\{x_u + y_u\})$, we have

$$(\alpha') \qquad \|\chi_{rQ'(n+1)+M_1} - f_{n+1}\|_{C(E_n')} \leqslant 2^{-n-5}$$

$$(\beta')_1 \qquad \|\chi_{rQ'(n+1)+M_2} - f_{n+1}\|_{C(L_n')} \leqslant 2^{-n-5}$$

$$(\beta')_2 \qquad \|\chi_{rQ'(n+1)+M_2} - f_{n+1}\|_{C(K_{n+1})} \leqslant 2^{-n-5}.$$

By the arguments of Lemma 5.2' (or by convolving 2 measures of the kind constructed in Lemma 5.2 in which case we must replace $(1-2^{-n-4})$ in (b) by $(1-2^{-n-3})$, or by any argument the reader constructs for himself based on Lemma 5.2' perhaps modifying the values of $N(2^{-4}\delta(Q(n)), 2^{n+5}\|\mu_n\|_M, 1-2^{-n-4}))$, we can find $m \geqslant 1$ and $N(2^{n-4}\delta(Q(n)), 2^{n+5}\|\mu_n\|_M, 1-2^{-n-4}) = P(n) < M_1(1) < M_1(2) < \ldots < M_1(m) < M_2(1) < M_2(2) < \ldots < M_2(m) < M_2(m+1) = Q(n+1)$ and $T_{n+1} \in M([-2^{-n-5}\delta(Q(n)), 2^{-n-5}\delta(Q(n))])$ with support E^*_{n+1} say such that $M_1(j)$ is congruent to M_1, $M_2(j)$ is congruent to M_2 and $Q(n+1)$ is congruent to zero, all modulo $Q'(n+1)$ $[1 \leqslant j \leqslant m]$ whilst

(a) $\quad M(m+1)E^*_{n+1} = 0$

(b) $\quad \|m^{-1}\sum_{j=1}^{m} \chi_{M_i(j)} - 1\|_{C(E^*_{n+1})} \leqslant 2^{-n-3}$

(c) $\quad \|T_{n+1}\|_{PM} = \hat{T}_{n+1}(0) = 1 \geqslant 2^{n+4}\|\mu_n\|_M \sup_{Q(n+1)-P(n) \geqslant r \geqslant P(n)} |\hat{T}_{n+1}(r)|.$

The arguments of Lemma 2.2 now show that, writing $E_{n+1} = E_n' + E^*_{n+1}$, $\mu_{n+1} = \mu_n' * T_{n+1}$, $L_{n+1} = L_n' + E^*_{n+1}$, the conclusions stated at the end of the last paragraph but one hold (set $R = M_2$).

To prove the lemma we construct a sequence $(E_{n+1}, \mu_{n+1}, Q(n+1), P(n+1), L_{n+1}, K_{n+1})$ as above $[n = 0, 1, 2, \ldots]$. Let E, L, K be the topological limits of the E_{n+1}, L_{n+1}, K_{n+1} and S a (in fact the) weak limit point of the μ_n.

Since $\inf_{R\in \underline{\underline{Z}}} \|\chi_R - f_{2n+1}\|_{C(L)} \to 0$ as $n \to \infty$, we have L Kronecker. Similarly K is Kronecker whilst by the arguments used in the proof of Theorem 1.1 $L \cup K$ and E are Helson but $0 \neq S \in PM(E)$, $S \notin M(E)$, so that E is not of synthesis.

Remark. A more spectacular way of stating Lemma 6.11 (in view of the remarks following Lemma 6.8) is the following. Either the sum of 2 disjoint independent Kronecker sets can be of non synthesis, or synthesis fails for tensor algebras. However, even if it were true that the sum of 2 Kronecker sets is of synthesis, a proof is unknown and probably very difficult. The best result known (an improvement by Drury of a result of Varopoulos) is that the sum of 2 closed subsets of a Kronecker set is always of synthesis [2]).

This concludes a section dealing more with what we cannot do than with what we can do. If we knew how to combine any 2 of the various constructions for thin sets (probabilistic, Baire category, tensor algebraic and direct) we might be able to go much further.

§ 7. THE UNION OF AA^+ SETS.

Recall that, if $\Lambda \subseteq \underline{\underline{Z}}$, we write $A_\Lambda(\underline{\underline{T}})$ for the Banach subspace of $A(\underline{\underline{T}})$ of $A(\underline{\underline{T}})$ given by $A_\Lambda(\underline{\underline{T}}) = \{f \in A(\underline{\underline{T}}) : \hat{f}(r) = 0 \text{ for } r \notin \Lambda\}$ ([4] p. 150). We say that a closed set E is AA_Λ if $A(E) = A_\Lambda(E)$ and ZA_Λ if there is a non zero $f \in A_\Lambda(\underline{\underline{T}})$ with $f(e) = 0$ for all $e \in E$. If $\Lambda = \underline{\underline{Z}}^+$, $A^+(\underline{\underline{T}}) = A_{\underline{\underline{Z}}^+}(\underline{\underline{T}})$ is a Banach algebra, we write AA^+, ZA^+ for $AA_{\underline{\underline{Z}}^+}$, $ZA_{\underline{\underline{Z}}^+}$. The reader is asked to re-read Lemma 4.1 as background. This section is devoted to the proof of the following theorems.

THEOREM 7.1.

(i) Given $C_1, C_2 \geq 1$ we can find closed sets E_1, E_2 such that $E_1 \cap E_2$ consists of 1 point, E_i is AA^+ with associated constant C_i [i = 1,2], but $E_1 \cup E_2$ is independent with AA^+ constant at least (so exactly) $C_1 + C_2 + C_1C_2$.

(ii) Given $C_1, C_2 \geqslant 1$, $\varepsilon > 0$ we can find disjoint closed sets E_1, E_2 such that E is AA^+ with associated constant at most C_i [i = 1,2], but $E_1 \cup E_2$ is independent with AA^+ with associated constant at least $C_1 + C_2 + C_1C_2 - \varepsilon$.

(iii) Given $\varepsilon > 0$, $n \geqslant 1$, we can construct n Dirichlet sets $E_1, E_2, \ldots$ $\ldots, E_n$ which are disjoint (respectively have $E_i \cap E_j = \{x\}$ $[i \neq j]$ for some $x \in \mathbb{T}$) such that $\bigcup_{i=1}^{n} E_i$ is independent with AA^+ constant $2^n - 1 - \varepsilon$ (respectively AA^+ constant $2^n - 1$).

THEOREM 7.2. We can find $\Lambda(1), \Lambda(2) \subseteq \mathbb{Z}^+ \setminus \{0\}$ together with E_1, E_2 closed sets with $E_1 \cap E_2 = \{x\}$ for some $x \in \mathbb{T}$ such that $E_1 \cup E_2$ is an independent set having the following properties : E_i is $AA_{\Lambda(i)}$ (indeed $\|f\|_{A_{\Lambda(i)}(E_i)} = \|f\|_{A(E_i)}$ for all $f \in A(E_i)$) [i = 1,2] yet $E_1 \cup E_2$ is not even $ZA_{\Lambda(1) \cup \Lambda(2) \cup -\Lambda(1) \cup -\Lambda(2) \cup \{0\}}$

THEOREM 7.3. Given any $1 > s \geqslant 0$ we can find an independent closed set E such that

$$\limsup_{m \to \infty} \sigma\{x \in E : |\chi_m(x) - 1| \leqslant \delta\} \geqslant s\|\sigma\|$$

for all $\sigma \in M^+(E)$, $\delta > 0$ with the following property : - Given any $R > 0$ we can find an $\varepsilon(R,s) > 0$ such that for any $\sum_{m=1}^{\infty} |a_m| \leqslant R$ we have $\|\sum_{m=1}^{\infty} a_m \chi_m - 1\|_{C(E)} \geqslant \varepsilon(R,s)$.

Moreover, we can also find an $n(R) \in \underline{\underline{Z}}^+$ such that $\sum_{|m| \geq n(R)} |a_m| \leq R$ implies $\|\sum_{|m| \geq n(R)} a_m \chi_m - 1\|_{C(E)} \geq \varepsilon(R,s)$.

LEMMA 7.4. Given $1 \geq s > 0$ and $\varepsilon > 0$, we can find an $R(\varepsilon,s) > 0$ with the following property : - If E is a closed set such that

$$\limsup_{m \to \infty} \sigma \left\{ x \in E : |\chi_m(x) - 1| \leq \delta \right\} \geq s\|\sigma\|$$

for all $\sigma \in M^+(E)$, $\delta > 0$, then

$$\limsup_{p \to \infty} \inf \left\{ \|\sum_{m \geq p} a_m \chi_m - 1\|_{C(E)} : \sum_{m \geq p} |a_m| \leq R(\varepsilon,s) \right\} \leq \varepsilon.$$

LEMMA 7.5. Given any $1 > s \geq 0$, we can find a $\Lambda \subseteq \underline{\underline{Z}}^+$ and an independent closed set E and for each $r \geq 1$ an $\varepsilon(r,s)$ such that

$$\limsup_{m \to \infty, m \in \Lambda} \sigma \left\{ x \in E : |\chi_m(x) - 1| \leq \delta \right\} \geq s\|\sigma\|$$

for all $\sigma \in M^+(E)$, $\delta > 0$, yet, writing $\Lambda(r) = \left\{ \sum_{i=1}^{r} n_i : n_i \in \Lambda \cup -\Lambda \right\} \setminus \{0\}$, we have $\|\sum_{m \in \Lambda(r)} a_m \chi_m - 1\|_{C(E)} \geq \varepsilon$ for all $\sum_{m \in \Lambda(r)} |a_m| < \infty$.

For completeness we rephrase the result of Björk and Kaufman given in Lemma 4.1.

LEMMA 7.6. If $\Lambda \subseteq \underline{\underline{Z}}^+$ and E is closed such that

$$\limsup_{m \to \infty, m \in \Lambda} \sigma \left\{ x \in E : |\chi_m(x) - 1| \leq \delta \right\} = \|\sigma\|$$

for all $\sigma \in M^+(E)$, $\delta > 0$, then

$$\limsup_{p \to \infty, p \in \Lambda} \inf \left\{ \|\sum_{m \geq p} a_m \chi_m - 1\|_{C(E)} : \sum_{m \geq p} |a_m| \leq 1 \right\} = 0.$$

It is an extremely instructive exercise to restate and prove the results for general collections of functions. For example :

LEMMA 7.4'. If $1 \geqslant s > 0$ and $\varepsilon > 0$ are given, we can find an $R(\varepsilon,s) > 0$ with the following property : - If X is a compact Hausdorff space and $g_m \in C(X)$ is such that $\|g_m\|_{C(X)} \leqslant 1$

$$\limsup_{m\to\infty} \sigma\left\{x : |g_m(x) - 1| \leqslant \delta\right\} \geqslant s\|\sigma\|$$

for all $\sigma \in M^+(X)$, $\delta > 0$ then

$$\limsup_{p\to\infty} \inf\Big\{\Big\|\sum_{s\geqslant 1}\sum_{m(1),m(2),\dots,m(s)\geqslant p} a_{m(1)m(2)\dots m(s)} g_{m(1)} g_{m(2)} \cdots g_{m(s)} - 1\Big\|_{C(X)} :$$

$$\sum_{s\geqslant 1}\sum_{m(1),m(2),\dots,m(s)\geqslant p} |a_{m(1)m(2)\dots m(s)}|\Big\} \leqslant R(\varepsilon,s) \leqslant \varepsilon .$$

LEMMA 7.5'. If $1 > s \geqslant 0$ we can find a compact Hausdorff space X and a sequence $g_m \in C(X)$ with $\|g_m\|_{C(X)} \leqslant 1$ such that

$$\limsup_{m\to\infty} \sigma\left\{x : |g_m(x) - 1| = 0\right\} \geqslant s\|\sigma\|$$

for all $\sigma \in M^+(X)$, but

$$\Big\|\sum_{q\geqslant s\geqslant 1}\sum_{m(1),m(2),\dots,m(s)} a_{m(1)m(2)\dots m(s)} g_{m(1)} g_{m(2)} \cdots g_{m(s)} - 1\Big\|_{C(X)} \geqslant \varepsilon(q) > 0$$

for all $\sum_{q\geqslant s\geqslant 1} \sum |a_{m(1)m(2)\dots m(s)}| < \infty \quad [q \geqslant 1]$.

Since Lemma 7.5' is a consequence of Lemma 7.5 we leave it as a recommended exercise for the reader to find the shortest and most illuminating direct proof. The gist of the matter is contained in the simpler version (with $q = 1$).

LEMMA 7.5". Let $p \geqslant 2$ be a positive integer. Let X be the finite discrete topological space $\{0, 1, \ldots, p\}$. Let $g_r \in C(X)$ be given by $g_r(s) = 1$ for $r \neq s$, $g_r(r) = -1$ $[1 \leqslant r \leqslant p]$. Then

$$\sup_{1 \leqslant r \leqslant p} \sigma\{s : |g_r(s) - 1| \neq 0\} \geqslant (p/p+1)\|\sigma\|$$

for all $\sigma \in M^+(X)$, yet

$$\|\sum_{1 \leqslant r \leqslant p} a_r g_r - 1\|_{C(X)} = \max_{0 \leqslant s \leqslant p} (|\sum_{1 \leqslant r \leqslant p} a_r g_r(s) - 1|)$$

$$= \max (|\sum_{1 \leqslant r \leqslant p} a_r - 1|, \max_{1 \leqslant s \leqslant p} |\sum_{1 \leqslant r \leqslant p} a_r - 1 - 2a_s|)$$

$$\geqslant 1/(p-1)$$

for all $\sum_{1 \leqslant r \leqslant p} |a_r| < \infty$.

Proof of Lemmas 7.4' and 7.4. (Lemma 7.4 is, of course, a special case of Lemma 7.4'.) Let us introduce the following temporary notations :

$$\mathcal{Y}_p = \{G : G = \sum_{s \geqslant 1} \sum_{m(1)m(2)\ldots m(s) \geqslant p} a_{m(1),m(2),\ldots,m(s)} g_{m(1)} g_{m(2)} \cdots g_{m(s)}$$

with $\sum\sum |a_{m(1)m(2)\ldots m(s)}| < \infty \}$, if $G \in \mathcal{Y}_p$ we write

$$|G|_p = \inf\{\sum\sum |a_{m(1)m(2)\ldots m(s)}| :$$

$$G = \sum_{s \geqslant 1} \sum_{m(r) \geqslant p} a_{m(1),m(2),\ldots,m(s)} g_{m(1)} g_{m(2)} \cdots g_{m(s)}\}.$$

We split the proof into a series of simple observations.

LEMMA 7.4".

(i) Given $1 \geqslant s > 0$, $\eta_1 > 0$, we can find $N(\eta_1, s)$ with the following property : - Given g_m as in Lemma 7.4', $p \geqslant 1$ and $\eta_2 > 0$, we can find, for each $\sigma \in M^+(X)$, closed sets $E_1, E_2, \ldots, E_N$ and integers $p \leqslant m(1) < m(2) < \ldots < m(N)$

such that $\sigma(\bigcup_{r=1}^{n} E_r) \geqslant (1-\eta_1)||\sigma||$ and $||g_{m(r)} - 1||_{C(E_r)} \leqslant \eta_2$ for each $1 \leqslant r \leqslant N$.

(ii) There exist integers $1 \leqslant q(1) < q(2) < \ldots$ such that given $\eta > 0$, we can find an $M(\eta)$ with the following property : - Suppose $g \in C(X)$, $||g||_{C(X)} \leqslant 1$. Then we can find $X_1(r)$, $X_2(r)$ closed with $X_1(r) \cup X_2(r) = X$ such that $||g^n - 1||_{C(X_2(r))} \leqslant 2^{-r-4}$ for all $0 \leqslant n \leqslant q(r)$ and, for every $\sigma \in M^+(X)$,

$$\operatorname{card}\left\{n \geqslant 1 : \int_{X_1(n)} \left|\frac{1}{q(n)} \sum_{r=1}^{q(n)} g^r \right| d\sigma \geqslant \eta\, ||\sigma||\right\} \leqslant M(\eta).$$

(iii) Given g_m as in Lemma 7.4', $p \geqslant 1$ and $\eta_1 > 0$, we can find, for each $\sigma \in M^+(X)$, a $G_\sigma \in \mathcal{Y}_p$ with $|G_\sigma|_p \leqslant 2^{N(\eta,s)} - 1$ such that $\int |G_\sigma - 1|\, d\sigma \leqslant 2\eta_1$.

(iv) Lemma 7.4' holds.

Remark. Although this is not strictly relevant we note that (ii) also gives

$$\operatorname{card}\left\{ n \geqslant 1 : \left|\operatorname{Im} \int_X \frac{1}{q(n)} \sum_{r=1}^{q(n)} g^r \, d\sigma \right| \geqslant 2\eta ||\sigma|| \right.$$

$$\text{and/or} \quad \left. \operatorname{Re} \int_X \frac{1}{q(n)} \sum_{r=1}^{q(n)} g^r \, d\sigma \leqslant \sigma\{x : g(x) = 1\} - 2\eta\, ||\sigma|| \right\} \leqslant M(\eta).$$

In other words, considered as a vector in the Argand diagram $\int \frac{1}{q(n)} \sum_{r=1}^{q(n)} g^r \, d\sigma$ points, most of the time, in the same direction (and/or tends to zero fairly rapidly).

Proof. (i) This is obvious. Suppose we have constructed closed sets $E_1, E_2, \ldots, E_k$ and integers $p \leqslant m(1) < m(2) < \ldots < m(k)$. Given $\eta_3 > 0$, we can find E'_{k+1} closed such that $E'_{k+1} \cap E_j = \emptyset$ $[1 \leqslant j \leqslant k]$ but $\sigma(E'_{k+1}) \geqslant \sigma(E \setminus \bigcup_{i=1}^{k} E_i) - \eta_3 ||\sigma||$. Writing $\sigma_{k+1} = \sigma | E'_{k+1}$ so that $\sigma_{k+1} \in M^+(X)$, we know by hypothesis, that there exists an $m(k+1) > m(k)$ and an E_{k+1} closed such that $||g_{m(k+1)} - 1||_{C(E_{k+1})} \leqslant \eta_2$

and $\sigma_{k+1}(E_{k+1}) \geqslant s\|\sigma_{k+1}\|$. (This last statement gives at once

$$\sigma(\bigcup_{i=1}^{k+1} E_i) \geqslant \sigma_{k+1}(E_{k+1}) + \sigma(\bigcup_{i=1}^{k} E_i) \geqslant s\|\sigma_{k+1}\| + \sigma(\bigcup_{i=1}^{k} E_i) \geqslant (s - \eta_3)\|\sigma\| + (1-s)\sigma(\bigcup_{i=1}^{k} E_i).)$$

We now restart the induction.

If $E_o, E_1, \ldots, E_n$ are defined in this way, we obtain $\sigma(\bigcup_{i=1}^{N} E_i) \geqslant (\sum_{r=1}^{N} s(1-s)^{r-1} - \eta_3)\|\sigma\| = (1 - \eta_3 - (1-s)^N)\|\sigma\|$. In particular, choosing $\eta_s = \eta_1/2$ and $N = N(\eta, s) = [(\log_2 \eta_3 - 2)/\log(1-s)] + 2$ we have the required result.

(ii) This is also trivial. Set $X_2(r) = \{x : |g(x) - 1| \leqslant 2^{-8r^2-4}\}$, $X_1(r) = \{x : |g(x) - 1| \geqslant 2^{-8r^2-4}\}$, $q(r) = 2^{8(r-1)^2+2r}$, $M(\eta) = 2([\eta^{-1}] + 1)$ (note that $X_1(r) \subseteq X_1(r-1)$). Then for all $\sigma \in M^+(X)$ we have (writing $X_o = X$)

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_{X_1(n)} \left|\frac{1}{q(n)}\sum_{r=1}^{q(n)} g^r\right| d\sigma &= \sum_{n=1}^{\infty}\sum_{t=1}^{n}\int_{X_1(t-1)\setminus X_1(t)} \left|\frac{1}{q(n)}\sum_{r=1}^{q(n)} g^r\right| d\sigma \\
&= \sum_{t=1}^{\infty}\sum_{n=t}^{\infty}\int_{X_1(t-1)\setminus X_1(t)} \left|\frac{1}{q(n)}\sum_{r=1}^{q(n)} g^r\right| d\sigma \\
&\leqslant \sum_{t=1}^{\infty}\Big(\sigma(X_1(t-1)\setminus X_1(t)) + \sum_{n=t+1}^{\infty}\int_{X_1(t-1)\setminus X_1(t)} \frac{2}{q(n)|1-g(x)|}d\sigma(x)\Big) \\
&\leqslant \sum_{t=1}^{\infty}\Big(\sigma(X_1(t-1)) - \sigma(X_1(t)) \\
&\qquad + \sum_{n=t+1}^{\infty}\|\sigma\| \sup_{x\in X_1(t-1)\setminus X_1(t)} \frac{2}{q(n)|1-g(x)|}\Big) \\
&= 2\|\sigma\| \leqslant M(\eta)\,\eta\|\sigma\|
\end{aligned}$$

and the result follows.

(iii) Choose $0 < \eta_2 < \eta_1 2^{-(N(\eta_1,s)+4)}$, $M_o \geqslant 1$ such that $2^{-M_o+4} < \eta_2$, $M_1 \geqslant M(\eta_2) + M_o + 1$ and $0 < \eta_3 < \eta_2/(M_1 2^{M_1+4})$. By (i) we can find $p \leqslant m(1) < m(2) < \ldots$ $\ldots < m(N(\eta, s))$ such that $\sigma(\bigcup_{r=1}^{N} E_r) \geqslant (1 - \eta_2)\|\sigma\|$ and $\|g_{m(r)} - 1\|_{C(E_r)} \leqslant \eta_3$ for all $1 \leqslant r \leqslant N(\eta_1, s)$. By (ii) we can find $q(1), q(2), \ldots, q(N(\eta_1, s))$ such that writing

$X_r = \{x : |g^t_{m(r)}(x) - 1| > 2^{-q(r)-4}$ for all $0 \leqslant t \leqslant q(r)\}$, $Y_r = X \setminus X_r$

(a) $M_o \leqslant q(r) \leqslant M_1$

(b) $\int_{X_r} \left| \frac{1}{q(r)} \sum_{t=1}^{q(r)} g^t_{m(r)} \right| d\sigma \leqslant \eta_2 ||\sigma||$

(c) $Y_r \cup X_r = X$, $Y_r \supseteq E_r$ $[1 \leqslant r \leqslant n]$.

Now set $G_\sigma = 1 - \prod_{r=1}^{N(\eta_1, s)} (1 - \frac{1}{q(r)} \sum_{t=1}^{q(r)} g^t_{m(r)})$.

(iv) Let $\Gamma_1 = \{G \in \mathcal{Y}_p : |G|_p \leqslant 2^{N(\eta_1, s)}\}$, $\Gamma_2 = \{f \in C(X) : ||f-1||_{C(X)} \leqslant 4\eta_1\}$. Suppose $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then, since Γ_1, Γ_2 are convex, there exists a separating hyperplane. Since Γ_1 is balanced and $\{\int (f-1) d\mu : f \in \Gamma_1\} = \{\lambda : |\lambda| \leqslant 4\eta_1 ||\mu||\}$ this shows the existence of a $\sigma \in M(X)$ with $\int 1 \, d\sigma = 1$, $\left|\int (G-1) d\sigma\right| \geqslant 4\eta_1 ||\sigma||$ for all $G \in \mathcal{Y}_p$. In particular $2\eta_1 ||\sigma|| \geqslant \int |G_\sigma - 1| d|\sigma| \geqslant \left|\int (G_\sigma - 1) d\sigma\right| \geqslant 4\eta_1 ||\sigma||$ which is absurd. Thus $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, i. e. there exists a $G \in \mathcal{Y}_p$ with $|G|_p \leqslant 2^{N(\eta_1, s)}$ and $||G - 1||_{C(X)} \leqslant 4\eta_1$. Setting $R(\varepsilon, s) = 2^{N(\eta_1/4, s)}$ we have the required result.

Why should we be interested in these results ? I think that they are important because they illuminate the conditions necessary to obtain good approximations to 1. Theorems 7.1 and 7.2, and Lemmas 7.4' and 7.5 show the difference between vector spaces of functions and algebras of functions, whilst Lemmas 7.4' and 7.6 show the difference between Weak Dirichlet and s-Weak Dirichlet $[1 > s \geqslant 0]$ for vector spaces of functions. Lemma 7.4' and Lemma 7.6 show the difference between Weak Dirichlet and s-Weak Dirichlet $[1 > s \geqslant 0]$ for algebras of functions. If the reader re-reads the introduction, he will see that I believe (or at least believed) that these differences are the fundamental reason why the methods of this paper work. In fact, it was Lemma 1.5

and Lemma 7.5" (with $p = 2$) which furnished the thread of Ariadne for this investigation.

We now start the proof of Theorem 7.1. We use the following construction :

LEMMA 7.7. We can find $E(0) = \{x\}$, $E(S,t)$ closed disjoint sets $[S \subseteq \{r : 1 \leqslant r \leqslant 2^{t+1}\},\ 1 \leqslant t]$, together with measures $\mu_{S,t} \in M^+(E(S,t))$, $\|\mu_{S,t}\| = 1$ $[S \subseteq \{r : 1 \leqslant r \leqslant 2^{t+1}\},\ 1 \leqslant t]$ and a sequence of integers $P(u,r,t)$ $[0 \leqslant u \leqslant t-1,\ 1 \leqslant r \leqslant 2^{t+1}+1,\ 1 \leqslant t]$ such that $10 + u_1 + r_1 + t_1 \leqslant P(u_1,r_1,t_1) < P(u_2,r_2,t_2)$ whenever $0 \leqslant u_1 < u_2$, $r_1 = r_2$, $t_1 = t_2$ or $r_1 < r_2$, $t_1 = t_2$ or $t_1 < t_2$ with the following properties : -

(i) $E(t) = \bigcup_{S \subseteq \{r : 1 \leqslant r \leqslant 2^{t+1}\}} E(S,t)$ tends topologically to $\{x\}$ as $t \to \infty$.

(ii) $\{x\} \cup \bigcup_{t=1}^{\infty} E(t)$ is independent.

Suppose $e \in E(S,t)$ $[S \subseteq \{1,2,\dots,2^{t+1}\},\ t \geqslant 1]$, $0 \leqslant u \leqslant k-1$, $1 \leqslant r \leqslant 2^{k+1}$, $1 \leqslant t$

(iii) If $k > t$ we have

$|\chi_{P(u,r,k)}(e) - 1| \leqslant 2^{-20(k+4)}$ if $u = 0$

$|\chi_{P(u,r,k)}(e) - 1| \geqslant 2^{-k-4}$ otherwise

(iv) If $t = k$ we have

$|\chi_{P(u,r,k)}(e) - 1| \leqslant 2^{-20(k+4)}$ if $u \leqslant t$, $1 + \left[\frac{r-1}{2^{k-t}}\right] \in S$

$|\chi_{P(u,r,k)}(e) - 1| \geqslant 2^{-k-4}$ otherwise.

Further

(v) $|\chi_{P(u,r,k)}(x) - 1| \leqslant 2^{-20(k+4)}$ if $u = 0$

$|\chi_{P(u,r,k)}(x) - 1| \geqslant 2^{-k-4}$ otherwise.

(vi) $|\hat{\mu}_{S,t}(v)| \geqslant 2^{-10t-10}$ implies $|\hat{\mu}_{S,t}(v) - \hat{\mu}_{T,t}(v)| \leqslant 2^{-10t-10}$

for all $S \subseteq T \subseteq \{1,2,\ldots,2^{t+1}\}$, $v \geqslant P(0,2^{t+1}+1, t)$

(vii) $|\hat{\mu}_{S\cap T,t}(v)| \geq \min(|\hat{\mu}_{S,t}(v)|, |\hat{\mu}_{T,t}(v)|) - 2^{-10t-10}$

for all $S,t \subseteq \{1,2, \ldots, 2^{t+1}\}$, $v \geqslant P(0,2^{t+1}+1 , t)$

(viii) $|\hat{\mu}_{\emptyset,t}(v)| \leqslant 2^{-10t}$ for all $v \geqslant P(0,2^{t+1}+1, t)$ $[t \geqslant 1]$

(ix) $|\hat{\mu}_{S,t}(v)| \leqslant 2^{-10t-10}$ whenever $|v - P(u,r,k)| \leqslant k$ and

$|\chi_{P(u,r,k)}(e) - 1| \geqslant 2^{-k-4}$ for $e \in S$.

Remark. It is only notational quirks which separate conditions (iii) and (v) and which put $P(0,2^{t+1}+1 , t)$ apparently on the same footing as $P(u,r,t)$ with $1 \leqslant r \leqslant 2^{t+1}$.

THEOREM 7.1'. Suppose $C_1, C_2 \geqslant 1$ and $n \geqslant 2$ an integer given. Choose $p_1(1) \leqslant p_1(2) \leqslant \ldots$, $p_2(1) \leqslant p_2(2) \leqslant \ldots$ positive integers such that $1 \geqslant p_i(t)2^{-t}$ and $p_i(t)2^{-t}$ decreases to C_i^{-1} as $t \to \infty$ $[i = 1,2]$. Construct $E(0)$, $E(S,t)$ as in Lemma 7.7.

(i) Write

$E_{jt} = \cup\{E(S,t) : \text{card } S = p_j(t) , S \subseteq \{(j-1)2^t + k : 1 \leqslant k \leqslant 2^t\}\}$

$E_{12t} = \cup\{E(S \cup T,t) : \text{card } S = p_1(t), \text{card } T = p_2(t), S \subseteq\{k : 1 \leqslant k \leqslant 2^t\}, T \subseteq \{2^t+k : 1 \leq k \leq 2^t\}\}$

$E_j = \bigcup_{t=1}^{\infty} E_{jt} \cup E(0)$ $[1 = 1,2]$

$E_{12} = \bigcup_{t=1}^{\infty} E_{12t} \cup E(0).$

Then $E_1, E_2 \cup E_{12}$ satisfy the conclusions of Theorem 7.1 (i).

(ii) Provided only $N \geqslant N_1(\varepsilon)$ for some N_1 depending only on ε, it follows that E_{1N}, $E_{2N} \cup E_{12N}$ satisfy the conclusions of Theorem 7.1 (ii).

(iii) Write

$$F_{jt} = \cup\{E(S,t) : \emptyset \neq S \subseteq \{r2^{t-n} : 1 \leqslant r \leqslant j\}\} \qquad [t \geqslant n]$$

$$F_j = \bigcup_{t \geqslant n}^{\infty} F_{jt} \cup E(0) \qquad [1 \leqslant j \leqslant n].$$

Then, provided only $N \geqslant N_2(\varepsilon)$ for some N_2 depending only on $\varepsilon > 0$, it follows that $F_{1N}, F_{2N}, \ldots, F_{nN}$ (respectively $F_1, F_2, \ldots, F_n$) satisfy the conclusions of Theorem 7.1 (iii).

Proof. The proof of parts (i) and (ii) splits as follows : -

Part A : We show that $E_j \cup E_{12}$ is AA^+ with constant at most C_j. To do this it suffices to show that, given any $w \in \mathbb{Z}$ and $\varepsilon > 0$, we can find a $g \in A(\mathbb{T})$ with $\|g\|_{A(\mathbb{T})} \leqslant C_j + \varepsilon$, $\hat{g}(v) = 0$ for $v \leqslant w$ and $\|g - 1\|_{A(E_j \cup E_n)} \leqslant \varepsilon$ (cf. the proof of Lemma 4.1 (i)).

Recall that f_δ is the trapezoidal function of height 1 with vertices at $-\delta, -\delta^2, \delta^2, \delta$. Thus $\|f_\delta\|_A \leqslant 1 + 4\delta$ (for $0 < \delta < 1/4$) and so $\|f_\delta \circ \chi_r\|_A = \|f\|_A = 1 + 4\delta$ (here $g \circ f(y) = g(f(y))$ for all $y \in \mathbb{T}$). Choose $k \geqslant 1$ in such a way that $2^{-k+4} \leqslant \varepsilon/(C_1+C_2)$ and $w \leqslant k$. Set $g_{u,r} = f_{2^{-k+4}} \circ \chi_{P(u,r,k)}$ $[0 \leqslant u \leqslant k-1,\ \ 1 \leqslant r \leqslant 2^{k+1}]$.

Writing $G_{u,j} = \sum_{r=(j-1)2^k+1}^{j2^k} g_{u,r}$ we see that $\|G_{u,j}\| \leqslant 2^k(1+2^{-k-2})$ and

$$\sum_{u=0}^{k} \lambda_u G_{u,j}(e) = \lambda_0 2^k \qquad \text{if} \qquad e \in E_j \setminus \bigcup_{1 \leqslant t \leqslant k} E(S,t)$$

$$\sum_{u=0}^{k} \lambda_u G_{u,j}(e) = \sum_{u=0}^{t} \lambda_u 2^{k-u} p_j(u) \qquad \text{if} \qquad e \in E_j \cap E(S,t), \quad 1 \leqslant t \leqslant k.$$

But $2^k \geqslant 2^{k-1} p_j(1) \geqslant 2^{k-2} p_j(2) \geqslant \ldots \geqslant p_j(k) \geqslant C_j^{-1} 2^k$, so that we can find $\lambda_{0,j}, \lambda_{1,j}, \ldots, \lambda_{k,j} \geqslant 0$ with $\sum_{u=0}^{k} \lambda_{u,j} \leqslant C_j 2^{-k}$, $\sum_{u=0}^{k} \lambda_{u,j} G_{u,j}(e) = 1$ for all

$e \in E_j \cup E_{12}$. Setting $G_j = \sum_{u=0}^{k} \lambda_{u,j} G_{u,j}$ we have $\|G_j\|_A \leqslant \sum_{u=0}^{k} \lambda_{u,j} \|G_{u,j}\|_A \leqslant$ $C_j(1 + 2^{-k-2})$ $[j = 1,2]$.

Now let us examine $g_{u,r}$ again. We know that $|\hat{f}_\delta(m)| \leqslant 8\delta^{-2}m^{-2}$ (in fact the actual numerical bounds are even more irrelevant than usual). Thus

$\sum_{m<2^{-4k-16}P(u,r,k)} |\hat{g}_{u,r}(m)| \leqslant \sum_{m<2^{-4k-16}} |\hat{f}_{2^{-k-4}}(m)| \leqslant 2^{-2k-8}$. Also, writing $U_r = \{e \in E : g_{u,r}(e) = 1\}$, $V_r = \{e \in E : g_{u,r}(e) = 0\}$, we have $\|\chi_{2^{-5k-20}P(u,r,k)} - 1\|_{C(U_r)} \leqslant 2^{-5(k+4)}$ so that $\|\chi_{2^{-5k-20}P(u,r,k)} - 1\|_{A(U_r)} \leqslant 2^{-2k-8}$ (see e. g. the proof of Lemma 4.1 (ii)).

Now $\|f\|_{A(E)} \leqslant (1 + 2^{-k-2})(\|f\|_{A(U_r)} + 2\|f\|_{A(V_r)})$ (for suppose $e_1, e_2 \in A(\underline{\underline{T}})$, $e_1|U_r = f|U_r$, $e_2|V_r = f|V_r$, then $(e_1 g_{u,r} + e_2(1-g_{u,r}))|E = f|E$ and $\|e_1 g_{u,r} + e_2(1-g_{u,r})\|_{A(\underline{\underline{T}})} \leqslant \|e_1\|_{A(\underline{\underline{T}})}\|g_{u,r}\|_{A(\underline{\underline{T}})} + \|e_2\|_{A(\underline{\underline{T}})}(1 + \|g_{u,r}\|_{A(\underline{\underline{T}})}))$.

Thus writing $h_{u,r} = \chi_{2^{5k+20}P(u,r,k)} g_{u,r}$ we have at once

$$\begin{aligned}\|h_{u,r} - g_{u,r}\|_{A(E)} &\leqslant (1+2^{-k-2})(\|h_{u,r} - g_{u,r}\|_{A(U_r)} + 2\|h_{u,r} - g_{u,r}\|_{A(V_r)}) \\ &\leqslant (1+2^{-k-2})(\|1 - \chi_{2^{5k+20}P(u,r,k)}\|_{A(U_r)}\|g_{u,r}\|_{A(\underline{\underline{T}})} + 0) \\ &\leqslant 2^{-2k-7}.\end{aligned}$$

Set $\ell_{u,r} = \sum_{m>w} \hat{h}_{u,r}(m)\chi_m$. By the estimate of the second sentence of the paragraph (and the fact that $P(u,r,k) \geqslant k + 10$) we have $\|\ell_{u,r}\| \leqslant \|g_{u,r}\|$, $\|h_{u,r} - \ell_{u,r}\|_{A(\underline{\underline{T}})} \leqslant$ 2^{-2k-8}: Thus $\|\ell_{u,r} - g_{u,r}\|_{A(E)} \leqslant 2^{-2k-6}$. Set $L_{u,j} = \sum_{r=(j-1)2^k+1}^{j2^k} \ell_{u,r}$, $L_j = \sum_{u=0}^{k} \lambda_{u,j} L_{u,j}$. We have at once $\|L_j - 1\|_{A(E_j \cup E_{12})} = \|L-G\|_{A(E_j \cup E_{12})} \leqslant$ $2^{-k} C_j \leqslant \varepsilon/4$, $\|L_j\|_{A(\underline{\underline{T}})} \leqslant C_j + \varepsilon$ (by the same calculations as for $\|G_j\|_{A(\underline{\underline{T}})}$ and

$\hat{L}_j(m) = 0$ for all $m \leqslant w$). Since $\varepsilon > 0$ and $w \in \underline{\underline{Z}}$ were arbitrary, we have shown that $E_j \cup E_{12}$ has AA^+ constant at most C_j $[j = 1,2]$.

Part B : We show that, given $\varepsilon > 0$, we can find an $N_o(\varepsilon)$ such that $E_{1N} \cup E_{2N} \cup E_{12N}$ has AA^+ constant at least $C_1 + C_2 + C_1C_2 - \varepsilon$ for all $N \geqslant N_o(\varepsilon)$.

Fixing N at some arbitrary positive value, we make the following observations. By Lemma 7.7 (vi), (vii), (viii) and (ix), writing $Q(N) = P(0, 2^{N+1}, N)$, we can find disjoint subsets $\Lambda(S) \subseteq \{v \in \underline{\underline{Z}} : v \geqslant Q(N)\}$ $[\emptyset \neq S \subseteq \{1, 2, \ldots, 2^{N+1}\} = U$ say$]$ such that

(a) $|\hat{\mu}_{S,N}(v)| \leqslant 2^{-3N-4}$ if $v \notin \bigcup_{S \supseteq R \neq \emptyset} \Lambda(R)$ $[v \geqslant Q(n)]$

(b) $|\hat{\mu}_{S,N}(v) - \hat{\mu}_{T,N}(v)| \leqslant 2^{-3N-4}$ if $v \in \bigcup_{T \cap S \supseteq R \neq \emptyset} \Lambda(R)$ $[\emptyset \neq S, T \subseteq U]$.

Write $\Sigma(j) = \{S \subseteq \{(j-1)2^N + 1 \leqslant k \leqslant j2^N\} : \text{card } S = p_N(j)\}$ $[j = 1,2]$

$\Sigma(1,2) = \{S \cup T : S \in \Sigma(1), \ T \in \Sigma(2)\}$ and put $\Lambda(j) = \{v : |\hat{\mu}_{S,N}(v)| > 2^{-3N-4}$ for some $S \in \Sigma(1)\}$ $\mu_j = \sum_{S \in \Sigma(j)} \mu_{S,N}/\text{card } \Sigma(j)$ $[1 = 1,2]$,

$\Lambda(1,2) = \{v : |\hat{\mu}_{S,N}(v)| > 2^{-3N-4}$ for some $S \in \Sigma(1,2) \setminus (\Lambda(1) \cup \Lambda(2))\}$

$\mu_{12} = \sum_{S \in \Sigma(1,2)} \mu_{S,N}/\text{card } \Sigma(1,2)$. The following facts are evident

(a) $|\hat{\mu}_j(v)| \leqslant 2^{-3N-4}$ if $v \notin \Lambda(j)$, $|\hat{\mu}_{12}(v)| \leqslant 2^{-3N-4}$ if $v \notin \Lambda(1,2) \cup \Lambda(1) \cup \Lambda(2)$ $[v \geqslant Q(n)]$

(b) $|\hat{\mu}_j(v) - \hat{\mu}_{12}(v)| \leqslant 2^{-3N-4}$ if $v \in \Lambda(j)$

(c) $\mu_j \in M^+(E_j)$, $\mu_{12} \in M^+(E_{12})$; $\|\mu_j\| = 1$, $\|\mu_{12}\| = 1$.

(d) $|\hat{\mu}_j(v)| \leqslant p_N(j) 2^{-N}$, $|\hat{\mu}_{12}(v)| \leqslant p_N(1) p_N(2)\, 2^{-2N}$ $[v \geqq P(0, 2^{N+1} + 1, N)]$.

Now suppose $\sum_{v \geqslant Q(n)} a_v \chi_v(e) = 1$ for all $e \in E_{1N} \cup E_{2N} \cup E_{12N}$.

If $\sum_{v \geqslant Q} |a_v| \geqslant C_1 + C_2 + C_1C_2$ then we have done, so suppose the contrary.

By (c) and (d) we have

$$(e_j)\quad |1 - \sum_{v\in\Lambda(j)} a_v \hat{\mu}_j(v)| \leqslant |1 - \sum_{v\geqslant Q(N)} a_v \hat{\mu}_j(v)| + \sum_{v\geqslant Q(N), v\notin\Lambda(j)} |a_v|\,|\hat{\mu}_j(v)|$$

$$\leqslant \left|\int (1 - \sum_{v\geqslant Q(N)} a_v \chi_v) d\mu_j\right| + \sum_{v\geqslant Q(N)} |a_v| 2^{-3N-4}$$

$$= \sum_{v\geqslant Q(N)} |a_v| 2^{-3N-4} \leqslant (C_1 + C_2 + C_1C_2)2^{-3N-4} \qquad [j = 1,2]$$

whilst similarly

$$(e_{12})\quad |1 - \sum_{v\in\Lambda(1,2)\cup\Lambda(1)\cup\Lambda(2)} a_v \hat{\mu}_{12}(v)| \leqslant (C_1 + C_2 + C_1C_2)2^{-3N-4}.$$

Further (b) gives

$$(f_{12})\quad \left|\sum_{v\in\Lambda(1)} a_r \hat{\mu}_1(v) + \sum_{v\in\Lambda(2)} a_r \hat{\mu}_2(v) - \sum_{v\in\Lambda(1,2)} a_r \hat{\mu}_{12}(v)\right|$$

$$\leqslant (C_1 + C_2 + C_1C_2)2^{-3N-4}$$

so, combining (e_1), (e_2), (e_{12}) and (f_{12}), we get

$$(e_{12})'\quad |1 + \sum_{v\in\Lambda(1,2)} a_r \hat{\mu}_{12}(v)| \leqslant (C_1 + C_2 + C_1C_2)2^{-3N-2}.$$

Applying (d) to (e_1), (e_2), $(e_{12})'$ we get

$$(g_j)\quad \sum_{v\in\Lambda(j)} |a_r| = (p_j(N)2^{-N})^{-1} \sum_{v\in\Lambda(j)} (p_j(N)2^{-N})\,|a_v|$$

$$\geqslant (p_j(N)2^{-N})^{-1} \sum_{v\in\Lambda(j)} |a_v|\,|\hat{\mu}_j(v)|$$

$$\geqslant (p_j(N)2^{-N})^{-1} \left|\sum_{v\in\Lambda(j)} a_v \hat{\mu}_j(v)\right|$$

$$\geqslant (p_j(N)2^{-N})^{-1}(1 - |1 - \sum_{v\in\Lambda(j)} a_v \hat{\mu}_j(v)|)$$

$$\geqslant (p_j(N)2^{-N})^{-1}(1 - (C_1+C_2+C_1C_2)2^{-3N-4})$$

and

$$(g_{12})\quad \sum_{v\in\Lambda(1,2)} |a_v| \geqslant (p_1(N)p_2(N)2^{-2N})^{-1}(1 - (C_1+C_2+C_1C_2)2^{-3N-2}).$$

Thus

$$\sum_{v \geqslant Q(N)} |a_r| \geqslant (2^N p_1^{-1}(N) + 2^N p_2^{-1}(N) + (2^N p_1^{-1}(N))(2^N p_2^{-1}(N)))(1 - (C_1+C_2+C_1C_2)2^{-3N-2})$$

$= \theta(N)$ say. By this argument we have shown that $\sum_{v \geq Q(N)} a_v \chi_v(e) = 1$ for all $e \in E_{1N} \cup E_{2N} \cup E_{12N}$ implies $\sum_{v \geqslant Q(N)} |a_v| \geqslant \theta(N)$, i. e. we have shown that $E_{1N} \cup E_{2N} \cup E_{12N}$ has AA^+ constant at least $\theta(N)$. Since $\theta(N) \to C_1 + C_2 + C_1C_2$, part B is proved.

Proof of (i) and (ii). Using the fact that if $E \subseteq F$ with E,F closed then, if F is AA^+ with constant C_3 then E is AA^+ with constant $C_4 \leqslant C_3$, we see from Part B that the AA^+ constant λ_{12} of $E_1 \cup E_2 \cup E_{12}$ is at least $C_1 + C_2 + C_1C_2$. But by Part A the AA^+ constant λ_1 of E_1 is at most C_1 and the AA^+ constant λ_2 of E_2 is at most C_2. Since $\lambda_{12} \leqslant \lambda_1 + \lambda_2 + \lambda_1\lambda_2$ (Lemma 4.1 (iv)), it follows that $\lambda_1 = C_1$, $\lambda_2 = C_2$, $\lambda_{12} = C_1 + C_2 + C_1C_2$.

Proof of (iii). This also splits into 2 parts but the proofs are so much simpler that we have not bothered to separate them formally. First we note that $\|\chi_{P(0,j2^{k-n},k)} - 1\|_{C(F_j)} \leq 2^{-20(k+4)} \to 0$ as $k \to \infty$ so that F_j is Dirichlet $[1 \leqslant j \leqslant n]$. (Thus in particular remembering Lemma 4.1 (ii) and (iv) we know that F_j is AA^+ with constant 1 and $\bigcup_{j=1}^{n} F_j$ has AA^+ constant at most $2^n - 1$).

On the other hand, fixing N temporarily, we can make (as in Part B) the following observations. By Lemma 7.7 (vi), (vii) and (viii), writing $Q(N) = P(0, 2^{N+1}+1, N)$, we can find disjoint subsets $\Lambda(S) \subseteq \{v \in \mathbb{Z}, v \geqslant Q(N)\}$ $\left[\emptyset \neq S = \{2^{N-n}, 2.2^{N-n}, \dots \dots, n.2^{N-n}\} = U \text{ say}\right]$ such that

(a) $|\hat{\mu}_{S,N}(v)| \leqslant 2^{-3N-4}$ if $v \notin \bigcup_{S \supseteq R \neq \emptyset} \Lambda(R)$

(b) $\quad |\hat{\mu}_{S,N}(v) - \hat{\mu}_{T,N}(v)| \leqslant 2^{-3N-4} \quad$ if $\quad v \in \bigcup_{S\cap T\supseteq R\neq\emptyset} \Lambda(R)$.

Now suppose $\sum_{v\not\geqslant Q(N)} a_v \chi_v(e) = 1$ for all $e \in \bigcup_{j=1}^{n} F_{jN}$.

If $\sum_{v\geqslant Q(n)} |a_v| \geqslant 2^{n-1}$ then we have done, so suppose not.

By (a) we have

$$(c)_S \quad \left|1 - \sum_{S\supseteq T\neq\emptyset} \sum_{v\in\Lambda(T)} a_v \hat{\mu}_S(v)\right| \leqslant \left|1 - \sum_{v\geqslant Q(N)} a_v \hat{\mu}_S(v)\right| + \sum_{v\notin\bigcup\{\Lambda(T):S\supseteq T\neq\emptyset\}} |a_v|\,|\hat{\mu}_S(v)|$$

$$\leqslant \left|\int (1 - \sum_{v\geqslant Q(N)} a_v \chi_v) d\mu_S\right| + \sum_{v\geqslant Q(N)} |a_r| 2^{-3N-4}$$

$$= \sum_{v\geqslant Q(N)} |a_v| 2^{-3N-4}$$

By (b) this gives

$$(d)_S \quad \left|1 - \sum_{S\supseteq T\neq\emptyset} \sum_{v\in\Lambda(T)} a_v \hat{\mu}_T(v)\right| \leqslant \left|1 - \sum_{S\supseteq T\neq\emptyset} \sum_{v\in\Lambda(T)} a_v \hat{\mu}_S(v)\right| + \sum_{S\supseteq T\neq\emptyset} \sum_{v\in\Lambda(T)} |a_v|\,|\hat{\mu}_S(v) - \hat{\mu}_T(v)|$$

$$\leqslant \sum_{v\geqslant Q(N)} |a_v| 2^{-3N-4} + \sum_{v\geqslant Q(N)} |a_v| 2^{-3N-3}$$

$$\leqslant \sum_{v\geqslant Q(N)} |a_v| 2^{-3N-2} \leqslant (2^n - 1) 2^{-3N-2}$$

$$\leqslant 2^{-2N} \qquad [\emptyset \neq S \subseteq U] .$$

Considering the $(d)_S$ as inequations in $\sum_{v\in\Lambda(T)} a_v \hat{\mu}_T(v)$ we may solve them step by step (first consider $(d)_S$ with card $S = 1$, then with card $S = 2$, and so on) to obtain

$$(e)_T \quad \left|1 - \sum_{v\in\Lambda(T)} a_r \hat{\mu}_T(v)\right| \leqslant n\, 2^{-2N} \qquad [\emptyset \neq T \subseteq U] .$$

Since $|\hat{\mu}_T(v)| \leqslant \|\mu_T\| = 1$ this gives

$$(e)'_T \quad \sum_{v\in\Lambda(T)} |a_v| \geqslant 1 - 2^{-2N}$$

whence

$$\sum_{v\geqslant Q(N)} |a_r| \geqslant \sum_{U\supseteq T\neq\emptyset} \sum_{v\in\Lambda(T)} |a_v| \geqslant (2^n - 1)(1 - n2^{-2N}).$$

We have thus shown that $\bigcup_{j=1}^{n} F_{jN}$ has AA^+ constant at least $(2^n-1)(1-n2^{-2N}) = \theta(N)$ say. Since $\theta(N) \to (2^n-1)$ as $N \to \infty$, (iii) is proved.

Remark 1. We draw the reader's attention to the fact that in Theorem 7.1 (i) we construct sets with any given AA^+ constant $C \geqslant 1$. Theorem 7.1 (ii) could thus be (uselessly) sharpened by replacing "at most C_i" by "exactly C_i".

Remark 2. Although Theorem 7.1 (iii) gives all we need from the point of view of the rest of the paper (i. e. "gets the biggest AA^+ constant from the fewest well behaved sets") it is clear that,combining the methods of (i) and (ii) with that of (iii) we can get the following improvement on (iii) : -

THEOREM 7.1 (iv). Given $n \geqslant 1$, $C_1, C_2, \dots, C_n \geqslant 1$, $\varepsilon > 0$, we can find closed sets $E_1, E_2, \dots, E_n$ such that $E_i \cap E_j = \{x\}$ $[i \neq j]$ for some $x \in \underline{\underline{T}}$ (respectively $E_i \cap E_j = \emptyset$ for $i \neq j$) such that $\bigcup_{i=1}^{n} E_i$ is independent, E_i has AA^+ constant C_i $[1 \leqslant i \leqslant n]$, yet $\bigcup_{i=1}^{n} E_i$ has constant $\sum C_i + \sum_{i\neq j} C_i C_j + \dots + C_1 C_2 \dots C_n$ (respectively AA^+ constant at least $\sum C_i + \sum_{i\neq j} C_i C_j + \dots + C_1 C_2 \dots C_n - \varepsilon$).

We now show that the sets described in Lemma 7.7 can in fact be constructed using Lemma 5.5. The reader will see firstly that the ideas used are simple and secondly that by removing some of the conditions (e. g. independence) we can get much simpler proofs.

Proof of Lemma 7.7. Choose $x \in 2\pi \mathbb{Q}$. Our construction is inductive. Suppose that at the n^{th} stage we have constructed $2^{-n} \geq \varepsilon(n) > 0$, $E(S,t)$ closed disjoint sets together with measure $\mu_{S,t} \in M^+(E(S,t))$, $\|\mu_{S,t}\| = 1$ $[S \subseteq \{r : 1 \leq r \leq 2^{t+1}\}]$ for all $1 \leq t \leq n$. Suppose further we have constructed a sequence of integers $P(u,r,t,n)$ $[1 \leq u \leq t-1,\ 1 \leq r \leq 2^{t+1}+1,\ 1 \leq t]$ such that

$$10 + u_1 + r_1 + t_1 \leq P(u_1,r_1,t_1,n) < P(u_2,r_2,t_2,n)$$

whenever $0 \leq u_1 < u_2$, $r_1 = r_2$, $t_1 = t_2$ or $r_1 < r_2$, $t_1 = t_2$ or $t_1 < t_2$ with the following properties (we write $E(t) = \bigcup_{\emptyset \neq S \subseteq \{1,2,\ldots,2^{t+1}\}} E(S,t))$: -

(ii)$_n$ $\{x\} \cup \bigcup_{t=1}^{n} E(t)$ is independent.

Suppose $e \in E(S,t)$ $[S \subseteq \{1,2,\ldots,2^{t+1}\}\ n \geq t \geq 1]$, $0 \leq u \leq k-1$, $1 \leq r \leq 2^{k+1}$, $1 \leq t \leq n$.

(iii)$_n$ If $k > t$ we have

$$|\chi_{P(u,r,k,n)}(e) - 1| \leq 2^{-20(k+4)} \qquad \text{if } u = 0$$
$$|\chi_{P(u,r,k,n)}(e) - 1| \geq 2^{-k-4} \qquad \text{otherwise.}$$

(iv)$_n$ If $t = k$ we have

$$|\chi_{P(u,r,k,n)}(e) - 1| \leq 2^{-20(k+4)} \qquad \text{if } u = t,\ 1 + \left[\frac{r-1}{2^{k-t}}\right] \in S$$
$$|\chi_{P(u,r,k,n)}(e) - 1| \geq 2^{-k-4} \qquad \text{otherwise.}$$

Further, if $|x-e| \leq \varepsilon(n)$, $n \geq k$

(v)$_n$
$$|\chi_{P(u,r,k,n)}(e) - 1| \leq 2^{-20(k+4)} \qquad \text{if } u = 0$$
$$|\chi_{P(u,r,k,n)}(e) - 1| \geq 2^{-k-4} \qquad \text{otherwise.}$$

Whilst for all k

$(v)^{*}_{k}$ $\quad |\chi_{P(u,r,k,n)}(x) - 1| \leqslant 2^{-20(k+4)}$ if $u = 0$

$|\chi_{P(u,r,k,n)}(x) - 1| \geqslant 2^{-k-4}$ otherwise.

Finally (iv), (vii), (viii) of Lemma 7.7 hold under the additional condition $n \geqslant t \geqslant 1$.

Applying Lemma 5.5 we can find integers $P(u,r,k,n+1)$ with disjoint closed sets $E(S,n+1) \subseteq [x-\varepsilon(n), x+\varepsilon(n)]$ (with $E(n+1) \cap E(t) = \emptyset$ for $n > t \geqslant 1$, $x \notin E(n+1)$), measures $\mu_{S,n+1} \in M^{+}(E(S,n+1))$ with $\|\mu_{S,n+1}\| = 1$ such that

$(ii)_{n+1}$ $\quad \{x\} \cup \bigcup_{t=1}^{n+1} E(t)$ is independent

$(a)_{n+1}$ $\quad P(u,r,t,n) = P(u,r,t,n+1)$ for $1 \leqslant u \leqslant t-1$, $1 \leqslant r \leqslant 2^{t+1}+1$, $1 \leqslant t \leqslant n$

$(b)_{n+1}$ $\quad P(u,r,t,n+1) = P(u,2^{k-t}r,k,n)$ for some $k > t$

$(c)_{n+1}$ $\quad P(u_1,r_1,t_1,n+1) < P(u_2,r_2,t_2,n+1)$ whenever $0 \leqslant u_1 < u_2$, $r_1 = r_2$ or $r_1 < r_2$, $t_1 = t_2$ or $t_1 < t_2$

$(iii)'_{n+1}$ $\quad$ Suppose $e \in E(S,n+1)$ $[S \subseteq 1,2,\dots,2^{n+1}]$, $0 \leqslant u \leqslant k-1$, $1 \leqslant r \leqslant 2^{k+1}$, $1 \leqslant t$, $k > n+1$ then

$|\chi_{P(u,r,k,n+1)}(e) - 1| \leqslant 2^{-20(k+4)}$ if $u = 0$

$|\chi_{P(u,r,k,n+1)}(e) - 1| \geqslant 2^{-k-4}$ otherwise.

$(iv)'_{n+1}$ $\quad$ If $t > k$, we have

$|\chi_{P(u,r,k,n+1)}(e) - 1| \leqslant 2^{-20(k+4)}$ if $u \leqslant t$, $1 + \left[\frac{r-1}{2^{k-t}}\right] \in S$

$|\chi_{P(u,r,k,n+1)}(e) - 1| \geqslant 2^{-k-4}$ otherwise.

Strictly speaking Lemma 5.5 does not imply (ix) directly but the reader can either verify that the proof of Lemma 5.5 given can also be made to give (ix) or make some other simple modification of our construction.

Further, (iv), (vii), (viii) and (ix) of Lemma 7.7 hold for $t = n+1$.

Observe now that $(\mathrm{iii})_n$, $(\mathrm{b})_{n+1}$ and $(\mathrm{iii})'_{n+1}$ give $(\mathrm{iii})_{n+1}$, $(\mathrm{iv})_n$, $(\mathrm{b})_{n+1}$ and $(\mathrm{iv})'_{n+1}$ give $(\mathrm{iv})_{n+1}$, and $(\mathrm{v})_n$ with $(\mathrm{b})_{n+1}$ give $(\mathrm{v})_{n+1}$. From $(\mathrm{v})_{n+1}$ and continuity it follows that $(\mathrm{v})^*_{n+1}$ will hold, provided only we take (as we immediately do) $\varepsilon(n+1) - \varepsilon(n)/2$ sufficiently small. We can restart the induction. (Incidentally we remark that to start the induction at the 0^{th} stage it suffices to note that, since $x \notin 2\pi\mathbb{Q}$, $\limsup_{n\to\infty} |1 - \chi_n(x)| = 2$, $\liminf_{n\to\infty} |1 - \chi_n(x)| = 0$ and it follows that $(\mathrm{v})^*_0$, the only non vacuous condition when $n = 0$, can be satisfied by a suitable choice of $P(u,r,k,0)$.)

Setting $P(u,r,t) = P(u,r,t,t)$ $[1 \leq u \leq t-1,\ \ 1 \leq r \leq 2^{t+1}+1]$ the conditions of Theorem 7.1 can be read off from the corresponding inductive conditions (with n sufficiently large).

We now turn to Theorem 7.2. To prove the full result we shall require arguments similar to those of Theorem 7.1, together with a very simple version (Lemma 7.9) of an argument which we shall use again in the last part of this section and in Sections § 8 and § 9. However, the reader may well be satisfied with the following simpler result (in which case he may resume reading after the conclusion of the proof of Theorem 7.2).

LEMMA 7.8. Given $K > 1$, we can find closed disjoint sets F_1, F_2 and subsets $\Lambda(1), \Lambda(2) \subseteq \mathbb{Z}^+ \setminus \{0\}$ such that F_i is $AA_{\Lambda(i)}$ with constant 1 $[i = 1,2]$, $F_1 \cup F_2$ is independent, but $\sum_{r\in\Lambda(1)\cup\Lambda(2)} |a_r| < K$ implies
$\| 1 - \sum_{r\in\Lambda(1)\cup\Lambda(2)} a_r \chi_r \|_{C(F_1\cup F_2)} > 0.$

Proof. Consider the statement of Lemma 5.5. Take $F = \emptyset$, $\eta = 10^{-2K}$, choose $\varepsilon > 0$, $x \in \mathbb{T}$ arbitrarily and set $m = 2$. Let $F_1 = E_1$, $F_2 = E_1 \cup E_{12}$. Take $\Lambda(i) = \{M_i(j)+s : |s| \leq j,\ j \geq 10K\}$ $[i = 1,2]$. Then by condition (i) $\|\chi_r - 1\|_{C(F_i)} \to 0$

as $r \to \infty$, $r \in \Lambda(i)$ and so, by the arguments of Lemma 4.1 (i), F_i is $AA_{\Lambda(i)}$ with constant 1.

On the other hand, by condition (i) again $\|\chi_r - 1\|_{C(F_i)} \leqslant 2^{-9K}$ so that $|\hat{\mu}_i(r) - 1| \leq 1/10K$, $|\hat{\mu}_{12}(r) - 1| \leqslant 1/10K$ for all $r \in \Lambda(i)$ $[i = 1,2]$ whilst by conditions (iv), (ix) and (v) $|\hat{\mu}_1(r)| \leqslant 1/10K$ for all $r \in \Lambda(2)$, $|\hat{\mu}_2(r)| \leqslant 1/10K$ for all $r \in \Lambda(1)$. Thus

$$3\|1 - \sum_{r \in \Lambda(1) \cup \Lambda(2)} a_r\chi_r\|_{C(F_1 \cup F_2)} \geqslant \left|\int (1 - \sum_{r \in \Lambda(1) \cup \Lambda(2)} a_r\chi_r) d\mu_1\right|$$

$$+ \left|\int (1 - \sum_{r \in \Lambda(1) \cup \Lambda(2)} a_r\chi_r) d\mu_2\right| + \left|\int (1 - \sum_{r \in \Lambda(1) \cup \Lambda(2)} a_r\chi_r) d\mu_{12}\right|$$

$$\geq |1 - \sum_{r \in \Lambda(1)} a_r \hat{\mu}_1(r)| + |1 - \sum_{r \in \Lambda(2)} a_r \hat{\mu}_2(r)| + |1 - \sum_{r \in \Lambda(1)} a_r \hat{\mu}_1(r) - \sum_{r \in \Lambda(2)} a_r \hat{\mu}_2(r)| - 2\sum |a_r|/10K$$

$$\geq 1 - 2/5 = 3/5 > 0.$$

and the proof is complete.

Theorem 7.2 is stronger than Lemma 7.8 not so much because we replace K by ∞, but because we claim not merely that no relation of the form $1 - \sum_{r \in \Lambda(1) \cup \Lambda(2)} a_r\chi_r(e) = 0$ for all $e \in E_1 \cup E_2$ can hold, but that no non trivial relation of the form $\sum_{r \in \{0\} \cup \Lambda(1) \cup \Lambda(2) \cup -\Lambda(1) \cup -\Lambda(2)} a_r\chi_r(e) = 0$ for all $e \in E_1 \cup E_2$ can hold. The reader may convince himself that the proof of Theorem 7.2 will require a new idea by seeing how the proof of Lemma 7.8 breaks down for $\sum_{r \in \Lambda(1) \cup \Lambda(2) \cup -\Lambda(1) \cup -\Lambda(2)} a_r\chi_r$ with $|a_r - a_{-r}|$ small, although it worked for $\sum_{r \in \Lambda(1) \cup \Lambda(2) \cup \{0\}} a_r\chi_r$ with $a_0 = 1$.

The new idea is, however, not very difficult.

LEMMA 7.9. Suppose $\varepsilon > 0$, $\delta > 0$, $R > 1$ and N a positive integer are given. Then we can find $\varkappa(\varepsilon, \delta, R) > 0$, $\varkappa(\varepsilon, \delta, R) \in \underline{\underline{Z}}^+$, and, given $x \in \underline{\underline{T}}$ together with some

$\eta > 0$, closed disjoint intervals $I_1, I_2, \ldots, I_k$ say with $\bigcup_{i=1}^{k} I_i \subseteq [x-\varepsilon, x+\varepsilon]$ such that for all $R \geq \sum_{r=-N}^{N} a_r \geq \delta$ we have

(i) $$\sup_{k \geq i \geq 1} \inf_{\lambda \in \mathbb{C}} \left\| \sum_{r=-N}^{N} a_r \chi_r - \lambda \right\|_{C(I_i)} \leq \eta$$

(ii) $$\sup_{k \geq i \geq 1} \left\| \sum_{r=-N}^{N} a_r \chi_r \right\|_{C(I_i)} \geq \varkappa(\varepsilon, \delta, R).$$

Proof. By translation, if necessary, we may suppose $x = 0$. Consider $\Gamma = \left\{ \sum_{r=-N}^{N} a_r \chi_r \mid [-\varepsilon, \varepsilon] : R \geq \sum |a_r| \geq \delta \right\}$. By elementary results on trigonometric polynomials $0 \notin \Gamma$. But Γ is a compact subset of $C([-\varepsilon, \varepsilon])$ (because, for instance, Γ is a bounded subset of a finite dimensional subspace). Thus

$\inf \left\{ \|f\|_{C(E)} : f \in \Gamma \right\} \geq 2\varkappa(\varepsilon, \delta, R)$ for some $\varkappa(\varepsilon, \delta, R) > 0$. Now we can find a finite set of points $x_1, x_2, \ldots, x_k$ say with $x_i \in (-\varepsilon, \varepsilon)$ $[1 \leq i \leq k]$ such that

$\inf_{k \geq i \geq 1} |t - x_i| \leq (RN)^{-1} \varkappa(\varepsilon, \delta, R)$ for all $t \in [-\varepsilon, \varepsilon]$. Under these circumstances

$$\inf_{k \geq i \geq 1} \left| \sum_{r=-N}^{N} a_r \chi_r(t) - \sum_{r=-N}^{N} a_r \chi_r(x_i) \right| \leq \sum_{r=-N}^{N} |a_r| N \sup_{k \geq i \geq 1} |x_i - t| \leq \varkappa(\varepsilon, \delta, R)$$

whenever $\sum_{r=-N}^{N} |a_r| \leq R$. It follows that

(ii)' $$\sup_{k \geq i \geq 1} \left| \sum_{r=-N}^{N} a_r \chi_r(x_i) \right| \geq \varkappa(\varepsilon, \delta, R)$$

for all $R \geq \sum_{r=-N}^{N} |a_r| \geq \delta$. Now choose $\gamma > 0$ so small that

$\gamma \leq \min\left(\min_{k \geq i > j \geq 1} |x_i - x_j|, \min_{k \geq i \geq 1} |x_i - \varepsilon|, \min_{k \geq i \geq 1} |x_i + \varepsilon| \right)/4$ and also $RN\gamma \leq \eta/2$.

Setting $I_i = [x_i - \gamma, x_i + \gamma]$ we have at once that the I_i are disjoint and $I_i \subseteq [-\varepsilon, \varepsilon]$. Since $x_i \in I_i$, (ii)' gives (ii) whilst we obtain (i) from the observation that

(i)' $$\sup_{t \in I_i} \left| \sum_{r=-N}^{N} a_r \chi_r(t) - \sum_{r=-N}^{N} a_r \chi_r(x_i) \right| \leq \sum_{r=-N}^{N} |a_r| N \sup_{t \in I_i} |x_i - t| \leq \eta$$

for all $R \geqslant \sum_{-N}^{N} |a_r|$.

Using this simple fact, we can now give the

Proof of Theorem 7.2. Choose $x \notin 2\pi\mathbb{Q}$. By Kronecker's theorem we can find $1 \leqslant P_{1,0}(0) < P_{2,0}(0) < P_{1,0}(1) < P_{2,0}(1) < P_{1,0}(2) < \dots\dots$ integers such that $|\chi_{P_j(r)}(x) - 1| \leqslant 2^{-r}$ $[j = 1,2 \; ; \; r \geqslant 0]$. Set $F_{1,0} = F_{2,0} = \{x\}$, $\varepsilon(0) = 1/(2^4 P_{2,0}(0))$. We shall obtain E_1, E_2 as the union of sets F_{1r}, F_{2r} $[r \geqslant 0]$ constructed inductively as follows : -

Suppose that at the beginning of the $n+1^{th}$ step we have constructed closed disjoint sets $F_{1,0}, F_{1,r}, F_{2,r}$ $[1 \leqslant r \leqslant n]$, integers $1 < P_{1,n}(0) < P_{2,n}(0) < P_{1,n}(1) < P_{2,n}(1) < P_{1,n}(2) < \dots$ and $\varepsilon(n) > 0$ such that writing $Q(n) = P_{2,n}(n)$ we have

$(i)_n$ $\quad F^n = \bigcup_{r=0}^{n} (F_{1,r} \cup F_{2,r})$ is independent

$(ii)_n$ $\quad \|\chi_{P_{j,n}(r)} - 1\|_{C(F_{j,r})} \leqslant 2^{-r}$ $\qquad [j = 1,2 \; ; \; r \geq 0]$

$(iii)_n$ $\quad \varepsilon(n)Q(n) \leqslant 2^{-n-4}$.

By Lemma 7.9 we can find $1 > \varkappa(n) > 0$ and closed disjoint intervals $I_{1,n}, I_{2,n}, \dots, I_{k(n),n}$ such that for all $n \geq \sum_{r=-Q(n)}^{Q(n)} |a_r| \geq n^{-1}$

$(iv)_n$ $$\sup_{k(n) \geqslant i \geqslant 1} \inf_{\lambda \in \mathbb{C}} \Big\| \sum_{r=-Q(n)}^{Q(n)} a_r \chi_r - \lambda \Big\|_{C(I_{i,n})} \leqslant \varkappa(n)/10$$

$(v)_n$ $$\sup_{k(n) \geqslant i \geqslant 1} \Big\| \sum_{r=-Q(n)}^{Q(n)} a_r \chi_r \Big\|_{C(I_{i,n})} \geqslant \varkappa(n) .$$

Using Lemma 5.5 repeatedly (with $\eta = \varkappa(n)/100$, $m = 2$, taking F to be successively $F^{(n)}, F^{(n)} \cup G_{1n}, F^{(n)} \cup G_{1n} \cup G_{2n}, \dots \subseteq [x-\varepsilon , x+\varepsilon]$ to be successively

$I_{1,n}, I_{2,n}, I_{3,n}, \dots,$ and extracting coarser and coarser sub-sequences of $P_{1,n}(r)$, $P_{2,n}(r))$, we can find $G_{in} = G_{in1} \cup G_{in2} \cup G_{in12}$ with $G_{in1}, G_{in2}, G_{in12}$ disjoint from each other, $G_{in} \subseteq I_{in}$, a sequence of integers

$1 < P_{1,n+1}(0) < P_{2,n+1}(0) < P_{1,n+1}(1) < \dots\dots$ and measures $\mu_{inj} \in M^+(G_{inj})$, $\|\mu_{inj}\| = 1$, $\mu_{in12} \in M^+(G_{inj})$, $\|\mu_{in12}\| = 1$ such that

$(i)_{n+1}$ If we set $F_{1,n+1} = \bigcup_{i=1}^{k(n)} G_{in1}$, $F_{2,n+1} = \bigcup_{i=1}^{k(n)} (G_{in2} \cup G_{in12})$

then F^{n+1} is independent

$(ii)'_{n+1}$ $P_{j,n+1}(r) = P_{j,n+1}(s(j,r))$ for some $s(j,r) \geq r$, further $s(j,r) = r$ for $0 \leqslant r \leqslant n+1$.

$(ii)''_{n+1}$ $\| \chi_{P_{j,n+1}(r)} - 1 \|_{C(F_{j,n+1})} \leqslant 2^{-r} K(n)/(100n)$ for all $r \geq n+1$

$(vi)_{n+1}$ $|\hat{\mu}_{in1}(\theta P_{2,n+1}(r)+s)| \leqslant \varkappa(n)/(10n)$ for all $|s| \leqslant r$, $r \geq n+1$, $\theta = \pm 1$

$|\hat{\mu}_{in2}(\theta P_{1,n+1}(r)+s)| \leqslant \varkappa(n)/(10n)$ for all $|s| \leqslant r$, $r \geq n+1$, $\theta = \pm 1$.

We note that as an immediate consequence of $(ii)''_{n+1}$ we have

$(vii)_{n+1}$ $|\hat{\mu}_{i\ n+1\ j}(\theta P_{j,n+1}(r)) - 1|$, $|\hat{\mu}_{i\ n+1\ 12}(\theta P_{j,n+1}(r)) - 1| \leqslant \varkappa(n)/(10n)$

for all $r \geq n+1$, $\theta = \pm 1$ $[j = 1,2\ ;\ 1 \leqslant i \leqslant k(n)]$.

Condition $(ii)_{n+1}$ holds (because of $(i)_n$, the first part of $(ii)'_{n+1}$ and $(ii)''_{n+1}$ for $r \geq n+1$; because of $(i)_n$, the second part of $(ii)'_{n+1}$ and $(iii)_n$ for $0 \leqslant r \leqslant n$). Thus setting $\varepsilon(n+1) = 2^{-n-5}/Q(n+1)$ (so that $(iii)_{n+1}$ is satisfied) we may restart the induction.

Since $F_{1,n} \cup F_{2,n} \subseteq [\ x-\varepsilon(n)\ ,\ x+\varepsilon(n)]$ and $\varepsilon(n) \to 0$ we see that

$E_1 = \bigcup_{r=0}^{\infty} F_{1,r}$, $E_2 = \bigcup_{r=0}^{\infty} F_{2,r}$ are closed. Since the $F_{i,r}$ are disjoint for $r \geq 1$

we have $E_1 \cap E_2 = \{x\}$. By condition $(i)_n$ $E_1 \cup E_2$ is independent. By condition $(ii)_n$ and the fact that (by the second part of condition $(iii)'_{n+1}$) $P_{j,m}(n) = P_{j,m}(m) = P_j(m)$ say for all $n \geq m$ we have $\| \chi_{P_j(m)+s} - \chi_s \|_{C(E_j)} \to 0$ and so, writing

$\Lambda(j) = \{P_j(m)+s : |s| \leqslant m,\ m \geq 0\}$, E_j is (by the arguments of Lemma 4.1 (i)) an $AA_{\Lambda(j)}$ set with constant 1 $[j = 1,2]$.

On the other hand $E_1 \cup E_2$ cannot be a ZA_Γ set with $\Gamma = \Lambda(1) \cup \Lambda(2) \cup -\Lambda(1) \cup -\Lambda(2) \cup \{0\}$. For suppose $\infty > \sum_{r\in\Gamma} |a_r| > 0$. Then we can find an n such that $n > \sum_{r\in\Gamma,\ |r| \leqslant Q(n)} |a_r| > n^{-1}$. By $(v)_n$ we can find an $k(n) \geqslant i \geqslant 1$ such that $\| \sum_{r\in\Gamma,\ |r| \leqslant Q(n)} a_r \chi_r \|_{C(I_{i,n})} \geq \varkappa(n)$.

To simplify the notation, we shall write $\varkappa = \varkappa(n)$, $Q = Q(n)$, $G = G_{in}$, $I = I_{in}$, $\mu_j = \mu_{inj}$ $[j = 1,2]$, $\mu_{12} = \mu_{in12}$. By $(iv)_n$ we can find a λ such that $\| \sum_{r\in\Gamma,\ |r| \leqslant Q} a_r \chi_r - \lambda \|_{C(I)} \leqslant \varkappa/10$ (and automatically $|\lambda| \geqslant 9\varkappa/10$).

We now argue as in Lemma 7.8

$$3\|\sum_{r\in\Gamma} a_r\chi_r\|_{C(G)} \geq 3(\|\lambda - \sum_{r\in\Gamma,\ |r|>Q} a_r\chi_r\|_{C(G)} - \|\sum_{r\in\Gamma,\ |r|>Q} a_r\chi_r - \lambda\|_{C(I)})$$

$$\geq 3(\|\lambda - \sum_{r\in\Gamma,\ |r|>Q} a_r\chi_r\|_{C(G)} - \varkappa/10)$$

$$\geq \left|\int(\lambda - \sum_{|r|\in\Lambda(1),\ |r|>Q} a_r\chi_r)d\mu_1\right| + \left|\int(\lambda - \sum_{|r|\in\Lambda(2),\ |r|>Q} a_r\chi_r)d\mu_2\right|$$

$$+ \left|\int(\lambda - \sum_{|r|\in\Lambda(1)\cup\Lambda(2),\ |r|>Q} a_r\chi_r)d\mu_{12}\right| - 3\varkappa/10$$

$$\geq \left|\lambda - \sum_{|r|\in\Lambda(1),\ |r|>Q} a_r\hat{\mu}_1(r)\right| + \left|\lambda - \sum_{|r|\in\Lambda(2),\ |r|>Q} a_r\hat{\mu}_2(r)\right|$$

$$+ \left|\lambda - \sum_{|r|\in\Lambda(1)} a_r\hat{\mu}_1(r) - \sum_{|r|\in\Lambda(2)} a_r\hat{\mu}_2(r)\right| - 2\sum_{r\in R} a_r\varkappa/(10n) - 3\varkappa/10$$

$$\geq |\lambda| - 5\varkappa/10 \geq 2\varkappa/5 > 0.$$

Hence $\|\sum_{r\in\Gamma} a_r\chi_r\|_{C(E_1\cup E_2)} \geqslant \|\sum_{r\in\Gamma} a_r\chi_r\|_{C(E)} > 0$ and the proof is complete.

The proof of Theorem 7.3 requires a result similar to, but more delicate than, Lemma 7.9.

LEMMA 7.10 (i) Suppose I is a closed interval, then $\widetilde{A^+(I)} = A^+(I)$;

(ii) If I is a closed interval (with non empty interior) than I is not a ZA^+ set

(iii) Suppose $\varepsilon > 0$ given. Then, for all $R > 0$, we can find $1 \geqslant \varkappa(\varepsilon,R) > 0$ with the following property. Suppose $N \geq 1$ given, then we can find a positive integer $k(\varepsilon,R,N)$ and a $\gamma(\varepsilon,k,N) > 0$ such that, given $x\in\underline{\underline{T}}$ together with some $\eta > 0$, we can find disjoint intervals $I_1, I_2, \ldots, I_k$ say with $\bigcup_{i=1}^{k} I_i \subseteq [x-\varepsilon , x+\varepsilon]$, $\inf_{k\geqslant i\geqslant 1} |I_i| \geqslant \gamma$ having the following property. If $\sum_{r=1}^{N} |a_r| \leqslant R$ then

(1) $$\sup_{k\geqslant i\geqslant 1}\ \inf_{\lambda\in\underline{\underline{C}}} \|(1 - \sum_{r=1}^{N} a_r\chi_r) - \lambda\|_{C(I_i)} \leqslant \eta$$

(2) $$\sup_{k\geqslant i\geqslant 1} \|1 - \sum_{r=1}^{N} a_r\chi_r\| \geq \varkappa(\varepsilon,R).$$

Remark. The important thing here is that $\varkappa$ does not depend on N and η does not depend on $\varkappa$.

Proof. (i) We follow word for word the standard proof ([5], p. 46) that $\widetilde{A(I)} = A(I)$. Note first that, if $PF^+(I) = \{T^+ : \langle T^+ , f\rangle = \langle T,f\rangle$ for all $f\in A^+(I)$ and some $T\in PF(E)\}$, $PM^+(I) = \{T^+ : \langle T^+,f\rangle = \langle T,f\rangle$ for all $f\in A^+(I)$ and some $T\in PM(E)\}$, then $A^+(I)$ is the dual of $PF^+(I)$ and $PM^+(I)$ is the dual of $A^+(I)$. If $f\in A^+(I)$, then

$$\left|\int f d\mu\right| \leqslant \|f\|_{\widetilde{A^+(I)}} \|\mu\|_{PM^+(I)} = \|f\|_{\widetilde{A^+(I)}} \sup_{n\geqslant 0} |\hat{\mu}(-n)|$$

for all $\mu \in M(I)$ and so for all $\mu \in M(I) \cap PF^+(I)$. By the theorem of Hahn-Banach applied to the dual of $PF^+(I)$ the linear continuous functional on $M(I) \cap PF^+(I)$

$$\mu \to \langle f, \mu \rangle = \int f \, d\mu$$

can be extended to a continuous linear functional on $PF^+(I)$, i.e. to a member of $A^+(I)$, and so there exists an $f_o \in A^+(I)$ with $\langle f, \mu \rangle = \langle f_o, \mu \rangle$ for all $\mu \in M(I) \cap PF^+(I)$. In particular, by the lemma of Riemann-Lebesgue $\int f \, d\mu = \int f_o d\mu$ for all absolutely continuous measures on I and so $f = f_o \in A^+(I)$.

(ii) This is an immediate consequence of Jensen's inequality (if $f(z) = 1 - \sum_{r=1}^{\infty} a_r z^r$ for $|z| \leqslant 1$ and $\sum_{r=1}^{\infty} |a_r| < \infty$ then, writing μ for Haar measure on $\underline{\underline{T}}$

$$0 = \log f(0) \leqslant \int_{\theta \in \underline{\underline{T}}} \log |f(e^{i\theta})| \, d\mu$$

and so in particular $\{\theta \in \underline{\underline{T}} : f(e^{i\theta}) = 0\} = \{\theta \in \underline{\underline{T}} : 1 - \sum a_r \chi_r(\theta) = 0\}$ must have Haar measure 0) but we may obtain the result by weaker means.

Suppose I a ZA^+ set, then we can find $\sum_{r=0}^{\infty} |a_r| > 0$ with $\sum a_r \chi_r(e) = 0$ for all $e \in I$. By multiplying by $\lambda \chi_{-n}$, if necessary, we may suppose $a_o = 1$. Choose $x_1, x_2, \ldots, x_m$ such that $\bigcup_{i=1}^{m} (x_i + I) = \underline{\underline{T}}$. Writing $f_i = \sum_{r=0}^{\infty} a_r \chi_{-r}(x_i) \chi_r$ we have $f_i \in A^+$ and $f_i(e) = \sum_{r=0}^{\infty} a_r \chi_r(e - x_i) = 0$ for all $e \in x_i + I$ $[1 \leqslant i \leqslant m]$.

Setting $g = f_1 \ f_2 \ \ldots \ f_m$ we have $g \in A$, $g(e) = 0$ for all $e \in \underline{\underline{T}}$ and $\hat{g}(0) = 1$ which, by the uniqueness of Fourier representations, is impossible.

(iii) By translation we may suppose $x = 0$ (thus, in particular, reducing the definition of y to a triviality). We remark first that $\inf \left\{ \| 1 - \sum_{r=1}^{\infty} a_r \chi_r \|_{C(I)} : \sum_{r=1}^{\infty} |a_r| \leqslant R \right\} \geqslant 2\varkappa(R)$ for some $\varkappa(R) > 0$. For suppose not. Then, since

$\|\chi_{-1} - \sum_{r=0}^{\infty} a_r\chi_r\|_{C(I)} = \|1 - \sum_{r=0}^{\infty} a_r\chi_r\|_{C(I)}$, we have $\chi_{-1}\in\widetilde{A^+(I)}$ (indeed $\|\chi_{-1}\|_{\widetilde{A^+(E)}} \leqslant R$) and so by (i) $\chi_{-1}\in A^+(I)$ and $\chi_{-1}(e) = \sum_{r=0}^{\infty} a_r\chi_r(e)$ for all $e\in I$ and some $\sum_{r=0}^{\infty} |a_r| < \infty$ (indeed $\sum_{r=0}^{\infty} |a_r| \leqslant R$). Thus $1 - \sum_{r=1}^{\infty} a_r\chi_r(e) = 0$ for al $e\in I$ and I is a ZA^+ set, contradicting (ii). The first sentence of this paragrap is thus true.

Now we can find a finite set of points $x_1, x_2, \dots, x_k$ say such that $\inf_{k\geqslant i\geqslant 1} |t-x_i| \leqslant (RN)^{-1}\varkappa(R)$ for all $t\in[-\varepsilon,\varepsilon]$. Automatically

$$(2)' \qquad \sup_{k\geqslant i\geqslant 1} |1 - \sum_{r=1}^{N} a_r\chi_r(x_i)| \geq \varkappa(R) \quad \text{for} \quad \sum_{r=1}^{N} |a_r| \leqslant R$$

whilst choosing

$$0 < \gamma < \min\Big(\min_{k\geqslant i>j\geqslant 1} |x_i-x_j|, \quad \min_{k\geqslant i\geqslant 1} |x_i-\varepsilon|, \quad \min_{k\geqslant i\geqslant 1} |x_i+\varepsilon|, \quad \eta/RN\Big)/4$$

we have, on setting $I_i = [x_i-\gamma, x_j+\gamma]$ that the I_i are disjoint, $I_i \subseteq [-\varepsilon,\varepsilon]$ and

$$(1)' \quad \|(1 - \sum_{r=1}^{N} a_r\chi_r) - (1 - \sum_{r=1}^{N} a_r\chi_r(x_i))\|_{C(I_i)} \leqslant \sum_{r=1}^{N} |a_r| N \sup_{t\in I} |x_i-t| \leqslant \eta.$$

Conditions (1) and (2) follow at once from (1)' and (2)'.

We shall also need the following complicated statement of a trivial fact :

LEMMA 7.10 (iv). Let E_1, E_2 be closed sets and $1 > s \geqslant 0$.

Suppose $\Lambda(p,t)$, $\Lambda(t)$ are infinite subsets of $\underline{\underline{Z}}^+$ with $\Lambda(p,t) \subseteq \Lambda(t)$ $[1\leqslant p\leqslant m, \ 1\leqslant t\leqslant k]$. If

$$\lim_{u\to\infty} \inf_{\sigma\in M^+(E_1)\,\|\sigma\|=1} \sup_{1\leqslant t\leqslant k} \inf_{r\in\Lambda(t),r\geqslant u} \sigma\{x\in E_1 : |\chi_r(x) - 1| \leqslant \alpha\} \geqslant s$$

and

$$\lim_{u\to\infty} \inf_{\sigma\in M^+(E_2)\,\|\sigma\|=1} \sup_{1\leqslant p\leqslant m} \inf_{r\in\Lambda(p,t),r\geqq u} \sigma\{x\in E_2 : |\chi_r(x) - 1| \leqslant \alpha\} \geqslant s$$

for all $\alpha > 0$, then

$$\lim_{u\to\infty} \inf_{\sigma\in M(E_1\cup E_2),\|\sigma\|=1} \sup_{1\leqslant t\leqslant k,\,1\leqslant p\leqslant m} \inf_{r\in\Lambda(p,t),r\geqslant u} \sigma\left\{x\in E_1\cup E_2 : |\chi_r(x)-1|\leqslant\alpha\right\} \geqslant s$$

for all $\alpha > 0$.

Proof. This consists simply in interpreting the statements. Suppose $\varepsilon > 0$ given. We can find a $u_o'(\varepsilon)$ such that

$$\inf_{\sigma_1\in M^+(E_1),\|\sigma_1\|=\beta_1} \sup_{1\leqslant t\leqslant k} \inf_{r\in\Lambda(p,t),r\geqslant u_o} \sigma_1\left\{x\in E_1 : |\chi_r(x) - 1| \leqslant \alpha\right\} \geqslant (s-\varepsilon)\beta_1$$

$$\inf_{\sigma_2\in M^+(E_2),\|\sigma_2\|=\beta_2} \sup_{1\leqslant p\leqslant m} \inf_{r\in\Lambda(p,t),r\geqslant u_o} \sigma_2\left\{x\in E_2 : |\chi_r(x) - 1| \leqslant \alpha\right\} \geqslant (s-\varepsilon)\beta_2$$

so that in particular, if $\sigma\in M^+(E_1\cup E_2)$, then taking $\sigma_1 = \sigma|E_1$, $\sigma_2 = \sigma - \sigma_1$ [i = 1,2] we know that there exists a $1\leqslant p\leqslant m$ and a $1\leqslant t\leqslant k$ such that

$$\inf_{r\in\Lambda(t_1),r\geqslant u_o} \sigma\left\{x\in E_1 : |\chi_r(x) - 1| \leqslant \alpha\right\} \geqslant (s-\varepsilon)|\sigma|(E_1)$$

$$\inf_{r\in\Lambda(t_1,p_1),r\geqslant u_o} \sigma\left\{x\in E_2\setminus E_1 : |\chi_r(x) - 1| \leqslant \alpha\right\} \geqslant (s-\varepsilon)|\sigma|(E_2\setminus E_1).$$

Since $\Lambda(t_1) \supseteq \Lambda(t_1,p_1)$ is infinite, we have at once

$$\inf_{r\in\Lambda(t_1,p_1),r\geqslant u_o} \sigma\left\{x\in E_1\cup E_2 : |\chi_r(x) - 1| \leqslant \alpha\right\} \geqslant (s-\varepsilon)|\sigma|(E_1\cup E_2)$$

and the required answer follows.

We now embark on the proof of Theorem 7.3. Since this is closely related to the proof of Theorem 8.1 and since we present the 2 proofs independently, each could be treated as heuristic for the other. To separate the various stages of the construction, we split it up into parts corresponding to the statement of the following lemma.

LEMMA 7.11 (i). Suppose we are given $1 > s \geqslant 0$, $K > 1$. Then we can find an

$1 \geq \varepsilon(s,K) > 0$ together with an $m(s,K) \geq 1$ with the following property : -

Given $\delta > 0$, we can find an $N(\delta,K,s) \geqslant 1$ with the following property : -

Given $x \in \mathbb{T}$, F a closed set in $\mathbb{T}$ such that $GpF \neq \mathbb{T}$ and $\Lambda(1), \Lambda(2), \ldots$ $\ldots, \Lambda(k)$ infinite subsets of $\mathbb{Z}^+$, we can find $\Lambda(p,t)$ $[1 \leqslant p \leqslant m(s,n)]$ infinite subsets of $\Lambda(t)$ $[1 \leqslant t \leqslant k]$ together with $E \subseteq [x-\delta, x+\delta]$ a closed independent set with $GpE \cap GpF = \{0\}$ such that

(1) $$\lim_{u\to\infty} \inf_{\sigma \in M^+(E), \|\sigma\|=1} \sup_{1 \leqslant p \leqslant m} \inf_{r \in \Lambda(p,t), r \geqslant u} \sigma\{x \in E : |\chi_r(x) - 1| \leqslant \alpha\} \geqslant s$$

for all $\alpha > 0$, $1 \leqslant t \leqslant k$.

(2) $$\inf\{\|\sum_{|r| \geqslant N} a_r \chi_r - 1\|_{C(E)} : \sum_{|r| \geqslant N} |a_r| \leqslant K\} \geqslant \varepsilon.$$

LEMMA 7.11 (ii). Suppose we are given $1 > s \geqslant 0$, $n > 1$, $\zeta > 0$. Then we can find an $\varkappa(s,n,\zeta) \geqslant 0$ with the following property : -

Given Q we can find a $Q'(Q,s,n,\zeta) \geqslant 0$ with the following property : -

Given $x \in \mathbb{T}$, F a closed set in $\mathbb{T}$ such that $GpF \neq \mathbb{T}$ and $\Lambda(1), \Lambda(2), \ldots$ $\ldots, \Lambda(k)$ infinite subsets of $\mathbb{Z}^+$, we can find a $k' \geqslant 1$ and $\Lambda(p,t)$ $[1 \leqslant p \leqslant k']$ infinite subsets of $\Lambda(t)$ $[1 \leqslant t \leqslant k]$ together with $E \subseteq [x-\zeta, x+\zeta]$ an independent closed set with $GpE \cap GpF = \{0\}$ such that

(1) $$\lim_{u\to\infty} \inf_{\sigma \in M^+(E), \|\sigma\|=1} \sup_{1 \leqslant p \leqslant k'} \inf_{r \in \Lambda(p,t), r \geqslant u} \sigma\{x \in E : |\chi_r(x) - 1| \leqslant \alpha\} \geqslant s$$

for all $\alpha > 0$, $1 \leqslant t \leqslant k$

(2) If $\sum_{r \geq 1} |a_r| \leq n$, $\sum_{r=Q}^{Q'-1} |a_r| \leq \varkappa(s,n)$ then $\|1 - \sum_{r=1}^{\infty} a_r \chi_r\|_{C(E)} \geq \varkappa(s,n)$

(2') If $\sum_{|r| \geq Q'} |a_r| \leq n$, then $\|1 - \sum_{|r| \geq Q'} a_r \chi_r\|_{C(E)} \geq \varkappa(s,n)$.

LEMMA 7.11 (iii). Suppose we are given $1 > s \geqslant 0$, $n > 1$, $\zeta > 0$. Then we can find a $\varkappa(n,s,\zeta)$ with the following property : -

Given Q we can find a $Q''(Q,s,n,\zeta)$ with the following property : -

Given $x \in \underline{\underline{T}}$, F a closed set in $\underline{\underline{T}}$ such that $GpF \neq \underline{\underline{T}}$ and $\Lambda(1), \Lambda(2), \ldots$ $\ldots, \Lambda(k)$ infinite subsets of $\underline{\underline{Z}}^+$ we can find a $k'' \geqslant 1$ and $\Lambda(p,t)$ $[1 \leqslant p \leqslant k'']$ infinite subsets of $\Lambda(t)$ $[1 \leqslant t \leqslant k]$ together with $E \subseteq [x-\zeta, x+\zeta]$ an independent closed set with $GpE \cap GpF = \{0\}$ such that

(1) $$\lim_{u \to \infty} \inf_{\sigma \in M^+(E), \|\sigma\|=1} \sup_{1 \leqslant p \leqslant k''} \inf_{r \in \Lambda(p,t), r \geqslant u} \sigma\{x \in E : |\chi_r(x) - 1| \leqslant \alpha\} \geqslant s$$

for all $\alpha > 0$, $1 \leqslant t \leqslant k$

(2) If $\sum_{r \geqslant 1} |a_r| \leqslant n$ then $\|1 - \sum_{r=1}^{\infty} a_r \chi_r\|_{C(E)} \geqslant \varkappa(s,n,\zeta)$

(2)' If $\sum_{|r| \geqslant Q''} |a_r| \leqslant n$ then $\|1 - \sum_{|r| \geqslant Q''} a_r \chi_r\|_{C(E)} \geqslant \varkappa(s,n,\zeta)$.

LEMMA 7.11 (iv). Given any $1 > s \geqslant 0$, we can construct an E satisfying the conclusions of Theorem 7.3.

Proof of (i). This is essentially a restatement of Lemma 5.8. To reduce confusion we label the quantities in the statement of Lemma 5.8 by a superscripted star (so e. g. $C(K,\lambda)$ becomes $C^*(K,\lambda)$).

Set $\varepsilon(s,K) = 1/(2C^*(2K,s))$, $m(s,K) = m^*(2K,s)$ and $N(\delta, K, s) = N^*(\delta, 2K, s)$. Since $\Lambda(1), \Lambda(2), \ldots, \Lambda(k)$ are infinite, we can select the $M_p^*(j)$ so that $\Lambda(p,t) = \{j : M_p^*(j) \in \Lambda(t)\}$ is infinite for all $1 \leqslant p \leqslant m$, $1 \leqslant t \leqslant k$. Taking $E = E^*$, $T = T^*$, we have the required construction ((1) follows from (ii)' and (2) from (iii)').

Proof of (ii). We use Lemma 7.10 (again adopting the convention of starring quantities appearing in the Lemma quoted). Set $\varkappa(s,n,\zeta) = \varkappa^*(\zeta,n)\varepsilon(s,n/\varkappa^*(\zeta,n))$, $w = k^*(\zeta,4n/\varkappa^*(s,n,\zeta),Q)$ and $Q' = N(\gamma^*(\zeta,n,Q),4n/\varkappa^*(s,n,\zeta)s)$ (where N,ε are as in part (i) of this lemma and $\eta = \varkappa(s,n)$). We know that we can find disjoint intervals $I_1, I_2, \ldots, I_w \subseteq [x-\zeta\ ,\ x+\zeta]$ such that $|I_i| \geqslant \gamma^*(\zeta,n,Q)$ and if $\sum_{r=1}^{Q} |a_r| \leqslant n$

(a) $\sup_{w \geqslant i \geqslant n} \inf_{\lambda \in \mathbb{C}} \|(1 - \sum_{r=1}^{Q} a_r \chi_r) - \lambda\|_{C(I_i)} \geqslant \varkappa(s,n,\zeta)$

(b) $\sup_{w \geqslant i \geqslant n} \|1 - \sum_{r=1}^{Q} a_r \chi_r\|_{C(I_i)} \geq 4\varkappa(s,n,\zeta)$.

Using part (1) we can construct sucessively closed independent sets $E_1, E_2, \ldots, E_w$ with $E_i \subseteq I_i$ and infinite subsets $\Lambda'(p_1, p_2, \ldots, p_i, t) \subseteq \Lambda'(p_1, p_2, \ldots, p_{i-1}, t)$ $[1 \leqslant t \leqslant k,\ 1 \leqslant p_j \leqslant m]$ (where $\Lambda'(t) = \Lambda(t)$, $m = m^*(s,4n/\varepsilon(s,n))$) such that

(c_i) $\quad Gp E_i \cap Gp(F \cup \bigcup_{1 \leqslant j < i} E_j) = \{0\}$ so $Gp(F \cup \bigcup_{1 \leqslant j \leqslant i} E_i) \neq \mathbb{T}$

$(1)_i$ $\quad \lim_{u \to \infty} \inf_{\sigma \in M^+(E_i), \|\sigma\|=1} \sup_{1 \leqslant p_i \leqslant m} \inf_{r \in \Lambda'(p_1,p_2,..p_i,t), r \geqslant u} \sigma\{x \in E_i : |\chi_r(x) - 1| \leqslant \alpha\} \geqslant s$

for all $\alpha > 0$, $1 \leqslant p_j \leqslant m$ $\quad [1 \leqslant j \leqslant i-1]$, $1 \leqslant t \leqslant k$.

$(2)_i$ $\quad \inf\left\{\|\sum_{r \geqslant Q'} a_r \chi_r - 1\|_{C(E_i)} : \sum |a_r| \leqslant n/\varkappa(s,n,\zeta)\right\}$

$\geqslant \inf\left\{\|\sum_{r \geqslant Q'} a_r \chi_r - 1\|_{C(E_i)} : \sum |a_r| \leqslant n/\varkappa(s,n,\zeta)\right\} \geq \varepsilon(s,4n/\varkappa(s,n,\zeta))$.

From $(2)_i$ we have at once that if $|\lambda| \geqslant \varkappa(s,n,\zeta)$ then (on multiplying $(2)_i$ through by λ) we have

$(2)_i''$ $\quad \inf\left\{\|\sum_{r \geq Q'} a_r \chi_r - \lambda\|_{C(E_i)} : \sum |a_r| \leqslant n\right\} \geqslant \varkappa(s,n,\zeta)$.

Write $E = \bigcup_{i=1}^{w} E_i$. Combining (a) and $(2)_i''$, and setting

$\lambda_i = \sum_{r=1}^{Q} a_r \chi_r(x_i)$ for some $x_i \in I_i$, we have that for all $\sum_{r=1}^{\infty} |a_r| \leqslant n$ and $\sum_{r=Q}^{Q'} |a_r| \leqslant \varkappa(s,n,\zeta)$ we know $\sup_{n \geqslant i \geqslant 1} |\lambda_i| \geqslant 3\varkappa(s,n,\zeta)$ and so

$$(2) \qquad \|1 - \sum_{r=1}^{\infty} a_r\chi_r\|_{C(E)} \geq \sup_{w \geqslant i \geqslant 1} \|1 - \sum_{r=1}^{\infty} a_r\chi_r\|_{C(E_i)}$$

$$\geq \sup_{w \geqslant i \geqslant 1} \|1 - \sum_{r=1}^{Q} a_r\chi_r - \sum_{r=Q'}^{\infty} a_r\chi_r\|_{C(E_i)} - \sum_{r=Q'}^{Q} |a_r|$$

$$\geq \sup_{w \geqslant i \geqslant 1} (|\lambda_i| - \|\lambda_i - \sum_{r=Q'}^{\infty} a_r\chi_r\|_{C(E_i)}) - \varkappa(s,n,\zeta)$$

$$\geq 3\varkappa(s,n,\zeta) - \varkappa(s,n,\zeta) - \varkappa(s,n,\zeta)$$

$$= \varkappa(s,n,\zeta)$$

as required.

If we set $k' = m^w$, $\Lambda(1+(p_1-1) + (p_2-1)m + \ldots + (p_w-1)m^{w-1}) = \Lambda'(p_1,p_2,\ldots,p_w,t)$ $[1 \leqslant t \leqslant k,\ 1 \leqslant p_i \leqslant m]$ then condition (1) follows from Lemma 7.10 (iv) and $(1)_i$. That E is independent and $\mathrm{Gp}F \cap \mathrm{Gp}E = \{0\}$ follows from $(c)_i$ and the independence of E_i. Condition (2)' follows on considering $(2)_i$ for any fixed i.

Proof of (iii). Choose v an integer with $(v-1)\varkappa(s,n,\zeta) \geq n+1$ (taking $\varkappa$ as in part (ii)). Set $Q(0) = Q$, $Q(r) = Q'(Q(r-1),s,n,\zeta)$ $[1 \leqslant r \leqslant v]$, $Q''(Q,s,n,\zeta) = Q(v)$. Using part (ii) we can construct successively closed independent sets $E_1, E_2, \ldots, E_v$ with $E_i \subseteq [x-\zeta, x+\zeta]$ and infinite subsets $\Lambda'(p_1,p_2,\ldots,p_i,t) \subseteq \Lambda'(p_1,p_2,\ldots,p_{i-1},t)$ $[1 \leqslant t \leqslant k,\ 1 \leqslant p_j \leqslant k_j'$ for some k_j' fixed$]$ (where $\Lambda'(t) = \Lambda(t)$) such that

$$(c)_i \qquad \mathrm{Gp}E_i \cap \mathrm{Gp}(F \cup \bigcup_{1 \leqslant j < 1} E_j) = \{0\} \quad \text{so} \quad \mathrm{Gp}(F \cup \bigcup_{1 \leqslant j < r} E_i) \neq \underline{\underline{T}}$$

$$(1)_i \quad \lim_{u \to \infty} \inf_{\sigma \in M^+(E_i), \|\sigma\|=1} \sup_{1 \leqslant p_i \leqslant k_i'} \inf_{r \in \Lambda'(p_1,p_2,\ldots p_i,t), r \geqslant u} \sigma\{x \in E_i : |\chi_r(x) - 1| \leqslant \alpha\} \geqslant s$$

for all $\alpha > 0$, $1 \leqslant p_j \leqslant k_i'$ $[1 \leqslant j \leqslant i-1]$, $1 \leqslant t \leqslant k$.

$(2)_i \quad \inf\left\{\|\sum_{r=1}^{\infty} a_r\chi_r - 1\|_{C(E_i)} : \sum_{r=1}^{\infty} |a_r| \leqslant n,\ \sum_{r=Q_{i-1}}^{Q_i - 1} |a_r| \leqslant \varkappa(s,n,\zeta)\right\} \geqslant \varkappa(s,n,\zeta).$

Now suppose $\sum_{r=1}^{\infty} |a_r| \leqslant n$. Then $\sum_{r=1}^{v}\sum_{r=Q_{i-1}}^{Q_i-1} |a_r| \leqslant n$ and so we can find $1 \leqslant j \leqslant v$ such that $\sum_{r=Q_{j-1}}^{Q_j-1} |a_r| \leqslant \varkappa(s,n,\zeta)$. But then we may use $(2)_j$ to obtain (on setting $E = \bigcup_{i=1}^{v} E_i$)

$(2) \qquad \inf\left\{\|\sum_{r=1}^{\infty} a_r\chi_r - 1\|_{C(E)} : \sum_{r=1}^{\infty} |a_r| \leqslant n\right\} \geqslant \varkappa(s,n,\zeta).$

If we set $k'' = k_1' \, k_2' \, \ldots \, k_v'$, $\Lambda(1+(p_1-1) + (p_2+1)k_1' + \ldots + (p_v-1)k_1' \, k_2' \ldots k_{v-1}', t) = \Lambda(p_1, p_2, \ldots, p_v t)$ $[1 \leqslant t \leqslant k,\ 1 \leqslant p_i \leqslant m]$ then condition (1) follows from Lemma 7.10 (iv) and $(1)_i$. That E is independent and $\mathrm{Gp}F \cap \mathrm{Gp}E = \{0\}$ follows from $(c)_i$ and the independence of E_i. Condition (2)' follows on considering $(2)_i$ for any fixed i.

Proof of (iv). Choose $x \notin 2\pi\mathbb{Q}$. By Dirichlet's theorem we can find $\Lambda(0) \subseteq \mathbb{Z}^+$ an infinite set such that $\sup_{r\in\Lambda(0), r\geqslant n} |\chi_r(x) - 1| \to 0$ as $n \to \infty$. By part (iii) we can construct inductively $\varkappa(n)$, $\zeta(n) > 0$, $N(n)$, $M(n)$, $Q(n)$, $m(n)$ positive integers, $E(n)$ a closed independent set and $\Lambda(n,1), \Lambda(n,2), \ldots, \Lambda(n,m(n))$ infinite subsets of $\mathbb{Z}^+$ such that writing $F(n) = \bigcup_{r=0}^{n} E(r)$ we have

$(a)_n \qquad E(n) \subseteq [x - \zeta(n-1), x + \zeta(n-1)]$

$(b)_n \qquad \mathrm{Gp}E(n) \cap \mathrm{Gp}F(n-1) = \{0\}$

$(c)_n \quad \lim_{u\to\infty} \inf_{\sigma\in M^+(F(n)), \|\sigma\|=1} \sup_{1\leqslant p\leqslant m(n)} \inf_{r\in\Lambda(n,p), r\geqslant u} \sigma\{y\in F(n): |\chi_r(y)-1| \leqslant \alpha\} \geqslant s$

for all $\alpha > 0$.

(We obtain $(c)_n$ from $(c)_{n-1}$, condition 1 of part (iii) and Lemma 7.10 (iv)).

$(d)_n$ If $\sum_{r=1} |a_r| \leqslant n$ then $\|1 - \sum_{r\geqslant 1} a_r \chi_r\|_{C(E(n))} \geqslant \varkappa(n)$

$(d)'_n$ If $\sum_{|r|\geqslant Q(n)} |a_r| \leqslant n$ then $\|1 - \sum_{|r|\geqslant Q(n)} a_r \chi_r\|_{C(E(n))} \geqslant \varkappa(n)$.

The conditions which follow define $M(n)$, $N(n)$ and $\zeta(n)$. By $(c)_n$ and the fact that the $\Lambda(w\ , v)$ are infinite, we can find $2^n \leqslant M(n) < N(n)$ such that

$(e)_n$ $\inf_{\sigma\in M^+(F(n)),\|\sigma\|=1} \sup_{1\leqslant p\leqslant m(n)} \inf_{r\in\Lambda(n,p),r\geqslant u} \sigma\{y\in F(n): |\chi_r(y)-1| \leqslant 2^{-n-1}\} \geqslant s.$

We choose $\zeta(n-1)/2 \geqslant \zeta(n) > 0$ such that

$(f)_n$ $\zeta(n)N(n) \leqslant 2^{-n-5}$

so that, in particular, if $F \subseteq F(n) + [\zeta(n), -\zeta(n)]$, then

$(e)'_n$ $\inf_{\sigma\in M^+(F),\|\sigma\|=1} \sup_{N(n)\geqslant r\geqslant M(n)} \sigma\{y\in F : |\chi_r(y) - 1| < 2^{-n}\} \geqslant s.$

We claim that $E = \bigcup_{n=0}^{\infty} E(n)$ satisfies the conditions of Theorem 7.3 with $\varepsilon(R,s) = \varkappa([R]+1)$. Observe first that by condition $(a)_n$ and the restriction $\zeta(n) \leqslant \zeta(n-1)/2$ we have E closed, whilst by $(b)_n$ (and the restriction $E(n)$ independent) we have E independent. That $\limsup_{m\to\infty} \sigma\{y : |\chi_m(y)-1| \leqslant \delta\} \geqslant s\|\sigma\|$ for all $\delta > 0$, $\sigma\in M^+(E)$ follows directly from $(e)'_n$. Finally, using $(d)_n$ and $(d)'_n$, we have for all $\sum_{r=-\infty}^{\infty} |a_r| \leqslant R \leqslant [R] + 1$ that

$$\|\sum_{r=1}^{\infty} a_r \chi_r - 1\|_{C(E)} \geq \|\sum_{r=1}^{\infty} a_r \chi_r - 1\|_{C(E([R]+1))} \geq \varkappa([R]+1) > 0$$

and, similarly,

$$\|\sum_{|r|\geqslant Q([R]+1)} a_r \chi_r - 1\|_{C(E)} \geq \|\sum_{|r|\geqslant Q([R]+1)} a_r \chi_r - 1\|_{C(E([R]+1))} \geq \varkappa([R]+1) > 0$$

as required.

The proof of Lemma 7.5 follows the pattern of the proofs above very closely. As usual, we split the demonstration into several parts.

LEMMA 7.12. (i) Suppose $1>\varepsilon>0$ and N a positive integer given. Then we can find a $\rho(\varepsilon,N)>1$ such that for any $x\in\mathbb{T}$ we have $\|\sum_{r=-N}^{N} a_r\chi_r\|_{C([x-\varepsilon,x+\varepsilon])}>2$ whenever $\sum_{r=-N}^{N}|a_r|\geqslant\rho(\varepsilon,N)$.

(ii) Suppose $1>\varepsilon>0$ and N a positive integer given. Then we can find a $1>\kappa(\varepsilon,N)>0$ such that for any $x\in\mathbb{T}$ we have $\|1-\sum_{1\leqslant|r|\leqslant N} a_r\chi_r\|_{C([x-\varepsilon,x+\varepsilon])}>\kappa(\varepsilon,N)$ for all $a_r\in\mathbb{C}$ $[1\leqslant|r|\leqslant N]$.

(iii) Suppose $1>\varepsilon>0$, $\eta>0$, $\rho>1$ and N a positive integer given. Then there exist $\gamma(\varepsilon,\eta,\rho,N)>0$ and $k(\varepsilon,\eta,\rho,N)\in\mathbb{Z}^+$ such that, given $x\in\mathbb{T}$, we can find closed disjoint intervals $I_1, I_2, \dots, I_k\subseteq[x-\varepsilon, x+\varepsilon]=I$ with $|I_i|\geqslant\gamma$ such that for all $\rho=\sum_{r=-N}^{N}|a_r|$

(1) $$\sup_{k\geqslant i\geqslant 1}\ \inf_{\lambda\in\mathbb{C}}\ \|\sum_{r=-N}^{N}a_r\chi_r-\lambda\|_{C(I_i)}\leqslant\eta$$

(2) $$\sup_{k\geqslant i\geqslant 1}\|\sum_{r=-N}^{N}a_r\chi_r\|_{C(I_i)}\geq\|\sum_{r=-N}^{N}a_r\chi_r\|_{C(I)}-\eta.$$

(iv) Suppose we are given $1>s>0$, $\kappa>0$ and $w\in\mathbb{Z}^+$. Then we can find an $\kappa\geqslant\eta(s,K,w)>0$ and an integer $m(s,\kappa,w)$ with the following property : -

Given $\gamma>0$, $x\in\mathbb{T}$, F a closed set in $\mathbb{T}$ such that $\mathrm{Gp}F\neq\mathbb{T}$ and $\Lambda(1), \Lambda(2), \dots, \Lambda(q')$ infinite subsets of $\mathbb{Z}^+$, we can find $\Lambda(p,t)$ $[1\leqslant p\leqslant m(s,K,w)]$ infinite subsets of $\Lambda(t)$ $[1\leqslant t\leqslant q']$ together with $E\subseteq[x-\gamma, x+\gamma]$ an independent closed set with $\mathrm{Gp}E\cap\mathrm{Gp}F=\{0\}$ such that, writing $\Lambda^*(w)=\{\sum_{i=1}^{w}n_i+v : n_i\in\Lambda(p,t)$ for some $1\leqslant p\leqslant m$, $1\leqslant t\leqslant q'$, $1\leqslant v\leqslant w\}$,

(1) $$\lim_{u\to\infty}\ \inf_{\sigma\in M^+(E),\|\sigma\|=1}\ \sup_{1\leqslant p\leqslant m(s,K,w)}\ \inf_{r\in\Lambda(p,t),r\geqslant u}\ \sigma\{x\in E: |\chi_r(x)-1|\leqslant\alpha\}\geq s$$

for all $\alpha>0$, $1\leqslant t\leqslant q'$

(2) $\quad \|\sum_{r\in\Lambda^*(w)} a_r\chi_r - \varkappa\| \geq \eta(s,\varkappa,w)$ for all $\sum_{r\in\Lambda^*(w)} |a_r| < \infty$.

(v) Suppose we are given $1 > \zeta > 0$, n and Q positive integers. Then we can find a $\varkappa_o(\zeta,n,Q) > 0$ with the following property :-

Given $x\in\underline{\underline{T}}$, F a closed set in $\underline{\underline{T}}$ such that $\mathrm{Gp}F \neq \underline{\underline{T}}$ and $\Lambda(1), \Lambda(2),\ldots$ $\ldots, \Lambda(q)$ infinite subsets of $\underline{\underline{Z}}^+$, we can find a $q'' \geqslant 1$ and $\Lambda(p,t)$ $[1\leqslant p\leqslant q'']$ infinite subsets of $\Lambda(t)$ $[1\leqslant t\leqslant q]$ together with $E \subseteq [x-\zeta, x+\zeta]$ an independent closed set such that

(1) $$\lim_{u\to\infty} \inf_{\sigma\in M^+(E),\|\sigma\|=1} \sup_{1\leqslant p\leqslant q''} \inf_{r\in\Lambda(p,t),r\geqslant u} \sigma\{x\in E : |\chi_r(x)-1| \leqslant \alpha\} \geq s$$

for all $\alpha > 0$, $1\leqslant t\leqslant q$

(2) $$\|\sum_{|r|\in\Lambda^{**}(n)} a_r\chi_r - 1\|_{C(E)} \geq \varkappa_o(\zeta,n,Q)$$

where $\Lambda^{**}(n) = \{\sum_{i=1}^{n} \ell_i + j : \ell_i\in\Lambda(p,t)$ for some $1\leqslant p\leqslant q''$, $1\leqslant t\leqslant q$ and $1\leqslant v\leqslant n\}$.

(vi) Given any $1 > s \geqslant 0$ we can construct an E satisfying the conclusions of Lemma 7.5.

Proof. We remark first that, by translation, we can and will take $x=0$ in the proof of (i), (ii), (iii) and (iv).

Proof of (i). Consider $\Gamma = \{\sum_{r=-N}^{N} a_r\chi_r|I : \sum_{r=-N}^{N} |a_r| = 1\}$ where $I = [-\varepsilon,\varepsilon]$. This is a compact subset of $(C(I), \|\ \|_{C(I)})$ (because, for example, Γ is a bounded subset of a finite dimensional subspace of $C(I)$.) Further $0\notin\Gamma$ (because, for example, the elements of Γ can take the value 0 at most $2N+1$ points). Thus $\inf\{\|f\|_{C(I)} : f\in\Gamma\} \geq \delta > 0$ for some $\delta > 0$ and, setting $\rho(\varepsilon,R) = 4\delta^{-1}$, we have the required result.

Proof of (ii). Set $\Gamma_1 \{(\sum_{1\leqslant |r|\leqslant N} a_r\chi_r - 1)|I : \sum_{1\leqslant |r|\leqslant N} |a_r| \leqslant \rho(\varepsilon,N)\}$ where $I = [-\varepsilon,\varepsilon]$. As above, Γ_1 is compact and $0 \notin \Gamma$, so there exists a $1 > \varkappa(\varepsilon,N) > 0$ with $\{\|f\|_{C(I)} : f \in \Gamma_1\} \geq \varkappa(\varepsilon,N)$. Since, if $f = (\sum_{1\leqslant |r|\leqslant N} a_r\chi_r - 1)|E$ with $\sum_{1\leqslant |r|\leqslant N} |a_r| \geqslant \rho(\varepsilon,N)$. Part (i) gives $\|f\|_{C(I)} \geqslant 1 > K(\varepsilon,N)$, we have the required result.

Proof of (iii). If $\rho \geq \sum_{r=-N}^{N} |a_r|$, then

$$|\sum_{r=-N}^{N} a_r\chi_r(y) - \sum_{r=-N}^{N} a_r\chi_r(z)| \leqslant \sum_{r=-N}^{N} |a_r|\,|r|\,|y-z| \leqslant \rho N|y-z| \leqslant \eta/8$$

whenever $|y-z| \leqslant 1/(8\rho N)$. Set $Q = [16\pi\rho N] + 4$, then, taking $\gamma = \pi/2Q$ and I_i to be those intervals $[2\pi r/Q - \pi/4Q\ ,\ 2\pi r/Q + \pi/4Q]$ which lie entirely within $I = [-\varepsilon\ ,\ \varepsilon]$ we have the required result.

Remark. Parts (i), (ii), (iii) taken together correspond to Lemma 7.9 in the proof of Theorem 7.2 and Lemma 7.10 (iii) in the proof of Theorem 7.3. To prove part (iv) which corresponds to the main part of the proof of Theorem 7.2 and to the main part of the proof of Lemma 7.11 (i) (itself the main step in the proof of Theorem 7.3) we require calculations along the lines of Section § 3. Note, however, that the case $r = 1$ can be handled directly in the manner of Theorem 7.2 which indeed gives the required result with $s = 1/2$, $\Lambda = \Lambda(1) \cup \Lambda(2)$. It is very possible that the reader will be satisfied by re-reading the proof of Theorem 7.2 and ignoring the details that follow. We adopt the notation of Section § 3 until further notice.

LEMMA 7.13. (i) Let m,q,w be integers such that $m \geqslant q+w-1$ and $q \geqslant w-1$. Then the system of equations

$$\sum_{S\supseteq T\neq\emptyset} A_T = 1 \qquad [S \in \Phi(m,q)]$$
$$A_T = 0 \qquad \text{for} \quad \text{card } T \geqslant w$$

has no solution.

Proof. Call the system of equations (*). By Lemma 3.5 the general solution of (*) is given by

$$A_T = (-1)^{\text{card } T-1}(1 - \sum_{\emptyset\neq U\subseteq T} B_U) \quad [\emptyset \neq T \subseteq \{1,2,\ldots,m\}]$$

where $B_U = 0$ if $\text{card } U > q$, but otherwise may be chosen freely subject to

$$A_T = 0 \qquad \text{for} \quad \text{card } T \geq w,$$

i. e. subject to

$$0 = 1 + \sum_{\emptyset\neq U\subseteq T} B_U \qquad \text{for} \quad \text{card } T \geq w.$$

(Here and in what follows the condition $T,U,V \subseteq \{1,2,\ldots,m\}$ and so on is implied).

Call the 2 conditions on the B_U (**). Then, writing Ξ for the permutation group on $\{1,2,\ldots,m\}$, we know that, if the B_U satisfy (**), then so do the B_U for all fixed $\sigma\in\Xi$ (here $\sigma U = \{\sigma u : u\in U\}$). Thus, by the linearity of the conditions (**) it follows that $C_U = \sum_{\text{card } V=\text{card } U} B_V/\text{card}\{V : \text{card } V = \text{card } U\} = \lambda_{\text{card } U}$ say also satisfies (**). We have thus shown that (*) has a solution if and only if

$$0 = \lambda_p \qquad \text{for} \quad m \geqslant p > q$$
$$0 = 1 + \sum_{t=1}^{s} \binom{s}{t}\lambda_t \qquad \text{for} \quad m \geqslant s \geqslant w$$

has a solution. In other words, (*) has a solution if and only if

$$0 = 1 + \sum_{t=1}^{q} \binom{s}{t}\lambda_t \qquad [m \geqslant s \geqslant w]$$

has a solution. But writing $K_{u,v}$ for the $u\times v$ matrix with $(s,t)^{th}$ element $\binom{u+v-s}{t}$ $[0\leqslant s,\ t\leqslant v]$, we have (on subtracting the i^{th} row from the $i+1^{th}$ row for

$i = v-1, v-2, \ldots, 1$ and using Pascal's triangle) that $\det K_{u,v} = \det K_{u,v-1}$ (if the reader writes out the case $v = 4$, $u = 1$ say, the result will be obvious), and so by induction $\det K_{u,v} = \det K_{u,1} = 1$. Thus $\det K_{m-q-1,q+1} = 1 \neq 0$ and the set of equations

$$0 = \sum_{t=1}^{q} \binom{s}{t} \lambda_t \qquad [m \geqslant s \geqslant m-q]$$

has a unique solution which is, obviously, $\lambda_1 = \lambda_2 = \ldots = \lambda_q = 0$. Thus the set of equations

$$0 = 1 + \sum_{t=1}^{q} \binom{s}{t} \lambda_t \qquad [m \geqslant s \geqslant w]$$

has no solutions, and so, retracing the argument,(*) cannot hold.

LEMMA 7.13. (ii) If m, q, w are given as in (i), then there exists a $\tau(m,q,w) > 0$ such that, if $|\varepsilon_{S,T}-1|$, $|\delta_{S,T}| \leqslant \gamma$ for all $\emptyset \neq S,T \subseteq \{1,2,\ldots,m\}$, then the system of inequations

$$\Big|\sum_{S \supseteq T \neq \emptyset} \varepsilon_{S,T} A_T + \sum_{S \not\supseteq T \neq \emptyset} \delta_{S,T} A_T - 1\Big| \leqslant \tau \quad [S \in \Phi(m,q)]$$

$$A_T = 0 \quad \text{for} \quad \text{card}\, T \geq w$$

has no solutions.

(iii) If m, q, w are given as in (i), then there exists a $\tau(m,q,w) > 0$ such that, given a collection of functions $f_{S,k} : \mathcal{P}\{1,2,\ldots,m\} \to \underline{\underline{C}}$ (from the set of subsets of $\{1,2,\ldots,m\}$ to $\underline{\underline{C}}$) with $|\lambda_{S,k} f_{S,k} - 1| \leqslant \tau$ if $S \supseteq T \neq \emptyset$, $|\lambda_{S,k} f_{S,k}(T)| \leqslant \tau$ otherwise for some $\lambda_{S,k} \in \underline{\underline{C}}$, then

$$\sup_{s \in \Phi(m,q), \text{card} T \leqslant w} \Big|\sum A_{T,k} f_{S,k}(T) - 1\Big| \geqslant \tau$$

for all choices of $A_{T,k} \in \underline{\underline{C}}$.

Proof of (ii). We use the fact that if a finite set of linear equations has no solution, then neither does any (sufficiently mildly) perturbed set. (This is obvious geometrically : - if an affine subspace of $\underline{\underline{R}}^n$ (i. e. a "plane") does not contain a point, then this situation persists under mild perturbation.) Alternatively, the algebraic proof of (i) will also show (ii).

Proof of (iii). This is a restatement of (ii).

Proof of (iv). If (iv) is true for $\varkappa = 1$, then a scaling argument shows that it is true for all $\varkappa > O$ (with $\eta(s,\varkappa,w) = \varkappa\eta(s,1,w)$, $m(s,\varkappa,w) = m(s,1,w)$ and $E,\Lambda(p,t)$ the same for all values of $\varkappa$). We therefore set $\varkappa = 1$, $m = m(s,1,w)$ and prove the result for this case.

Choose m, q in such a way that $q/m \geqslant s$, $m \geqslant q+w-1$ and $q \geqslant w-1$ (this is certainly true if we put $q = m-w$ and choose m sufficiently large). Take $\gamma \geq \gamma' \geq 0$ such that $n\gamma' \leqslant \tau 2^{-20(m+8)}$ (where $\tau = \tau(m(s,1,w),q,w)$ with the notation of Lemma 7.13).

By Lemma 5.5 we can find integers $N \geq n$, $M_p(j) \geq N$ $[1 \leqslant p \leqslant m,\ 0 \leqslant j]$ closed sets $E \subseteq [x-\gamma', x+\gamma']$ and measures $\mu_S \in M^+(E_S)$ with $\|\mu_S\| = 1$ $[S \subseteq \{1,2,\ldots,m\}]$ such that

(i) $|\chi_{M_p(j)}(e)-1| \leqslant \tau 2^{-20(m+8)}$

(ii) $|\hat{\mu}_S(r)| \geq \tau 2^{-20(m+8)}$ implies $|\hat{\mu}_S(r) - \hat{\mu}_T(r)| \leqslant \tau 2^{-20(m+8)}$

(iii) $|\hat{\mu}_{S\cap T}(r)| \geqslant \max(|\hat{\mu}_S(r), \hat{\mu}_T(r)|) - \tau 2^{-20(m(s,w)+8)}$

for all $S,T \subseteq \{1,2,\ldots,m\}$, $|r| \geq N$

(iv) $|\hat{\mu}_\emptyset(r)| \leqslant \tau 2^{-20(m(s,w)+8)}$ for all $|r| \geq N$

(v) Writing $E^* = \bigcup_{S \subseteq \{1,2,\ldots,m\}} E_S$ we have E^* independent and

$GpE^{*} \cap GpF = 0$

(vi) For each $1 \leqslant p \leqslant m$, $1 \leqslant t \leqslant q'$ we have $\Lambda(p,t) = \{M_p(j) \in \Lambda(t) : j \geq 1\}$ infinite.

The essential content of (i), (ii), (iii), (iv) and the condition $\gamma' \leqslant \tau 2^{-20(m+8)}$ from our point of view is

(vii) $\left| e^{\pm ixu} \hat{\mu}_S(\pm(M_{p(1)}(j) + M_{p(2)}(j) + \ldots + M_{p(u)}(j)) + v) - \varepsilon_{S,\{p(1),p(2),\ldots,p(w)\}} \right| \leq \tau$

where $1 \leqslant p(1) \leqslant p(2) \leqslant \ldots \leqslant p(u) \leqslant m$, $1 \leqslant v \leqslant w$, $\varepsilon_{S,T} = 1$ if $S \supseteq T$, $\varepsilon_{S,T} = 0$ otherwise.

Now writing $E = \cup\{E_S : S \in \Phi(q,m)\}$ (so that all the conditions of Lemma 7.12 (iv) - except perhaps (2) - are satisfied, for example (1) follows from condition (i) on the list in the paragraph above), we have for all $S \in \Phi(m,q)$

$$\int \Big(\sum_{r \in \Lambda^*(w)} a_r \chi_r - 1\Big) d\mu_S = \sum_{r \in \Lambda^*(w)} a_r \hat{\mu}_S(r) - 1$$

$$= \sum_{\operatorname{card} T \geqslant w} \sum_{v=-n} A_{T,v}\, \delta_{S,T}$$

where $A_{T,\mp v}\, e^{\pm e\pi ivx}$ is the sum of all those a_r with $r \in \mp \sum_{p \in T} \bigcup_{t=1}^{q} \Lambda(p,t) + v$ and $|\delta_{S,T} - \varepsilon_{S,T}| \leqslant \tau$. By Lemma 7.13 (iii) we have, therefore

$$\sup_{S \in \Phi(m,q)} \left| \int \Big(\sum_{r \in \Lambda^*(w)} a_r \chi_r - 1\Big) d\mu_S \right| \geqslant \tau$$

and so (since $\|\mu_S\| = 1$)

$$\Big\| \sum_{r \in \Lambda^*(w)} a_r \chi_r - 1 \Big\|_{C(E)} \geq \tau$$

for all $\sum |a_r| < \infty$. Setting $\eta(s,w) = \tau$ we have the result.

Remark. From now on the proof is plain sailing. Parts (v) and (vi) correspond almost exactly to parts (iii) and (iv) of Lemma 7.11 and to the remainder of the proof of Theorem 7.2.

Proof of (v). Using the notation of parts (i), (ii), (iii), (v), set $\varkappa_o = \varkappa_o(\zeta, n, Q) = \eta(s, \varkappa(\zeta, Q)/8, n)/8$ and put $k = k(\zeta, \varkappa_o/4, \rho(\zeta, Q), n)$. By (iii) we can find disjoint intervals $I_1, I_2, \ldots, I_k \subseteq [x-\zeta, x+\zeta]$ such that for all $\rho \geq \sum_{r=-Q}^{Q} |a_r|$

$$(1)^* \quad \sup_{k \geqslant i \geqslant 1} \inf_{\lambda \in \underline{\underline{C}}} \|\sum_{r=-Q}^{Q} a_r \chi_r - \lambda\|_{C(I_i)} \leqslant \varkappa(\zeta, Q)/8$$

$$(2)^* \quad \sup_{k \geqslant i \geqslant 1} \|\sum_{r=-Q}^{Q} a_r \chi_r\|_{C(I_i)} \geq \|\sum_{r=-Q}^{Q} a_r \chi_r\|_{C([x-\zeta, x+\zeta])} - \varkappa(\zeta, Q)/8.$$

Using (iv) we can construct, successively, closed independent sets $E_1, E_2, \ldots, E_k$ with $E_i \subseteq I_i$ and infinite subsets $\Lambda'(p_1, p_2, \ldots, p_i, t) \subseteq \Lambda'(p_1, p_2, \ldots, p_{i-1}, t)$ $[1 \leqslant t \leqslant k,\ 1 \leqslant p_j \leqslant m]$ (where $\Lambda'(t) = \Lambda(t)$, $m = m(s, \varkappa(\zeta, Q)/8, n)$) such that

$$(3)_i \quad GpE_i \cap Gp(F \cup \bigcup_{1 \leqslant j < i} E_j) = \{0\} \qquad (\text{so } Gp(F \cup \bigcup_{i \leqslant j \leqslant i} E_i) \neq \underline{\underline{T}})$$

$$(4)_i \quad \lim_{u \to 0} \inf_{\sigma \in M^+(E_i), \|\sigma\|=1} \sup_{1 \leqslant p_i \leqslant m} \inf_{r \in \Lambda'(p_1, p_2, \ldots, p_i, t), r \geqslant u} \sigma\{x \in E_i : |\chi_r(x) - 1| \leqslant \alpha\} \geq s$$

for all $\alpha > 0$, $1 \leqslant p_j \leqslant m$ $[1 \leqslant j \leqslant i-1]$, $1 \leqslant t \leqslant k$.

$$(5)_i \quad \|\sum_{|r| \in \Lambda^*(n,i)} a_r \chi_r - \varkappa(\zeta, Q)/8\|_{C(E_i)} \geq 8\varkappa_o \quad \text{for all } \sum_{r \in \Lambda^*(n,i)} |a_r| < \infty$$

where $\Lambda^*(n,i) = \{\sum_{j=1}^{n} \ell_j + u : \ell_j \in \bigcup \Lambda(p_1, p_2, \ldots, p_i, t),\ 1 \leqslant u \leqslant Q\}$.

We are particularly interested in the consequences of $(5)_i$.

Set $E = \bigcup_{i=1}^{k} E_i$, $q'' = m^k$, $\Lambda(1 + (p_1 - 1) + (p_2 - 1)m + \ldots + (p_k - 1)m^k) = \Lambda'(p_1, p_2, \ldots, p_k, t)$ $[1 \leqslant t \leqslant k,\ 1 \leqslant p_i \leqslant m]$ (so that condition (1) of part (v) follows at once by repeated use of Lemma 7.10 (iv) and $(4)_i$). Suppose $\sum_{|r| \in \Lambda^{**}(n)} |a_r| < \infty$.

If $\sum_{1 \leqslant |r| \leqslant Q} |a_r| \geq \rho$, then, setting $\theta = \rho / \sum_{1 \leqslant r \leqslant n} |a_r|$ (so that $\sum_{i \leqslant |r| \leqslant Q} |\theta a_r| = \rho$), we have that, since $\theta \leq 1$

$$\|\theta(\sum_{|r|\in\Lambda^{**}(n)} a_r\chi_r - 1)\|_{C(E)} = |\theta|\,\|\sum_{|r|\in\Lambda^{**}(n)} a_r\chi_r - 1\|_{C(E)}$$

$$\leqslant \|\sum_{|r|\in\Lambda^{**}(n)} a_r\chi_r - 1\|_{C(E)}$$

and since $\theta(\sum_{1\leqslant |r|\leqslant Q} |a_r| + 1) \geq \rho$

$$\|\theta(\sum_{|r|\in\Lambda^{**}(n)} a_r\chi_r - 1)\|_{C([x-\zeta,x+\zeta])} \geq \varkappa(\zeta,Q).$$

On the other hand, if $\sum_{1\leqslant |r|\leqslant N} |a_r| \leqslant \rho$, then automatically

$$\|\sum_{|r|\in\Lambda^{**}(n)} a_r\chi_r - 1\|_{C([x-\zeta,x+\zeta])} \geq K(\zeta,Q).$$

Thus, if we can prove that $\sum_{r=-n}^{n} |b_r| \leqslant \rho$, $\|\sum_{r=-n}^{n} b_r\chi_r\|_{C([x-\zeta,x+\zeta])} \geq \varkappa(\zeta,Q)$

and $\sum_{|r|\in\Lambda^*(n)} |b_r| < \infty$ (where $\Lambda^*(n) = \Lambda^{**}(n) \setminus \{r : 1 \leqslant r \leqslant Q\}$) together imply

$\|\sum_{|r|\in\Lambda^{**}(n)\cup\{0\}} b_r\chi_r\|_{C(E)} \geq \varkappa_o$, we shall have proved condition (2) of (v).

Let us therefore suppose that b_r are given in accordance with the hypotheses of the last sentence. By $(1)^*$ and $(2)^*$ we can find an $1 \leqslant i \leqslant k$ and a $\lambda \in \underline{\underline{C}}$ such that $\lambda \geq \varkappa(\zeta,Q) - 2\varkappa(\zeta,Q)/8 = 3K(\zeta,Q)/4$ and $\|\sum_{r=-Q}^{Q} b_r\chi_r - \lambda\|_{C(I_i)} \leqslant \varkappa_o$. Since $\Lambda^*(w) \subseteq \Lambda^*(n,i)$, $(5)_i$ gives

$$(5)_i' \quad \|\sum_{|r|\in\Lambda^*(n)} b_r\chi_r - \lambda\|_{C(E_i)} \geq 8\varkappa_o\lambda(\varkappa(\zeta,Q)/8)^{-1} \geq 32\varkappa_o,$$

so

$$(5)_i'' \quad \|\sum_{|r|\in\Lambda^{**}(n)\cup\{0\}} b_r\chi_r\|_{C(E_i)} \geq \|\sum_{|r|\in\Lambda^*(n)} b_r\chi_r - \lambda\|_{C(E_i)} - \|\sum_{r=-Q}^{Q} b_r\chi_r - \lambda\|_{C(E_i)}$$

$$\geq 32\varkappa_o - \varkappa_o \geq \varkappa_o$$

and we have proved condition (2) of (v). Since the fact that E is independent and $GpF \cap GpE = \{0\}$ are immediate consequences of $(3)_i$ and the independence of the E_i, we have completed the proof of (v).

Proof of (vi). This follows the proof of Lemma 7.1 (iv) almost word for word. Choose $x \notin 2\pi\mathbb{Q}$ and write $E(0) = \{x\}$, $K(0) = \zeta(0) = 1$. By Dirichlet's theorem we can find $\Lambda(1,0) \subseteq \mathbb{Z}^+$ an infinite set such that $\sup_{r\in\Lambda(1,0)} |\chi_r(x) - 1| \to 0$ as $n \to \infty$. By part (v) we can construct inductively $\kappa(n)$, $\zeta(n) > 0$, $N(n)$, $M(n)$, $m(n)$ positive integers, $E(n)$ a closed independent set and $\Lambda(n,1)$, $\Lambda(n,2)$, ..., $\Lambda(n,m(n))$ infinite subsets of $\mathbb{Z}^+$ such that, writing $F(n) = \bigcup_{r=0}^{n} E(r)$, we have

$(a)_n$ $E(n) \subseteq [x-\zeta(n-1), x+\zeta(n-1)]$

$(b)_n$ $\mathrm{Gp}E(n) \cap \mathrm{Gp}F(n-1) = \{0\}$

$(c)_n$ $\lim_{u\to\infty} \inf_{\sigma\in M^+(F(n)),\|\sigma\|=1} \sup_{1\leq p\leq m(n)} \inf_{r\in\Lambda(n,p),r\geq u} \sigma\{y\in F(n) : |\chi_r(y)-1| \leq \alpha\} \geq s$

for all $\alpha > 0$.

(We obtain $(c)_n$ from $(c)_{n-1}$, condition (1) from part (v) and Lemma 7.10 (iv)).

$(d)_n$ Writing $\Lambda^{**}(n) = \{\sum_{i=1}^{n} \ell_i + u : \ell_i \in \Lambda(p,n)$ for some $1 \leq p \leq m(n)$, $1 \leq u \leq N(n-1)\}$ we have $\|1 - \sum_{|r|\in\Lambda^{**}(n)} a_r\chi_r\|_{C(E(n))} \geq \kappa(n)$ for all $\sum_{|r|\in\Lambda^{**}(n)} |a_r| < \infty$.

The conditions that follow define $M(n)$, $N(n)$, $\zeta(n)$. By $(c)_n$ and the fact that the $\Lambda(n,r)$ are infinite, we can find $N(n) + 2^n \leq M(n) < N(n)$ such that

$(e)_n$ $\inf_{\sigma\in M^+(F(n)),\|\sigma\|=1} \sup_{1\leq p\leq m(n)} \inf_{r\in\Lambda(n,p)} \sigma\{y\in F(n) : |\chi_r(y)-1| < 2^{-n-1}\} \geq s.$

We choose $\zeta(n-1)/2 \geq \zeta(n) > 0$ such that

$(f)_n$ $\zeta(n)\,N(n) \leq 2^{-n-5}$

so that, in particular, if $F \subseteq F(n) + [\zeta(n), -\zeta(n)]$, then

$(e)'_n$ $\inf_{\sigma\in M^+(F),\|\sigma\|=1} \sup_{N(n)\geq r\geq M(n)} \sigma\{y\in F : |\chi_r(y) - 1| < 2^{-n}\} \geq s$.

We claim that $E = \bigcup_{n=0}^{\infty} E(n)$ satisfies the conditions of Lemma 7.5, if we put

$\Lambda = \bigcup_{n=1}^{\infty} \bigcup \{N(n) \geq r \geq M(n) : r \in \Lambda(p,n) \text{ for some } 1 \leqslant p \leqslant m(n)\}$, $\varepsilon(n,s) = \varkappa(n)$. Observe first that by condition $(a)_n$ and the restriction $\zeta(n) \leqslant \zeta(n-1)/2$ we have E closed, whilst by $(b)_n$ (and the restriction $E(n)$ independent) we have E independent. That $\limsup_{m \to \infty} \sigma\{y : |\chi_m(y) - 1| \leqslant \delta\} \geq s\|\sigma\|$ for $\delta > 0$, $\sigma \in M^+(E)$ follows directly from $(e)_n'$. Finally, using $(d)_n$ and the observation that (in the notation of Lemma 7.5) $\Lambda(n) \subseteq \Lambda^{**}(n)$, we have

$$\|1 - \sum_{|r| \in \Lambda(n)} a_r \chi_r\|_{C(E)} \geq \|1 - \sum_{|r| \in \Lambda(n)} a_r \chi_r\|_{C(E(n))} \geq \varkappa(n) = \varepsilon(n,s) > 0$$

for all $\sum_{|r| \in \Lambda^{**}(n)} |a_r| < \infty$ as required.

This completes the section. The author hopes that, although the results in it are separately not immensely interesting, together they will be found to give rather exact information about how the structure of $A_\Lambda(E)$ can depend on Λ and in particular about the relations between the cases $\Lambda = \mathbb{Z}$, $\Lambda = \mathbb{Z}^+$ and Λ arbitrary.

§ 8. N SETS AND ZERO SETS.

The object of this section is to prove the existence of weak Dirichlet sets (i.e. N sets) which are not zero sets for $A^+(\mathbb{T})$. We shall prove two versions of the result, a strong one : -

THEOREM 8.1. There exists an independent weak Dirichlet set which is not ZA^+,

and a weak one : -

THEOREM 8.1'. There exists a weak Dirichlet set which is not ZA^+.

Our proof of Theorem 8.1' is technically much neater than that of Theorem 8.1, and depends only on the first 3 sections (indeed almost entirely on those parts of the first 3 sections needed to prove Lemma 1.11) and the very simple remarks placed under the heading of Lemma 7.10 (i), (ii), (iii). The version of Lemma 7.10 we shall need is

LEMMA 8.2. (i) For all $\varepsilon > 0$, $R > 0$ there exists a $1 \geq \varkappa_o(\varepsilon,R) > 0$ such that

$$\|\sum_{r=1}^{\infty} a_r \chi_r - 1\|_{C([-\varepsilon,\varepsilon])} \geq \varkappa_o(\varepsilon,R)$$

for all $\sum_{r=1}^{\infty} |a_r| \leqslant R$.

Proof. This can be proved as a direct consequence of Lemma 7.10 (iii) or, more neatly, by repeating and simplifying the argument used there.

We shall, therefore, prove Theorem 8.1' first. Then we shall briefly discuss the relation of the result and the proof to the results of the first 3 sections, and finally prove Theorem 8.1. The proof of Theorem 8.1 uses some of the results of the first part of this section together with some of the results of Section § 5. The inductive construction required is quite complicated. However, although a knowledge of the contents of Section § 7 would be helpful, it is not needed. It would, perhaps, also be helpful if the reader briefly tries to prove the results himself, using the existence of a weak Dirichlet set supporting a true pseudofunction. In this way he will see that a new idea is required (the more so as we know the existence of weak Kronecker and thus both weak

Dirichlet and ZA^+ sets which support a true pseudofunction). However, the new idea is very simple and the reader, particularly if he has read Section § 7, may already have seen it.

LEMMA 8.3. Suppose E is a closed set with the following properties : -

Given $n \geq 1$ we can find $\varkappa(n) > 0$, a positive integer $m(n)$ with $m(n)\varkappa(n) \geq 4n$ and a sequence of integers $0 = N(0,n) < N(1,n) < \dots < N(m(n),n)$ with the following properties : -

Given $0 \leqslant j \leqslant m(n)-1$ and $\sum_{r=1}^{N(j,n)} |a_r| \leqslant n$, we can find $T \in (A(E))'$ (i. e. T in the dual of $A(E)$) with

(i) $\|T\|_{PM} \leqslant 2$

(ii) $|\langle T\,,\ 1 - \sum_{r=1}^{N(j,n)} a_r \chi_r \rangle| \geq \varkappa(n)$

(iii) $|\hat{T}(r)| \leqslant \varkappa(n)/4n$ for all $r \geq N(j+1,n)$.

Then, under these conditions, E cannot be a ZA^+ set.

Proof. Suppose E is a ZA^+ set. Then we can find $0 < \sum_{r=0}^{\infty} |a_r| < \infty$ such that $\sum_{r=0}^{\infty} a_r \chi_r(e) = 0$ for all $e \in E$. Let s be the smallest t such that $a_t \neq 0$. We have, on multiplying by $a_s^{-1}\chi_{-s}$, $1 - \sum_{r=1}^{\infty} (-a_{r-s} a_s^{-1}) \chi_r(e) = 0$ for all $e \in E$. So without loss of generality we may assume $a_o = 1$.

Consider now some $\sum_{r=1}^{\infty} |a_r| < \infty$. There exists an $n \geq 1$ with $\sum_{r=1}^{\infty} |a_r| \leqslant n$. Using the notation of the theorem we have

$$\sum_{j=0}^{m(n)-1} \sum_{r=N(j,n)+1}^{N(j+1,n)} |a_r| \leqslant n$$

and so there exists a $0 \leqslant k \leqslant m(n)-1$ with

$$\sum_{r=N(k,n)+1}^{N(k+1,n)} |a_r| \leqslant n/m(n) \leqslant \varkappa(n)/4.$$

But we know that we can find a $T \in PM(E)$ with

(i) $\|T\|_{PM} \leqslant 2$

(ii) $|\langle T , 1 - \sum_{r=1}^{N(k,n)} a_r \chi_r \rangle| \geq \varkappa(n)$

(iii) $|\hat{T}(r)| \leqslant \varkappa(n)/4n$ for all $r \geq N(k+1,n)$

and so, in particular with

$$\begin{aligned} |\langle T, 1 - \sum_{r=1}^{\infty} a_r \chi_r \rangle| &\geq |\langle T, 1 - \sum_{r=1}^{N(k,n)} a_r \chi_r \rangle| - \sum_{r=N(k,n)+1}^{N(k+1,n)} |\langle T, a_r \chi_r \rangle| - \sum_{r=N(k+1,n)+1}^{\infty} |\langle T, a_r \chi_r \rangle| \\ &\geq \varkappa(n) - \sum_{r=N(k,n)+1}^{N(k+1,n)} |a_r| \, \|T\|_{PM} - \sum_{r=N(k+1,n)+1}^{\infty} |a_r| \, |\hat{T}(r)| \\ &\geq \varkappa(n) - 2\varkappa(n)/4 - n\varkappa(n)/4n \\ &\geq \varkappa(n)/4. \end{aligned}$$

Since $T \in (A(E))'$, we have $(1 - \sum_{r=1}^{\infty} a_r \chi_r)|E \neq 0$ and, since the a_r were arbitrary, we have shown that E cannot be a zero set for A^+.

Remark. It is very important that T belongs not merely to $PM(E)$ but to the subset $(A(E))'$ the dual of $A(E)$. For if $S \in PM(E) \setminus (A(E))'$, then we know that there exist $f \in A$ with $f(e) = 0$ for $e \in E$, yet $\langle f, S \rangle = 0$ (this being another way of saying that $(A(E))'$ is the set of synthesisable pseudomeasures), and the last sentence of the proof above would not necessarily apply with S in place of T. We shall return to this point in the remark following the proof of Theorem 8.1'.

We now commence the series of lemmas which will put us in a position to use Lemma 8.3.

LEMMA 8.2. (ii) Given $1 > \varepsilon > \zeta > 0$ and M a positive integer, we can find an $N_o(\zeta, M)$ such that for all $P \geq 2N_o(\zeta, M)$ we know that the set $E(x,\varepsilon,P) = \{2\pi r/P : 2\pi r/P \in [x-\varepsilon, x+\varepsilon]\}$ has the following property : -

Given $y \in [x-\varepsilon, x+\varepsilon]$ we can find a $\mu \in M^+(E(x,\varepsilon,P))$, $||\mu||_M = 1$ such that

(1) $|\hat{\mu}(r) - \hat{\delta}_y(r)| \leqslant \zeta$ for all $|r| \leqslant M$

(2) $|\hat{\mu}(r)| \leqslant \zeta$ for all $N_o(\zeta, M) \leqslant r \leqslant P - N_o(\zeta, M)$.

LEMMA 8.2. (iii) Given $1 > \varepsilon > \zeta > 0$ and m a positive integer, we can find a sequence of integers $0 = N_1(0,\zeta) < N_1(1,\zeta) < \ldots < N_1(m,\zeta)$ with the following property : -

If $P \geq 2N_1(m,\zeta)$, then the set $E(x,\varepsilon,P) = \{2\pi r/P : 2\pi r/P \in [x-\varepsilon, x+\varepsilon]\}$ has the following property : -

Given $y \in [x-\varepsilon, x+\varepsilon]$ and $0 \leqslant j \leqslant m-1$, we can find a $\mu \in M^+(E(x,\varepsilon,P))$, $||\mu||_M = 1$ such that

(1) $|\hat{\mu}(r) - \hat{\delta}_y(r)| \leqslant \zeta$ for all $|r| \leqslant N(j,\zeta)$

(2) $|\hat{\mu}(r)| \leqslant \zeta$ for all $N_1(\zeta, j+1) \leqslant r \leqslant P - N_1(\zeta, j+1)$.

Proof of (ii). This is an immediate consequence of Lemma 3.1. Set $N_o(\zeta,M) = [160\zeta^{-2}(M+1)] + 1$. If $P \geq 2N_1(\zeta,M)$, then we can certainly find a $w \in E(x,\varepsilon,P)$ such that $y \in [w-\zeta/4(m+1), w+\zeta/4(m+1)] \subseteq [x-\varepsilon, X+\varepsilon]$.

Set $\mu = \mu_{\zeta/4(m+1),P} * \delta_w$ (in the notation of Lemma 3.1). By Lemma 3.1 (i) $\mu \in M^+(E(x,\varepsilon,P))$, $||\mu|| = 1$. Since $\operatorname{supp} \mu \subseteq [w-\zeta/4(M+1), w+\zeta/4(M+1)]$, we have

$|\chi_r(y) - \chi_r(v)| \leqslant |r|\ |y - v| \leqslant |r|\ \zeta/2(M+1) \leqslant \zeta$ for all $|r| \leqslant M$ and so

(1) $|\hat{\mu}(r) - \hat{\delta}_y(r)| \leqslant \zeta$ for all $|r| \leqslant M$

whilst by Lemma 1.8 (ii)

(2)' $|\hat{\mu}(r)| \leqslant \zeta$ for all $40.(\zeta/4(M+1))^{-1}\zeta^{-1} \leqslant r \leqslant P-40.(\zeta/(M+1))^{-1}\zeta^{-1}$

and so in particular

(2) $|\hat{\mu}(r)| \leqslant \zeta$ for all $N_0(\zeta, M) \leqslant r \leqslant P-N_0(\zeta, M)$.

Remark. As usual, the fact that P may be chosen freely will play an important role in what follows.

Proof of (iii). Setting $N_1(0,\zeta) = 0$, $N_1(r,\zeta) = N_0(N_1(r-1,\zeta)+1\ ,\ \zeta)$ for $1 \leqslant r \leqslant m$, we obtain (iii) as an immediate consequence of (ii).

Combining Lemmas 8.2 (i) and (iii) we obtain a finite version of the hypotheses of Lemma 8.3.

LEMMA 8.2. (iv) Given $1 > \varepsilon > 0$ and $n \geq 1$ we can find $\varkappa(\varepsilon,n) > 0$, a positive integer $m(\varepsilon,n)$ with $m(\varepsilon,n)\varkappa(\varepsilon,n) \geq 4n$ and a sequence of integers $0 = N(0,\varepsilon,n) < N(1,\varepsilon,n) < \ldots < N(m(\varepsilon,n),\varepsilon,n)$ with the following properties : -

Choose $P \geq 2N(m(\varepsilon,n),\varepsilon,n)$, $x \in \underline{\underline{T}}$ and set $E(x,\varepsilon,P) = \{2\pi r/P :$ $2\pi r/P \in [x-\varepsilon\ ,\ x+\varepsilon]\}$. Then, given $0 \leqslant j \leqslant m(\varepsilon,n)-1$ and $\sum_{r=1}^{N(j,\varepsilon,n)} |a_r| \leqslant n$, we can find a $\mu \in M^+(E(x,\varepsilon,P))$ such that

(i) $\|\mu\|_M = 1$

(ii) $|\langle \mu\ ,\ 1 - \sum_{r=1}^{N(j,\varepsilon,n)} a_r \chi_r \rangle| \geq 2K(\varepsilon,n)$

(iii) $|\hat{\mu}(r)| \leqslant K(\varepsilon,n)/4n$ for all $N(j+1,\varepsilon,n) \leqslant r \leqslant P-N(j+1,\varepsilon,n)$.

Proof. (We remind the reader of the convention $\sum_{r=1}^{0} a_r = 0$.) With the notation of (i) set $\varkappa(\varepsilon,n) = \varkappa_0(\varepsilon,n)/3$. Take $\zeta = \min(\varepsilon/2\ ,\ \varkappa(\varepsilon,n)/4n)$, $m(\varepsilon,n)=4n([\varkappa(\varepsilon,n)^{-1}]+1)$ and (with the notation of (iii)) $N(j,\varepsilon,n) = N_1(j,\zeta)$ for $0 \leqslant j \leqslant m(\varepsilon,n)$.

Suppose now $P \geq 2N(m(\varepsilon,n),\varepsilon,n)$, $x \in \mathbb{T}$, $0 \leqslant j \leqslant m(\varepsilon,n) - 1$, $\sum_{r=1}^{N(j,\varepsilon,n)} |a_r| \leqslant n$ chosen. By (i) and compactness, we can find a $y \in [x-\varepsilon\ ,\ x+\varepsilon]$ with $\left|\sum_{r=1}^{N(j,\varepsilon,n)} a_r\chi_r(y) - 1\right| \geqslant 3\varkappa(\varepsilon,n)$, i. e. with $\left|\langle\sum_{r=1}^{N(j,\varepsilon,n)} a_r\chi_r - 1\ ,\ \delta_y\rangle\right| \geq 3\varkappa(\varepsilon,n)$.

By (iii) we can find a $\mu \in M^+(E(x,\varepsilon,P))$ such that

(i) $\|\mu\|_M = 1$

(iii) $|\hat{\mu}(r)| \leqslant \zeta \leqslant \varkappa(\varepsilon,n)/4n$ for all $N(j+1,\varepsilon,n) \leqslant r \leqslant P-N(j+1,\varepsilon,n)$

(ii)' $|\hat{\mu}(r) - \hat{\delta}_y(r)| \leqslant \zeta \leqslant \varkappa(\varepsilon,n)/4n$ for all $|r| \leqslant N(j+1,\varepsilon,n)$.

Using (ii)' we have, at once,

$$\text{(ii)}\qquad |\langle\sum_{r=1}^{N(j,\varepsilon,n)} a_r\chi_r - 1,\mu\rangle| \geq |\langle\sum_{r=1}^{N(j,\varepsilon,n)} a_r\chi_r - 1,\delta_y\rangle| - \sum_{r=1}^{N(j,\varepsilon,n)} |a_r|\,|\hat{\mu}(r)-\hat{\delta}_y(r)|$$
$$\geq 3\varkappa(\varepsilon,n) - n\varkappa(\varepsilon,n)/4n$$
$$\geq 2\varkappa(\varepsilon,n)$$

as required.

At this point the proofs of Theorems 8.1 and 8.1' diverge. We want to obtain,from the measures of Lemma 8.2 (iv) with support contained in a finite set,the pseudomeasures of Lemma 8.3 with support contained in a weak Dirichlet set. In the proof of Theorem 8.1 we shall attack this problem directly, but the fact that we do not require independence enables us to use a slicker but very much more indirect method in the proof of Theorem 8.1'. The purpose of the next 3 lemmas (Lemmas 8.4, 8.5 and 8.6 (i)) may not become clear until the proof of the fourth (Lemma 8.6 (ii)), containing the proof of Theorem 8.1') has been read.

LEMMA 8.4. We can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to 2\underline{\underline{Z}}^+$ (such that $h(r) > 2r$) and a monotonic decreasing function $\varkappa_1 : \underline{\underline{Z}}^+ \to \underline{\underline{R}}^+$ with the following property : -

Suppose we choose $0 < P(0) < h(P(0)) < M^*(1) < P(1) < h(P(1)) < M^*(2) < P(2) < h(P(2)) < M^*(3) < \ldots\ldots$ integers such that $M^*(r)$ is an integral multiple of $P(r-1)$ and $P(r)$ an integral multiple of $M^*(r)$ $[1 \leq r]$. Then we can find a closed countable set E with 0 as unique accumulation point, having the following properties (for all $n \geq 1$) : -

$(a)_n$ $\|\chi_{s\,M^*(n)} - 1\|_{C(E)} \leq 2^{-n}$ for all $0 \leq s\,M(n) \leq P(n)$

$(b)_n$ We can find a positive integer $m(n)$ with $m(n)\varkappa_1(P(n-1)) \geq 4n$ and a sequence of integers $0 = N(0,n) < N(1,n) < \ldots\ldots < N(m(n),n) < h(P(n-1))/2$ such that, given $0 \leq j \leq m(n) - 1$ and $\sum_{r=1}^{N(j,n)} |a_r| \leq n$, we can find a $\mu \in M^+(E)$ such that

$(i)_n$ $\|\mu\|_M = 1$

$(ii)_n$ $|\langle \mu, 1 - \sum_{r=1}^{N(j,n)} a_r \chi_r \rangle| \geq 2\varkappa_1(P(n-1))$

$(iii)_n$ $|\hat{\mu}(r)| \leq \varkappa_1(P(n-1))/4n$ for all $N(j+1,n) \leq r \leq M^*(n) - N(j+1,n)$

and so, in particular,

$(iii)'_n$ $|\hat{\mu}(r)| \leq \varkappa_1(P(n-1))/4n$ for all $h(P(n-1))/2 \leq r \leq M^*(n) - h(P(n-1))/2$.

Proof. Set $\varepsilon(r) = 2^{-r-4}/r$, $h(r) = 2N(m(\varepsilon(r),r),\ \varepsilon(r),\ r) + 2r + 2$ and $\varkappa_1(r) = \varkappa(\varepsilon(r),r)$ (in the notation of Lemma 8.2 (iv)). We shall construct E inductively. At the beginning of the n^{th} step we have a sequence of integers

$0 < P(0) < h(P(0)) < M^*(1) < P(1) < h(P(1)) < \ldots < P(n-1)$ and a finite set $E(n-1)$ satisfying conditions $(a)_q$ and $(b)_q$ for $n-1 \geq q \geq 1$ and additionally having $M^*(n-1)E(n-1) = 0$. (Of course, if $n = 1$, then we can take $E(0) = 0$ and have the conditions vacuously satisfied). Choose for $M^*(n)$ any integral multiple of $P(n-1)$ having $M^*(n) \geq h(P(n-1))$ and set (again in the notation of Lemma 8.2 (iv)) $F(n) = E(0, \varepsilon(P(n-1)), M(n))$, $E(n) = E(n-1) \cup F(n)$ (so $E(n)$ is finite). We note that

$$(c)_n \quad E(n) \setminus E(n-1) \subseteq F(n) \subseteq [-2^{-n-3}/P(n-1), 2^{-n-3}/P(n-1)] \subseteq [-2^{-n-3}, 2^{-n-3}].$$

In particular, therefore, $\|\chi_r - 1\|_{C(F(n))} \leqslant 2^{-n}$ for all $|r| \leqslant P(n-1)$ and so $(a)_q$ is true for $E(n)$ and $1 \leqslant q \leqslant n-1$. On the other hand, the definitions of $\varepsilon(r)$, $h(r)$ and $\varkappa_1(r)$ in terms of quantities defined in Lemma 8.4 (iv) imply that $(b)_n$ is true for $F(n)$ and so for $E(n)$. Finally, if $P(n)$ is chosen to be a positive integral multiple of $M^*(n)$, then, since $M^*(n)E(n) = 0$, condition $(a)_n$ follows at once for $E(n)$ (indeed $\|\chi_{sM^*(n)} - 1\|_{C(E(n))} = 0$ for all $s \in \mathbb{Z}$).

Set $E = \bigcup_{n=0}^{\infty} E(n)$. Using $(c)_n$ and the finiteness of $E(n)$, we have E closed and countable with 0 as unique limit point. Since $(a)_q$ is true for all $E(n)$, $(a)_q$ is true for E $[1 \leqslant q]$. Since $E(n) \subseteq E$, the fact that $(b)_n$ is true for $E(n)$ implies the truth of $(b)_n$ for E $[1 \leqslant n]$ and E satisfies the conditions of the Lemma.

Lemma 8.4 parodies, in some sense, half of the conditions of Lemma 8.2 with E Dirichlet. In Lemma 8.5 (ii) we imitate the other half and in Lemma 8.6 we combine the two halves. But first we require a version of the Central Lemma.

LEMMA 8.5. (i) Given $K > 1$, $1 > \lambda > 0$ we can find an $1 > \varepsilon_0(K, \lambda) > 0$ and an $m(K,\lambda)\in \underline{\underline{Z}}^+$ with the following property : -

Suppose P a positive integer and $1 > \zeta$, $\varepsilon_1 > 0$ given with $P\varepsilon_1 \geq 1$, then $M(0) = 1 + [P\zeta^{-1}\varepsilon_o^{-1}(K,\zeta)]$ has the following property : -

Given $\delta > 0$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ (such that $h(r) > r$) with the following property : -

Given $M(0) < h(M(0)) < M(1) < h(M(1)) < M(2) < \ldots < h(M(m)) < M(m+1)$ such that $M(r+1)$ is an integral multiple of $M(r)$ $[1 \leqslant r \leqslant m]$ we can find a finite set $E \subseteq [-\varepsilon_1, \varepsilon_1]$ and $T \in M(E) = PM(E)$ such that

(i) $M(m+1)E = 0$

(ii) $\|m^{-1}\sum_{r=1}^{m} \chi_{M(r)} - 1\|_{C(E)} \leqslant \delta$

(iii) $\|T\|_{PM} = \hat{T}(0) = 1 \geq K \sup_{M(m+1)-M(0) \geq r \geq M(0)} |\hat{T}(r)|$

(iv) $|\hat{T}(r) - 1| \leqslant \zeta$ for all $|r| \leqslant P$.

Remark 1. If we used Lemma 5.2' instead of Lemma 2.1 in the proof below, we could drop the condition $M(r+1)$ a multiple of $M(r)$.

Remark 2. Looking at the first sentence of the proof of Lemma 1.7 we see that we can take $N(\varepsilon,K,\lambda) = ([-\varepsilon^{-1}] + 1)N_2(K,\lambda)$ (i.e. $N(\varepsilon,K,\lambda) = \varepsilon^{-1}N_2(K,\lambda)$) in the statement of Lemma 2.1. If we choose not to use this fact then, a priori, all we know is that the $M(0)$ of Lemma 8.5 (i) depends on P, ζ, K and λ. However, the resulting weaker statement remains strong enough to prove Lemma 8.5 (ii) and this is all we need.

Proof. Set $\varepsilon_o(K,\lambda) = (200(C(K,\lambda)+1)(N_2(K,\lambda)+1))^{-1}$ and take $m(K,\lambda)$ as in

Lemma 2.1. Suppose P , ζ, ε_1 , $M(0)$ taken as indicated. Take $\varepsilon = \zeta/(20C(K,\lambda)P)$ and choose h as in Lemma 2.1. We have $N(\varepsilon,K,\lambda) = ([\varepsilon^{-1}]+1)N_2(K,\lambda) \leqslant (20C(K,\lambda)P\zeta^{-1}+1)N_2(K,\lambda) \leqslant \varepsilon_o(K,\lambda)^{-1}P\zeta^{-1}/10 \leqslant M(0)/2$ so that by Lemma 2.1 it follows that, given $M(0) < h(M(0)) < M(1) < h(M(1)) < M(2) < \ldots < h(M(m)) < M(m+1)$ we can find a finite set $E \subseteq [-\varepsilon , \varepsilon] \subseteq [-\varepsilon_1 , \varepsilon_1]$ and $T \in M(E) = PM(E)$ such that

(i) $M(m+1)E = 0$

(ii) $\|m^{-1} \sum_{r=1}^{m} \chi_{M(r)} - 1\|_{C(E)} \leqslant \delta$

(iii) $\|T\|_{PM} = \hat{T}(0) = 1 \geq K \sup_{M(m+1)-M(n)\geq r \geq M(m)} |\hat{T}(r)|$

(iv) $\|T\|_M \leqslant C(K,\lambda)$

whence

(iv)
$$
\begin{aligned}
|\hat{T}(r) - 1| &\leqslant \|T\|_M \sup_{x\in E} |1 - \chi_r(x)| \\
&\leqslant |r| \|T\|_M \sup_{x\in E} |x| \\
&= |r| \varepsilon \|T\|_M \\
&= \zeta|r|/20P \leqslant \zeta \qquad \text{for all} \quad |r| \leqslant P
\end{aligned}
$$

and we have shown that E has all the required properties.

Remark. Here the important thing is not (as in the construction of a pseudofunction on a Helson set) that the mass of T is bounded independently of $M(m+1)$, but that the mass is bounded independently of the interval $[-\varepsilon , \varepsilon]$ in which the support of T is made to lie. In particular, by taking ε small, we can make T behave like a point mass at the origin, at least with respect to characters χ_r with long wavelength (i.e. for r not too large). This control is vital in what follows.

We now prove a version of Lemma 1.11 incorporating this element of control (and, as promised,imitating half of the conditions of Lemma 8.2 with E weak Dirichlet).

LEMMA 8.5. (ii) Given $\tau(n) \to 0$, we can find a monotonic increasing function $h : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ (such that $h(r) > r$) and a function $p : \underline{\underline{Z}}^+ \to \underline{\underline{Z}}^+$ with the following property : -

Suppose we choose $0 < P(0) < h(P(0)) < M(1) = M(1,0) < h(M(1,0)) < M(1,1) < h(M(1,1)) < \dots < M(1,p(1)+1) = P(1) < h(P(1)) < M(2) = M(2,0) < h(M(2,0)) < M(2,1) < \dots < h(M(2,p(2))) < M(2,p(2)+1) = P(2) < h(P(2)) < M(3) = M(3,0) < \dots$ integers such that $M(r,0) = M(r) > h(P(r-1))$ and $M(r,0) = M(r)$ is an integral multiple of $P(r-1)$, $M(r,s) > h(M(r,s-1))$ and $M(r,s)$ is an integral multiple of $M(r,s-1)$, $P(r) = M(r,p(r)+1)$ $[1 \leqslant s \leqslant p(r)+1, 1 \leqslant r]$. Then we can find a closed set E with the following properties (for all $n \geq 1$) : -

$(a)_n$ $\quad \|p(n)^{-1} \sum_{s=1}^{p(n)} \chi_{M(n,s)} - 1\|_{C(E)} \leqslant 2^{-n}$

$(b)_n$ There exists a $T_n \in PF(E) \cap (A(E))'$ (the set of synthesisable pseudo-functions on E) such that

$(i)_n$ $\quad \hat{T}_n(0) = \|T_n\|_{PM} = 1$

$(ii)_n$ $\quad |\hat{T}_n(r) - 1| \leqslant \tau(n)$ for all $|r| \leqslant P(n)$

$(iii)_n$ $\quad |\hat{T}_n(r)| \leqslant \tau(n)$ for all $|r| \geqslant M(n+1)$.

Proof. Without loss of generality suppose τ decreasing. By Lemma 8.5 (i) we can find for each n a $p(n) \in \underline{\underline{Z}}^+$ (such that $h(r) > r$) with the following property:-

Suppose we choose $0 < P(n-1) < h_n(P(n-1)) < M(n) = M(n,0) < h_n(M(n,0)) < M(n,1) < h_n(M(n,1)) < M(n-2) < \dots < M(n,p(n)+1) = P(n) < h_n(P(n)) < M^*(n+1)$ such that $M(n,r+1)$

is an integral multiple of $M(n,r)$ and $M^*(n+1)$ an integral multiple of $P(n)$ $[0 \leqslant r \leqslant p(n)]$. Then we can find a finite set $E(n) \subseteq [-2^{-n-5}/P(n-1)]$ and $\mu_n \in M(E(n)) = PM(E(n))$ such that

$(1)_n \quad M(n+1)E(n) = 0$

$(2)_n \quad \|p(n)^{-1} \sum_{r=1}^{p(n)} \chi_{M(n,r)} - 1\| \leqslant 2^{-n-6}$

$(3)'_n \quad \|\mu_n\|_{PM} = \hat{\mu}_n(0) = 1 \geq 2^{n+6}\tau(n)^{-1} \sup_{M^*(n+1)-M(n)\geq r\geq M(n)} |\hat{\mu}_n(r)|$

$(4)_n \quad |\hat{\mu}_n(r) - 1| \leqslant 2^{-n-6}\tau(n) \quad$ for all $\quad |r| \leqslant P(n-1)$.

Set $h(n) = 2 \max_{n+1\geq r\geq 1} h_r(n)$. Then, if $P(r)$, $M(r)$, $M(r,s)$ are chosen in accordance with the hypotheses of the lemma with $M(r) = 2M^*(r)$, they automatically satisfy the hypotheses of the paragraph above (with, for example, $P(r-1) \geq r-1$ so that $M^*(r) = M(r) > h(P(r-1))/2 \geq h_r(P(r-1))$ as required). We may therefore choose $E(n)$, μ_n as above. Note that $(3)'_n$ gives at once

$(3)_n \quad \|\mu_n\|_{PM} = \hat{\mu}_n(0) = 1 \geq 2^{n+6} \tau(n)^{-1} \sup_{M(n+1)\geq r\geq M(n)} |\hat{\mu}_n(r)|$.

It is clear that $E(n+1) + \ldots + E(n+r)$ converges topologically to a closed set $F(n) \subseteq [-2^{-n-4}/P(n), 2^{-n-4}/P(n)]$ and (using $(4)_n$ and $(3)'_n$) that $\mu_{n+1} * \ldots * \mu_{n+r}$ converges weakly to a pseudomeasure T_n, having properties $(i)_n$, $(ii)_n$ and $(iii)_n$. T_n has support in $F(n) \subseteq F(0) = E$ say and $\text{supp}\, \mu_m \subset F(n-1) \subseteq E$, so $T_n \in PF(E) \cap (A(E))'$. By $(i)_n$ $M(n,s)(E(1) + E(2) + \ldots + E(n-1)) = 0$, and by the fact that $F(n) \subseteq [-2^{-n-4}/P(n), 2^{-n-4}/P(n)]$ we have $\|\chi_{P(n)} - 1\|_{C(F(n))} = 0$ so $(a)_n$ follows at once from $(2)_n$ and we are done.

We are now in a position to marry the 2 halves of Lemma 8.4 and Lemma 8.5.

LEMMA 8.6. (i) We can find $\varkappa(n) > 0$, positive integers $m(n)$, $p(n)$ with $M(n)\varkappa(n) \geq 4n$, a sequence of integers $0 < P(0) < M(1) = M(1,0) < M(1,1) < M(1,2) < \ldots$ $\ldots < M(1,p(1)+1) = P(1) < M(2) = M(2,0) < M(2,1) < \ldots < M(2,p(2)+1) < P(2) < \ldots$, a closed set E_1 and a closed countable set E_2 with 0 as unique accumulation point with the following properties (for all $n \geq 1$) : -

$(a)_{n1}$ $\quad \|p(n)^{-1} \sum_{s=1}^{p(n)} \chi_{M(n,s)} - 1\|_{C(E_1)} \leqslant 2^{-n}$ $\qquad$ for all $\quad 1 \leqslant s \leqslant p(n)$

$(b)_{n1}$ There exists a $T_n \in PF(E_1) \cap (A(E_1))'$ such that

$(i)_{n1}$ $\qquad \hat{T}_n(0) = \|T_n\|_{PM} = 1$

$(ii)_{n1}$ $\qquad |\hat{T}_n(r) - 1| \leqslant K(n)/4n$ $\qquad$ for all $\quad |r| \leqslant P(n-1)$

$(iii)_{n1}$ $\qquad |\hat{T}_n(r)| \leqslant K(n)/4n$ $\qquad$ for all $\quad |r| > M(n+1)$

$(a)_{n2}$ $\quad \|\chi_{M(n,s)} - 1\|_{C(E)} \leqslant 2^{-n}$ $\qquad$ for all $\quad 1 \leqslant s \leqslant p(n)$

$(b)_{n2}$ There exist positive integers $0 = N(0,n) < N(1,n) < \ldots < N(m(n),n) < P(n)$ such that, given $0 \leqslant j \leqslant m(n)-1$ and $\sum_{r=1}^{N(j,n)} |a_r| \leqslant n$, we can find a $\mu \in M^+(E_2)$ such that

$(i)_{n2}$ $\quad \|\mu\|_M = 1$

$(ii)_{n2}$ $\quad |\langle \mu , 1 - \sum_{r=1}^{N(j,n)} a_r \chi_r \rangle| \geq 2\varkappa(n)$

$(iii)_{n2}$ $\quad |\hat{\mu}(r)| \leqslant \varkappa(n)/4n$ $\qquad$ for all $\quad P(n-1) \leqslant r \leqslant M(n+1)$.

Proof. This follows at once from Lemmas 8.4 and 8.5 (ii). (Take $\varkappa(n) = \varkappa_1(P(n-1))$ as in Lemma 8.4, $m(n)$ as in Lemma 8.4, take $\tau(n) = \varkappa(n)/4n$ in applying Lemma 8.5 (ii) (since $P(n-1)$ depends only on $\tau(n-1)$ this is not circular), take $p(n)$ as in Lemma 8.5 (ii), choose $M^*(m+1) \geq 2M(m+1)$ in applying Lemma 8.4 (so that $(ii)'_n$

of Lemma 8.4 yields $(iii)_{n2}$ above), and observe that certain of the $sM^*(n)$ of Lemma 8.4 $(a)_n$ coincide with the $M(n,s)$ of Lemma 8.5 (ii) $(a)_n$ giving $(a)_{n1}$ and $(a)_{n2}$.)

LEMMA 8.6. (ii) If E_1, E_2 are in (i), then $E = E_1 + E_2$ is weak Dirichlet but not a ZA^+.

Proof. If $e_1 \in E_1$, $e_2 \in E_2$, then by $(a)_{n1}$ and $(a)_{n2}$

$$|p(n)^{-1} \sum_{s=1}^{p(n)} \chi_{M(n,s)}(e_1 + e_2) - 1| \leq |p(n)^{-1} \sum_{s=1}^{p(n)} \chi_{M(n,s)} - 1| \leq 2^{-n+1}$$

so for all $\sigma \in M^+(E)$

$$\inf_{1 \leq s \leq p(n)} \int |\chi_{M(n,s)} - 1| d\sigma \leq \int |p(n)^{-1} \sum_{s=1}^{p(n)} \chi_{M(n,s)} - 1| d\sigma \leq 2^{-n+1} ||\sigma||.$$

Thus E is weak Dirichlet. (Alternatively we could use the fact that if F_1 and F_2 are weak Dirichlet, so is F_1+F_2 ; the proof of this is easy.)

On the other hand suppose $0 \leq j \leq m(n)-1$ and $\sum_{r=1}^{N(j,n)} |a_r| \leq n$. By $(b)_{n2}$ we can find a $\mu \in M^+(E_2)$ satisfying $(i)_{n2}$, $(ii)_{n2}$, $(iii)_{n2}$. Set $T = \mu * T_n$. Using $(i)_{n1}$, $(ii)_{n1}$ and $(iii)_{n1}$, we obtain

(i) $||T||_{PM} \leq ||\mu||_{PM} ||T_n||_{PM} = 1 \leq 2$

(ii) $$|\langle T , 1 - \sum_{r=1}^{N(j,n)} a_r \chi_n \rangle| \geq |\langle \mu , 1 - \sum_{r=1}^{N(j,n)} a_r \chi_r \rangle| - \sum_{r=1}^{N(j,n)} |a_r| \, |\hat{\mu}(r) - \hat{T}(r)|$$

$$\geq 2\varkappa(n) - \sum_{r=1}^{N(j,n)} |a_r| \, ||\mu||_{PM} \, |1 - \hat{T}_n(r)|$$

$$\geq 2\varkappa(n) - 2n\varkappa(n)/4n \geq \varkappa(n)$$

(iii) $|\hat{T}(r)| \leq \min(|\hat{T}_n(r)| \, ||\mu||_{PM}, \; |\hat{\mu}(r)| \, ||T||_{PM}) \leq \varkappa(n)/4$ for all $|r| \geq P(n)$.

Thus the conditions of Lemma 8.3 hold and E is not a zero set for A^+.

Remark. The above proof of Theorem 8.1' shows strong connections between the existence of a weak Dirichlet set which supports a true pseudofunction and the existence of a weak Dirichlet set which is not a ZA^+ set. However, it does so at the expense, at least in this presentation, of obscuring the motivation of the proof. The proof may be expressed differently and in a natural manner in terms of measures only (and indeed was first so obtained, though in a cruder way ; we remind the reader of the discussion in the introduction running from Lemma 1.9 to Lemma 1.11). Lemma 8.3 is replaced by

LEMMA 8.3'. As for Lemma 8.3 but with the penultimate sentence reading : - Given $0 \leq j \leq m(n)-1$, $\sum_{r=1}^{N(j,n)} |a_r| \leq n$ and Q, we can find $\sigma \in M(E)$ with

(i) $\|\sigma\|_{PM} \leq 2$

(ii) $|\langle \sigma, 1 - \sum_{r=1}^{N(j,n)} a_r \chi_r \rangle| \geq \varkappa(n)$

(iii) $|\hat{\sigma}(r)| \leq \varkappa(n)/4n$ for all $Q \geq r \geq N(j+1,n)$.

Proof. Since $\sum_{r=1}^{\infty} |a_r| < \infty$ we can find a Q with $\sum_{r=Q}^{\infty} |a_r| < \varkappa(n)/8$. The proof now runs as for Lemma 8.3 until the 3rd sentence of the 2nd paragraph which now reads : -

But we know that we can find a $\sigma \in M(E)$ with

(i) $\|\sigma\|_{PM} \leq 2$

(ii) $|\langle \sigma, 1 - \sum_{r=1}^{N(j,n)} a_r \chi_r \rangle| \geq \varkappa(n)$

(iii) $|\hat{\sigma}(r)| \leq \varkappa(n)/4n$ for all $Q \geq r \geq N(j+1,n)$.

By the arguments of Lemma 8.3

$$|\langle \sigma, 1 - \sum_{r=1}^{\infty} a_r \chi_r \rangle| \geq |\langle \sigma, 1 - \sum_{r=1}^{Q} a_r \chi_r \rangle| - \sum_{r=Q+1}^{\infty} |a_r| \, |\hat{\sigma}(r)|$$

$$\geq \varkappa(n)/4 - \|\sigma\|_{PM}\varkappa(n)/16 \geq \varkappa(n)/16 > 0$$

so $1 - \sum_{r=1}^{\infty} a_r \chi_r$ is not identically 0 on E and the proof of Lemma 8.3' is complete.

The proof of Lemma 8.6 (ii) (and so of Theorem 8.1') goes through as before with the pseudomeasure T replaced by the measure $\sigma = \mu * \mu_{n+1} * \mu_{n+2} * \dots * \mu_{n+m}$ for m large enough.

The reader may feel that we have replaced one proof by an almost equivalent one. In reply I would remark that such a procedure is not always possible so trivially (consider Theorem 1.1). By giving the two proofs we have shown that the problem of the existence of a weak Dirichlet non ZA^+ set lies on the borderline between those problems only involving measures (e. g. those concerning the union of Helson sets and the existence of independent Helson sets) and those appropriately treated using pseudo-functions (e. g. those concerning the existence of E with $A(E)$ not closed in $\widetilde{A(E)}$ (for an illuminating example, see Varopoulos' proof in [19]), and the existence of Helson sets not of synthesis). Possibly the force of this remark may appear greater after our proof of Theorem 8.1, but since many readers will skip this we have placed the remark here.

We conclude this section with the

Proof of Theorem 8.1. Choose $x \notin 2\pi\mathbb{Q}$. We construct inductively the following objects :

$(a)_n$ $\varepsilon(n+1) > 0$

$(b)_n$ $P(n)$, $M(n)$ positive integers with $P(n) \geq 4M(n)$, $N(n) = P(n) - M(n)$

$(c)_n$ $\varkappa(n) > 0$, $m(n)$ a non negative integer with $m(n)\varkappa(n) \geq 4n$

$(d)_n$ A sequence of integers $0 = N(0,n) < N(1,n) < \dots < N(m(n),n) < M(n) < N(n)$

$(e)_n$ A finite set $E(n)$ with $P(n)E(n) = 0$

such that $\varkappa(r)$ decreases and $M(n+1) > P(n)$, satisfying the following conditions :

$(1)_n$ For each $1 \leqslant r \leqslant n$ we know that

$(1)_{nr}$ For each $r \leqslant k \leqslant n$ we have an A_{rk} independent of n such that

$(1)_{nrk}$ Given $0 \leqslant j \leqslant m(r)-1$, $\sum_{s=1}^{N(j,r)} |a_s| \leqslant r$ we can find $\mu_{nrk} \in M(E(n))$ with

$(i)_{nrk}$ $\|\mu_{nrk}\|_{PM} \leqslant 2 - 2^{-n}$, $\|\mu_{nrk}\|_M \leqslant A_{rk}$

$(ii)_{nrk}$ $|\langle \mu_{nrk}, 1 - \sum_{s=1}^{N(j,r)} a_s \chi_s \rangle| \geq \varkappa(r)(1 + 2^{-n})$

$(iii)_{nrk}$ $|\hat{\mu}_{nrk}(s)| \leqslant \varkappa(r)(1-2^{-n})/4r$ for all $N(k) \geq r \geq N(j+1, r)$

$(2)_n$ Taking $\varepsilon_o(K,\lambda)$ as in Lemma 8.5 (i) we have $M(n)\varepsilon(n+1) \geq 1$,

$N(n) \geq 1 + \left[M(n)2^{n+8}n(\varkappa(n))^{-1}\varepsilon_o(2^{n+8}n(\varkappa(n))^{-1}, 1-2^{-n-1})^{-1}\right]$.

To start the induction we set $\varepsilon(1) = 1/10$, $P(0) = 16$, $Q(0) = 2$, $M(0) = 1$, $\varkappa(0) = 1/10$, $m(0) = 0$, $E(0) = \emptyset$. The n^{th} step is as follows. By Lemma 8.5' (and Dirichlet's Theorem applied to $\{x\}$ to give $(4)_n$), we can find $\varepsilon(n+1)/4 > \varepsilon'(n+1) > 0$, $P'(n) > M'(n) > P(n)$ with $M'(n)$ an integral multiple of $P(n)$, $N'(n)$ of $M'(n)$, a set $0 \in F(n+1) \subseteq [-\varepsilon(n+1)/2, \varepsilon(n+1)/2]$ and a $T_{n+1} \in M(F(n+1))$ such that

$(3)_n$ $P'(n)F(n+1) = 0$

$(4)_n$ We can find $q(n+1)$ an integer and $P(n) < M(n,1) < M(n,2) < \ldots < M(n,q(n+1)) < M'(n)$ such that

$$\|q(n+1)^{-1} \sum_{s=1}^{q(n+1)} \chi_{M(n,s)} - 1\|_{C(F(n+1))} \leqslant 2^{-n+1} \quad \text{and} \quad |\chi_{M(n,s)}(x) - 1| \leqslant 2^{-n}$$

for all $1 \leqslant s \leqslant q(n+1)$

$(5)_n$ $\|T_{n+1}\|_{PM} = \hat{T}_{n+1}(0) = 1 \geq 2^{n+8} n(\varkappa(n))^{-1} \sup_{P'(n)-N(n) \geq s \geq N(n)} |\hat{T}_{n+1}(s)|$

$(6)_n$ $|\hat{T}_{n+1}(s) - 1| \leqslant 2^{-n-8}\varkappa(n)/n$

$(7)_n$ $M'(n)\varepsilon'(n+1) \leqslant 10^{-2}.2^{-n-8} \min \varkappa(n)/n$

$(8)_n$ $P(n) - N(n) \geq 12800\ [2^{n+12}\ \varepsilon'(n+1)^{-1} n/\varkappa(n)]$

(conditions $(7)_n$ and $(8)_n$ will be used when we apply Lemma 5.6).

Set $E'(n+1) = E(n) + F(n+1)$. We note that

$(9)_n$ $E'(n+1) \subseteq E(n) + [-\varepsilon(n+1)/2\ ,\ \varepsilon(n+1)/2]$.

If $E \subseteq (E'(n+1) \cup x) + [-\varepsilon'(n+1)\ ,\ \varepsilon'(n+1)]$ then, by $(e)_n$, $(4)_n$ and $(7)_n$ we have

$(10)_n$ $\left\| q(n+1)^{-1} \sum_{s=1}^{q(n+1)} \chi_{M(n,s)} - 1 \right\|_{C(E)} \leqslant 2^{-n+2}$.

With the notation of the inductive condition set $\mu_{n\,r\,n+1} = \mu_{n\,r\,n} * T_{n+1}$.

Conditions $(1)_{nrn}$, $(5)_n$ and $(6)_n$ give

$(i)_{n\,r\,n+1}$ $\|\mu_{n\,r\,n+1}\|_{PM} \leqslant 2 - 2^{-n}$, $\|\mu_{n\,r\,n+1}\|_M \leqslant \|T_{n+1}\|_M A_{rn}$

$(ii)_{n\,r\,n+1}$ $|\langle \mu_{n\,r\,n+1}\ ,\ 1 - \sum_{s=1}^{N(j,r)} a_s \chi_s \rangle| \geq \varkappa(r)(1+2^{-n} - 2^{-n-2})$

$(iii)_{n\,r\,n+1}$ $|\hat{\mu}_{n\,r\,n+1}(s)| \leqslant \varkappa(r)(1-2^{-n} + 2^{-n-2})/4r$ for all $P(n)-N(n) \geq r \geq N(j+1,r)$.

By Lemma 5.6 (using $(7)_n$, $(8)_n$) we can find an $\varepsilon''(n+1) \leqslant \varepsilon'(n+1)/8$ such that, provided only we take $P(n+1)$ large enough, we can find a linear map $L_n : M(\underline{\underline{T}}) \to M(\underline{\underline{T}})$ such that for all $\sigma \in M(\underline{\underline{T}})$

$(11)_n$ $P(n+1)\ \mathrm{supp}\ L_n\sigma = 0$

$(12)_n$ $\inf_{e' \in \mathrm{supp} L_n \sigma}\ \sup_{e \in \mathrm{supp}\ \sigma} |e - e'| \leqslant \varepsilon'(n+1)/8$

$(13)_n$ $|(L_n\sigma - \sigma)\hat{}\,(r)| \leqslant 2^{-n-8}\varkappa(n)\|\sigma\|/8$ for all $|r| \leqslant M'(n)$

$(14)_n$ $|(L_n\sigma)\hat{}\,(r)| \leqslant |\hat{\sigma}(r)| + 2^{-n-8}\varkappa(n)\|\sigma\|/8$ for all r

$(15)_n$ $\quad |(L_n\sigma)\hat{}(r)| \leq 2^{-n-8}\varkappa(n)\|\sigma\|/8$ for all $P(n+1)-P'(n)+N(n) \geq r \geq P'(n)-N(n)$

$(16)_n$ $\quad \|L_n\sigma\|_M = \|\sigma\|_M$

$(17)_n$ Given $1 \leq \ell \leq n$, $y_1, y_2, \ldots, y_\ell \in \operatorname{supp}\sigma$, with $|y_i - y_j| \geq 2^{-n}$ for $i \neq j$, we have, for all $|y_i - x_i| \leq \varepsilon''(n+1)$, $|\omega_0| \leq n$, $0 \neq |\omega_i| \leq n$ integers, $\sum_{i=1}^{\ell} \omega_i x_i + \omega_0 x \neq 0$.

By Lemma 8.2 (iv) we can find $0 < \varkappa(n+1) < \varkappa(n)$, $m(n+1)$ a positive integer with $m(n+1)\ (n+1) \geq 4(n+1)$, and a sequence of integers $0 = N(0,n+1) < N(1,n+1) < N(2,n+1) < \ldots < N(m(n+1),n+1)$ such that, provided only we take $P(n+1)$ large enough, we have

$(18)_{n+1\ n+1\ n+1}$ Given $0 \leq j \leq m(n+1)-1$, $\sum_{s=1}^{N(j,n+1)} |a_s| \leq n+1$, we can find a $\mu_{n+1\ n+1\ n+1} \in M(\{2\pi r/P(n+1) : |x - 2\pi r/P(n+1)| \leq \varepsilon(n+1)/8\})$ with

$(i)_{n+1\ n+1\ n+1}$ $\quad \|\mu_{n+1\ n+1\ n+1}\|_{PM} = 1 \leq 2 - 2^{n+1}$,

$\quad \|\mu_{n+1\ n+1\ n+1}\|_M = A_{n+1\ n+1}$ say

$(ii)_{n+1\ n+1\ n+1}$ $\quad |\langle \mu_{n+1\ n+1\ n+1}, 1 - \sum_{s=1}^{N(j,n+1)} a_s \chi_s \rangle| \geq \varkappa(n+1)(1+2^{-n-1})$

$(iii)_{n+1\ n+1\ n+1}$ $\quad |\hat{\mu}_{n+1\ n+1\ n+1}(s)| \leq \varkappa(n+1)(1-2^{-n-1})/4(n+1)$

for all $P(n+1)-N(j+1,n+1) \geq r \geq N(j+1\ ,\ n+1)$.

Select $M(n+1)$ so that

$(19)_{n+1}$ $\quad M(n+1) \geq N(m(n+1)\ ,\ n+1) + P(n) + 64(\ [(\varepsilon''(n+1))^{-1}] + 8)$.

Then, provided only that $P(n+1)$ is large enough, we have $P(n+1) \geq 4M(n+1)$ and

$(20)_{n+1}$ $\quad N(n+1) = P(n+1) - M(n+1)$

$$\geq 1 + M(n+1)2^{n+9}(n+1)(\varkappa(n+1))^{-1}(\varepsilon_0(2^{n+9}(n+1)(\varkappa(n+1))^{-1}, 1-2^{-n-2}))^{-1}.$$

Each of the last 3 paragraphs is true subject only to $P(n+1)$ being taken large enough. We take $P(n+1)$ to be sufficiently large in the sense of the last 3 paragraphs and choose $\varepsilon(n+2) = \varepsilon''(n+1)/4$, so that

$(21)_{n+1}$ $\quad \varepsilon(n+2) \leqslant \varepsilon(n+1)/32$.

We set $E^*(n+1) = \cup \{\text{supp } L_{n+1}\sigma : \sigma \in M(E'(n+1))\}$, $E^{**}(n+1) = \{2\pi r/P(n+1) : |x - 2\pi r/P(n+1)| \leqslant \varepsilon(n+1)\}$ and $E(n+1) = E^*(n+1) \cup E^{**}(n+1)$. Checking through $(a)_{n+1}$, $(b)_{n+1}$, $(c)_{n+1}$, $(d)_{n+1}$, $(e)_{n+1}$ we see that the quantities have the right form. Using $(20)_{n+1}$ we see that $(2)_{n+1}$ is satisfied, and all that remains is to check $(1)_{n+1}$. Note first that $(18)_{n+1\ n+1\ n+1}$ gives $(1)_{n+1\ n+1\ n+1}$. To obtain $(1)_{n+1\ r\ k}$ with $r \leqslant k \leqslant n+1$ we set $\mu_{n+1\ r\ k} = L_{n+1}\,\mu_{n\ r\ k}$. We obtain $(i)_{n+1\ r\ k}$ from $(i)_{n\ r\ k}$, $(14)_n$ and $(16)_n$. We obtain $(ii)_{n+1\ r\ k}$ from $(ii)_{n\ r\ k}$ and $(13)_n$. We obtain $(iii)_{n+1\ r\ k}$ from $(iii)_{n\ r\ k}$, $(15)_n$ and $(19)_{n+1}$. (Note that the conditions with $k = n+1$ are not found under $(i)_n$, but in the paragraph after $(10)_n$). Before restarting the induction we observe

$(22)_{n+1}$ $\quad E^{**}(n+1) \subseteq [x - \varepsilon(n+1)\ ,\ x + \varepsilon(n+1)]$

$(23)_{n+1}$ $\quad E(n+1) \subseteq E(n) + [-3\varepsilon(n+1)/4\ ,\ 3\varepsilon(n+1)/4]$.

We can now begin the next inductive step.

Let E be the topological limit of the $E(n)$. By $(23)_{n+1}$ and $(21)_{n+1}$ $E \subseteq E(n) + [-\varepsilon(n+1)\ ,\ \varepsilon(n+1)]$ so by $(10)_n$ we have, for all n,

$$\Big\|q(n+1)^{-1} \sum_{s=1}^{q(n+1)} \chi_{M(n,s)} - 1\Big\|_{C(E)} \leqslant 2^{-n+2}$$

so that E is weak Dirichlet. Using $(23)_{n+2}$, $(17)_n$ and the definition of $\varepsilon(n+2)$ together with $(22)_{n+1}$, we see that E is independent. (For suppose $x_o = x_1, x_2, \ldots$

$\ldots, x_\ell \in E$ distinct and $\sum_{r=0}^{\ell} m_r x_r = 0$. Then, provided only n is so large that $n \geq \ell + \sum_{r=0}^{\ell} |m_r| + 1$ and $\inf |x_i - x_j| \geq \varepsilon(n-2), 2^{-n-8}$, we can, by $(23)_{n+1}$ and $(22)_{n+1}$, find $y_i \in E^*(n)$ $[1 \leqslant i \leqslant n]$ with $|y_i - y_j| \geq 2^{-n}$ for $1 \leqslant i < j \leqslant \ell$ and $|y_i - x_i| \leqslant \varepsilon''(n+1)$. Using $(17)_n$ we have at once $m_o = m_1 = m_2 = \ldots = m_\ell = 0$).

We claim also that E is not a ZA^+ set. To show this we use the ideas of Lemma 8.3 and the remark following Lemma 8.6 (ii) (including Lemma 8.3'). Suppose $r \geq 1$, $0 \leqslant j \leqslant m(r)-1$ and $\sum_{s=1}^{N(j,r)} |a_s| \leqslant r$ fixed once and for all. Fix $k \geq r$ temporarily and consider the $\mu_{n\,r\,k}$ given in $(i)_{n\,r\,k}$ for $n \geq k$. We have $\|\mu_{n\,r\,k}\|_M \leqslant A_{rk}$ so that $(\mu_{n\,r\,k})$ has a weak $*$ limit point $\mu_{r\,k} \in M(\underline{\underline{T}})$. Since $\mu_{n\,r\,k} \in E(n)$, we have $\mu_{r\,k} \in M(E)$. Further, using $(1)_{n\,r\,k}$ we have

$(\alpha)_{rk}$ $\quad \|\mu_{r\,k}\|_{PM} \leqslant 2, \quad \|\mu_{r\,k}\|_M \leqslant A_{rk}$

$(\beta)_{rk}$ $\quad |\langle \mu_{r\,k}, 1 - \sum_{s=1}^{N(j,r)} a_s \chi_s \rangle| \geq \varkappa(r)$

$(\gamma)_{rk}$ $\quad |\hat{\mu}_{r\,k}(s)| \leqslant \varkappa(r)/4r$ for all $N(k) \geq r \geq N(j+1,r)$.

Since $N(k) \to \infty$ as $k \to \infty$ Lemma 8.3' shows that E is not a ZA^+ set and Theorem 8.1 is proved by construction.

Remark 1. Let us see how the proof works with pseudomeasures. Consider the $\mu_{r\,k}$ given above with k allowed to vary. Since $\|\mu_{r\,k}\|_{PM} \leqslant 2$ the $(\mu_{r\,k})$ has a weak limit point $S_r \in PM$. Since $\mu_{r\,k} \in M(E)$, we have $S_r \in (A(E))'$ (i.e. we once again emphasize the fact that S is synthesisable on E) and using $(\alpha)_{rk}$, $(\beta)_{rk}$, $(\gamma)_{rk}$ we have

(α) $\quad \|S_r\| \leqslant 2$

(β) $\quad |\langle S_r \, , \, 1 - \sum_{s=1}^{N(j,r)} a_s \chi_s \rangle| \geq \varkappa(r)$

(γ) $\quad |\hat{S}_r(s)| \leqslant \varkappa(k)/4k$ for all $s \geq N(j+1 \, , \, r)$ $\quad S_r \in PF(E)$

and we can apply Lemma 8.3.

Remark 2. As usual it is easy to see that by being a little more precise we could ensure $\mu_{n\,r\,k} \to \mu_{r\,k}$, as $n \to \infty$, $\mu_{r\,k} \to S_r$ as $k \to \infty$ (in the weak * topology) instead of using general considerations to prove the existence of limit points.

§ 9. TILDE ALGEBRAS.

Varopoulos has shown

LEMMA 9.1. (i) There exist disjoint closed sets E_i of bounded synthesis with constant 1 such that $\bigcup_{i=0}^{\infty} E_i$ is closed and of synthesis but not of bounded synthesis.

(In fact he stated a slightly weaker result [19]

LEMMA 9.1'. (i) There exists a closed set E of synthesis but not of bounded synthesis.

However it is not difficult to extract from his proof a demonstration of the stronger result ; in the tradition of the cited author we leave this as an exercise).

We shall give a proof of Lemma 9.1' based on different and rather more direct techniques. If the reader only wants to see the ideas on which this section is based and is prepared to take various obvious or easily believed results (which we prove in Lemmas 9.5 and 9.6) on trust he should skip at once to the proof of Lemma 9.1 (which is found after Lemma 9.6) and read it through quickly.

We shall then obtain

LEMMA 9.2. There exists a translational set F with $A(F)$ not closed in $(\tilde{A}(F)\,,\,\|\ \|_{\tilde{A}(F)})$.

The interest of this result lies in the fact that every symmetric set (ensemble symétrique, see [5], Chapter 1) with constant of dissection a Pisot number is by a result of Meyer necessarily of bounded synthesis (see reference to question 1 in the Appendix). It is unknown whether every symmetric set is of synthesis and possible that some might be of synthesis but not of bounded synthesis.

We conclude the section with our main result on bounded synthesis. We prove :

THEOREM 9.3. (i) There exist sets E_1, E_2 of bounded synthesis with constant 1 such that $E_1 \cap E_2 = \{0\}$ yet $E_1 \cup E_2$ is not of synthesis,

by proving the stronger result :

THEOREM 9.3. (ii) Given $\varepsilon(n) \to 0$ we can find $Q(n) \to \infty$ and sets E_1, E_2 of bounded synthesis with constant 1 such that $\operatorname{card} E_1 \cap E_2 = 1$, $E_1 \cup E_2$ is independent, $E_1 \cup E_2 \subseteq \bigcap_{n=1}^{\infty} \bigcup_{r=1}^{Q(n)} [2\pi r/Q(n) - \varepsilon(Q(n))\,,\, 2\pi r/Q(n) + \varepsilon(Q(n))]$ yet $E_1 \cup E_2$ is not of bounded synthesis.

We shall also obtain in passing

LEMMA 9.1. (ii) Given $\varepsilon(n) \to 0$ we can find $Q(n) \to \infty$ and disjoint closed sets E_i of bounded synthesis with constant 1 such that $\bigcup_{i=0}^{\infty} E_i$ is closed, independent with $\bigcup_{i=0}^{\infty} E_i \subseteq \bigcap_{n=1}^{\infty} \bigcup_{r=1}^{Q(n)} [2\pi r/Q(n)-\varepsilon(n) , 2\pi r/Q(n)+\varepsilon(n)]$ and of synthesis but not of bounded synthesis.

Observe that we have also obtained

LEMMA 9.1'. (ii) There exists an independent set which is of synthesis but not of bounded synthesis.

Proof. This follows at once from Lemma 9.1 (ii) or more instructively from Theorem 9.3 (i) and Lemma 9.4 (ii) below.

LEMMA 9.4. (i) Suppose $x \in E$ a closed set and $E \setminus (x-\varepsilon , x+\varepsilon)$ is of synthesis for all $\varepsilon > 0$. Then E is of synthesis.

(ii) If E_1, E_2 are sets of synthesis with $E_1 \cap E_2 = \{x\}$. Then $E_1 \cup E_2$ is of synthesis.

(iii) If E_1, E_2 are closed disjoint sets then $\tilde{A}(E_1 \cup E_2) = \{f : f | E_i \in \tilde{A}(E_i)\}$.

(iv) Suppose E_1, E_2 are disjoint closed sets which are of synthesis (respectively bounded synthesis). Then $E_1 \cup E_2$ is of synthesis (respectively bounded synthesis).

(v) If E_i is of synthesis and $E_i \to E_o = \{0\}$ topologically as $i \to \infty$ then $\bigcup_{i=0}^{\infty} E_i$ is of synthesis.

Proof. (i) Let $T \in PM(E)$. Let h_n be the trapezoidal function which is 0 outside $[x - 1/n , x + 1/n]$, 1 on $[x - 1/n^3 , x + 1/n^3]$, linear on $[x - 1/n , x - 1/n^3]$, $[x + 1/n^3 , x - 1/n]$. Since $||h_n T|| \leq 3||T||$ we may suppose extracting a subsequence, that $h_n T \to a\delta_x$ weakly. But $(1-h_n)T \in PM(E \setminus (x - 1/n^3, x + 1/n^3))$ so we can find a net $(\{\mu_\alpha^{(n)}\}_{\alpha\in A(n)}, \succ_n)$ of measures with $\mu_\alpha^{(n)} \to T(1-h_n)$ weakly. Now define a new net $(\{\mu_{\underline{\alpha}}\}_{\underline{\alpha}\in A}, \succ)$ where $A = \mathbf{Z}^+ \times \prod_{n=1}^{\infty} A(n)$ and writing $\underline{\alpha} = (\alpha(0), \alpha(1), \ldots)$ we have $\mu_{\underline{\alpha}} = \mu^{(\alpha(0))}_{\alpha(\alpha(0))}$ and $\underline{\alpha} \succ \underline{\beta}$ if and only if $\alpha(0) \geq \beta(0)$ and $\alpha(i) \succ_i \beta(i)$ $[i \geq 1]$. We claim that $\mu_{\underline{\alpha}} + a\delta_{x_0} \to T$ weakly. For suppose $f \in A(\mathbf{T})$, $\varepsilon > 0$ given. Then for each $i \geq 1$ we can find $\beta(i) \in A(i)$ such that $|\langle (1-h_i)T - \mu_\alpha^{(i)}, f\rangle| \leqslant \varepsilon/2$ for all $\alpha \succ_i \beta$ $[\alpha \in A(i)]$. Further by the definition of a we can find a $\beta(0) \in \mathbf{Z}^+$ such that $|\langle h_n T - a\delta_x, f\rangle| \leqslant \varepsilon/2$ for all $\alpha \geq \beta(0)$ $[\alpha \in \mathbf{Z}^+]$. Thus $|\langle \mu_{\underline{\alpha}} + a\delta_{x_0} - T, f\rangle| = |\langle \mu^{(\alpha(0))}_{\alpha(\alpha(0))} + a_{x_0} - T, f\rangle|$

$\leqslant |\langle \mu^{(\alpha(0))}_{\alpha(\alpha(0))} - (1-h_{\alpha(0)})T, f\rangle| + |\langle a\delta_0 - h_{\alpha(0)}T, f\rangle| \leqslant \varepsilon$ for all $\underline{\alpha} \succ \underline{\beta}$.

(ii) The proof runs as for (i).

(iii) Since E_1, E_2 are closed and disjoint we can find an $h \in A(\mathbf{T})$ with $h = 1$ in a neighbourhood of E_1, $h = 0$ in a neighbourhood of E_2. Now suppose $f_1 \in \tilde{A}(E_1)$, $f_2 \in \tilde{A}(E_2)$. Then we can find $g_i^{(n)} \in A(\mathbf{T})$, $\|g_i^n\|_{A(\mathbf{T})} \leqslant \|f_i\|_{\tilde{A}(E_i)}$ such that $\|g_i^{(n)}|E_i - f_i\|_{C(E_i)} \to 0$ as $n \to \infty$ $[i = 1,2]$. Set $g^{(n)} = hf_1^{(n)} + (1-h)f_2^{(n)}$ and define $f \in C(E)$ by $f|E_i = f_i$. Then $g^{(n)} \in A(\mathbf{T})$, $\|g^{(n)}\|_{A(\mathbf{T})} \leqslant$ $\|h\|_{A(\mathbf{T})}\|f_1\|_{\tilde{A}(E_1)} + (1 + \|h\|_{A(\mathbf{T})})\|f_2\|_{\tilde{A}(E_2)}$ and $\|g^{(n)} - f\|_{C(E)} \to 0$. Thus $f \in A(E_1 \cup E_2)$ as required.

(iv) By (iii) it suffices to prove the result for synthesis. Suppose $T \in PM(E_1 \cup E_2)$. Take h as in (ii). Then $hT \in PM(E_1)$, $(1-h)T \in PM(E_2)$. (For suppose $f \in A(\mathbf{T})$, supp $f \cap E_1 = \emptyset$. Then, using the fact that h is zero in a neighbourhood of E_2, supp $fh \cap (E_1 \cup E_2) = \emptyset$ and $\langle hT, f\rangle = \langle T, hf\rangle = 0$. Thus supp $hT \subseteq E_1$ and similarly $\operatorname{supp}(1-h)T \subseteq E_2$). It follows that we can find nets $(\{\mu_\alpha^{(1)}\}_{\alpha\in A}, >_1)$, $(\{\mu_\beta^{(2)}\}_{\beta\in B}, >_2)$ with $\mu_\alpha^{(1)}\in M(E_1)$, $\mu_\beta^{(2)}\in M(E)$, $\mu_\alpha^{(1)} \to hT$, $\mu_\beta^{(2)} \to (1-h)T$. Write $(\alpha,\beta) > (\alpha',\beta')$ if and only if $\alpha >_1 \alpha'$, $\beta >_1 \beta'$ $[\alpha,\alpha'\in A\ ;\ \beta,\beta'\in B]$. Then $(\{\mu_\alpha^{(1)} + \mu_\beta^{(2)}\}, >)$ is a net and if $f\in A(T)$ we have $\langle \mu_\alpha^{(1)} + \mu_\beta^{(2)}, f\rangle = \langle \mu_\alpha^{(1)} + \mu_\beta^{(2)}, hf + (1-h)f\rangle = \langle \mu_\alpha^{(1)}, hf\rangle + \langle \mu_\alpha^{(2)}, (1-h)f\rangle = \langle h\mu_\alpha^{(1)}, f\rangle + \langle (1-h)\mu_\beta^{(2)}, f\rangle$

$= \langle \mu_\alpha^{(1)}, f\rangle + \langle \mu_\beta^{(2)}, f\rangle \to \langle hT, f\rangle + \langle (1-h)T, f\rangle = \langle T, hf\rangle + \langle T, (1-h)f\rangle = \langle T, f\rangle$.

Thus $\mu_\alpha^{(1)} + \mu_\beta^{(2)} \to T$ and we have the required result.

(v) This follows from (1) and (iv).

Remark. I strongly advise the reader to run through the proof of Lemma 9.4 (i) in the particular case when $E \setminus (x-\delta, x+\delta)$ is of bounded synthesis. (In our construction for Theorem 9.3 (ii) this is indeed the case). The nets $(\mu_\alpha^{(n)}, >_\alpha)$ may then be replaced by sequences $(\mu^{(n)})$ but none the less (since $E = E_1 \cup E_2$ is not of bounded synthesis) there exists a pseudomeasure on E obtainable only by the use of nets. The proof that E is of synthesis consists in constructing such nets.

Let us prove some further useful results in the same spirit (Lemma 9.4 (viii) is one of the folklore versions of a famous result of Herz [3])

LEMMA 9.4. (vi) Suppose integers $1 \leq q(1) < q(2) < \dots$ and closed sets $E_n \subseteq [x+2^{-(4q(n)+3)}, x+2^{-(4q(n)+1)}]$ given with $B_n = \sup\{\|f\|_{A(E_n)} : \|f\|_{\tilde{A}(E_n)} \leq 1, f\in A(E_n)\} \to \infty$. Then setting $E = \{x\} \cup \bigcup_{n=1}^{\infty} E_n$ we have $A(E)$ not closed

(vii) Given $1 > \delta > 0$, $B \geq 1$, $Q\in \mathbf{Z}^+$ we can find an $\varepsilon(B,Q,\delta) > 0$ such that if $F \subseteq \{2\pi r/Q : 1 \leq r \leq Q\}$ and $E \subseteq F + [-\varepsilon,\varepsilon]$ is closed the relations $\sum_{|r|\leq Q} a_r\chi_r = 0$, $\sum |a_r| \leq B$ imply $\|\sum_{|r|\leq Q} a_r\chi_r |E\|_{A(E)} \leq \delta$.

(viii) (Herz) We can find $\varepsilon_n \to 0$ with the following property : -

Suppose E is a closed set such that we can find $Q(n) \to \infty$ with $E \subseteq [-\varepsilon_{Q(n)}, \varepsilon_{Q(n)}] + \{2\pi r/Q(n) : 2\pi r/Q(n) \in E, r \in \mathbf{Z}\}$. Then E is of bounded synthesis.

Proof (vi). Let h_n be the trapezoidal function which is 0 outside $(x + 2^{-(4q(n)+3)}, x+2^{-(4q(n)+1)})$, 1 on $[x+2^{-(4q(n)+2)}, x+3.2^{-(4q(n)+3)}]$ and linear on $[x+2^{-(4q(n)+3)}, x+2^{-(4q(n)+2)}]$ and $[x+3.2^{-(4q(n)+3)}, x+2^{-(4q(n)+1)}]$. We know that $\|h_n\|_{A(\mathbf{T})} \leqslant 2$. By the definition of B_n we can find an $f_n \in A(\mathbf{T})$ such that $\|f_n | E_n\|_{A(E_n)} \geq B_n - 1$ yet there exist $g_n^m \in A(\mathbf{T})$ with $\|g_n^m\|_{A(\mathbf{T})} \leqslant 1$ and $\|g_n^m | E_n - f_n | E_n\|_{C(E_n)} \to 0$ as $m \to \infty$. Set $F_n = \frac{1}{2} f_n h_n | E$, $G_n^m = g_n^m h_n | E$. Clearly $F_n \in A(E)$, $\|F_n\|_{A(E)} \geq \|F_n | E_n\|_{A(E_n)} = \|\frac{1}{2} f_n | E_n\|_{A(E_n)} \geq \frac{1}{2}(B_n - 1) \to \infty$ as $n \to \infty$. On the other hand $G_n^m \in A(E)$, $\|G_n^m\|_{A(E)} \leqslant \|\frac{1}{2} h_n\|_{A(E)} \|g_n^m | E\|_{A(E)} \leqslant$ $\|\frac{1}{2} h_n\|_{A(\mathbf{T})} \|g_n^m\|_{A(\mathbf{T})} \leqslant 1$ and $\|G_n^m - F_n\|_{C(E)} = \|g_n^m | E_n - f_n | E_n\|_{C(E_n)} \to 0$. Thus $\|F_n\|_{\tilde{A}(E)} \leqslant 1$ and so $\sup\{\|F\|_{A(E)} : \|F\|_{\tilde{A}(E)} \leqslant 1\} = \infty$ as required.

(vii) Take $\varepsilon(B,Q,\delta) = Q^{-7} B^{-4} \delta^{-4} 10^{-12}$. Since $\|f | E\|_{A(E)} \leqslant$ $\|f | F + [-\varepsilon, \varepsilon]\|_{A(F + [-\varepsilon, \varepsilon])}$ for all $f \in A(\mathbf{T})$ we may suppose $E = F + [-\varepsilon, \varepsilon]$. Let h_x be the trapezoidal function which is 0 outside $(x-2\varepsilon, x+2\varepsilon)$, 1 on $(x-\varepsilon, x+\varepsilon)$ and linear on $(x-2\varepsilon, x-\varepsilon), (x+\varepsilon, x+2\varepsilon)$. Given $f \in A(E)$, $\eta > 0$ we can find for each $x \in F$ an $f_x \in A(\mathbf{T})$ with $f_x | x + [-\varepsilon, \varepsilon] = f | x + [-\varepsilon, \varepsilon]$ and $\|f_x\|_{A(\mathbf{T})} \leqslant \|f | x + [-\varepsilon, \varepsilon]\|_{A(x + [-\varepsilon, \varepsilon])} + \eta$. Set $g = \sum_{x \in F} h_x f_x$. Then $g | E = f$ and $\|g\|_{A(\mathbf{T})} \leqslant \sum_{x \in F} \|h_x\|_{A(\mathbf{T})} \|f_x\|_{A(\mathbf{T})} \leqslant 2 \sum_{x \in F} \|f_x\|_{A(\mathbf{T})} \leqslant 2Q \sup_{x \in F} \|f_x\|_{A(\mathbf{T})}$. Thus $\|f\|_{A(E)} \leqslant 2Q \sup \|f | x + [-\varepsilon, \varepsilon]\|_{A(x + [-\varepsilon, \varepsilon])}$. In particular taking $\sum_{|s| \leqslant Q} a_s \chi_s$ as given and writing $g = \sum_{|r| \leq Q} a_s \chi_s | E$ we have

$$\|g\|_{A(E)} \leq 2Q \sup_{x \in F} \| \sum_{|s| \leq Q} a_s \chi_s | x + [-\varepsilon, \varepsilon] \|_{A(x+[-\varepsilon,\varepsilon])}$$

$$= 2Q \sup_{x \in F} \|(\sum_{|s| \leq Q} a_s \chi_s - \sum a_s \chi_s(x)) | x + [-\varepsilon, \varepsilon] \|_{A(x+[-\varepsilon,\varepsilon])}$$

$$\leq 2Q \sup_{x \in F} \sum_{|s| \leq Q} |a_s| \|\chi_s - \chi_s(x) | x + [-\varepsilon, \varepsilon] \|_{A(x+[-\varepsilon,\varepsilon])}$$

$$= 2Q \sum |a_s| \|(\chi_s - 1) | [-\varepsilon, \varepsilon] \|_{A([-\varepsilon,\varepsilon])}$$

$$\leq 2QB \|(\chi_1 - 1) | [-Q\varepsilon, Q\varepsilon] \|_{A([-Q\varepsilon, Q\varepsilon])}$$

$$\leq \delta$$

using the formula of Lemma 4.1 (i).

(viii) We define a number $B(n)$ as follows. Suppose $F \subseteq \{2\pi r/n : 1 \leq r \leq n\}$ and $X \subseteq \{r : 0 \leq r \leq n-1\}$ is such that $A(F)$ is spanned by $\{\chi_r | F : r \in X\}$. For each $f \in A(F)$ $B(F,X,f) = \inf \{\sum_{s \in X} |b_s| : \sum_{s \in G} b_s \chi_s | F = f\}$. Since $A(F)$ is finite dimensional $B(F,X) = \sup \{B(F,X,f) : \|f\|_{A(F)} \leq 1\} < \infty$. Thus $B(n) = \sup_F(\sup_X B(F,X)) < \infty$. We claim that the lemma is true provided only we take $\varepsilon_n = \varepsilon(B(n)+4, n, 2^{-n})/2$.

For suppose $T \in PM(E)$ where E satisfies the hypotheses above. Set $F_n = F(Q(n)) = E \cap \{2\pi r/Q(n) : 1 \leq r \leq Q(n)\}$. Take a basis $\chi_{\alpha(1)} | F_n$, $\chi_{\alpha(2)} | F_n$, ..., $\chi_{\alpha(r)} | F_n$ for F_n $[0 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(r) < Q(n)]$. Set

$S_n(\sum_{t=1}^{r} a_t \chi_{\alpha(t)} | F_n) = T(\sum_{t=1}^{r} a_t \chi_{\alpha(t)})$. Then S_n is a continuous linear map $S_n : A(F_n) \to \mathbb{C}$, i.e. (since F_n is finite) $S_n \in M(F_n)$. It follows that S_n can be extended to a measure μ_n with $\langle \mu_n, f \rangle = \langle S_n, f | F_n \rangle$ for all $f \in C(T)$.

Now we wish to estimate $\langle \mu_n, \chi_m \rangle$ for $|m| \leq Q(n)$. By the definition of

$B(Q(n))$ we can find $a_1, a_2, \ldots, a_r$ with $\sum_{t=1}^{r} |a_t| \leqslant B(Q(n)) + 1$ and $\chi_m | F = \sum_{t=1}^{r} a_t \chi_{\alpha(t)} | F$. By (vii) $\|(\chi_m - \sum_{t=1}^{r} \alpha_t \chi_t) | (E - [-\varepsilon_n, \varepsilon_n])\|_{A(E+[-\varepsilon_n,\varepsilon_n])}$ $\leqslant 2^{-Q(n)}$ and so $|\langle T, \chi_n\rangle - \langle \mu_n, \chi_n\rangle| = |\langle T, \chi_m - \sum_{t=1}^{r} \alpha_t \chi_t\rangle| \leqslant 2^{-Q(n)}\|T\|_{PM}$. Since $\hat{\mu}_n(m)$ is periodic with period $Q(n)$ we have

$$\|\mu_n\|_{PM} = \sup_{1 \leqslant m \leqslant Q} |\hat{\mu}_n(m)| \leqslant \sup_{1 \leqslant m \leqslant Q} |\hat{T}(m)| + 2^{-Q(n)}\|T\|_{PM} \leqslant (1+2^{-Q(n)})\|T\|_{PM} \quad \text{so}$$

$\limsup_{n\to\infty} \|\mu_n\|_{PM} = \|T\|_{PM}$. On the other hand fixing m and letting $n \to \infty$ we obtain $|\hat{T}(m) - \hat{\mu}_n(m)| \leqslant 2^{-Q(n)}\|T\|_{PM} \to 0$. Thus $\mu_n \to T$ weakly and T is synthesised boundedly with constant 1 by measures $\mu_n \in PM(E)$. The lemma is proved.

Remark. We have already remarked when considering Theorem 1.1' how delicate Herz's result is. Our construction for Theorem 9.3 will give further evidence of this. It is therefore profitable to look carefully at proofs of this result to see why we might expect such a delicate balance between the hypotheses and the conclusion. In the proof above for example we used the fact that F was finite to define an $\varepsilon(n)$ such that for a certain class of $f \in A(F)$ all extensions $g \in A(F + [-\varepsilon(n), \varepsilon(n)])$ satisfying certain conditions are close in $A(F + [-\varepsilon(n), \varepsilon(n)])$ norm. We used it again to see that any $S_n \in (A(F))'$ is in fact a measure. The fact that F_n was an arithmetical progression enabled us to bound $\|\mu_n\|_{PM}$ by the supremum of $|\hat{\mu}_n(m)|$ for m close to 0 (in fact $|m| \leq Q/2$). Finally without the condition $F_n \subseteq E$ the fact that the μ_n synthesised T would not have shown that T was synthesised by measures on E.

Another version of Herz's result is

LEMMA 9.4. (xi) (Herz) Suppose E is a closed set such that there exist $Q(n) \to \infty$ with $\inf_{e \in E} |2\pi r/Q(n) - e| < 2\pi/Q(n)$ if and only if $2\pi/Q(n) \in E$. Then E

is of synthesis and for any given $T \in PM(E)$ we can find $\mu_n \in M(E \cap \{2\pi r/Q(n) : 1 \leq r \leq Q(n)\})$ with $\mu_n \to T$, $\|\mu_n\|_{PM} \leq \|T\|_{PM}$.

Proof. See [5] pp. 122-124 for a demonstration.

Having collected together these results for later use we are now in a position to start proving the results announced at the beginning of this section.

To prove Lemma 9.1' all we shall need is the following refinement of Lemma 1.11.

LEMMA 9.5. (i) Given $\varepsilon > 0$, there exists a closed set $E \subseteq [-\varepsilon, \varepsilon]$ and a sequence $3 < M(1) < M(2) < \ldots < M(r) < \ldots$ of integers with $M(r+1)$ and integral multiple of $2M(r)$ and $M(r+1) \geq 3^{3^r} M(r)$ such that

(a) $\liminf_{r \to \infty} \int |\chi_{M(r)} - 1| \, d\mu = 0$ for all $\mu \in M^+(E)$.

(b) E carries a true pseudomeasure S, synthesisable on E (i.e. $S \in (A(E))'$).

(ii) We can choose E in (i) such that additionally

(c) E is of synthesis.

Proof. The proof of Lemma 1.11 consisted precisely of constructing a set E with the property (a) (i.e. E weak Dirichlet) and (b) (from which we deduced E not of uniqueness). We use the note at the end of the first paragraph of the proof to get $M(r+1) \geq 3^{3^r} M(r)$. To get $E \subseteq [-\varepsilon, \varepsilon]$ we replace the quantities in the first sentence of the proof given in Section § 1 by $T_o = \delta_o$, $\varepsilon_1 = \varepsilon/10$, $M_o(1) = 4N(\varepsilon_1, 2, 2^{-1})$.

(ii) Let F satisfy (a), $F \subseteq [-\varepsilon/2, \varepsilon/2]$, $M(t) \geq 100\varepsilon^{-1}$. Set $F^* = \bigcup_{r \geq t}^{\infty} \{2\pi s/M(r) : \inf_{x \in E} |x - 2\pi s/M(r)| < 2\pi/M(r)\}$. F^* is countable and its limit points are precisely the points of F. Thus $F \cup F^*$ is closed and, if $\mu \in M(F \cup F^*)$, then, for each $\eta > 0$ we can find $\mu_1 \in M^+(F)$, $\mu_2 \in M^+(F^*)$ with $\operatorname{supp} \mu_2$ finite, and $\mu_3 \in M^+(F^*)$ with $\|\mu_3\|_M \leq \eta$ such that $\mu = \mu_1 + \mu_2 + \mu_3$. Since every point of F^* has the form $e = 2\pi s/M(k)$ and thus satisfies $|\chi_{M(r)}(e) - 1| = 0$ for all r sufficiently large, and since $\operatorname{supp} \mu_2$ is finite we have $\int |\chi_{M(r)} - 1| \, d\mu_2 = 0$ for all r sufficiently large. Hence

$$\liminf_{r\to\infty} \int |\chi_{M(r)} - 1| \, d\mu \leq \liminf_{r\to\infty} \left\{ \int |\chi_{M(r)} - 1| \, d\mu_1 + \int |\chi_{M(r)} - 1| \, d\mu_2 + \int |\chi_{M(r)} - 1| \, d\mu_3 \right\}$$

$$\leq \liminf_{r\to\infty} \int |\chi_{M(r)} - 1| \, d\mu_1 + 2\eta = 2\eta$$

and since η, μ were arbitrary $E = F \cup F^*$ satisfies (a). Since $E \subseteq F$, (b) is trivial and (c) itself follows from Lemma 9.4 (iv) (Herz's condition).

Proof of Lemma 9.1'. By Lemma 9.4 (vi) (and the fact that A is a translational invariant algebra) and Lemma 9.4 (v) it suffices to show that, given $\varepsilon > 0$, $K > 1$ we can find a closed $F \subseteq [-\varepsilon, \varepsilon]$ of synthesis with $\sup_{0 \neq f \in A(F)} \|f\|_{A(F)} / \|f\|_{\tilde{A}(F)} \geq K$.

Take E and S as in Lemma 9.5 (ii) with $E \subseteq [-\varepsilon/2, \varepsilon/2]$. We may suppose without of loss of generality that $\hat{S}(0) = 1 = \|S\|_{PM}$. (If not, use the existence of a $k \in \mathbf{Z}$ with $\hat{S}(k) = \|S\|_{PM}$ and consider $\chi_{-k} S / \|S\|_{PM}$). Since $\hat{S}(n) \to 0$ as $|n| \to \infty$ there exists a k_o with $|\hat{S}(k)| \leq K^{-1}$ for all $|k| \geq k_o$. Choose an r_o such that $M(r_o) \geq 100 \max(\varepsilon^{-1}, Kk_o)$. Set $x = \sum_{r=r_o}^{\infty} \pi/M(r)$. Automatically $\chi_{M(r)}(x) \to -1$ as $r \to \infty$.

Now by the proof of the lemma of Kaufman and Björk (Lemma 4.1 (iii)) applied to E we can find for each r $a^{(r)}, a^{(r)}, \ldots \geq 0$ with $\sum_{s=r}^{\infty} a_s^{(r)} = 1$ and $\left\| \sum_{s=r}^{\infty} a_s^{(r)} \chi_{M(s)} - 1 \right\|_{C(E)} \to 0$. By the last sentence of the last paragraph

$\|\sum_{s=r}^{\infty} a_s^{(r)} \chi_{M(s)} + 1\|_{C(E+x)} \to 0$ and so writing $f(e) = 1$ for $e \in E$, $f(e) = -1$ for $e \in E+x$ we have $\sup_{e \in F} |\sum_{s=r}^{\infty} a_s^{(r)} \chi_{M(s)}(e) - f(e)| \to 0$. We deduce at once that $f \in \tilde{A}(E \cup (E+x))$, $\|f\|_{\tilde{A}(E \cup (E+x))} \leq 1$, $f \in C(E \cup (E+x))$, E , $(E+x)$ are closed disjoint sets.

Set $F = E \cup (E+x)$. Since $E \cap (E+x) = \emptyset$ and E, $E+x$ are closed we know that $f \in A(F)$. We wish to estimate $\|f\|_{A(F)}$. To this end consider the pseudomeasure $T = S - \delta_x * S$. We have at once $\langle f, T \rangle = \langle f, S - \delta_x * S \rangle = \langle 1, S \rangle + \langle -1, -\delta_x * S \rangle = \langle 1, S \rangle + \langle 1, S \rangle = 2$. On the other hand if $|k| \leq k_o$ then $|\hat{T}(k)| = |\hat{S}(k) - \hat{\delta}_x(k)\hat{S}(k)| = |\hat{S}(k)|\,|1 - \hat{\delta}_x(k)| = |\hat{S}(k)|\,|1 - \chi_k(x)| \leq |kx|\,|\hat{S}(k)| \leq 2k_o\,|x|\,\|S\|_{PM} \leq 2K^{-1}$ whilst if $|k| > k_o$ we have $|\hat{T}(k)| \leq |\hat{S}(k)| + |\hat{\delta}_x(k)|\,|\hat{S}(k)| \leq 2K^{-1}$. But since $S \in A(E)'$ (and so by translation $\delta_x * S \in A(E+x)'$) it follows by arguments similar to Lemma 9.4 (iv) that $T \in A(E \cup (E+x))'$. Thus $\|f\|_{A(F)} \geq |\langle f, T \rangle| / \|T\|_{PM} \geq K$ and we are done.

Remark. Suppose E is a Helson set. Then $F = E \cup (y+E)$ is also Helson (if we demand $E \cap (x+E) = \emptyset$ this is trivial) and so, even if F supports a true pseudo-function, we have $A(F) = C(F) = \tilde{A}(F)$. The argument above works because S is synthesisable on F.

The set E which we constructed in Lemma 9.5 and with which we worked above is not of translation because we had to adjoin points to make it of synthesis. However if we drop the condition E of synthesis we can certainly take E a translational set. Under these circumstances $F = E \cup (x+E)$ becomes a translational set with $\sup_{0 \neq f \in A(E)} \frac{\|f\|_{A(F)}}{\|f\|_{\tilde{A}(E)}} \geq K$. Using this hint we can construct a translational set with

$\sup_{0\neq f\in A(F)} \frac{\|f\|_{A(F)}}{\|f\|_{A(F)}} = \infty$. Let us give yet another form of Lemma 1.11.

LEMMA 9.5'. There exists a translational set E together with sequences $0 = q(1) < q(2) < \ldots$ and $3 < M(1) < M(2) < \ldots < M(r) < \ldots$ of integers with $M(r+1)$ integral multiple of $2M(r)$ and $M(r+1) \geq 3^{3^r} M(r)$ and a sequence (a_r) of positive numbers such that

(a) $\sum_{r=q(s)+1}^{q(s+1)} a_r = 1, \quad \|\sum_{r=q(s)+1}^{q(s+1)} a_r\chi_r - 1\|_{C(E)} \leq 2^{-s} \qquad [s \geq 1]$

(b) E carries a true pseudofunction synthesisable on S.

Proof. The set E constructed in Lemma 1.11 is indeed a translational set. By the note at the end of the first paragraph of the proof we can certainly find $M(r+1) \geq 3^{3^r} M(r)$ with $\liminf_{r\to\infty} \int |\chi_{M(r)} - 1|\,d\mu = 0$ for all $\mu \in M^+(E)$. But by the arguments of Lemma 4.1 (iii) given a set and a $q(s) \geq 0$ we can always find $q(s+1)$ and $a_{q(s)+1}, a_{q(s)+2}, \ldots, a_{q(s+1)} \geq 0$, $\sum_{r=q(s)+1}^{q(s+1)} a_r = 1$ with $\|\sum_{r=q(s)+1}^{q(s+1)} a_r\chi_{M(r)} - 1\|_{C(E)} \leq 2^{-s}$. The lemma is proved.

Proof of Lemma 9.2. Take E, $q(s)$, $M(r)$, a_r as in Lemma 9.5'. As in the proof of Lemma 9.1' we may take $\hat{S}(0) = \|S\|_{PM} = 1$. Now take $\Lambda(1), \Lambda(2), \ldots$ infinite subsets of $\mathbf{Z}^+$ with $\Lambda(i) \cap \Lambda(j) = \emptyset$ for $i \neq j$ and $r \geq i+2$ for all $r \in \Lambda(i)$. Set $x_i = \sum_{s\in\Lambda(i)} \sum_{r=q(s)+1}^{q(s+1)} \pi/M(r)$. Since $\sum_{i=1}^{\infty} |x_i| \leq \sum_{r\in Z^+} \pi/M(r) < \pi$ the set $E^* = \{\sum_{i=1}^{\infty} \varepsilon_i x_i : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1\}$ is a well defined translational set and so $F = E + E^*$ is a translational set.

Now if $q(s)+1 \leq r \leq q(s+1)$, $s \in \Lambda(j)$ we have

$|\chi_{M(r)}(\sum_{i=1}^{\infty} \varepsilon_i x_i) - 1| = |\prod_{i=1}^{\infty} (\chi_{M(r)}(x_i))^{\varepsilon_i} - 1| \leq 3^{-q(s)}$. In particular writing

$F_{j1} = E + \{\sum_{i=1}^{\infty} \varepsilon_i x_i : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1, \quad \varepsilon_j = 0\}$

$F_{j2} = E + \{\sum_{i=1}^{\infty} \varepsilon_i x_i : \varepsilon_i = 0 \text{ or } \varepsilon_i = 1, \quad \varepsilon_j = 1\}$ and taking $f_j(e) = 1$ for $e \in F_{j1}$, $f_j(e) = -1$ for $e \in F_{j2}$ we have $\sup_{e \in F} |\sum_{r=q(s)+1}^{q(s+1)} a_r \chi_{M(r)}(e) - f_j(e)| \leq 3^{-q(s)} \to 0$ as $s \to \infty$, $s \in \Lambda(j)$. Thus $f_j \in \tilde{A}(F)$, $\|f_j\|_{\tilde{A}(F)} \leq 1$, $f_j \in C(F)$, F_{j1} and F_{j2} are disjoint closed sets, $f_j \in A(E)$.

We now wish to estimate $\|f_j\|_{A(F_j)}$. In order to do this, consider the pseudomeasure $T = S - \delta_{x_j} * S \in PM(F)$. Since $|x_j| \leq \sum_{r \geq j+2} \pi/M(r) \leq 3^{-2j}$ we have at once $|\hat{T}(k)| = |\hat{S}(k)|\,|1 - \chi_k(x)| \leq 3^{-2j}|k| \leq 3^{-j}$ for all $|k| \leq 3^j$ whilst $|\hat{T}(k)| \leq |\hat{S}(k)| + |\hat{\delta}_{x_j}(k)|\,|\hat{S}(k)| \leq \sup_{|k| \geq 3^j} |\hat{S}(k)|$. Since S is synthesisable on E, T is synthesisable on $E \cup (x_j + E) \subseteq F$. Thus

$$\|f_j\|_{A(F)} \geq \frac{\langle T, f_j \rangle}{\|T\|_{PM}} = \frac{2}{\|T\|_{PM}} \geq \frac{2}{\max(3^{-j}, \sup_{|k| \geq 3^j} |\hat{S}(k)|)} \to \infty \quad \text{as } j \to \infty.$$ It follows

that, as stated, $A(F)$ is not closed in $(\tilde{A}(E), \| \|_{\tilde{A}(F)})$.

To illustrate further the kind of proof we shall use we give a proof of Lemma 9.1. Since we shall obtain much stronger results later the reader should not bother with the details of the proof of Lemma 9.6.

LEMMA 9.6. Suppose we are given $w \in \mathbf{T}$, $\delta, \varepsilon, \eta > 0$ a sequence $\varepsilon(r) > 0$ decreasing with $\varepsilon(r) \to \infty$ and integers $3 < P(1) < P(2) < \ldots$ such that $3P(k)$ is a factor of $P(k+1)$. Then we can find a sequence $q(1) < q(2) < \ldots$ of integers, a translational set E, points $x, y \in \mathbf{T}$ and a pseudomeasure T synthesisable on

E such that

(i) $\|T * \delta_x - T * \delta_y\|_{PM} \le \eta/2$, $\hat{T}(0) = 1$

(ii) $(x+E) \cup (y+E) \subset [w-\delta, w+\delta]$

(iii) $(x+E) \cap (y+E) = \emptyset$

(iv) Writing $f(t) = 1$ for $t \in x+E$, $f(t) = -1$ for $t \in y+E$ we have $f \in \tilde{A}(E)$, $\|f\|_{\tilde{A}(E)} \le 1$ (and so trivially $\|f\|_{\tilde{A}(E)} = 1$).

(v) $E \subseteq [-\varepsilon_{P(q(n))}, \varepsilon_{P(q(n))}] + \{2\pi r/P(q(n)) : 2\pi r/P(q(n)) \in E, \ r \in Z\}$

Before constructing such a set let us see how by using this result we can prove Lemma 9.1.

LEMMA 9.7. Suppose $\varepsilon_n > 0$ chosen as in Lemma 9.4 (viii). With the notation of Lemma 9.6 $x+E$, $y+E$ are of isometric synthesis but writing $F = (x+E) \cup (y+E)$ $f \in A(F)$, $\|f\|_{A(F)} \ge \eta^{-1}$ and so is a translational set of bounded synthesis but with constant greater than η^{-1}.

Proof. That $x+E$, $y+E$ are of isometric synthesis follows from condition (v) and Lemma 9.4 (viii). That $(x+E) \cup (y+E)$ is of bounded synthesis then follows from condition (iii) and Lemma 9.4 (iv). To show $\|f\|_{A(E)} \ge \eta^{-1}$ we argue as in Lemma 9.1' and Lemma 9.2 that since $S = T * \delta_x - T * \delta_y \in A(F)'$ we have

$$\|f\|_{A(F)} \ge \frac{|\langle T, S\rangle|}{\|S\|_{PM}} \ge \frac{2}{2\eta} = \eta^{-1}.$$

Proof of Lemma 9.1. By condition (ii) of Lemma 9.6 and the conclusions of Lemma 9.7 we can find sets $F_n \subseteq [2^{-(4n+3)}, 2^{-(4n+1)}]$ such that F_n is the union of 2 disjoint sets of isometric synthesis F_{n1}, F_{n2} but $\sup\{\|f\|_{A(F_n)} : \|f\|_{\tilde{A}(F_n)} \leq 1, f\in A(F_n)\} \geq 2^n \to \infty$. Thus $E = \bigcup_{n=1}^{\infty} F_n \cup \{0\}$ is the closed union of disjoint sets of isometric synthesis $\{0\}, F_{11}, F_{12}, F_{21}, F_{22}, F_{31}, \ldots$ (so by Lemma 9.4 (v) of synthesis) but (by Lemma 9.4 (vi)) not of bounded synthesis.

It remains to show how to construct the set E of Lemma 9.6.

Proof of Lemma 9.6. Suppose $\delta > 3^{-n}$. By considering only $\{P(r) : r \geq 2n+2\}$ we may suppose 3^{2n+10} a factor of $P(r)$. Then if E_o, x_o, y_o satisfy conditions (i), (ii), (iii), (iv), (v) with w replaced by 0 and δ replaced by $\delta/2$ it would follow that (setting $k = \lfloor w3^{-(n+6)}/2\pi \rfloor$) $E = E_o + 2\pi k/3^{n+6}$, $x = x_o$, $y = y_o$ satisfy the original conditions (i), (ii), (iii), (iv), (v). We may therefore suppose $w = 0$ (we can then take $y = 0$).

Our proof now runs much as Lemma 1.11. We construct inductively T_n, E_n, $q(n)$, x_n, δ_n such that

(i) $P(q(n))E_n = 0$, $E_n \subseteq \delta(1-2^{-n})/2$

(ii) $T_n \in M(E_n)$

(iii) $\hat{T}_n(0) = 1 = \|T\|_{PM}$

(iv) $\|T_n - T_n * \delta_{x_n}\|_{PM} \leq \eta(1 - 2^{-n})/2$

(v) $P(n)x_n = 0$, $|x_n| \leq (1-2^{-n})\delta/2$

(vi) $\delta_n P(q(n)) \leq 2^{-2n-10}\varepsilon_{P(q(n))}\delta$.

By Lemmas 1.9 and 1.9' we can find an $m(n+1)$, a $q(n+1) > q(n)$ and $Q(n) = M_{n+1}(0) < M_{n+1}(1) < M_{n+1}(2) < \ldots < M_{n+1}(m(n+1)+1) = P(n+1)$ together with a measure S_{n+1} such that

(vii) $M_{n+1}(m(n+1)+1) = P(q(n+1))$

(viii) $|\hat{S}_{n+1}(u)| \leq \eta/8$ for all $Q(n) \leq |u| \leq R(n+1) - Q(n)$

(ix) $M_{n+1}(1) \geq 2^{10} + 10\eta^{-1}Q(n)$

(x) $M_{n+1}(r)$ is a factor of $M_{n+1}(r+1)$

(xi) $M_{n+1}(r+1) \geq 10^8 \eta^{-1} M_{n+1}(r)$ $[0 \leq r \leq m(n+1)]$

(xii) $R(n+1)\text{supp } S_{n+1} = 0$

(xiii) $\text{supp } S_{n+1} \subseteq \delta_n/8$

(xiv) $\|S_{n+1}\|_{PM} = \hat{S}_{n+1}(0)$

(xv) $\left\|(m(n+1))^{-1} \sum_{r=1}^{m(n+1)} \chi_{M_{n+1}(r)} - 1\right\|_{C(\text{supp } S_{n+1})} \leq 2^{-n-4}$.

Set

(a) $x_n = \sum_{r=1}^{m(n+1)} \pi/M_{n+1}(r) + x$

(b) $E_{n+1} = E_n + \text{supp } S_{n+1}$

(c) $T_{n+1} = T_n * S_{n+1}$.

Conditions (i), (ii), (iii), (v) of the induction are satisfied more or less trivially (using (xiii), (xiv), (xv) and (x)). To check (iv) we consider the value of $\hat{T}_{n+1}(u) - (\delta_{x_{n+1}} * T_{n+1})\hat{\ }(u)$ in the 2 cases $|u| \leq Q(n)$, $P(n+1)/2 \geq |u| > Q(n)$.

If $|u| < Q(n)$ then (using (ix) and (a))

$$\begin{aligned}|\hat{T}_{n+1}(u) - (T_{n+1} * \delta_{x_{n+1}})\hat{\ }(u)| &\leq \|S_{n+1}\|_{PM} \, |\hat{T}_n(u) - (T_n * \delta_{x_{n+1}})\hat{\ }(u)| \\ &\leq \|S_{n+1}\|_{PM} \, |\hat{T}_n(u) - (T_n * \delta_{x_n})\hat{\ }(u)| \\ &\quad + \|S_{n+1}\|_{PM} \|T_n\|_{PM} |1 - \hat{\delta}_{x_n}(u)| \\ &\leq |\hat{T}_n(u) - (T_n * \delta_{x_n})\hat{\ }(u)| + \eta 2^{-n-4} \\ &\leq \eta(1-2^{-n-1})/2.\end{aligned}$$

If $|P(n+1)| \geq |u| > Q(n)$ then (using (viii))

$$|\hat{T}_{n+1}(u) - (T_{n+1} * \delta_{x_{n+1}})\hat{}(u)| \leq 2\|S_{n+1}\|_{PM}\|T_n\|_{PM} \leq \eta/4 \leq \eta(1-2^{-n-1})/2.$$

Thus (iv) is satisfied and taking δ_{n+1} to satisfy (vi) the induction may continue.

We note that by (b) (vi) and (xiii)

(xvi) $\quad E_{n+1} \subseteq [-\epsilon_{P(q(n))}/2 , \epsilon_{P(q(n))}/2] + \{2\pi r/P(q(n)) \in E_{n+1}, \ r \in Z\}$

whilst for the same reasons (and by condition (ii))

(xvii) $\quad E_n \subseteq E_{n+1} \subseteq E_n + [-\epsilon_{P(q(n-1))}/2 , \epsilon_{P(q(n-1))}/2]$

(xviii) If $f \in A(\mathbf{T})$, $\|f\|_{A(\mathbf{T})} = 1$, $\hat{f}(r) = 0$ for $r \geq Q(n) = P(q(n))$

then $\|f\|_{C(E_{n+1})} \leq \|f\|_{C(E_n)} + 2^{-2n-2}$, $\|f\|_{C(E_{n+1}+x_{n+1})} \leq \|f\|_{C(E_n+x_n)} + 2^{-2n-2}$

Finally we note from (x), (xi), (a) and (xv)

(xix) $\quad \|m(n+1)^{-1} \sum_{r=1}^{m(n+1)} \chi_{M_{n+1}(r)} - 1\|_{C(E_{n+1})} \leq 2^{-n-3}$

(xx) $\quad \|m(n+1)^{-1} \sum_{r=1}^{m(n+1)} \chi_{M_{n+1}(r)} - 1\|_{C(E_{n+1}+x_{n+1})} \leq 2^{-n-3}.$

Now let E be the topological limit of the E_n and let $x_n \to x$ (note that $|x_n - x_{n-1}| \leq 2^{-n}$). By (i) and (v) E , $x + E \subseteq [-\delta, \delta]$. Let T be a weak limit point of the T. From (ii) $T \in (A(E))'$, from (iii) $\hat{T}(0) = 1 = \|T\|_{PM}$, from (iv) $\|T - T * \delta_x\|_{PM} \leq \eta/2$. From (xvi) and (xvii) $E \subseteq [\epsilon_{-P(q(n))} , \epsilon_{P(q(n))}] + \{2\pi r/P(q(n)) : 2\pi r/P(q(n)) \in E\}$. Finally using (xix), (xx) and (xviii) we see that $E \cap (E+x) = \emptyset$ and that writing $f|E = 1$, $f|E+x = -1$ we have $f \in C(E \cup (E+x))$ and $\|m(n)^{-1} \sum_{r=1}^{m(n)} \chi_{M_n(r)} - f\|_{C(E\cup(E+x))} \leq 2^{-n} \to 0$ as $n \to \infty$ so $f \in \tilde{A}(E \cup (E+x))$, $\|f\|_{\tilde{A}(E\cup(E+x))} \leq 1$.

The reader may have wondered if the proof of Lemma 9.1 given above could not be modified to give Theorem 9.3 (i). His suspicions would be justified, and originally

I intended to present a proof of Theorem 9.3 (i) rather than Lemma 9.1. However as written it turned out to depend on manipulations which though based on principles simpler than those needed in Theorem 9.3 (ii) are just as delicate. Possibly the reader may find a simpler presentation. Mine depended on

LEMMA 9.4. (x) We can find $\varepsilon_1(n) \to 0$, $\varepsilon_2(n) \to 0$ with the following property. Suppose E is a closed set such that we can find $x_n \to 0$, $Q(n) \to \infty$ with the following properties

(a) $\varepsilon_1(n) \geq |x_n| \geq \varepsilon_1(n)/2$

(b) If $e \in E$ then either $|e| \leq \varepsilon_2(n)$ or $|e - x_n - 2\pi r/Q(n)| \leq \varepsilon_2(n)$ for some $r \in \mathbf{Z}$.

Lemma 9.4 (x) and the results we shall give below show that although Herz's criterion is already so fine that no useful deep generalization (except to Pisot numbers) is available, there are a large number of ad hoc modifications which may be used to help construct particular thin sets. Let us give as an example.

LEMMA 9.7. (i) Given E a closed set with $\mathrm{Gp}E \neq T$, $\delta, \varepsilon > 0$, $Q \in \mathbf{Z}^+$, and $F \subseteq \{2\pi r/Q : 1 \leq r \leq Q\}$ we can find a closed independent set F^* with $F^* \subseteq F + [-\varepsilon, \varepsilon]$ and $\mathrm{Gp}F^* \cap \mathrm{Gp}E = \{0\}$ such that given $\mu \in M(F)$ we can find a $\mu^* \in M(E)$ such that

(a) $\|\mu^*\|_{PM} \leq \|\mu\|_{PM}$, $\|\mu^*\|_M \leq \|\mu\|_M$

(b) $\hat{\mu}^*(r) \to 0$ as $|r| \to \infty$

(c) $|\hat{\mu}(r) - \hat{\mu}^*(r)| \leq \delta ||\mu||_M$ for $|r| \leq Q$.

Moreover given a closed interval I with $\text{int } I \cap F^* \neq \emptyset$ we can find a $\sigma \in M^+(I \cap F^*)$ with $\hat{\sigma}(r) \to 0$ as $|r| \to \infty$.

(ii) Given E an independent closed set and $\varepsilon > 0$ we can find E^* an independent closed set with $E \subseteq E^* \subseteq E + [-\varepsilon, \varepsilon]$ and E^* of isometric synthesis.

(iii) There exists an independent closed set E^* of isometric synthesis such that given I a closed interval with $\text{Int } I \cap E^* \neq \emptyset$ we can find $0 \neq \sigma \in M^+(I \cap E^*)$ with $|\hat{\sigma}(r)| \to 0$ as $|r| \to \infty$.

Proof (i). By Lemma 5.6 we can construct inductively, $N_n \in \mathbf{Z}$ $E_n \subseteq \{2\pi r/N_n : 1 \leq r \leq N_n\}$, $\varepsilon_n > 0$ and a mapping $L_n : M(E_{n-1}) \to M(E_n)$ such that $E_o = F$, $\varepsilon_o = \varepsilon/4$

(1) $||L_n\mu||_{PM} \leq ||\mu||_{PM}$, $||L_n\mu||_M \leq ||\mu||_M$ for all $\mu \in M(E_{n-1})$

(2) $|(L_n\mu)\hat{}(r)| \leq 2^{-n}||\mu||_M$ for all $N_{n-1} - N_{n-2} \leq |r| \leq N_n - N_{n-1}$, $\mu \in M(E_{n-1})$, $n \geq 2$.

(3) $|(L_n\mu - \mu)\hat{}(r)| \leq 2^{-2n-8}\delta$ for all $|r| \leq nN(0)$, $\mu \in M(E_{n-1})$

(4) $|(L_n\mu)\hat{}(r)| \leq |\hat{\mu}(r)| + 2^{-n}||\mu||_M$ for all r, $\mu \in M(E_{n-1})$

(5)(a) $\text{supp } L_n\mu \subseteq \text{supp } \mu + [-\varepsilon_{n-1}/2, \varepsilon_{n-1}/2]$ for all $\mu \in M(E_{n-1})$

(5)(b) $E_n \subseteq E_{n-1} + [-\varepsilon_{n-1}/2, \varepsilon_{n-1}/2]$

(6) $\varepsilon_n \leq \varepsilon_{n-1}/4$

(7) If $x_1, x_2, \ldots, x_m \in E$ are distinct, $y_1, y_2, \ldots, y_m \in E_n + [-\varepsilon_n, \varepsilon_n]$ $|y_i - y_j| \geq \varepsilon_{n-1}$ for $i \neq j$, then $\Sigma n_i x_i + \Sigma n_i' y_i = 0$, $\Sigma|n_i|$, $\Sigma|n_j'| \leq n$ imply $n_1 = n_2 = \ldots = n_m = n_1' = n_2' = \ldots = n_m' = 0$.

Clearly $E_n \to F^*$ topologically where F^* is a closed independent set with $F^* \subseteq F + [-\varepsilon, \varepsilon]$, $\mathrm{Gp}F^* \cap \mathrm{Gp}\, E = \emptyset$.

Again if $\mu \in M(E)$ then $L_n L_{n-1} L_{n-2} \dots L_1 \sigma$ converges weakly to a measure $\mu^* = L\mu$ with properties (a), (b), (c). Finally note that if $\mathrm{Int}\, I \cap F^*$ we can find $x \in E_n$ with $x + [-\varepsilon(n), \varepsilon(n)] \subseteq I$. Setting $\sigma_m = L_m L_{m-1} \dots L_{n+1} \delta_x$ we see that σ_m converges to $0 \neq \sigma \in M(F^* \cap I)$ with $\hat{\sigma}(r) \to 0$ as $|r| \to \infty$.

Remark. The reader can easily construct a rather simpler proof using the ideas of [10] Section § 7.

(ii) By Lemma 9.4 (ix) for conditions (1), (2), (3) and the result just proved for (4), (5) and (6) we can construct inductively $Q(n)$, $\varepsilon(n)$, $K(n)$, $F(n)$, $F^*(n)$ such that writing $E^*(n-1) = F^*(1) \cup F^*(2) \cup \dots \cup F^*(n-1) \cup E$

(1) $\varepsilon(n) < \varepsilon'(n) < \min(\varepsilon(n-1),\ \varepsilon'(n-1) - \varepsilon(n-1)/(2^n Q(n-1)),\ \varepsilon(1) \leq \varepsilon/4$

(2) $F(n) \subseteq (E^*(n-1) + [-\varepsilon(n), \varepsilon(n)]) \cap \{2\pi r/Q(n) : 1 \leq r \leq Q(n)\}$

(3) If $T \in PM(E^*(n-1) + [-\varepsilon'(n), \varepsilon'(n)])$ and $\|T\|_{PM} \leq n$ then we can find $\mu \in M(F(n))$ with $\|\mu\|_{PM} \leq \|T\|_{PM}$, $\|\mu\|_M \leq K(n)$ such that $|\hat{T}(r) - \hat{\mu}(r)| \leq 2^{-n}$ for all $|r| \leq n$.

(4) If $\mu \in M(F(n))$ and $\|\mu\|_M \leq K(n)$ then we can find $\mu^* \in M(F^*(n))$ such that $\|\mu^*\|_{PM} \leq \|\mu\|_{PM}$ and $|\hat{\mu}^*(r) - \hat{\mu}(r)| \leq 2^{-n}$ for all $|r| \leq n$

(5) $F^*(n) \subseteq F(n-1) + \left[\frac{-\varepsilon'(n+1) - \varepsilon(n+1)}{2}, \frac{\varepsilon'(n+1) + \varepsilon(n+1)}{2}\right]$

(6) $E^*(n)$ is independent and closed.

Clearly $E^*(n)$ tends to a topological limit E^* say with $E \subseteq E^* \subseteq E + [-\varepsilon, \varepsilon]$, E^* independent. Since $E^* \subseteq E + [-\varepsilon'(n), \varepsilon'(n)]$ conditions (3) and (4) show that E is of isometric synthesis.

(iii) Set $E = \{e\pi\}$ and perform the construction of (ii). Note that by (i) we may take the $F(n)$ of local strong multiplicity and so have E^* of local strong multiplicity as desired. The reader faced with result (ii) above may reasonably feel that we have simply drowned a set E in a much larger set E^* which bears no relation to E at all. The point, after all, of the classical Herz theorem is that given any closed set E we can find $E^* \supseteq E$ of isometric synthesis such that E^* contains only countable many points not contained in E. But we are not seeking illuminating results but simply tools for particular constructions. In this case we obtain the result of Lemma 9.7 (iii) which is new (even if not deep). The reader unconvinced of the utility of ad hoc adaptations of Herz's theorem should try and use the standard tools to obtain (iii).

However the use of such adaptations does depend on establishing a strong enough connection between E and E^*. For our purposes the following results suffice.

LEMMA 9.8 (i). Suppose $M(1) < M(2) < M(3) < \dots$, $q(1) < q(2) < \dots$ increasing sequences of positive integers and a sequence $\varepsilon(r) > 0$ given with $M(r+1)$ a multiple of $M(r)$ and $M(r+1) \geq 10^{10^{10r}} M(r)$. Suppose E a closed set with $\mathrm{Gp}E \neq \mathbf{T}$, F a finite set $Q \geq 1$ with $F \subseteq \{2\pi r/Q : r \in \mathbf{Z}\}$ and $\delta, \varepsilon > 0$ given. Then we can find $h : \mathbf{Z}^+ \to \mathbf{Z}^+$ strictly increasing such that given $Q \leq p(0) < h(p(0)) < p(1) < h(p(1)) < p(2) < \dots$ can find a closed independent set $F^* \subseteq F + [-\varepsilon, \varepsilon]$ such that $\mathrm{Gp}F^* \cap \mathrm{Gp}E = \{0\}$ and given $\mu \in M(F)$ we can find a $\mu^* \in M(E)$ with

(a) $\|\mu^*\|_{PM} \leq \|\mu\|_M$, $\|\mu^*\|_M = \|\mu\|_M$

(b) $|\hat{\mu}^*(r) - \hat{\mu}(r)| \leq \delta \|\mu\|_M$ for $|r| \leq Q$.

Further we have

(c) $\|\chi_{M(r)} - (-1)^s\|_{C(F^*)} \le \varepsilon(r)/(10^3M(r))$ for all $q(s)\le r\le q(s+1)-1$, $h(p(t)) \le s \le p(t+1)-1$.

(ii) Suppose $M(j)$, $q(j)$, $\varepsilon(j)$, $\varepsilon > 0$, E given as in (i) such that $\|\chi_{M(q(s))}-1\|_{C(E)} \le \varepsilon(q(s))/4$ whenever s is a multiple of 6. Suppose further $\varepsilon(j)$ satisfies the conditions of Lemma 9.4 (viii). Then we can find $p(0) < p(0)+5 < p(1) < p(1)+5 < p(2) < \dots$ with $p(r)$ a multiple of 6 and a closed independent set $E \subseteq E^* \subseteq [-\varepsilon,\varepsilon]$ which is of isometric synthesis and such that for each $e\in E^*$ and each $p(t) \le s \le p(t)+3$ we have either

(i) $|\chi_{M(r)}(e) - (-1)^s| \le \varepsilon(r)/(10^3M(r))$ for all $q(s) \le r \le q(s+1)-1$

or

(ii) $|e - e'| \le \varepsilon(r)/(10^3M(r))$ for some $e'\in E$ and all $q(s) \le r \le q(s+1)-1$.

Moreover if $E \subseteq \bigcup_{n=1}^{\infty} \left[x + 2^{-4n-7/4}, x+2^{-4n-3/2}\right]\cup\{x\}$ we can ensure that $E^* \subseteq \bigcup_{n=1}^{\infty} \left[x+2^{-4n-3}, x+2^{-4n-1}\right]$.

Proof (i). We shall leave the details of the construction of h to the reader. We construct inductively finite sets $E_n \subseteq \{2\pi r/M(q(n)) : r\in\mathbf{Z}\}$ and $\delta_n > 0$ in the following manner (taking $E_0 = F$, $\varepsilon_0 = \varepsilon/4$). By Lemma 5.6 we can find $h'(q(n))$ and $E'_{n+1} \subseteq \{2\pi r/M(h'(q(n))) : r\in\mathbf{Z}\}$ together with a mapping $L'_n : M(E_n) \to M(E'_{n+1})$ such that

(1) $\|L'_n\mu\|_{PM} \le \|\mu\|_{PM}$, $\|L'_n\mu\|_M \le \|\mu\|_M$ for all $\mu\in M(E_{n-1})$

(2) $|(L'_n\mu - \mu)\hat{}(r)| \le 2^{-2n-8}\delta$ for all $|r| \le M(p(n))$

(3) $E'_n \subseteq E_{n-1} + [-\delta_{n-1}/2, \delta_{n-1}/2]$

(4) $\delta'_n < \delta_{n-1}/8$, $M(h'(q(n)))\delta'_n \ge 100$

(5) If $x_1, x_2, \ldots, x_m \in E$ are distinct, $y_1, y_2, \ldots, y_m \in E'_n + [-\varepsilon_n, \varepsilon_n]$, $|y_i - y_j| \geq \varepsilon_{n-1}$ for $i \neq j$ then $\Sigma n_i x_i + \Sigma n'_i y_i = 0$, $\Sigma |n_i|$, $\Sigma |n'_i| \leq n$ imply $n_1 = n_2 = \ldots = n_m = n'_1 = n'_2 = \ldots = n'_m = 0$.

Now choose $p(n+1) > h'(p(n))$, set $E_{n+1} = E'_{n+1} + x$ where $x = \sum_{s=h'(p(n))}^{p(n+1)-1} \sum_{r=q(s)}^{q(s+1)-1} \pi(1-(-1)^s)/2M(r)$, set $L_{n+1}\mu = L'_{n+1}\mu * \delta_x$, choose $\delta_{n+1} = \min(\varepsilon(p(n)), \delta'_n)/(10^{10}M(p(n+1)))^2$ and restart the induction.

On taking F^* to be the topological limit of the E_n and μ^* to be the topological limit of the E_n and μ^* to be the weak limit of $L_n L_{n-1} \ldots L_1\mu$ the results desired may be read off as in Lemma 9.7 (i).

(ii) Assume without loss of generality that $\varepsilon(n+1) \leq \varepsilon(n)/200$. Using Lemma 9.4 (viii) and the result just proved we can construct inductively $p(n)$, $F(n)$, $K(n) \geq 1$, $F^*(n)$ and $\Lambda(n) \subseteq \mathbf{Z}^+$ such that $\Lambda(n)$ contains arbitrarily long sequences such that writing $E^*(n-1) = F^*(1) \cup F^*(2) \cup \ldots \cup F^*(n-1) \cup E$ we have $p(n) > p(n-1)+5$, $p(n) + r \in \Lambda(n-1)$ for $6 \geq r \geq 0$, $p(n)$ a multiple of 6

(1) $F(n) \subseteq (E^*(n-1) + [-\varepsilon(p(n))/4\ ,\ \varepsilon(p(n))/4]) \cap \{2\pi r/M(p(n)) : r \in \mathbf{Z}\}$

(2) If $T \in PM(E^*(n-1) + [-\varepsilon(p(n))\ ,\ \varepsilon(p(n))])$ and $\|T\|_{PM} \leq n$ then we can find a $\mu \in M(F(n))$ with $\|\mu\|_M \leq K(n)$, $\|\mu\|_{PM} \leq \|T\|_{PM}$ such that $|\hat{T}(r) - \hat{\mu}(r)| \leq 2^{-n}$ for all $|r| \leq n$.

(3) If $\mu \in M(F(n))$ and $\|\mu\|_M \leq K(n)$ then we can find $\mu^* \in M(F^*(n))$ such that $\|\mu^*\|_{PM} \leq \|\mu\|_{PM}$ and $|\hat{\mu}^*(r) - \hat{\mu}(r)| \leq 2^{-n}$ for all $|r| \leq n$

(4) $F^*(n) \subseteq F(n) + [-\varepsilon(p(n))/2\ ,\ \varepsilon(p(n))/2]$

(4)' If $E \subseteq \{x\} \cup \bigcup_{r=1}^{\infty} [x+2^{-4r-7/4}\ ,\ x+2^{-4r-3/2}]$ then we can ensure

$F^*(n) \subseteq \bigcup_{r=1}^{\infty} [x+2^{-4r-3}, x+2^{-4r-1}]$

(5) $\Lambda(n) \subseteq \Lambda(n-1)$ is such that $\Lambda(n)$ contains arbitrarily long sequences and $\| \chi_{M(r)} - (-1)^s \|_{C(F(n))} \leq \varepsilon(r)/(10^3 M(r))$ for all $q(s) \leq r \leq q(s+1)-1$, $s \in \Lambda(n)$.

The required results now follow as in Lemma 9.7 (ii).

Remark. We note in passing the corollary that given E Dirichlet and independent we can find $E^* \supseteq E$ independent Dirichlet and of isometric synthesis. This should be compared with Theorem 1.1'. Note that if $E^* \supseteq E$ and E^* is Helson then if E supports a pseudomeasure E^* cannot be of synthesis. Thus every Helson set of synthesis is of resolution and no similar embedding process is possible.

To apply Lemma 9.8 we need a complementary Lemma.

LEMMA 9.9. (i) Suppose $M(1) < M(2) < M(3) < \ldots$, $q(1) < q(2) < \ldots$ increasing sequences of positive integers with $q(r) - q(r-1) \to \infty$ and a sequence $\varepsilon(r) > 0$ given with $M(r+1)$ a multiple of $M(r)$ and $M(r+1) \geq 10^{10^{10r}} M(r)$. Suppose E a closed set with $GpE \neq T$, $x \in E$ given together with an integer $k > 2$. Then we can find $h : \mathbf{Z}^+ \to \mathbf{Z}^+$ strictly increasing such that given $3 \leq p(0) < h(p(0)) < p(1) < h(p(1)) < \ldots$ we can find a closed independent set $F \subseteq x + [2^{-4k-7/4}, 2^{-4k-3/2}]$ with $GpF \cap GpE = \{0\}$ such that $F = F_1 \cup F_2$ where F_1, F_2 are closed and disjoint having the following properties

(1) $\| \chi_{M(q(p(r)))} - 1 \|_{C(F)} \leq \varepsilon(r)/(10^3 M(r))$

(2) There exist $\ell(1) < \ell(2) < \ldots$ such that

$$\Big\| (q(\ell(2t))-1-q(\ell(2t-1)))^{-1} \sum_{r=q(\ell(2t-1))+1}^{q(\ell(2t))-1} \chi_{M(r)} - 1 \Big\|_{C(F_1)} \leq 2^{-t}$$

$$\|(q(\ell(2t))-1-q(\ell(2t-1)))^{-1}\sum_{r=q(\ell(2t-1))+1}^{q(\ell(2t))-1}\chi_{M(r)}-1\|_{C(F_2)}\leq 2^{-t}.$$

(3) There exist T_j pseudomeasures on F_j $[j = 1,2]$ such that $\|T_1 - T_2\|_{PM} \leq \delta$, $\hat{T}_1(0) = \hat{T}_2(0) = \|T_1\|_{PM} = \|T_2\|_{PM} = 1$.

(ii) Suppose $M(1) < M(2) < \dots$ given as in (i). Then given $x \notin 2\pi Q$ we can find an x, $k(n)$ strictly increasing, $F_{1n}, F_{2n} \subseteq [x+2^{-4k(n)-4-7/8}, x+2^{-4k(n)-4-3/2}]$ together with an increasing sequence of positive integers $q(1) < q(2) < \dots$ such that writing $F = \{x\} \cup \bigcup_{n=1}^{\infty}(F_{1n} \cup F_{2n})$ we have F independent and

(1) $\|\chi_{M(q(s))}-1\|_{C(F)} \leq \varepsilon(q(s))/4$ whenever s is a multiple of 6

(2) For each n we can find a sequence $\ell(t) \to \infty$ such that

$$\|(q(\ell(t)+1)-1-q(\ell(t)))^{-1}\sum_{r=q(\ell(t))}^{q(\ell(t)+1)-1}\chi_{M(r)}-1\|_{C(F_{1n})}\leq 2^{-t}$$

$$\|(q(\ell(t)+1)-1-q(\ell(t)))^{-1}\sum_{r=q(\ell(t))}^{q(\ell(t)+1)-1}\chi_{M(r)}+1\|_{C(F_{2n})}\leq 2^{-1}.$$

(3) There exist $T_{jn} \in PM(F_{jn})$ such that $\|T_{1n} - T_{2n}\| \leq 2^{-n}$, $\hat{T}_{1n}(0) = \hat{T}_{2n}(0) = \|T_{1n}\|_{PM} = \|T_{2n}\|_{PM} = 1$.

Proof (i). This is along the same lines as those given earlier so we merely sketch it. We construct inductively $\eta(n) > 0$, measures S_{1n}, S_{2n} with $\operatorname{supp} S_{1n} = E_{1n}$, $\operatorname{supp} S_{2n} = E_{2n}$, an integer $p(n)$ with $(p(n))(E_{1n} \cup E_{2n}) = 0$ such that

$$E_{1n} \cup E_{2n} \subseteq x + 2^{-4k-13/8} + [-2^{-4k-20}(1-2^{-k}), 2^{-4k-20}(1-2^{-k})]$$

and $\|S_{1n} - S_{2n}\|_{PM} \leq \delta(1-2^{-k})$, $\hat{S}_{1n}(0) = \hat{S}_{2n}(0) = 1 = \|S_{1n}\|_{PM} = \|S_{2n}\|_{PM}$.
(To start the induction we could take $S_{1m} = S_{2m} = \delta_{2\pi/Q}$ for a suitable Q and m).

Now by Lemma 5.6 we can certainly find $\eta'(n+1) > 0$ and $h'(p(n))$ and

S'_{1n}, S'_{2n} with supp $S'_{1n} = E'_{1n}$, supp $S'_{2n} = E_{2n}$ $q(h'(p(n)))(E'_{1n} \cup E'_{2n}) = 0$ such that $E'_{jn} \subseteq E_{jn} + [-\eta(n), \eta(n)]$, $E'_{1n} \cup E'_{2n} \subseteq x + 2^{-4k-13/8} + [-2^{-4k-20}(1-2^{-k-1/2}),$ $2^{-4k-20}(1-2^{-k-1/2})]$, $\|S'_{1n} - S'_{2n}\|_{PM} \leq \delta(1-2^{-k-1/2})$,
$\hat{S}'_{1n}(0) = \hat{S}'_{2n}(0) = 1 = \|S'_{1n}\|_{PM} = \|S'_{2n}\|_{PM}$, and further given $x_1, x_2, \ldots, x_m \in E$ distinct, $y_1, y_2, \ldots, y_m \in E' + [-\eta'(n+1), \eta'(n+1)]$, $z_1, z_2, \ldots, z_m \in E'_{2n}$ with $|y_i - y_j|$, $|z_i - z_j| \geq 4\eta(n)$ for $i \neq j$ we know that $\Sigma k_i x_i + \Sigma k'_i y_i + \Sigma k''_i z_i = 0$, $\Sigma |k_i| + \Sigma |k'_i| + \Sigma |k''_i| \leq n$ implies $k_1 = k_2 = \ldots = k_m = k'_1 = \ldots = k'_m = k''_1 = \ldots = k''_m = 0$.

Next using qualitative improvement of Lemma 1.9 found in Section § 5 we know that given any $\varkappa(n+1) > 0$ we can find $\ell(2n) > \ell(2n-1) \geq h'(p(n)) + 3$ and a measure σ_{n+1} such that

(i) $M(q(\ell(2n)+1))$ supp $\sigma_{n+1} = 0$

(ii) $|\hat{\sigma}_{n+1}(u)| \leq \delta/8$ for all $M(q(\ell(2n)+1)) - M(h'(p(n))+1) \geq u \geq M(h'(p(n))+1))$

(iii) $\left\|(q(\ell(2n)) - q(\ell(2n-1)-1))^{-1} \sum_{r=q(\ell(2n-1))}^{q(\ell(2n)-1)} \chi_{M(r)} - 1\right\| \leq 2^{-n-5}$

(iv) supp $\sigma_{n+1} \subseteq [-\varkappa(n+1), \varkappa(n+1)]$.

Choose $\varkappa(n+1) \leq 2^{-10k-10}\eta'(n+1)$. Set $y_{n+1} = \sum_{r=q(\ell(2n-1))}^{q(\ell(2n)-1)} \pi/M(r)$ and $S_{1n+1} = S'_{1n}$, $S_{2n+1} = S'_{2n+1} * \delta_{y_{n+1}}$. Set $\eta(n+1) = \eta'(n+1)\varepsilon(q(\ell(2n)))/(2^{20n+20}M(q(\ell(n))))$ and choose $p(n+1) \geq q(\ell(2n))+1$. (With a little work this gives the definition of h). We check as in Lemma 9.6 that the induction can be restarted.

It is trivial that $F_{1n} \to F_1$, $F_{2n} \to F_2$ topologically and $S_{1n} \to T_1$, $S_{2n} \to T_2$ weakly where F_1, F_2, T_1, T_2 satisfy the conclusions of the lemma.

(ii) Choose $x \notin 2\pi Q$ with $|\chi_{M(r)}(x) - 1| \leq \varepsilon(r)/4$ for all r. By repeated use of (1) we can define $\Lambda(n) \subseteq \mathbf{Z}^+$, $\Gamma_1(n) \subseteq \mathbf{Z}^+$, $\Gamma_2(n) \subseteq \mathbf{Z}^+$ and F_{1n}, F_{2n} closed sets such that

(1) $\Lambda(n)$ contains arbitrarily long sequences, $\Lambda(n) \subseteq \Gamma_2(n)$.

(2) $F_{1n+1} \cup F_{2n+1} \subseteq \left[x+2^{-4k(n+1)-4-7/8}, x+2^{-4k(n+1)-4-3/2}\right]$

where $k(n+1) > k(n)$ and we can find $q(6n+6) \in \Gamma_1(n)$ such that $M(q(6n+6))^2(\varepsilon(q(6n+6)))^{-1} \leq 2^{k(n+1)}$.

(3) $\{x\} \cup F_{11} \cup F_{12} \cup \ldots \cup F_{2n+1}$ is independent.

(4) We can find finite sequences $\{q(\ell(t)), q(\ell(t))+1, \ldots, q(\ell(t)+1)\} \subseteq \Lambda(n)$ such that $\Lambda'(n+1) = \Lambda(n) \setminus \bigcup_{t=1}^{\infty} \{q(\ell(t)), q(\ell(t))+1, \ldots, q(\ell(t)+1)\}$ contains arbitrarily long sequences and

$$\left\|(q(\ell(t)+1)-1-q(\ell(t)))^{-1} \sum_{r=q(\ell(t))}^{q(\ell(t)+1)} \chi_{M(r)} - 1\right\|_{C(F_{1n})} \leq 2^{-t}$$

$$\left\|(q(\ell(t)+1)-1-q(\ell(t)))^{-1} \sum_{r=q(\ell(t))}^{q(\ell(t)+1)} \chi_{M(r)}+1\right\|_{C(F_{2n})} \leq 2^{-t}.$$

(5) There exist $T_{jn} \in PM(F_{jn})$ such that $\|T_{1n} - T_{2n}\|_{PM} \leq 2^{-n}$, $\hat{T}_{1n}(0) = \hat{T}_{2n}(0) = \|T_{1n}\|_{PM} = \|T_{2n}\|_{PM} = 1$.

(6) We can find $\Gamma_1(n+1) \subseteq \Gamma_1(n) \setminus \{q(6n+6)\}$ such that $\Lambda(n+1) = \Gamma_1(n+1) \cap \Lambda'(n+1)$ contains arbitrarily long sequences and $\|\chi_{M(s)} - 1\|_{C(F_{1n+1} \cup F_{2n+1})} \leq \varepsilon(s)$ whenever $s \in \Gamma_1(n+1)$.

We can restart the induction. The stated results are easily read off. (Actually not all the $q(n)$ are explicitly defined but the reader can easily correct this).

Combining Lemmas 9.8 and 9.9 we get

LEMMA 9.10. Suppose $\varepsilon(n) \to 0$ given. We can find closed sets E_1, E_2 with

$E_1 \cap E_2 = \{x\}$ such that $E_1 \cup E_2$ is independent, together with a sequence of integers $3 \leq k(1) < k(2) < \ldots$ having the following properties

(a) $E_1 \cup E_2 \subset \{x\} \cup \bigcup_{r=1}^{\infty} \left[x+2^{-4k(r)-3}, x+2^{-4k(r)-1}\right]$

(b) Writing $E_{jn} = E_j \cap \left[x+2^{-4k(r)-3}, x-2^{-4k(r)-1}\right]$ $[j = 1,2]$

we know that writing $f_n \mid E_{jn} = (-1)^{j+1}$ we have $f_n \in \tilde{A}(E_{1n} \cup E_{2n})$ $\|f\|_{\tilde{A}(E_{1n} \cup E_{2n})} = 1$.

(c) There exist $T_{jn} \in PM(E_{jn})$ with $\|T_{1n} - T_{2n}\|_{PM} \leq 2^{-n}$

$\hat{T}_{1n}(0) = \hat{T}_{2n}(0) = \|T_{1n}\|_{PM} = \|T_{2n}\|_{PM} = 1$

(d) E_1, E_2 are of isometric synthesis

(e) We can find $Q(s) \to \infty$ with $\|\chi_{M(Q(s))} - 1\|_{C(E_1 \cup E_2)} \leq \varepsilon(M(Q(s)))$.

Proof. Choose $M(r)$ as in the statements of Lemmas 9.8, 9.9. Construct x, F_{jn}, $k(n)$ and so on as in Lemma 9.9 (ii). Set $F_j = \bigcup_{j=1}^{\infty} F_{jn} \cup \{x\}$. Using condition (1) of Lemma 9.9 (ii) we may apply Lemma 9.8 (ii) to obtain $E_1 \supseteq F_1$, $E_2 \supseteq F_2$ satisfying (a), (e) and (d). But combining Lemma 9.9 (ii) condition (2) with Lemma 9.8 (ii) conditions (i) and (ii) we see that we can also satisfy (b). That condition (c) holds is trivial.

Proof of Theorem 9.3. We claim that E_1, E_2 in Lemma 9.10 have the properties stated in the conclusions of the Theorem. This is obvious except possibly for the statement $E_1 \cup E_2$ not of bounded synthesis. To show this we observe $E_{jn} = E_j \cap \left[x+2^{-k(n)-3}, x+2^{-k(n)-1}\right]$ is of bounded synthesis. (In fact looking at the construction we see that E_{jn} is of isometric synthesis. This proves Lemma 9.1"). Thus $T_{jn} \in (A(E_{jn}))'$ and $\|f_n\|_{A(E_{1n} \cup E_{2n})} \geq \dfrac{\langle T_{1n} - T_{2n}, f_n \rangle}{\|T_{1n} - T_{2n}\|_{PM}} \geq 2^n$. Using

Lemma 9.4 (v) we see that $E_1 \cup E_2$ is not of bounded synthesis. This concludes the proof, the section and the paper.

I should like, as so many have done before me, to pay tribute to Madame Dumas for her excellent typing and for her invariable helpfulness. The errors that remain are my own.

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APPENDIX. The state of play.

The object of this appendix is to summarize the work contained in [A4], [A5], [A6], [A7], [A8]. In the last part we give some open questions. Where necessary for completeness the work of others has been quoted, but this should not mislead the reader into thinking that this is a general survey of results on thin sets, let alone of any wider field. If he wants a fair, balanced and complete picture of work on thin sets he should consult [A1], [A2], [A3], [A9] and [A10]; we shall give him nothing of the sort.

A.1 DEFINITIONS. We work on the circle group $\mathbf{T} = R/2\pi\mathbf{Z}$ unless otherwise stated.

$\chi_n(t) = \exp int$ $[t \in \mathbf{T}]$

$A(\mathbf{T}) = \left\{\sum_{r=-\infty}^{\infty} a_r\chi_r : \sum_{r=-\infty}^{\infty} |a_r| < \infty\right\}$. This is a Banach algebra under the norm $\|\sum_{r=-\infty}^{\infty} a_r\chi_r\| = \sum_{r=-\infty} |a_r|$

$A^+(\mathbf{T}) = \left\{f \in A(\mathbf{T}) : \hat{f}(n) = 0 \quad \forall n < 0\right\}$ a subalgebra of $A(\mathbf{T})$

$A_\Lambda(\mathbf{T}) = \left\{f \in A(\mathbf{T}) : \hat{f}(n) = 0 \quad \forall n \notin \Lambda\right\}$ a subspace but not necessarily a subalgebra of $A(\mathbf{T})$

$S(\mathbf{T}) = \left\{f \in C(\mathbf{T}) : |f(t)| = 1 \quad \forall t \in \mathbf{T}\right\}$

If $E \subseteq \mathbf{T}$ (not necessarily closed) then

E is an <u>N_0 set</u> if we can find $\Lambda \subseteq \mathbf{Z}$ infinite such that $\sum_{r \in \Lambda} |\sin rt| < \infty$ $\forall t \in E$.

E is an <u>R set</u> if we can find $a_r \geq 0$, $\theta_r \in \mathbf{T}$ such that $\limsup_{r\to\infty} |a_r| > 0$, $\sum_{r=1}^{\infty} a_r(\sin rt + \theta_r)$ converges $\forall t \in E$.

E is an <u>N set</u> if we can find $a_r \geq 0$ such that $\sum_{r=1}^{\infty} a_r = \infty$, $\sum a_r |\sin rt| < \infty$ $t \in E$.

We write $PM(\mathbf{T})$ for the dual of $A(\mathbf{T})$. Since $A(\mathbf{T}) \supseteq C^{\infty}(\mathbf{T})$ the members of $PM(\mathbf{T})$ have a well defined support. If E is closed then we write $PM(E) = \{S \in PM(\mathbf{T}) : \text{supp } S \subseteq E\}$. We say that

E is <u>without true pseudomeasure</u> (WT) if $PM(E) = M(E)$.

E is of <u>synthesis (S)</u> if every $S \in PM(E)$ is the weak limit of $\mu_{\alpha} \in M(E)$.

E is of <u>bounded synthesis</u> (BS) with constant at most C if every $S \in PM(E)$ is the weak limit of a bounded sequence $\mu_n \in M(E)$ with $\|\mu_n\|_{PM} \leq C$.

E is of <u>resolution (RE)</u> if every closed subset of E is of synthesis.

E is of <u>uniqueness (U)</u> if there exists no non zero $S \in PM(E)$ with $\langle S, \chi_r \rangle \to 0$ as $|r| \to \infty$.

E is of uniqueness in the broad sense (U_0) if there exists no non zero $\mu \in M(E)$ with $\langle \mu, \chi_r \rangle - 0$ as $|r| \to \infty$.

Continuing to confine ourselves to closed sets E we say that

E is <u>Kronecker (K)</u> if $\inf_{r \in \mathbf{Z}} \|f - \chi_r\|_{C(E)} = 0 \quad \forall f \in S(E)$

E is <u>Dirichlet (D)</u> if $\liminf_{|r| \to \infty} \|1 - \chi_r\|_{C(E)} = 0$

E is <u>Weak Kronecker (WK)</u> (respectively Weak Dirichlet) if given $\mu \in M^+(E)$, $\varepsilon > 0$ we can find $E_1 \subseteq E$ closed with $|\mu|(E \setminus E_1) < \varepsilon$ and E_1 Kronecker (respectively Dirichlet)

E is <u>Helson</u> if $C(E) = A(E)$.

The <u>Helson constant</u> s of a Helson set E is given by $s = \inf_{0 \neq f \in C(E)} \frac{\|f\|_{C(E)}}{\|f\|_{A(E)}}$ (some authors, more sensibly, use s^{-1}). We use the definition

$s = \inf_{\mu \in M(E), \|\mu\|=1} \limsup_{n \to \infty} |\hat{\mu}(n)|$ which is not known to be equivalent numerically (except, of course, for $s = 1$) for all Helson sets but may be made so for the sets we construct (see [A5] § 7) and is more easily handled.

E is a Helson-1 (H_1) (respectively a Helson-s set) if it is a Helson set with constant 1 (respectively a constant s).

Turning now to the algebra $A^+(\mathbf{T})$, suppose E is again a closed set

E is AA^+ if $A(E) = A^+(E)$

An AA^+ set E has AA^+ constant $C = \sup_{0 \neq f \in A(E)} \frac{\|f\|_{A^+(E)}}{\|f\|_{A(E)}}$.

E is peak set (P) if we can find an $0 \neq f \in A^+(\mathbf{T})$ with $f(e) = 1$ for all $e \in E$, $|f(t)| < 1$ for all $t \notin E$.

E is an exact zero set (EZ) if we can find an $0 \neq f \in A^+(\mathbf{T})$ with $f(e) = 0$ for all $e \in E$, $|f(t)| < 1$ for all $t \notin E$.

E is a zero set (Z) if we can find an $0 \neq f \in A^+(\mathbf{T})$ with $f(e) = 0$ for all $e \in E$.

Finally miscellaneous notations.

A set E (closed or not) is said to be independent if it is independent over Q (i. e. if $x_1, \ldots, x_k \in E$ are distinct then $\sum_{j=1}^{k} m_j x_j = 0$, $m_j \in \mathbf{Z}$ implies $m_1 = m_2 = \ldots = m_k = 0$).

If E is closed and countable we write $E^{(1)}$ for the set of limit points of E (clearly closed), $E^{(\lambda+1)} = E^{(\lambda)(1)}$ for λ an ordinal, $E^{(\mu)} = \bigcap_{\lambda<\mu} E^{\lambda}$ for μ a limit ordinal.

A.2. RESULTS. Each result is labelled by a number which in turn gives a reference in A3. First we give a table of results on closed sets. It should be read as follows "every set in the right hand column is (Y) (respectively need not be (N)) a set in the upper row". Except for the result labelled by an asterisk the addition of the condition "independent" makes no difference. The notations FU, CU refer to stability under finite union and under closed countable union.

	WT	RE	S	BS	U	U_o	K	H_1	H	D	N_o	R	N	AA^+	P	EZ	Z	FU	CU
WT	Y	Y	Y	Y	Y	Y	N	N	Y	N	N	N	N	Y	Y	Y	Y	Y	N
RE	N	Y	Y	?	Y	Y	N	N	N	N	N	N	N	N	N	N	N	?	?
S	N	N	Y	N	N	N	N	N	N	N	N	N	N	N	N	N	N	?	?
BS	N	N	Y	Y	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
U	N	N	N	N	Y	Y	N	N	N	N	N	N	N	N	N	N	N	Y	Y
U_o	N	N	N	N	N	Y	N	N	N	N	N	N	N	N	N	N	N	Y	Y
K	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N
H_1	N	N	N	N	N	Y	N	Y	Y	N	N	N	Y	Y	Y	Y	Y	N	N
H	N	N	N	N	N	Y	N	N	Y	N	N	N	N	Y	Y	Y	Y	Y	N
D	N	N	N	N	Y	Y	N	N	N	Y	Y	Y	Y	Y	Y	Y	Y	N	N
N_o	N	N	N	N	?	Y	N	N	N	N	Y	Y	Y	?	?	?	?	N	N
R	N	N	N	N	?	Y	N	N	N	N	N	Y	Y	?	?	?	?	N	N
N	N	N	N	N	N	Y	N	N	N	N	N	N	Y	N	N	N	N	N	N
AA^+	N	N	N	N	N	Y	N	N	N	N	N	N	N	Y	?	?	Y	Y	N
P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	Y	Y	Y	?	N
EZ	N	N	N	N	N	N	N	N	N	N	N	N	N	N	?	Y	Y	Y	N
Z	N	N	N	N	N	N	N	N	N	N	N	N	N	N	?	?	Y	Y	N

Note : WK $\overset{*}{=}$ H_1, WD = N

	WT	RE	S	BS	U	U_o	K	H_1	H	D	N_o	R	N	AA^+	P	EZ	Z	FU	CU
WT	0	0	0	0	3	3	4	4	3	4	4	4	4	3	3	3	3	5	6
RE	6	0	0	?	7	7	4	4	6	4	4	4	4	6	6	6	6	?	?
S	6	8	0	9	10	10	4	4	6	4	4	4	4	6	6	6	6	?	?
BS	6	8	0	0	10	10	4	4	6	4	4	4	4	6	6	6	6	9	9
U	8	8	8	8	0	0	4	4	6	4	4	4	4	6	6	6	6	11	11
U_o	8	8	8	8	12	0	4	4	6	4	4	4	4	6	6	6	6	13	13
K	14	14	14	14	14	15	0	0	16	0	0	0	0	17	18	18	18	4	4
H_1	8	8	8	8	12	15	19	0	0	19	19	19	1	17	18	18	18	4	4
H	8	8	8	8	12	15	4	4	0	4	4	4	4	17	18	18	18	20	6
D	8	8	8	8	21	0	22	22	22	0	0	0	0	23	24	24	24	4	4
N_o	8	8	8	8	?	0	22	22	22	25	0	0	1	?	?	?	?	4	4
R	8	8	8	8	?	0	22	22	22	25	26	0	1	?	?	?	?	4	4
N	8	8	8	8	12	0	22	22	22	25	26	19	0	27	27	27	27	4	4
AA^+	8	8	8	8	12	28	4	4	22	4	4	4	4	0	?	?	0	29	6
P	8	8	8	8	12	30	4	4	22	4	4	4	4	6	0	0	0	?	6
EZ	8	8	8	8	12	30	4	4	22	4	4	4	4	6	?	0	0	0	6
Z	8	8	8	8	12	30	4	4	22	4	4	4	4	6	?	?	0	0	6

Note (1) , (2).

We give some further results on closed sets.

SUM QUESTIONS $(E_1 + E_2 = \{e_1 + e_2 : e_1 \in E_1 \ , \ e_2 \in E_2\})$.

(31) The sum of 2 Kronecker sets can be the whole of $\mathbf{T}$ even if one is fixed arbitrarily (it must however be uncountable).

(32) For every $1 > s \geq 0$ there exists an E with Helson constant at least s and $GpE = \mathbf{T}$.

(33) There exists a non Dirichlet set every proper closed subset of which is Kronecker.

(34) There exists a Dirichlet non Kronecker set every proper closed subset of which is Kronecker.

(35) For every $1 \geq s > 0$ there exists an independent Helson set E of constant s, a $\mu \in M^+(E)$ with $\mu(E \cap N) > 0$ whenever N is open and $E \cap N \neq \emptyset$ such that every closed subset of E with positive μ measure has Helson constant s.

(36) For every $1 > s > 0$ there exists a Helson set of constants such that every closed subset of E has higher Helson constant.

(37) Given $1 > s \geq 0$ we can find an independent weak Dirichlet set with Helson constant greater than s.

RESULTS ON A_Λ.

(38) The conditions $A_{\Lambda(1)}(E_1) = C(E_1)$, $A_{\Lambda(2)}(E_2) = C(E_2)$ do not imply $A_{\Lambda(1)\cup\Lambda(2)}(E_1 \cup E_2) = C(E_1 \cup E_2)$.

(39) If $\liminf_{n\to\infty} \operatorname{card}\{\Lambda \cap [-n,n]\}/n = 0$ then we can find a Kronecker set E with $A_\Lambda(E) \neq C(E)$.

(40) If Λ is an arithmetical progression then $A(E) = C(E)$ implies $A_\Lambda(E) = C(E)$.

(41) Write, in an obvious notation, $A_\Lambda(T^2) = \{\sum a_{rs} \chi_{rs} ,\ a_{rs} = 0\ \ \forall (r,s) \notin \Lambda\}$. If Λ is a half plane of $\mathbf{Z}^2$, $E \subseteq \mathbf{T}^2$ then $A(T^2)(E) = C(E)$ implies $A_\Lambda(T^2)(E) = C(E)$.

(42) This result is best possible.

RESULTS ON CLOSED COUNTABLE SETS.

(43) There exist countable independent Dirichlet sets which are not Kronecker.

(44) Every countable closed set is AA^+ with constant 1.

(45) The independent union of disjoint Kronecker sets need not be Dirichlet even if one is fixed arbitrarily.

(46) If $E^{(n)} = \emptyset$ and E is independent, then E is the union of a finite number of Kronecker sets. This result is false for general E with $E^{(\omega+1)} = \emptyset$.

(47) If $E^{(n)} = \emptyset$ then E is peak.

(48) If $E^{(\omega+n)} = \emptyset$ then E is an exact zero.

(49) There exists an E independent with $E^{(\omega+1)} = \emptyset$ such that

$$\sup_{f \in A(E) \cap S(E)} (\inf \{\|f - \sum_{r=-\infty}^{\infty} a_r \chi_{p(r)}\|_{C(E)} : p(r) \in \mathbf{Z}\}) = 1 \quad \text{for every} \quad \sum_{r=-\infty}^{\infty} |a_r| < \infty .$$

MISCELLANEOUS.

(50) If $E \subseteq \mathbf{T}$ is closed of measure 0 then we can find $a_r \to 0$ and $1 < q(1) < q(1)+1 < q(2) < q(2)+1 < q(3) < \ldots$ such that given any $f \in A(D)$ (the uniform algebra of functions analytic on the interior of the unit disk D, continuous on D) we can find $p(r)$ with value $q(r)$ or $q(r)+1$ and $\sum_{r=1}^{\infty} a_r \chi_{p(r)} \in A(D)$

$\sum_{r=1}^{\infty} a_r \chi_{p(r)} | E = f | E.$

(51) There exists a (not closed) independent R set which is not an N set.

REMARK. Whenever the statement of a result can be extended to a locally compact Abelian group of an appropriate type without becoming trivially false its proof will also extend without difficulty.

A.3. REFERENCES AND REMARKS.

(0) Obvious.

(1) Salem proved that every N set is weak Dirichlet (see e. g. Lemma 1.7 [A4] or Chapter XIII [A9]). Björk and Kaufman proved independently that every weak Dirichlet set is an N set (see e. g. Lemma 4.2 [A7] or [A9], Chapter XIII). If E is an R set then automatically there exists a sequence $m(j) \to \infty$ with $\chi_{m(j)}(e) \to 1$ for all $e \in E$ and thus by Lebesgue's dominated convergence theorem E is weak Dirichlet. (Recall that all our sets are closed).

(2) Every Helson 1 set is the translate of a weak Kronecker set, every translate of a weak Kronecker set is Helson 1, every independent Helson 1 set is weak Kronecker and every weak Kronecker set is an independent Helson 1 set. (Lemma 1.4 [A5] and Lemma 1.7 [A4]; similar results were obtained independently by Lafontaine). Note that since the translate of a Dirichlet set is Dirichlet (evident) every Helson-1 set is weak Dirichlet.

(3) See [A3] p. 139.

(4) There exist 2 disjoint Kronecker sets whose union E is independent and carries a positive measure μ with $\limsup_{n\to\infty} |\hat{\mu}(n)| = \frac{1}{2}$, $\|\mu\| = 1$. Clearly E is without true pseudomeasure (since it is the union of 2 disjoint sets without true pseudomeasure) but cannot be a weak Dirichlet set. (Theorem 7 § 7 [A4]).

We note also the following results.

(4a) The union of n independent Kronecker sets has Helson constant at least $1/n$. (Varopoulos [A9] Chapter X §1).

(4b) There exist n independent Kronecker sets (one of which may be fixed in advance whose union has Helson constant $1/n$ (Theorem 7 § 7 [A5] for a partial result, Theorem 4 § 8 [A4] for the full result).

(5) Varopoulos ([A9] Chapter X, Lemma 2.9).

(6) Consider a sequence $\{x_o\} = E_o, E_1, E_2, \ldots$ of disjoint Kronecker sets with $E_n \to E_o$ topologically and $E = \bigcup_{r=o}^{\infty} E_r$ independent. Automatically E is of resolution and if E is AA^+ then E is an exact zero (see [A8]). However we can construct E such that E is not a zero set for $A^+(T)$ (proof in [A8], a weaker result is given as Theorem 8 7 [A4]). Note also

(6a) We can construct E so that E is peak but not AA^+ [A8].

(6b) We can construct E so that E is Dirichlet but not Helson (Lemma 8.6 [A5]).

(7) Malliavin [A3] Chapter V § 8.

(8) By Theorem 1.1' [A7] given $1 \geq 10n\, \varepsilon(n) > 0$ we can find $m(n) \to \infty$ and an $E \subseteq \bigcap_{n=1}^{\infty} \bigcup_{r=1}^{m(n)} [2\pi r/m(n)-\varepsilon(m(n)),\ 2\pi r/m(n)+\varepsilon(m(n))]$ such that E is weak

Kronecker but not of synthesis. Provided only we demand that $n\varepsilon(n) \to 0$ we know that E is Dirichlet. Again we can find $E^* \subset \bigcap_{n=1}^{\infty} \bigcup_{r=1}^{m(n)} \{I = [2\pi r/m(n) - \varepsilon(m(n)), 2\pi r/m(n) + \varepsilon(m(n))] : E \cap I \neq \emptyset\}$ such that $E \cup E^*$ is independent and of bounded synthesis with constant 1 (Lemma 9.9 [A7]) but not of resolution (since E is not of synthesis).

(9) The independent union E of 2 sets of bounded synthesis with constant 1 intersecting at one point only (so that E is readily seen to be of synthesis) need not be of bounded synthesis (Theorem 9.3 (ii) [A7]). The first example of a set of synthesis not of bounded synthesis is due to Varopoulos (Chapter XII [A9]).

(10) By Lemma 9.7 [A7] there exist independent closed sets E locally of strong multiplicity (so not U_o) which are of bounded synthesis with constant 1.

(11) This a result of Bary ([A1] Volume 2, p. 355).

(12) By Theorem 1.1 [A6] there exists a weak Kronecker set carrying a true pseudomeasure (i.e. which is not a U set).

(13) Suppose $E_1, E_2, \ldots$ are closed U_o sets and $E = \bigcup_{i=1}^{\infty} E_i$ is closed but not U_o. Then we can find $\mu \in M(E)$ with $\hat{\mu}(n) \to 0$ as $n \to \infty$. Now we can find E_k such that $|\mu|(E_k) > 0$ and we can find a sequence of $f_n \in C^{\infty}(\mathbf{T})$ such that $\|f_n\mu - \mu|E\| \to 0$. But $C^{\infty}(\mathbf{T}) \supseteq A(\mathbf{T})$ so that $\lim_{|r| \to \infty} \int f_n \chi_r d\mu = 0$ for each fixed n and thus $\limsup_{|r| \to \infty} \int_{E_k} \chi_r d\mu \leq \|f_n\mu - \mu|E\|$. It follows that $(\mu|E_k)\hat{}\,(r) \to 0$ as $r \to \infty$ contrary to the hypothesis.

(14) Varopoulos ([A9] Chap. VIII).

(15) Easy. For this and other results, see e. g. [A3] p. 139.

(16) Use successive approximation. Or see e. g. [A9] Chap. I.

(17) Wik. See e. g. [A9] Chap. II.

(18) Varopoulos [A9] Chap. X.

(19) Theorem 3 § 3 [A4].

(20) By work of Varopoulos [A11] extending work of Drury we know that the union of 2 Helson sets is Helson : we know that the independent union of a Helson s and a Helson t set is Helson with constant at least $\frac{t^2 s^2}{t^2 + s^2}$. We have examples ([A4] Lemma 7.12) for all $1 \geq t, \quad s > 0$ of independent disjoint Helson s and Helson t sets whose union has Helson constant $\frac{ts}{t+s}$.

(21) Kahane. See e. g. Lemma 4.1 (ii) [A7].

(22) Theorem 9 § 8 [A4] or use (6b).

(23) Varopoulos. See e. g. Lemma 4.1 (iii) [A7].

(24) Drury ; an improvement of his method due to Varopoulos is given in [A6], where we use it to prove that the finite union of Dirichlet sets is peak.

(25) Lemma 4.3 [A4].

(26) Theorem 6 § 10 [A5].

(27) Theorem 8.1 [A7].

(28) Easy. For this and other results see [A2].

(29) The union of AA^+ sets with constants C_1, C_2 is an AA^+ set with constant at most $C_1+C_2+C_1C_2$ and this value can be attained for all $C_1, C_2 \geq 1$ (§ 7 of [A7], [A6]).

(30) [A8].

(31) Lemma 3.4 [A4], Lemma 8.4 [A5]. First proved by Varopoulos.

(32) Lemma 7.10 [A4].

(33) Theorem 2 § 3 [A4].

(34) Corollary 5.1 § 5 [A4].

(35) Theorem 1 § 3 [A5].

(36) Lemma 4.7 [A5].

(37) Lemma 7.11 [A4].

(38) Theorem 7.2 [A7].

(39) Theorem 3 § 7 [A5].

(40) Easy consequence of (20).

(41) Bernard ; see e. g. Chap. II [A9]. This is a generalization of Wik's result (17).

(42) Hedberg ; an improvement of his result due to Katznelson forms Theorem 2 § 5 [A5].

(43) Theorem 4 § 5 [A4].

(44) p. 148 [A2].

(45) Theorem 1 § 3 [A4]. First proved by Bernard and Varopoulos.

(46) Salinger C.R.A.S.P. 272, p. 786. Similarly if $E^{(n)} = \emptyset$ then E is the union of a finite number of Dirichlet sets.

(47) Use the remarks following (46) and (24).

(48) [A8].

(49) [A8].

(50) [A8].

(51) Theorem 5 § 9 [A5].

A.4. OPEN QUESTIONS.

The following is a list of problems on thin sets. An attempt has been made to grade them according to interest and "estimated difficulty". Thus (A) represents a problem which is believed to be both very important and very difficult, (B) a problem which may well be important and might be difficult, grades (C) and (D) are given to problems which either seem to be within the range of present techniques or, although likely to be difficult are not considered likely to prove important.

(1) (A) Is every symmetric set of constant ratio of dissection of synthesis ?

(2) (B) Is every symmetric set of synthesis ? of bounded synthesis ?

(For the best results known on these 2 questions see [A 10]).

(3) (A) Is the union of 2 (respectively the closed countable union of) sets of synthesis of synthesis ?

(4) (B) Is the union of 2 (respectively the closed countable union of) sets of resolution of resolution ?

(5) (A) If E is a closed set such that non real analytic functions operate on $A(E)$ is E necessarily Helson ?

(6) (B) Is every zero set an exact zero set ?

(7) (B) Is every closed countable set an exact zero set ?

(8) (C) Is every zero set (respectively exact zero set, AA^+ set, AA^+ set with constant 1) a peak set ?

(9) (B) Is every AA^+ set an exact zero set ?

(10) (C) Is the union of 2 peak sets necessarily peak ?

(11) (C) Suppose E is a closed set such that we can find $n(1) < n(2) < \dots$

with $\chi_{n(r)}(e) \to 1$ for all $e \in E$. Is E necessarily a zero set ? Can E carry a non zero pseudo function ?

(12) (D) Is every N_o set (respectively R set) a set of uniqueness ? An AA^+ set ? A peak set ? An exact zero set ? A zero set ?

(13) (B) Is every set of resolution necessarily of bounded synthesis ?

(14) (C) Does there exist a closed set E with $A(E)$ dense in $(A(E), \| \|_{A(E)})$ but $A(E) \neq A(E)$?

(15) (B) Is the basis conjecture always true for $A(E)$?

(16) (B) Is the sum of 2 Kronecker sets (respectively 2 sets of synthesis) always of synthesis ?

(17) (C) Is every set of synthesis Ditkin ? Is any uncountable closed set Ditkin ?

(For a short discussion see the appendix "Note sur des recherches récentes et en cours" [A3]).

(18) (D) Find $\inf \{\text{Helson constant } E_1 \cup E_2 : E_i \text{ has Helson constant } \alpha_i\}$. Find $\inf \{\text{Helson constant } E_1 \cup E_2 : E_i \text{ has Helson constant } \alpha_i \ \ GpE_i \cap GpE_i = \{0\}\}$.

(19) (C) Write $h_1(E) = \inf\limits_{\mu \in M(E), \|\mu\|=1} \sup |\hat{\mu}(n)|$, $h_2(E) = \inf\limits_{\mu \in M(E), \|\mu\|=1} \limsup\limits_{|n| \to \infty} |\hat{\mu}(n)|$ and $c(\alpha) = \inf\limits_{h_1(E) = \alpha} h_2(E)$. Can $c(\alpha)$ be less than 1 ?

(20) (A) Characterize countable Helson sets by their arithmetical properties.

(21) (B) Is the Kahane-Salem necessary "Maille condition" (see [A2] pp.30-34) also sufficient for a closed countable set to be Helson ?

The last two questions concern the tensor algebras $V(D^\infty)$, $V(T)$ (see e. g. [A2] Chap VIII).

(22) (C) Is every set of interpolation for V of synthesis ?

(23) (C) Is every closed independent set of interpolation for V ?

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