Astérisque

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Astérisque, tome 5 (1973), p. 231-239

<http://www.numdam.org/item?id=AST_1973__5__231_0>

A PSEUDOFUNCTION ON A HELSON SET. II.

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ABSTRACT. A simpler proof is given of the existence of a pseudofunction on a Helson set.

This note is devoted to the bitter sweet task of replacing the contents of sections 1 to 3 of the paper above by a short demonstration of the main result. This is achieved by a much simpler demonstration of the main combinatorial lemma (Lemma 1 below) without using Conway's lemma and by passing directly from the result for weak Dirichlet to the result for weak Kronecker sets.

LEMMA 1. Let $\Psi(m) = \{\emptyset \neq S \subseteq \{1, 2, \ldots, m\}\}$ and set

$f_S(T) = 1$ if $S \subseteq T$

$f_S(T) = 0$ otherwise.

If $1 > \lambda > 0$ write

$$B(\lambda, m) = \inf\Big\{\sum_{S\in\Psi(m)} |a_S| : \sum_{S\in\Psi(m)} a_S f_S(T) = 1 \quad \text{for all} \quad T\in\Psi(m), \text{ card } T \geq \lambda m\Big\}.$$

Then $B(\lambda, m) \to \infty$ as $m \to \infty$.

Proof. Suppose $\sum_{S\in\Psi(m)} a_S f_S(T) = 1$ for all $T\in\Psi(m)$, card $T \geq \lambda m$.
Then if $\Sigma(m)$ is the permutation group on $\{1, 2, \dots m\}$ it follows that

$\sum_{S\in\Psi(m)} a_S f_S(\sigma T) = 1$ for all $T \in \Psi(m)$, card $T \geq \lambda m$, $\sigma\in\Sigma(m)$ and so

$\sum_{\sigma\in\Sigma(m)} \sum_{S\in\Psi(m)} a_S f_S(\sigma T) = \sum_{\sigma\in\Sigma(m)} 1$ for all $T \in \Psi(m)$, card $T \geq \lambda m$. Thus

$\sum_{s=1}^{m} \Big(\sum_{S\in\Psi(m),\ \mathrm{card}\, S = s} a_S \Big)\gamma_{s,t,m} = 1$ for all $m \geq t \geq \lambda m$ where

$$\gamma_{s,t,m} = \frac{\sum_{\sigma\in\Sigma(m)} f_S(\sigma T)}{\sum_{\sigma\in\Sigma(m)} 1} = \frac{t}{m}\,\frac{(t-1)}{(m-1)} \cdots \frac{(t-s+1)}{(m-s+1)} \quad \left[\mathrm{card}\ T = t, \quad 1 \leq t\ ,\ s \leq m\right].$$

Thus, noting that $\sum_{1\leq s\leq m} \left|\sum_{\mathrm{card}\, S = s} a_S\right| \leq \sum |a_S|$, we see that if

$B(\lambda\ , m) \not\to \infty$ we can find a $B > 0$ and $m(1), m(2), \dots$ together with

$\alpha_{s,m(j)}$ such that

$$\sum_{s=1}^{m(j)} |\alpha_{s,m(j)}| \leq B$$

and $\sum_{s=1}^{m(j)} \alpha_{s,m(j)}\gamma_{s,t,m(j)} = 1$ for all $m \geq t \geq \lambda m$. Now, since $\sum_{s=1}^{m(j)} |\alpha_{s,m(j)}| \leq B$
it follows that we can find $j(k) \to \infty$ such that $\alpha_{s,m(j(k))} \to \alpha_s$ and since

$|\gamma_{s,t,m(j)}| \leq \left(\frac{t}{m(j)}\right)^s$ it follows that allowing $\frac{t}{m(j)} \to x$ for some $1 > x > \lambda$ we have

$$\sum_{s=1}^{\infty} \alpha_s x^s = 1.$$

Thus $\sum_{s=1}^{\infty} |\alpha_s| \leq B$ and $\sum_{s=1}^{\infty} \alpha_s x^s = 1$ for all $1 \geq x \geq \lambda$ which is absurd. It follows that $B(\lambda\ , m) \to \infty$ as $m \to \infty$ and the lemma is proved.

LEMMA 2. Let $1 > \lambda > 0$, $m \geq 1$. Then, with the notation of Lemma 1, we can find $b_T\in\mathbb{C}$ $\left[T \in \Psi(m)\ ,\ \mathrm{card}\ T \geq \lambda m\right]$ such that

(i) $\sum_{T\in\Psi(m),\mathrm{card}\, T\geq\lambda m} b_T = 1$

(ii) $\left|\sum_{T\in\Psi(m),\mathrm{card}\, T\geq\lambda m} b_T f_S(T)\right| \leq B(\lambda\ ,m)^{-1}$ for all $S\in\Psi(m)$.

Proof. Write $E = \{T \in \Psi(m) : \text{card } T \geq \lambda m\}$ and observe that

$\Gamma = \{\sum_{S \in \Psi(m)} a_S f_S | E : \sum_{S \in \Psi(m)} |a_S| < B(\lambda, m)\}$ is a convex balanced subset of $C(E)$ which does not contain 1. Thus by the theorem of Hahn-Banach there exists a $\mu \in M(E)$ such that

(i) $\langle \mu, 1 \rangle = 1$

(ii)' $|\langle \mu, g \rangle| \leq B(\lambda, m)^{-1}$ for all $g \in \Gamma$

and so in particular

(ii) $|\langle \mu, f_S | E \rangle| \leq B(\lambda, m)^{-1}$ for all $S \in \Psi(m)$.

Writing $b_T = \mu(\{T\})$ we have the result.

Next let us establish some notation. Let D be the direct product of a countable number of copies of the group $\{-1, 1\}$ on 2 elements. We shall write the element $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \ldots) \in D$ $[\alpha_i = \pm 1]$ as $\sum_{i=1}^{\infty} 2\alpha_i/3^i$. The dual $\hat{D}$ of D consists of all strings $\underset{\sim}{\beta} = (\beta_1, \beta_2, \ldots)$ with $\beta_i = \pm 1$ and only a finite number of β_i equal to -1. We shall write $\underset{\sim}{\beta}$ as $\chi_{\sum_{i=1}^{\infty} \beta_i 2^i}$. Thus for example

$\chi_5(2/3 + 2/9) = \langle(-1, 1, -1, 1, 1, \ldots), (-1, -1, 1, 1, \ldots)\rangle = -1.$

LEMMA 3. Let $1 \leq n_1 < n_2 < \ldots < n_{m+1}$, $1 > \lambda > 0$. Set

$\rho_i = \mathop{*}_{j=n_i}^{n_{i+1}-1} (\delta_{2/3^j} + \delta_o)/2$ (where δ_t is the Dirac point mass at $t \in D$) and

$\sigma_T = \mathop{*}_{i \notin T} \rho_i$ $[T \in \Psi(m)]$. Then if, with the notation of Lemma 2, we set

$\mu = \sum_{T \in \Psi(m)} b_T \sigma_T$ we obtain

(i) $\hat{\mu}(r) = 1$ for all $0 \leq r < 2^{n_1}$

(ii) $|\hat{\mu}(r)| \leq B(\lambda, m)^{-1}$ for all $2^{n_1} \leq r < 2^{n_{m+1}}$

whilst setting $E = \operatorname{supp} \mu$ we have

(iii) $\|m^{-1} \sum_{i=1}^{m} \chi_{2^{n_i}} - 1\|_{C(E)} \leq 2(1-\lambda)$.

Proof. Since $\hat{\sigma}_T(r) = \prod_{i \notin T} \hat{\rho}_i(r) = 1$ for $0 \leq r < 2^{n_1}$ condition (i) of Lemma 3 follows directly from condition (i) of Lemma 2. On the other hand, suppose $2^{n_1} \leq r < 2^{n_m+1}$. Then $r = \sum_{j=1}^{n_{m+1}-1} \gamma_j 2^j$ where $\gamma_j = 0, 1$ and $S(r) = \{i : \exists n_i \leq j < n_{i+1} \text{ with } \gamma_j \neq 0\} \in \Psi_m$. Clearly $\hat{\rho}_i(r) = 0$ if $i \in S(r)$, $\hat{\rho}_i(r) = 1$ otherwise so that $\hat{\sigma}_T(r) = \prod_{i \notin T} \hat{\rho}_i(r) = f_{S(r)}(T)$ and condition (ii) of Lemma 3 follows directly from condition (ii) of Lemma 2.

Finally suppose $x \in E$. Then $x \in \operatorname{supp} \sigma_T$ for some $T \in \Psi_m$, card $T \geq \lambda m$. automatically $\chi_{2^{n_i}}(x) = 1$ if $i \notin T$, $\chi_{2^{n_i}}(x) = \pm 1$ in general and so $\left| m^{-1} \sum_{i=1}^{m} \chi_{2^{n_i}}(x) - 1 \right| \leq 2(1-\lambda)$.

LEMMA 4. We can find $1 = k(1) < k(2) < \ldots$ and $n(1) < n(2) < \ldots$ together with a closed set E such that E supports a pseudofunction T with $\hat{T}(0) = 1 = \|T\|_{PM}$ and

$$\|(k(j+1) - k(j))^{-1} \sum_{i=k(j)}^{k(j+1)-1} \chi_{2^{n(i)}} - 1\|_{C(E)} \leq 2^{-j} \qquad [j \geq 1].$$

Proof. By Lemma 1 we can find $k(1) < k(2) < \ldots$ such that $B(1-2^{j+1}, k(j+1)-k(j)) \leq 2^{-j}$. Now choose integers $n(1) < n(2) < \ldots$ By Lemma 3 we can find measures μ_j such that

(i) $\hat{\mu}_j(r) = 1$ for all $0 \leq r < 2^{n(k(j))}$

(ii) $|\hat{\mu}_j(r)| \le 2^{-j}$ for all $2^{n(k(j))} \le r < 2^{n(k(j+1))}$

whilst setting $E_j = \operatorname{supp} \mu_j$ we have

(iii) $\|(k(j+1) - k(j)) \sum_{i=k(j)}^{k(j+1)-1} \chi_{2^i} - 1\|_{C(E_j)} \le 2^{-j}$

and (iv) $\| \chi_{2^i} - 1\|_{C(E_j)} = 0$ whenever $0 \le i < k(j)$ or $k(j+1) \le i$. Note that (i), (ii) and (iv) show that $\|\mu_j\|_{PM} = 1$.

Now set $T_j = \mathop{*}_{i=1}^{j} \mu_j$. It is clear that $\|T_j\|_{PM} = \hat{T}_j(0) = 1$ and $\hat{T}_j(r) = \hat{T}_{j+1}(r)$ for all $r < k(j)$. Thus T_j converges weakly to a pseudomesure T with $\|T\|_{PM} = \hat{T}(0) = 1$. Since $|\hat{T}_j(r)| = \prod_{i=1}^{j} |\hat{\mu}_i(r)| \le 2^{-\ell}$ for all $2^{k(\ell)} \le r < 2^{k(\ell+1)}$ $[1 \le \ell \le j]$ it follows that $|\hat{T}(r)| \le 2^{-\ell}$ for all $2^{k(\ell)} \le r < 2^{k(\ell+1)}$ and so T is a pseudofonction. Using (iv) we see that $F_j = E_1 + E_2 + \ldots + E_j$ converges (in the topological sense) to a closed set E.

We want to show that $T \in PM(E)$. To this end suppose $f \in A(D)$, $\operatorname{supp} f \cap E = \emptyset$. Then $\operatorname{supp} f \cap E_j = \emptyset$ for j sufficiently large and so (since $T_j \in M(E_j)$) $\langle T_j, f\rangle = 0$ for j sufficiently large. Thus $\langle T, f\rangle = 0$ and $\operatorname{supp} T \subseteq E$ as required.

On the other hand, suppose $e \in E$. Then we can write $e = e_1 + e_2 + \ldots$ where $e_j \in E_j$. In particular, using (iv) we obtain $\chi_{2^{n(i)}}(e) = \chi_{2^{n(i)}}(e_j)$ for all $k(j) \le i < k(j+1)$. Thus by (iii)

$$\left|(k(j+1) - k(j))^{-1} \sum_{i=k(j)}^{k(j+1)-1} \chi_{2^{n(i)}}(e) - 1\right| = \left|(k(j+1) - k(j))^{-1} \sum_{i=k(j)}^{k(j+1)-1} \chi_{2^{n(j)}}(e_j)\right| \le 2^{-j}$$

and the full result is proved.

In effect we have constructed a Weak Dirichlet set supporting a true non zero pseudofunction. But any such set can be perturbed to give a Weak Kronecker set supporting a true non zero pseudofunction. (We shall give a proof of this in the simple special case given in Lemma 4 but the general proof is hardly more complicated).

LEMMA 5. Suppose E and T are given as in Lemma 4. Suppose further $j_o \geq 1$ and an $f \in C(E)$ with $f(e) = \pm 1$ for all $e \in E$ is given. Then $E_1 = \{e \in E : f(e) = 1\}$ and $E_2 = \{e \in E : f(e) = -1\}$ are closed and we can find a $j > j_o$ and an x such that writing $T' = T|E_1 + (T|E_2) * \delta_x$ $E' = E_1 \cup (E_2 + x)$ we have

(i) $\chi_{2^i}(x) = 1$ for all $i < k(j_1)$ and for all $i \geq k(j_1+1)$

(ii) $\chi_{2^i}(x) = -1$ for all $k(j_1) \leq i < k(j_1+1)$

so in particular, setting $f_o|E_1 = 1$, $f_o|E_2+x = -1$ we have $f_o \in C(E')$ and

(i)' $\|(k(j+1) - k(j))^{-1} \sum_{i=k(j)}^{k(j+1)-1} \chi_{2^{n(i)}} - 1\|_{C(E')} \leq 2^{-j}$ $[j \geq 1, j \neq j_1]$

(ii)' $\|(k(j_1+1) - k(j))^{-1} \sum_{i=k(j_1)}^{k(j_1+1)-1} \chi_{2^{n(i)}} - f_o\|_{C(E')} \leq 2^{-j_1}$

whilst on the other hand T' is a pseudo function with $T' \in PM(E')$, $\hat{T}'(0) = 1 = \|T\|_{PM}$ and

(iii) $|\hat{T}'(r)| \leq |\hat{T}(r)| + 2^{-j_o}$.

Proof. Since E_1 and E_2 are disjoint closed sets we can find $g_\ell \in A(D)$ with $g_\ell(e) = 1$ if e lies in a neighborhood of E_ℓ, $g_\ell(e) = 0$ if e lies in a neighborhood of $E_{3/2-(-1)^\ell/2}$ $[\ell = 1,2]$. Thus since $\hat{T}(r) \to 0$ as $r \to \infty$ it follows that $\hat{T}_\ell(r) = T|E_\ell^\wedge(r) = Tg_\ell^\wedge(r) \to 0$ as $r \to \infty$.

Choose $j_1 \geq j_o$ such that $|\hat{T}_\ell(r)| \leq 2^{-j_o-3}$ for all $r \geq k(j_1)$. Set

$x = \sum_{i=k(j_1)}^{k(j_1+1)-1} 2/3^{n(i)}$. Conditions (i) and (ii) (and so (i)' and (ii)') follow trivially.
Further since $\hat{\delta}_x(r) = 1$ we have $\hat{T}(r) = \hat{T}'(r)$ for all $r < k(j_1)$. On the other hand if $r \geq k(j_1)$ we know that $\hat{\delta}_x(r) = \pm 1$, $|\hat{T}_\ell(r)| \leq 2^{-j_o-2}$ $[\ell = 1,2]$ so that $|(T-T')^\wedge(r)| \leq 4.2^{-j_o-2} = 2^{-j_o}$ so condition (iii) is proved. Finally since $\hat{T}_1(r), \hat{T}_2(r) \to 0$ as $r \to \infty$ it follows that $\hat{T}'(r) \to 0$ as $r \to \infty$.

THEOREM. There exists a closed set $F \subseteq D$ which is of interpolation for $A(D)$ but carries a non zero pseudo measures.

Proof. This is an easy consequence of Lemmas 4 and 5. Take E as in Lemma 4. We can find a sequence of partitions $P_n = \{E_{n1}, E_{n2}, \ldots, E_{n2^n}\}$ such that E_{nr} is closed $[1 \leq r \leq 2^n]$, $E_{nr} \cap E_{ns} = \emptyset$ $[1 \leq r < s \leq 2^n]$, $\bigcup_{r=1}^{2^n} E_{nr} = E$, $E_{n+1\,2t-1} \cup E_{n+1\,2t} = E_{nt}$ $[1 \leq t \leq 2^n]$.

By repeated use of Lemma 5 we can find $Q(1) < Q(2) < \ldots$, trigonometric polynomials $f_{n\varepsilon_1\varepsilon_2\ldots\varepsilon_{2^n}}$ $[\varepsilon_i = \pm 1]$ and points $x_{nr} \in D$ $[1 \leq r \leq 2^n]$ such that setting $E_n = \bigcup_{r=1}^{2^n} (E_{nr} + x_{nr})$, $T_n = \sum_{r=1}^{2^n} (T|E_{nr}) * \delta_{x_{nr}}$ we have

(i) $\|f_{n\,\varepsilon_1\,\varepsilon_2\ldots\varepsilon_{2^n}} - \varepsilon_r\|_{C(E_{nr}+x_{nr})} \leq 2^{-n}$

(ii) $\|f_{n\,\varepsilon_1\,\varepsilon_2\ldots\varepsilon_{2^n}}\|_{A(D)} = 1$

(iii) $\hat{f}_{n\,\varepsilon_1\,\varepsilon_2\ldots\varepsilon_{2^n}}(s) = 0$ for all $s \geq 2^{Q(n)}$

(iv) $\chi_{2^p}(x_{n+1\,2t-1} - x_{nt}) = \chi_{2^p}(x_{n+1\,2t} - x_{nt}) = 1$ for all $0 \leq p \leq Q(n)$, $1 \leq t \leq 2^n$

(v) $|\hat{T}_n(r)| \leq |\hat{T}_{n-1}(r)| + 2^{-n}$ for all r $[n \geq 1]$ where $T_o = T$

(vi) T_n, E_n satisfy the conditions of Lemma 4 for a suitable choice of $k(j)$, $n(j)$.

Under these conditions it is clear that T_n converges weakly to a pseudofunction S with $\|S\| = \hat{S}(0) = 1$, and that E_n converges topologically to a set F with $S \in PM(F)$. (Use argument of the paragraph before last of Lemma 4). It only remains to show that F is of interpolation.

To prove this suppose $\varepsilon > 0$ given, $f \in C(D)$ and f takes only the values 1 and -1. Then we can find an $n \geq 1$ such that $\varepsilon \leq 2^{-n}$, f is constant on each $E_{nr} + x_{nr}$ $[1 \leq r \leq 2^n]$ and $f(x+y) = f(y)$ whenever $\chi_{2^p}(x) = 1$ for all $0 \leq r \leq Q(n)$. Set $\varepsilon_{2t} = \varepsilon_{2t-1} = f(E_{nt} + x_{nt})$ $[1 \leq t \leq 2^n]$. It follows from (i), (ii), (iii) and (iv) that $\| f_{n+1\,\varepsilon_1 \ldots \varepsilon_{2^{n+1}}} \|_{A(D)} = 1$ and $\| f_{n+1\,\varepsilon_1 \ldots \varepsilon_{2^{n+1}}} - f \|_{C(F)} \leq \varepsilon$. Thus F is of interpolation.

Remark. The work above was done after but in ignorance of Kaufman's elegant work reported above. However it may be useful to have a simple version of my original method to compare with that of Kaufman and the earlier results of Piatecki-Shapiro.

In particular it prompts the following remark. Consider the set
$E = \left\{ \sum_{r=1}^{\infty} \varepsilon_r 2^{-r}\pi : \sum_{r=1}^{s} |\varepsilon_r| \leq s2^{-10000}, \quad \varepsilon_r = 0,1 \right\} \subset \mathbf{T}$. By the theorem of Piatecki-Shapiro E supports a non zero pseudofunction T. But it is clear that given $n(0)$ we can find an $n(1)$ sufficiently large that $\| (n(1)-n(0))^{-1} \sum_{r=n(0)}^{n(1)-1} \chi_{2^{100r}} - 1 \|_{C(E)} \leq 2^{-10}$. Thus perturbing E and T as in Lemmas 4 and 5 we obtain such that

$$\inf_{f \in A(T), \|f\|_{A(T)}=1} \|f - g\|_{C(E')} \leq 2^{-8} \quad \text{for all } g \in C(\mathbf{T}) \text{ with } |g(t)| = 1 \quad [t \in \mathbf{T}].$$

Such a set is a Helson set and we have another proof of the existence of Helson sets not of synthesis.