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TRACES OF ANALYTIC SOLUTIONS OF THE HEAT EQUATION

by N. Aronszajn.

Table of Contents:

§1. Definition of traces, their basic properties and identification with functions, distributions, and analytic functionals.

In what follows the complete version of the talk at the Colloquium is given. Notation, terminology, and results stated in the preliminary notes (which will be refered to by P. N.) will be used. However, we will slightly change one notation. It will be convenient to write the spaces $\mathfrak{A}\mathcal{H}(D)$, $\mathfrak{A}\mathcal{H}$ and $\mathfrak{A}\mathcal{H}_R$ as $\widetilde{\mathfrak{A}\mathcal{H}}(D)$, $\widetilde{\mathfrak{A}\mathcal{H}}$, and $\widetilde{\mathfrak{A}\mathcal{H}}_R$. The preceding notation will be reserved for the corresponding spaces of traces which are the main topic of our talk.

Consider a space $\widetilde{\mathfrak{A}\mathcal{H}}(D)$, a point $t^0 \in \partial D$ and a function $\tilde{u}(x,t) \in \widetilde{\mathfrak{A}\mathcal{H}}(D)$. If this function is regular for $t = t^0$ the section $\tilde{u}(x,t^0)$ restricted to $x \in \mathbb{R}^n$ will be called the trace of $\tilde{u}$ at t^0. Such traces will be called regular traces; they are entire functions on $\mathbb{R}^n$ of Laplacian order 2 and finite type.

By formula 2, P.N. Ch. III, §1 such a regular trace determines uniquely the corresponding solution $\tilde{u}(x,t)$.

In the general case when $\tilde{u}(x,t)$ is not regular at t^0 we will consider at first the trace of $\tilde{u}$ at t^0 as an abstract object in one-one correspondence with the function $\tilde{u} \in \widetilde{\mathfrak{A}\mathcal{H}}(D)$. The set of all these objects will be called the trace-space corresponding to D and $t^0 \in \partial D$ and denoted by $\mathfrak{A}\mathcal{H}(D;t^0)$. The one-one correspondence between $\widetilde{\mathfrak{A}\mathcal{H}}(D)$ and $\mathfrak{A}\mathcal{H}(D;t^0)$ allows one to transfer the topological vector space structure from $\widetilde{\mathfrak{A}\mathcal{H}}(D)$ on $\mathfrak{A}\mathcal{H}(D;t^0)$ making the space of traces a Montel space.

If $f \in \mathcal{GN}(D;t^0)$ we will denote by $\tilde{f}(x,t)$ the corresponding function in $\widetilde{\mathcal{GN}}(D)$. We will be mostly concerned with the space $\widetilde{\mathcal{GN}} = \widetilde{\mathcal{GN}}_\infty = \widetilde{\mathcal{GN}}(\mathbb{C}^1_+)$, and sometimes with $\widetilde{\mathcal{GN}}_R = \widetilde{\mathcal{GN}}(B^2_R(R))$ and their corresponding trace-spaces at $t = 0$ which will be denoted briefly by $\mathcal{GN} = \mathcal{GN}_\infty$ and $\mathcal{GN}_R$ respectively. For brevity's sake we will call the elements of $\mathcal{GN}$ "<u>traces</u>" and those of $\mathcal{GN}_R$ "<u>R-traces</u>".

If $R < R' \leqq \infty$ we will identify the trace $f \in \mathcal{GN}_{R'}$ with the trace $g \in \mathcal{GN}_R$ such that $\tilde{g}(x,t)$ is a restriction of $\tilde{f}(x,t)$. In this way we have clearly

(1) $$\mathcal{GN}_{R'} \hookrightarrow \mathcal{GN}_R \,.$$

We will see a little later that $\mathcal{GN}_{R'}$ is dense and non-closed in $\mathcal{GN}_R$.

Our next task is to identify the spaces $\mathcal{GN}_R$ with topological duals of suitable proper functional spaces. To this effect we use the spaces $\mathcal{V}_R$ introduced in P.N., Ch. III, §3, which we will now denote by $\tilde{\mathcal{V}}_R$. We recall that $\tilde{\mathcal{V}}_R$ is the class of all finite linear combinations of functions of the form $D^k_{\zeta,\tau} E(x-\zeta, t+\tau)$ for different values of the parameters $\zeta \in \mathbb{C}^n$, and $\tau \in B^2_R(R)$. Since these functions belong to $\widetilde{\mathcal{GN}}_R$ and are all regular at $t = 0$ the class of the corresponding traces which we will denote by $\mathcal{V}_R$ is composed of regular traces which are functions of $x \in \mathbb{R}^n$, finite linear combinations of functions of the form $D^k_{\zeta,\tau} E(x-\zeta,\tau)$. Every $v(x) \in \mathcal{V}_R$ determines uniquely the function $\tilde{v}(x,t) \in \tilde{\mathcal{V}}_R$. By this correspondence we transfer the topology of $\tilde{\mathcal{V}}_R$ on $\mathcal{V}_R$ and using Theorem V of P.N., Ch. III, §3, we get

<u>THEOREM I.</u> <u>The trace-space</u> $\mathcal{GN}_R$ <u>is identified with the topological dual of</u> $\mathcal{V}_R$, $\mathcal{GN}_R \equiv \mathcal{V}'_R$. <u>For every</u> $f \in \mathcal{V}'_R$ <u>the corresponding</u> $\tilde{f}(x,t)$ <u>is given by</u>

(2) $$\tilde{f}(z,t) = \langle f, E(x-z,t) \rangle \,.$$

For any $u \in \mathfrak{GH}_R$ and $v \in \mathcal{V}_R$ the scalar product can be written in the form

(3) $$< u, v > = \int \tilde{u}(x, t')\tilde{v}(x-t')\,dx$$

for sufficiently small positive t'.

This theorem allows us to identify a large class of familiar objects with traces. Since every function $v(x) \in \mathcal{V}_R$ multiplied by $e^{\alpha x^2}$ belongs to the class $\mathcal{S}$ of L. Schwartz for sufficiently small positive α, we obtain immediately:

THEOREM II. Every distribution T such that $e^{-\alpha x^2} T \in \mathcal{S}'$ for every $\alpha > 0$ is a trace. The scalar product with any $v \in \mathcal{V}_R$ is given by

$$< T, v > = e^{-\alpha x^2} T(e^{\alpha x^2} v(x)) = T(v(x))$$

for sufficiently small positive α (depending on v). The function $\tilde{T}(z, t) \in \widetilde{\mathfrak{GH}}_R$ is given by

$$\tilde{T}(z, t) = e^{-\alpha x^2} T(e^{\alpha x^2} E(x-z, t)) = T(E(x-z, t)).$$

Corollary III. Functions in the class $\mathcal{D}$ of L. Schwartz, distributions with compact support and tempered distributions are traces.

THEOREM IV. (Density Theorem.)

a) The regular R-traces are dense in $\mathfrak{GH}_R$;

b) The linear combinations of functions of the form $E(x-\zeta, \tau)$ for $\zeta \in \mathbb{C}^n$, $\tau \in B^2_R(R)$ are dense in $\mathfrak{GH}_R$;

c) Polynomials are dense in $\mathfrak{GH}_R$;

d) The class $\mathcal{D}$ of L. Schwartz is dense in $\mathfrak{GH}_R$.

Proof. a) For $R = \infty$ with any function $\tilde{u}(x, t) \in \widetilde{\mathfrak{GH}}$ the function $\tilde{u}_\epsilon(x, t) = \tilde{u}(x, t+\epsilon)$ is also in $\widetilde{\mathfrak{GH}}$ for $\epsilon > 0$. Its trace is $u_\epsilon = \tilde{u}(x, \epsilon)$. It is a regular trace and obviously $u_\epsilon \to u$ for $\epsilon \searrow 0$ in the topology of $\mathfrak{GH}$.

For $0 < R < \infty$ the isomorphism $\mathcal{D}\begin{pmatrix} 1 & , 0 \\ -\frac{1}{2R} & , 1 \end{pmatrix}$ (see P.N., Ch. III, §1 and §3) transforms $\widetilde{\mathfrak{A}\mathcal{K}}$ onto $\widetilde{\mathfrak{A}\mathcal{K}}_R$. Using this isomorphism we prove our statement for all R's.

b) It suffices to show that for regular $u \in \mathfrak{A}\mathcal{K}_R$ the corresponding solution $\tilde{u}(z,t)$ is uniformly approximable on compacts in z and t by linear combinations of functions of the form $E(z-\zeta, t+\tau)$. Since $\tilde{u}(z,t)$ is regular for $t \in B_R^2(R) \cup B_\rho^2(0)$ for certain $\rho > 0$, for any compact $K \subset B_R^2(R)$ we can find $\rho' > 0$ and $R' > 0$ so that $K \subset B^2_{R'}(R') \subset B^2_{\rho'+R'}(R') \subset B_R^2(R) \cup B_\rho^2(0)$. Since the function $\tilde{u}(z, t-\rho') \subset \widetilde{\mathfrak{A}\mathcal{K}}_{R'+\rho'}$, we can apply formula (1) of Theorem III of P.N., Ch. III, §3, with $u(z,t)$ replaced by $\tilde{u}(z,t-\rho')$, θ by 0, t^0 by ρ', R by $R'+\rho'$. Noticing that now the circle $B(R'+\rho', \rho', 0) = B^2_{R'}(R'+\rho')$, from the formula (1) we see that $\tilde{u}(z,t-\rho')$ can be approximated uniformly for $t \in K+\rho'$ and z in any compact of $\mathbb{C}^n$ by a linear combination of functions $E(z-x, t-\rho')$ for finite number of values of x which proves our assertion.

c) Using Proposition 2 of P.N., Ch. III, §2, and Theorem IV of P. N., Ch. III, §3 we prove immediately statement c).

d) In view of c) it is enough to show that if $p(x)$ is any polynomial and $\varphi(x) \in \mathcal{D}$ with $\varphi(x) = 1$ in a neighborhood of 0 then $\varphi(x/\lambda)p(x)$ for $1 < \lambda \nearrow \infty$ converges in the topology of traces to $p(x)$ but this is clear if one writes by the formula of Theorem II

$$(\varphi(x/\lambda)p(x))^{\sim}(z,t) = \int_{\mathbb{R}^n} \varphi(x/\lambda)p(x)\,E(z-x, t)dx\,.$$

which for $\lambda \nearrow \infty$ converges uniformly on compacts in z and t to

$$\tilde{p}(z,t) = \int_{R^n} p(x)\,E(z-x, t)dx\,.$$

THEOREM V. All analytic functionals on $\mathbb{C}^n$ are traces.

Proof. Since all functions $v(x) \in \mathcal{V}$ are entire functions on $\mathbb{R}^n$, hence also on $\mathbb{C}^n$, for any analytic functional we can consider the scalar product $< F, v >$. It is immediately seen that this scalar product is continuous in v in the topology of $\mathcal{V}$. Furthermore $< F, v >$ cannot vanish identically on $\mathcal{V}$ since by Theorem IV b) and c) $\mathcal{V}$ is dense in $\mathcal{H}(\mathbb{C}^n)$. This identifies F with an element of $\mathfrak{A}\mathcal{H}$. The corresponding $\tilde{F}(z,t) = < F, E(z-x, t) >$. Furthermore, being identified with traces, the analytic functionals are automatically identified with R-traces.

We can ask ourselves which functions u(x) defined on $\mathbb{R}^n$ can be identified as traces. The elementary case is the case when the convolution $\tilde{u}(z,t) = u(x) * E(x,t)$ is valid for all $z \in \mathbb{C}^n$ and $t \in \mathbb{C}^1_+$ in the ordinary sense i.e., with Lebesgue integration. One sees immediately that for this it is necessary and sufficient that $u(x)e^{-\alpha x^2}$ be integrable for all $\alpha > 0$. u(x) will be an R-trace if and only if $u(x)e^{-\alpha x^2}$ is integrable for $\alpha > \frac{1}{2R}$.

However, we know already a large class of functions, namely the regular R-traces for which the convolution is valid not in the ordinary sense, since the integral has to be taken in general as the symbolic integral. To include both cases we will make the following definition:

Definition. A function u(x) defined on $\mathbb{R}^n$ will be identified as a trace if and only if u(x) is locally integrable and the

$$\tilde{u}(z,t) = \mathfrak{S}\!\!\!\int u(x)\, E(z-x, t)dx$$

exists and the uniformity conditions (see P.N., Ch. I, §5) are satisfied with respect to the parameters $z \in \mathbb{C}^n$, $t \in \mathbb{C}^1_+$. Replacing $\mathbb{C}^1_+$ by $B^2_R(R)$ we obtain the definition of functions identifiable with R-traces.

It is obvious that $\tilde{u}(z,t)$ is the corresponding solution of the heat equation. It is easy to prove also that in this case the scalar product $< u, v >$ for any $v \in \mathcal{V}$ is given by $\mathfrak{S}\!\!\int u(x)v(x)dx$. The condition of existence of the symbolic integral is a very special kind of condition on the behavior of the function $u(x)$ at ∞ which is not in general connected with the behavior of $|u(x)|$.

For a function $f(x)$ defined on $\mathbb{R}^n$ which is a trace or R-trace, the corresponding solution $\tilde{f}(x,t)$ has $f(x)$ as initial values in the usual sense as described in

THEOREM VI. *If* $f(x)$ *defined on* $\mathbb{R}^n$ *is a* R-*trace,* $0 < R \leqq \infty$, *then for every* $x^0 \in \mathbb{R}^n$ *for which* $\lim_{\rho=0} \rho^{-n} \int_{B^n_\rho(x^0)} |f(x)-f(x^0)|dx = 0$, *hence for almost all* $x^0 \in \mathbb{R}^n$ *we have*

(4) $$\tilde{f}(x^0, t) \to f(x^0) \quad \text{for} \quad t \searrow 0 .$$

Proof. By definition of the symbolic integral (in elementary case, see P.N., Ch. I, §5)

$$\tilde{f}(x^0, t) = \mathfrak{S}\!\!\int_{x^0} f(x)\, E(x^0-x, t)dx = \int_0^\infty \omega_n r^{n-1} M_f(x^0, r) \frac{1}{(4\pi t)^{n/2}} e^{\frac{-r^2}{4t}} dr.$$

By the classical argument of Fatou-type theorems we obtain then (4).

Remark. The existence for a solution $\tilde{f}(x,t) \in \widetilde{\mathfrak{A}\mathscr{K}}_R$ of the limit $f(x^0)$ in (4) for almost all $x^0 \in \mathbb{R}^n$ does not assure in general that $f(x)$ is the trace corresponding to $\tilde{f}(x,t)$ (see next section, Theorem I).

§2. Noncompatibility of $\mathfrak{A}\mathscr{K}$ with $\mathscr{K}(\mathbb{C}^n)$ and $\mathscr{D}'$.

In the preceding section we established an identification of a class of entire functions with traces (the regular traces) and of a class of distributions with traces. Each time we have an identification of a subspace of one topological vector space with a subspace of another the question arises whether this identification is compatible with the topologies of the two spaces.

Let A_1 and B_1 be linear subspaces of the topological vector spaces A and B respectively, and let J be the identification mapping of A_1 onto B_1, i.e. a linear 1-1-mapping of A_1 onto B_1. This identification is said to be compatible with the topologies of A and B if J considered as operator from A into B is closable with a closable inverse. A necessary and sufficient condition for this is that if the nets (or sequences) $\{a_\alpha\}$ and $\{Ja_\alpha\}$ converge in their respective spaces A and B and if one of these nets converges to zero then the other does also.

The importance of compatibility lies in two facts:

1° We can extend the identification canonically to the closure of J which is then also an identification mapping.

2° By considering the graph of $-\bar{J}$, $G(-\bar{J})$, which is a closed subspace of the direct topological sum $A \dot{+} B$ we can consider the topological quotient space $(A \dot{+} B)/G(-\bar{J})$ in which the elements $x \in A$ and $y \in B$ are identified with $(x,0) + G(-\bar{J})$ and $(0,y) + G(-\bar{J})$ respectively. In this way $(A \dot{+} B)/G(-\bar{J})$ becomes $A + B$ (not a direct sum) with A and B continuously embedded in $A + B$. Thus, a compatible identification leads to a common enlargement of A and B to a topological space $A + B$ where the identification is the natural one.

In most cases of identification the compatibility is trivially true.

As an interesting example of non-compability consider the class $\mathcal{M}$ of all measurable functions on $\mathbb{R}^1$ and the class $\mathcal{D}'$ of all distributions on $\mathbb{R}^1$. The class $L^1_{loc} \subset \mathcal{M}$ is identified with a subspace of $\mathcal{D}'$ and the sequence of functions $n\chi_n(x)$, where χ_n is the characteristic function of $(-1/2n, 1/2n)$, converges to zero in the intrinsic topology of $\mathcal{M}$ and converges to the Dirac measure δ_0 in $\mathcal{D}'$. Hence the identification is not comaptible.

THEOREM I. The identifications between $\mathcal{GN}$ and $\mathcal{M}(\mathbb{C}^n)$ and also between $\mathcal{GN}$ and $\mathcal{D}'$ are not compatible.

Proof. We use an idea suggested to us by F. Treves. We consider the function $h(t) = e^{\log t/t}$. One proves without much difficulty that for every θ, with $0 < \theta < \pi/2$, and every $\varepsilon > 0$ there exists a function $C_{\varepsilon,\theta}(t) > 0$ defined in the angle $|\mathrm{Arg}\, t| \leqq \theta$ and independent of p such that

(1) $$|h^{(p)}(t)| \leqq C_{\varepsilon,\theta}(t)\,\varepsilon^p (2p)!\,,\quad p = 0,1,2,\ldots\,.$$

Furthermore for fixed ε and θ

(2) $$C_{\varepsilon,\theta}(t) \to 0 \ \underline{\text{when}}\ t \to 0\ \underline{\text{in the angle}}\ |\mathrm{Arg}\, t| \leqq \theta\,.$$

It is immediately seen that the function

(3) $$\tilde{u}(z,t) = \sum_{p=0}^{\infty} h^{(p)}(t)\,\frac{z^{2p}}{(2p)!}$$

belongs to $\widetilde{\mathcal{GN}}$ in one space-variable z .

If $|z| < \frac{1}{2\sqrt{\epsilon}}$ and $|\mathrm{Arg}\, t| \leq \theta$, we have, according to (1), $|\tilde{u}(z, t)| \leq 2C_{\epsilon,\theta}(t)$ which tends to zero with t according to (2). If follows that $\tilde{u}(z,\tau)$ as entire functions of z converge to zero in $\mathcal{K}(\mathbb{C}^1)$ for $\tau \to 0$ in any angle. However, $\tilde{u}(z,\tau)$ is a regular trace corresponding to the solution $\tilde{u}(z, t+\tau)$ and in $\mathfrak{G}\mathcal{K}$ it converges to the trace $u \neq 0$ corresponding to the solution $\tilde{u}(z, t)$. Hence, incompatibility.

Since the functions $\tilde{u}(z,\tau)$ are also distributions and they converge to 0 in $\mathcal{D}'$, our theorem is completely proved.

Remark 1. As a counterpart to the somewhat negative result of Theorem I it should be noticed that it is easy to prove that the identification of the spaces $\mathcal{D}$, $\mathcal{S}'$ and $\mathcal{K}'(\mathbb{C}^n)$, with a subspace of traces is compatible with the usual topology of these spaces since the identification mapping is a continuous mapping of $\mathcal{D}$, $\mathcal{S}'$ and $\mathcal{K}'(\mathbb{C}^n)$ into $\mathfrak{G}\mathcal{K}$.

§3. Some elementary operations on traces.

We will use the transformations defined in P.N. Ch. III, §1, to establish certain elementary operations on traces.

a) The transformation $\mathfrak{G}(\underline{a} + T)$ is a topological automorphism of $\widetilde{\mathfrak{G}\mathcal{K}}_R$, hence transferred to $\mathfrak{G}\mathcal{K}_R$ it becomes an automorphism of $\mathfrak{G}\mathcal{K}_R$. It doesn't change the variable t in $\widetilde{\mathfrak{G}\mathcal{K}}_R$ and corresponds to a change of variables in C_n obtained by an orthogonal transformation T followed by a translation $\underline{a}$. Since the traces are attached to $\mathbb{R}^n$ the significance of this transformation can only be given via the corresponding solution of the heat equation. However, if the trace is regular, i.e. is an entire function $u(x) = \tilde{u}(x, 0)$, the significance can be directly pictured by considering the function $u(a + Tx)$ restricted to $x \in \mathbb{R}^n$.

Remark 1. There is an interesting interpretation of the transformation $\mathcal{A}(a)$ for any complex vector $\underline{a}$. Since $\mathbb{R}^n$ is considered as a vector space it has a fixed origin 0. The translation with a real vector $\underline{a}$ shifts the origin to the point $\underline{a}$. If, however, $\underline{a}$ is a complex vector, then $\mathbb{R}^n$ is shifted to a parallel hyperplane in $\mathbb{C}^n$ with origin at $\underline{a}$. We can then consider the original trace u as determining an "analytic" function defined on $\mathbb{C}^n$ with values in $\mathcal{A}\mathcal{H}_R$, the value for $a = 0$ being u itself and for any complex $a \in \mathbb{C}^n$ the value is obtained from u by the transformation $\mathcal{A}(a)$. This interpretation is sometimes useful.

Remark 2. $\mathcal{A}(a)$ transforms a regular trace into a regular trace. In general, however, even if u is a very regular function the transform of u by $\mathcal{A}(a)$ is not a function for $\underline{a}$ complex.

b) The transformation $\mathcal{B}(\tau)$ is a topological isomorphism of $\widetilde{\mathcal{A}\mathcal{H}}(D)$ onto $\widetilde{\mathcal{A}\mathcal{H}}(D-\tau)$. Notice that for $0 < \tau < 2R$, $B^2_R(R)-\tau \supset B^2_{R-(\tau/2)}(R-(\tau/2))$. We can then accept as transform $\mathcal{B}(\tau)\tilde{u}$, for $\tilde{u}(x,t) \in \widetilde{\mathcal{A}\mathcal{H}}_R$, the restriction of the actual transform to $B^2_{R-(\tau/2)}(R-(\tau/2))$. With this convention $\mathcal{B}(\tau)$ $(0 < \tau < 2R)$ becomes a continuous isomorphism of $\widetilde{\mathcal{A}\mathcal{H}}_R$ <u>into</u> $\widetilde{\mathcal{A}\mathcal{H}}_{R-(\tau/2)}$ and by transfer $\mathcal{B}(\tau)$ is a continuous isomorphism of $\mathcal{A}\mathcal{H}_R$ into $\mathcal{A}\mathcal{H}_{R-(\tau/2)}$ (for $0 < \tau < 2R$) and of $\mathcal{A}\mathcal{H}$ into $\mathcal{A}\mathcal{H}$ (for $\tau > 0$). It is clear that for an R-trace u $\mathcal{B}(\tau)u = \tilde{u}(x,\tau)$.

c) We will use the transformations $\mathcal{C}(c)$ only for real $c \neq 0$. They transform $\mathcal{C}\mathcal{H}_R$ isomorphically onto $\mathcal{C}\mathcal{H}_{R/c^2}$.

d) The transformations $\mathcal{D}\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ will be used especially in the following two cases (see P.N., Ch. III, §1).

1° $\mathfrak{s}\begin{pmatrix} 0 & , & i/2 \\ 2i & , & 0 \end{pmatrix}$. If $\varphi \in \mathfrak{s}$ the Fourier transform $\hat{\varphi}(x) = \int_{\mathbb{R}^n} \varphi(\xi)\, e^{-i(\xi x)} d\xi$ belongs to $\mathfrak{S}$ and a direct computation shows that

$$\tilde{\hat{\varphi}} = (2\pi)^{n/2} e^{\frac{n\pi i}{4}} \mathfrak{s}\begin{pmatrix} 0 & , & i/2 \\ 2i & , & 0 \end{pmatrix} \tilde{\varphi} = \frac{(2\pi)^{n/2}}{(2t)^{n/2}} e^{-x^2/4t} \tilde{\varphi}\left(\frac{x}{2ti}, \frac{1}{4t}\right).$$

$\mathfrak{s}\begin{pmatrix} 0 & , & i/2 \\ 2i & ; & 0 \end{pmatrix}$ being a topological automorphism of $\widetilde{\mathfrak{G}\mathfrak{X}}$ and $\mathfrak{s}$ being dense in $\mathfrak{G}\mathfrak{X}$ the mapping $\varphi \to \hat{\varphi}$ extends by continuity to the whole of $\mathfrak{G}\mathfrak{X}$ and we obtain thus the Fourier transform $u \to \hat{u}$ transforming the whole of $\mathfrak{G}\mathfrak{X}$ onto itself by the formula

$$\tilde{\hat{u}}(x,t) = \frac{(2\pi)^{n/2}}{(2t)^{n/2}} e^{-x^2/4t} \tilde{u}\left(\frac{x}{2ti}, \frac{1}{4t}\right), \tag{1}$$

The inverse Fourier transform $\widetilde{\hat{u}^{-1}}$ is given by

$$\widetilde{\hat{u}^{-1}}(x,t) = (2\pi)^{-n/2} e^{\frac{-n\pi i}{4}} \mathfrak{s}\begin{pmatrix} 0 & , & -i/2 \\ -2i & , & 0 \end{pmatrix} \tilde{u} = \frac{(2\pi)^{-n/2}}{(2t)^{n/2}} e^{-x^2/4t} \tilde{u}\left(\frac{x}{-2ti}, \frac{1}{4t}\right). \tag{1'}$$

<u>Remark 3.</u> Formula (1) for the Fourier transform leads to an interesting (but somewhat vague) interpretation if we restrict t to be positive and introduce the notion of traces at $+\infty$. We could say that for a function $\tilde{u} \in \widetilde{\mathfrak{G}\mathfrak{X}}$ the trace at 0 on the real hyperplane $\mathbb{R}^n$ is transformed by the Fourier transform into the trace of $\tilde{u}$ at $+\infty$ on the purely imaginary hyperplane $i\mathbb{R}^n$.

2° $\mathfrak{s}\begin{pmatrix} 1 & , & 0 \\ -4\alpha & , & 1 \end{pmatrix}$. If $\varphi \in \mathfrak{s}$, then $e^{\alpha x^2}\varphi \in \mathfrak{s}$ and a direct computation shows that $(e^{\alpha x^2}\varphi)^{\sim}$ is given by

$$(e^{\alpha x^2}\varphi)^{\sim}(x,t) = \mathfrak{s}\begin{pmatrix} 1 & , & 0 \\ -4\alpha & , & 1 \end{pmatrix}\tilde{\varphi} =$$
$$= (1-4\alpha t)^{-n/2} e^{\frac{\alpha x^2}{1-4\alpha t}} \tilde{\varphi}\left(\frac{x}{1-4\alpha t}, \frac{t}{1-4\alpha t}\right)$$

If $\alpha = \frac{1}{8R'} - \frac{1}{8R}$ with R and R' positive, $\leq \infty$, then $\mathfrak{s}\begin{pmatrix} 1 & , & 0 \\ -4\alpha & , & 1 \end{pmatrix}$ is a topological isomorphism of $\widetilde{G\mathcal{N}}_R$ onto $\widetilde{G\mathcal{N}}_{R'}$. Since $\mathcal{D}$ is dnese in $G\mathcal{N}_R$ as well as $G\mathcal{N}_{R'}$ we can extend by continuity the multiplication by $e^{\alpha x^2}$ to the whole of $G\mathcal{N}_R$ and thus the multiplication by $e^{\alpha x^2}$ is an isomorphism of $G\mathcal{N}_R$ onto $G\mathcal{N}_{R'}$ given by

$$(2) \qquad (e^{\alpha x^2} u)^{\sim}(x,t) = \mathcal{D}\begin{pmatrix} 1, & 0 \\ -4\alpha, & 1 \end{pmatrix}\tilde{u} = (1-4\alpha t)^{-n/2}\, e^{\frac{\alpha x^2}{1-4\alpha t}}\, \tilde{u}\left(\frac{x}{1-4\alpha t}, \frac{t}{1-4\alpha t}\right).$$

Remark 4. It is actually the formulas for $(e^{\alpha x^2}\varphi)^{\sim}$ and $\hat{\tilde{\varphi}}$ with $\varphi \in \mathcal{D}$ which lead to the establishment of the whole group of transformations $\mathcal{D}\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$. We could consider multiplication by $e^{\alpha x^2}$ with arbitrary complex α but this would necessitate the consideration of more general trace-spaces $G\mathcal{N}(D;t^0)$.

Remark 5. If $\alpha = \frac{1}{8R'} - \frac{1}{8R} < 0$, i.e. $R < R'$ then using the identification of R'-traces with R-traces we obtain that multiplication by $e^{\alpha x^2}$ transforms $G\mathcal{N}_{R'}$ into itself continuously.

e): $\mathcal{E}(b)$. Again we take $\varphi \in \mathcal{D}$, multiply it by $e^{(bx)}$, and by elementary computation obtain an expression for $(e^{(bx)}\varphi)^{\sim}$ which then by continuity, is extended to all traces. We define thus multiplication by $e^{(bx)}$ for all R-traces. It is given by

$$(3) \qquad (e^{(bx)}u)^{\sim}(x,t) = \mathcal{E}(b)\tilde{u} = e^{(bx) + b^2 t}\tilde{u}(x + 2tb, t) .$$

The multiplication by $e^{(bx)}$ is a topological automorphism of each $G\mathcal{N}_R$.

Remark 6. The formula for $(e^{(bx)}\varphi)^{\sim}$ is the one which led to the establishment of the group of transformations $\mathcal{E}(b)$.

We give now a list of formulas connecting the Fourier transform with other transformations introduced in this section. Many of them are similar to the usual formulas for Fourier transforms of functions. The proofs are omitted since they consist in simple application of preceeding formulas given in this section.

In all formulas $u \in \mathfrak{G}\mathcal{H}$.

(4) $$\hat{u}^{-1} = (2\pi)^{-n}\mathcal{C}(-1)\hat{u},$$

(5) $$(\mathfrak{G}(u)u)^{\wedge} = e^{i(ax)}\hat{u} \text{ for any } a \in \mathbb{C}^n,$$

(5') $$(e^{(bx)}u)^{\wedge} = \mathfrak{G}(ib)\hat{u} \text{ for any } b \in \mathbb{C}^n,$$

(6) $$(B(\tau)u)^{\wedge} = e^{-\tau x^2}\hat{u} \quad \text{for} \quad \tau > 0,$$

(6') $$(e^{-\tau x^2}u)^{\wedge} = \mathfrak{B}(\tau)\hat{u} \quad \text{for} \quad \tau > 0,$$

(7) $$(\mathcal{C}(c)u)^{\wedge} = \frac{1}{|c|^n}\,\mathcal{C}(1/c)\hat{u} \quad \text{for real} \quad c \neq 0 .$$

§4. Convolutable traces and multipliable traces.

If we take two functions φ and ψ in $\mathcal{D}$ then $\tilde{\varphi}(z,t) = \mathop{\mathrm{S}\!\!\int} \varphi(x)\, E(z-x,t)dx$, $\tilde{\psi}(z,t) = \mathop{\mathrm{S}\!\!\int} \psi(x)\, E(z-x,t)dx$ where the symbolic integrals are actually ordinary integrals. By the convolution properties of $E(x,t)$ it is then clear that $(\varphi * \psi)^{\sim}(z,t'+t'') = (\tilde{\varphi}(x,t') * \tilde{\psi}(x,t''))(z)$. Here the convolution on the right side is again given by the ordinary integral $\int_{\mathbb{R}^n} \tilde{\varphi}(x,t')\tilde{\psi}(z-x,t'')dx$. We can transfer this property to general traces to define convolutable traces namely:

Definition. Two traces u, v are convolutable if for every positive t' and t'' the convolution (taken with symbolic integrals)

$$\tilde{u}(x,t') * \tilde{v}(x,t''))(x) = \mathop{\mathrm{S}\!\!\int} \tilde{u}(x,t')\tilde{v}(z-x,t'')dx$$

exists relative to the three parameters z, t', t''.

THEOREM I. If the traces u and v are convolutable then the expression $\mathop{\mathrm{S}\!\!\int} \tilde{u}(x,t')\tilde{v}(z-x,t'')dx = \tilde{w}_1(z,t',t'')$ is a function $\tilde{w}(z,t'+t'')$ and $\tilde{w}(z,t) \in \mathfrak{G}\mathcal{H}$.

Proof. By the properties of $\tilde{u}$ and $\tilde{v}$ to be analytic solutions of the heat equation, the function $w_1(z,t',t'')$ is an analytic solution of the heat equation in variables z and t'', i.e. $\Delta_z \tilde{w}_1 = \frac{\partial}{\partial t''} w_1$. By the commutativity of the convolution (see P.N., Ch. I, §5) we can also write $\tilde{w}_1(z,t',t'') = \int \tilde{v}(x,t'')\tilde{u}(z-x,t')dx$. Hence $\Delta_z \tilde{w}_1 = \frac{\partial}{\partial t'} \tilde{w}_1$. The equation $\frac{\partial}{\partial t''} \tilde{w}_1 = \frac{\partial}{\partial t''} \tilde{w}_1$ for analytic functions $\tilde{w}_1(z,t',t'')$ means that $\tilde{w}_1$ depends only on the sum $t' + t''$. Hence we can write $\tilde{w}_1(z,t',t'') = \tilde{w}(z,t'+t'')$ and obviously $\tilde{w}(z,t)$ is an analytic solution of the heat equation.

In view of Theorem I we can define:

Definition 2. If u and v are convolutable traces then the trace w corresponding to the solution $\tilde{w}(z,t)$ defined in Theorem I will be called the convolution of u and v and denoted by $w = u * v$.

Our definition gives immediately the following theorem:

THEOREM II. If u, v are convolutable traces then

(1) $u * v = v * u$

(2) For any complex vector b, $e^{(bx)}u$ and $e^{(bx)}v$ are also convolutable and $(e^{(bx)}u) * (e^{(bx)}v) = e^{(bx)}(u * v)$.

(3) For any two complex vectors a' and a'', $\mathcal{G}(a')u$ is convolutable with $\mathcal{G}(a'')v$ and $(\mathcal{G}(a')u) * (\mathcal{G}(a'')v) = \mathcal{G}(a' + a'')(u * v)$.

(4) For positive τ' and τ'', $\mathcal{B}(\tau')$ and $\mathcal{B}(\tau'')v$ are convolutable and $(\mathcal{B}(\tau')u) * (\mathcal{B}(\tau'')v) = \mathcal{B}(\tau' + \tau'')(u * v)$.

Since for φ and ψ belonging to $\mathcal{D}$ we have

$$(\varphi * \psi)^\wedge = \hat{\varphi}\hat{\psi}$$

we will use the Fourier transform to define multipliability of traces.

Definition 3. If for two traces u, v their inverse Fourier transforms

$\hat{u}^{-1}$ and $\hat{v}^{-1}$ are convolutable then we will say that u, v are multipliable and we will put

$$uv = (\hat{u}^{-1} * \hat{v}^{-1})^{\wedge} .$$

Comparing the formulas (4)-(7) of the preceding section with Theorem II we obtain immediately:

THEOREM III. If the traces u and v are multipliable then

1° $uv = vu$

2° $e^{(b'x)}u$ and $e^{(b''x)}v$ are multipliable for any complex vectors b' and b'' and $(e^{(b'x)}u)(e^{(b''x)}v) = e^{(b' + b'')x}uv$.

3° $\mathfrak{a}(a)u$ and $\mathfrak{a}(a)v$ are multipliable for a..y complex vector a and $(\mathfrak{a}(a)u)(\mathfrak{a}(a)v) = \mathfrak{a}(a)uv$.

4° For any positive τ' and τ'', $e^{-\tau' x^2}u$ and $e^{-\tau'' x^2}v$ are multipliable and $(e^{-\tau' x^2}u)(e^{-\tau'' x^2}v) = e^{-(\tau' + \tau'')x^2}uv$.

Remark.1. If u and v are functions in $\mathfrak{D}$, parts 1°, 2° and 4° are obvious since for such functions the multiplication as defined by us coincides with the usual multiplication. Part 3° is also obvious in the case of a real vector a since then $\mathfrak{a}(a)$ is the change of variables x by translation by the vector a. However, for more general functions identified with traces the statements in Theorem III are not obvious since we don't know if, in general, the multiplication as defined by us corresponds to the usual multiplication of functions. More precisely, the following problem is open.

Problem. If u and v are functions identified with traces and if their product uv (as functions) is also identified with a trace, is it true that the two functions are multipliable by our Definition 3 and that their product by Definition 3 is identical to their product as functions?

In the next section we will give a few instances where the answer to this problem is affirmative.

§ 5. Convolutors and multipliers.

Definition 1. A trace f which convolutes with every trace $u \in \mathfrak{a}\mathscr{K}$ is called a convolutor. A trace g which is multipliable with every $u \in \mathfrak{a}\mathscr{K}$ is called a multiplier.

Obviously the multipliers are Fourier transforms of convolutors and vice versa.

Theorem I. If f is a convolutor then the operator $f * u$ transforms continuously $\mathfrak{a}\mathscr{K}$ into $\mathfrak{a}\mathscr{K}$. If g is a multiplier then gu is a continuous operator of $\mathfrak{a}\mathscr{K}$ into $\mathfrak{a}\mathscr{K}$.

Proof. It is enough to prove the first statement since the second follows by Fourier transform.

Let f be a convolutor. Denote for $N = 1, 2, \dots$ by χ_N the characteristic function for $B_N^{2n}(0) \subset \mathbb{C}^n$. Consider then the operators $F_{N,\varepsilon}$, $\varepsilon > 0$, defined for $\tilde{u} \in \widetilde{\mathfrak{a}\mathscr{K}}$ by

$$(F_{N,\varepsilon}\tilde{u})(z,t) = \int_{\mathbb{R}^n} \tilde{f}\left(x, \varepsilon + \frac{1}{N}\right) \chi_N(x)\, \tilde{u}(z-x,t)dx .$$

It is obvious that $F_{N,\varepsilon}$ transforms $\widetilde{\mathfrak{a}\mathscr{K}} \to \widetilde{\mathfrak{a}\mathscr{K}}$ continuously. For ε fixed it is clear also that

$$\lim_{N=\infty} \int \tilde{f}\left(x, \varepsilon + \frac{1}{N}\right) \chi_N(x)\, \tilde{u}(z-x,t)dx =$$

$$\int \tilde{f}(x,\varepsilon)\tilde{u}(z-x,t)dx = (f * u)^{\sim}(z, t+\varepsilon) = ((\mathcal{B}(\varepsilon) f) * u)^{\sim}(z,t) .$$

By transfer and by uniform boundedness theorem extended to Frechet spaces the convolution operator $\mathcal{B}(\varepsilon)f = \lim_{N=\infty} F_{N,\varepsilon}$ is a continuous operator $\mathfrak{a}\mathscr{K} \to \mathfrak{a}\mathscr{K}$. This, however is sufficient to show the continuity of the convolution operator f.

Consider now a convolutor f and the corresponding $\tilde{f}(x, t^0)$, $t^0 \in \mathbb{C}_+^1$.

For any $\tilde{u}(x,t) \in \widetilde{\mathfrak{A}\mathcal{X}}$, consider

$$\int \tilde{u}(x,t')\,\tilde{f}(-x,t^0-t')dx = (f * u)^{\sim}(0,t^0) \text{ for } 0 < \operatorname{Re} t' < \operatorname{Re} t^0 .$$

This is obviously a continuous linear functional on $\widetilde{\mathfrak{A}\mathcal{X}}$.

Proposition 1. If f is a convolutor then for every $t^0 \in \mathbb{C}^1_+$, $(\mathfrak{B}(t^0)f)^{\sim} \in \widetilde{\mathfrak{A}\mathcal{X}}'$.

Proposition 2. If for a trace f, $(\mathfrak{B}(t^0)f)^{\sim} \in \mathfrak{A}\mathcal{X}''$ for every $t^0 \in \mathbb{C}^1_+$ then f is a convolutor.

In fact, by Theorem VI' of P.N., Ch. III, §3, $\tilde{f}(x,t^0-t')$ is convolutable with any $\tilde{u}(x,t')$ for $0 < \operatorname{Re} t' < \operatorname{Re} t^0$ and since this is true for every $t^0 \in \mathbb{C}^1_+$, f is convolutable with every $u \in \mathfrak{A}\mathcal{X}$.

THEOREM II. In order that f be a convolutor it is necessary and sufficient that for each $t^0 \in \mathbb{C}^1_+$, $\tilde{f}(x,t^0)$ be a trace corresponding to a solution $\tilde{f}(x,t+t^0) \in \widetilde{\mathfrak{A}\mathcal{X}}'$.

By transfer we obtain immediately:

THEOREM II'. The dual $\mathfrak{A}\mathcal{X}'$ of $\mathfrak{A}\mathcal{X}$ is composed of all regular traces which are sections $\tilde{f}(x,t^0)$ of convolutors f.

Using Theorem II of §4 we obtain:

THEOREM III. If f is a convolutor then

1° $e^{(bx)}f$ is a convolutor for every $b \in \mathbb{C}^n$,

2° $\mathfrak{C}(a)f$ is a convolutor for any $a \in \mathbb{C}^n$,

3° $\mathfrak{B}(\tau)f$ is a convolutor for every $\tau > 0$.

For multipliers we obtain (see Th. III, §4):

THEOREM III'. If g is a multiplier then

1° $e^{(bx)}g$ is a multiplier for any $b \in \mathbb{C}^n$,

2° $\mathfrak{C}(a)g$ is a multiplier for any $a \in \mathbb{C}^n$

3° $e^{-\tau x^2}g$ is a multiplier for any $\tau > 0$.

Examples of convolutors and multipliers.

Every analytic functional is a convolutor; every entire function of exponential order one is a multiplier. The second part follows from the first by Fourier transform. On the other hand, the first part results from the identification of an analytic functional F as a trace of the solution

$$\tilde{F}(z,t) = \langle F(x), E(z-x), t \rangle$$

and the scalar product is given by an integral over a compact set in x (see P.N., Ch. I, §4, (1)).

It follows that all distributions with compact support, in particular, functions in $\mathcal{D}$ are convolutors.

For all these convolutors we can write the convolution in much simpler form than in Definition 2. If F is an analytic functional then

$$(1) \qquad (F * u)^{\sim}(z,t) = \langle F(x), \tilde{u}(z-x,t) \rangle .$$

In particular, for a derivative of the Dirac measure, $D^k\delta_0$ which is a distribution with compact support we have

$$(D^k\delta_0 * u)^{\sim}(x,t) = D_x^k\tilde{u}(x,t) .$$

The Fourier transform of $D^k\delta_0$ is the monomial $i^{|k|}\, x|^k$; hence all polynomials are multipliers.

Baouendi's theorem (see P.N., Ch. III, §3, Th. VII) leads to the following statement:

THEOREM IV. (Baouendi). a) f is a convolutor if and only if for every $\tau > 0$ there exist non-negative constants A_τ, B_τ, C_τ and M_τ, $A_\tau > B_\tau$, such that $|\tilde{f}(x,\tau)| \leqq C_\tau \, e^{M_\tau |x| + B_\tau |x^2| - A_\tau \operatorname{Re} x^2}$.

b) g is a multiplier if and only if g is an entire function such that for each $\varepsilon > 0$ there exists non-negative constants A_ε, B_ε, C_ε and M_ε, $A_\varepsilon \geqq B_\varepsilon - \varepsilon$, with the property that $|g(x)| \leqq C_\varepsilon \, e^{M_\varepsilon |x| + B_\varepsilon |x^2| - A_\varepsilon \operatorname{Re} x^2}$.

Remark 1. It is clear by virtue of this theorem that each multiplier is a function identifiable with a trace in elementary case (see §1). If $u(x)$ is any function identifiable as a trace then by definition of multipliers gu is well-defined as a trace but we don't know if the function $g(x)u(x)$ is always identifiable as a trace. However, if we know that $g(x)u(x)$ is identifiable as a trace then we will prove in the next theorem that the trace corresponding to the function $g(x)u(x)$ is actually gu. This theorem will solve in a special case the problem stated in the preceding section.

THEOREM V. If $g(x)$ is a multiplier and the functions $u(x)$ and $g(x)u(x)$ are identifiable as traces then the trace corresponding to $g(x)u(x)$ is actually gu.

Proof. Consider χ_N, $N = 1, 2, \ldots$, the characteristic function of $B^n_N(0) \subset \mathbb{R}^n$ and let $\varphi \in \mathcal{D}$ be non-negative with $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. Then the function $v_{N,\varepsilon}(x) = (u\chi_N) * \left(\frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right)\right) \in \mathcal{D}$. An elementary computation shows then that $(g(x)u_{N,\varepsilon}(x))^\wedge = (2\pi)^{-n/2} \hat{g} * \hat{v}_{N,\varepsilon}$. The function $v_{N,\varepsilon}(x)$ and $v_N(x) = u\chi_N$ are identifiable as traces in elementary cases. Hence it is clear that in trace topology $\lim_{\varepsilon=0} v_{N,\varepsilon} = v_N$. Since the multiplier g as operator is continuous on $\mathfrak{A}$, it follows that $\lim_{\varepsilon=0} gv_{N,\varepsilon} = gv_N$, $g(x)v_{N,\varepsilon}(x)$ and $g(x)v_N(x)$ are functions identifiable by traces in ordinary case. Hence the limit of traces corresponding to the first function, as $\varepsilon \to 0$, is the trace corresponding to the second. Thus gv_N corresponds to the function $g(x)v_N(x)$. Finally, since $g(x)u(x)$ is identifiable with a trace, that means that

$$\oint g(x)u(x)\,E(z-x,t)dx = (gu)^{\sim}(z,t)$$

exists. Hence it is equal to, in particular

$$\oint_0 g(x)u(x)\,E(z-x,t)dx = \int_0^{\infty} R^{n-1}\left[\int_{S_1^{n-1}} g(R\theta)u(R\theta)E(z-R\theta,t)d\theta\right] dR$$

by the same procedure one sees immediately that

$$(gv_N)^{\sim}(z,t) = \int_0^N R^{N-1}\left[\int_{S_1^{n-1}} g(R\theta)u(R\theta)E(z-R\theta,\ t)d\theta\right]dR\ .$$

Since similar expansions can be written for $\tilde{u}(z,t)$ and $\tilde{v}_N(z,t)$ it is clear that the trace u corresponding to $u(x)$ is the trace limit of v_N. The trace corresponding to $g(x)u(x)$ is the trace limit of gu_N. Finally by continuity of the multiplier-operator g we have that the trace corresponding to $g(x)u(x)$ is actually gu.

Corollary V'. If g is a multiplier and $u(x)$ is a function identifiable as a trace in elementary case then $g(x)u(x)$ is also a trace in elementary case and is equal to the trace gu.

Remark 2. Except for functions, the largest class of distributions which we identified with traces are the distributions T such that $e^{-\alpha x^2}T \in \mathcal{S}'$ for every $\alpha > 0$. This identification is elementary (i.e. does not use symbolic integrals). By procedures essentially similar but somehow simpler than the ones in the proof of Theorem V, it can be shown that if g is a multiplier the trace gT is identifiable with the distribution $g(x)T$.

Remark 3. A very convenient method applied, in paticular, in partial differential equations is the method of localization based on multiplication of functions and distributions by a function $\varphi(x) \in \mathcal{D}$. This method cannot be applied in all generality when dealing with traces since functions in $\mathcal{D}$ are not multipliers. However, a kind of "almost" localization may perhaps be useful based on the fact that a function f(x) entire of exponential order 1 when multiplied by $e^{-\tau x^2}$, $\tau > 0$, becomes a convolutor (that is checked easily by using Baouendi's criterion). It follows by Fourier transform that if $\varphi \in \mathcal{D}$, or more generally, if φ is an analytic functional then for every $\tau > 0$, $\tilde{\varphi}(x, \tau)$ is a multiplier. For any trace u, $\tilde{\varphi}(x,\tau)u$ is well defined and without being of compact support it has heuristically a very rapid decrease at ∞.

§6. Developments in Hermite polynomials.

If u is an arbitrary trace, formula 2 of P.N., Ch. III, §3, Th. IV suggests the development:

$$u = \sum_{p=0}^{\infty} \sum_{|k|=p} A_k (t^0)^{|k|/2} H_k \left(\frac{x}{2\sqrt{t^0}}\right) \tag{1}$$

for any $t^0 > 0$. The A_k's can be determined by the formula (3) of the above mentioned theorem

$$A_k = \frac{1}{k!} D_x^k \tilde{u}(0, t^0) . \tag{2}$$

In order to know in which sense equality in formula (1) is valid, we consider each term in the development as a trace (which we may, since each polynomial is not only a trace but a multiplier). By Proposition 2 of P.N., Ch. III, §3, we have .

$$\left(A_k (t^0)^{|k|/2} H_k \left(\frac{x}{2\sqrt{t^0}}\right)\right)^{\sim}(z,t) = A_k (t^0 - t)^{k/2} H_k \left(\frac{z}{2\sqrt{t^0 - t}}\right)$$

Since (again by the above-mentioned Th. IV) we have the development

$$\tilde{u}(z,t) = \sum_{p=0}^{\infty} \sum_{|k|=p} A_k (t^0-t)^{|k|/2} H_k \left(\frac{z}{2\sqrt{t^0-t}}\right)$$

the convergence of the series being uniform on compacts in z and t, the last series converges in the topology of $\widetilde{\mathfrak{a}\mathscr{H}}$, hence the series (1) converges to u in the topology of $\mathfrak{a}\mathscr{H}$. We obtain thus

THEOREM I. For every $t^0 > 0$, the series in (1) with coefficients A_k given by (2) converges in the topology of $\mathfrak{a}\mathscr{H}$ to u.

The interest of this theorem lies in the fact that it allows us to approximate canonically any trace u by polynomials, namely the partial sums $\sum_{p=0}^{N}$ of the series in (1).

§7. Application to inversion of convolutors.

It is easily proved that the convolutors form a commutative algebra if the convolution is taken as the multiplication-operator. This algebra has a unit which is the Dirac measure at 0, δ_0. It can be proved that in this algebra the inverse exists only in very few exceptional cases such as $f = \delta_a = \mathfrak{a}(a)\delta_0$. However, we may ask for the existence of $f^{*-1} \in \mathfrak{a}\mathscr{H}$ such that

$$f * f^{*-1} = f^{*-1} * f = \delta_0 . \tag{1}$$

Such f^{*-1} is in general not unique. We will call f^{*-1} a quasi-inverse of f. If $f = \sum_{|k|\leq N} A_k D^k \delta_0$ with A_k constants, f is a differential operator with constant coefficients and f^{*-1} is then a corresponding elementary solution.

Since f^{*-1} is in general not a convolutor we cannot solve an equation of the form $f * v = u$ by $v = f^{*-1}u$; this will give a solution if and only if f^{*-1} is convolutable with u.

Baouendi proved the solvability of the equations $f * v = u$ for large classes of convolutors f and traces u. What we propose to do here is to use the preceding developments to give a simple construction of f^{*-1} which seems to be valid in quite general cases and may be developped further to cover even more general cases.

Consider a convolutor f and its Fourier transform $g = \hat{f}$. g is a multiplier and by Baouendi's characterization it is an entire function $g(x)$ of an exponential order at most 2. If we denote by h the Fourier transform of the looked-for quasi-inverse of f, we must have the equality $gh = 1$ in the sense of traces. Heuristically then h should be the function $1/g$ of x. This function, however, is not in general a trace for two reasons:

1° It may not be locally integrable because of the null manifolds of $g(x)$.

2° Its behavior at infinity may not be consistent with our definition of functions identifiable as traces (see definition at the end of §1).

However, there may exist complex vectors $a \in \mathbb{C}^n$ such that the function $\frac{1}{g(x+a)}$ restricted to $x \in \mathbb{R}^n$ is identifiable as a trace. This is the case in which our construction would be valid. By putting $h_a = 1/g(x+a)$ we see that as a function $h_a(x)$ satisfies $g(x+a)h_a(x) = 1$. Since $g(x+a)$ is a multiplier and 1 is a function identifiable with a trace (in elementary case) we get by Th. V of §5 that the trace corresponding to $g(x+a)$ which is $\mathfrak{a}(a)g$ satisfies $(\mathfrak{a}(a)g)h_a = 1$. Applying Th. III, 3° of §4 with the transformation $\mathfrak{a}(-a)$, we obtain that

$$(\mathfrak{a}(-a)\mathfrak{a}(a)g)(\mathfrak{a}(-a)h_a) = \mathfrak{a}(-a)1$$

i.e. $g(\mathfrak{a}(-a)h_a) = 1$ and the inverse Fourier transform of $\mathfrak{a}(-a)h_a$ gives us the desired f^{*-1}.

Examples.

I) Consider one space variable z and let f be the analytic functional expressed by $e^{(1/i)D}\delta_0$. Its Fourier transform is $g(x) = e^x$. Since its reciprocal is e^{-x} which is identifiable with a trace (in elementary case) we have that $(e^{-x})^{\wedge -1} = e^{-(1/i)D}\delta_0$ is a quasi-inverse which is actually an inverse (it is a convolutor).

II) Consider now for one space variable $f = \sin(\frac{1}{i}D)\delta_0$. Now $g(x) = \sin x$ and the function $\frac{1}{\sin x}$ is not identifiable as a trace. However, if we take any non-real vector $a \in \mathbb{C}^1$, $\frac{1}{\sin(x+a)}$ is identifiable with a trace, again in elementary case. Hence, this trace transformed by $\mathfrak{A}(-a)$ and by inverse Fourier transform will give us the desired quasi-inverse. However, this quasi-inverse is not an inverse since it is not a convolutor and depending on the vector $\underline{a}$ we may get several such quasi-inverses.

III) Consider now $z \in \mathbb{C}^n$ and take $f = \sum_{|k| \leq N} A_k D^k \delta_0$ with constant coefficients A_k, i.e. a differential operator with constant coefficients. Then $g(x) = \hat{f} = \sum_{|k| \leq N} A_k i^{|k|} x|^k$. If the polynomial $g(x)$ has no multiple null-manifolds in $\mathbb{C}^n$, there will exist vectors $a \in \mathbb{C}^n$ such that on the hyperplane $a + \mathbb{R}^n$ the null-manifolds of $g(x)$ are simple and of dimension at most $n-2$. For such a vector the function $\frac{1}{g(x+a)}$ is locally integrable for $x \in \mathbb{R}^n$ and its behavior at infinity allows one to prove that it is identifiable with a trace in elementary case. Hence again $\left(\mathfrak{A}(-a)\frac{1}{g(x+a)}\right)^{\wedge -1}$ gives us the desired quasi-inverse which is an elementary solution for the differential operator.

As illustration let us take the case of two space variables with $f = -\Delta\delta_0$, $g(x) = x_1^2 + x_2^2$. Then $1/g(x)$ is not locally integrable around the origin. However, if we take the vector $a = (ia_1, 0)$ with $a_1 \neq 0$, a_1 real, then

$g(x + a) = (x_1 + ia_1)^2 + x_2^2$. The null-space of this polynomial on $\mathbb{R}^2$ is the couple of single points $x_1 = 0$, $x_2 = \pm a_1$ and the function $1/g(x + a)$ is a trace. Taking $\left(\mathfrak{G}(-a)\frac{1}{g(x+a)}\right)^{\wedge -1}$ we obtain an elementary solution for the Laplacian. However, it will be a different elementary solution for $a_1 > 0$ and for $a_1 < 0$.

§8. General operators and differential operators on the trace space $\mathfrak{G}\mathcal{X}$.

Most of the applications of traces to differential operators were investigated by M. S. Baouendi and the reader will find them in the text of his lecture. We will limit ourselves here to some general developments which will not be found in Baouendi's text. We will start by a general setting.

Let A be a linear operator in $\mathfrak{G}\mathcal{X}$ defined on some subspace B of $\mathfrak{G}\mathcal{X}$. Since the mapping $u \to \tilde{u}(x,t)$ for any fixed $t \in \mathbb{C}^1_+$ is the continuous isomorphism $\mathfrak{B}(t)$ whose inverse can be denoted by $\mathfrak{B}(-t)$, for any regular trace $v \in \mathfrak{B}(t)(B)$ we can define the operator

(1) $$A_t v = \mathfrak{B}(t)\ A\mathfrak{B}(-t)v\,.$$

Obviously

(1') If $v = \mathfrak{B}(t)u$ then $A_t v = (Au)^{\sim}(x, t)$.

It follows that a solution $u \in \mathfrak{G}\mathcal{X}$ of the equation $Au = v$, $v \in \mathfrak{G}\mathcal{X}$ is obtainable by solving the equation $A_t u_t = v_t$ for $t \in \mathbb{C}^1_+$ where v_t is the regular trace $\mathfrak{B}(t)v$ and u_t is a regular trace $u_t(x)$ satisfying the equation.

$$\Delta_x u_t(x) = \frac{\partial}{\partial t}\, u_t(x)\ .$$

Hence the general problem of solvability of the equation $Au = v$ in traces reduces to a problem where the given data and the required solutions are regular traces, hence entire functions in x. In the case of differential operators

A with polynomial coefficients, A_t turns out to be also a differential operator with polynomial coefficients of special type and for the solution of the equation for u_t the Cauchy-Kowalewski type theorems can be applied. This was one of the main tools used by Baouendi to prove that certain standard differential operators A for which the equation $Au = v$ is not in general solvable in distributions (or hyperfunctions) are solvable in traces.

For special operators A we can give A_t in a more concrete form than (1). To simplify we will restrict our operators A more than strictly needed and we will assume that

$$Au = g(f * u) \tag{2}$$

where f is a convolutor and g is a multiplier. Hence, $A: \mathcal{G}\mathcal{N} \to \mathcal{G}\mathcal{N}$. Before we continue we will give a proposition which belongs to the general theory of convolutions and follows immediately from the definition of convolutability.

Proposition 1. *If* u *and* v *are convolutable then so is* u *and* $\tilde{v}(x, t)$ *for any fixed* $t \in \mathbb{C}^1_+$, *where* $\tilde{v}(x, t)$ *is considered as the regular trace* $\mathcal{B}(t)v$. *Furthermore*

$$(u * v)^{\sim}(x, t) = (u * \mathcal{B}(t)v)(x). \tag{3}$$

Using Fourier transforms and Proposition 1 we can write

$$(Au)^{\wedge -1} = \hat{g}^{-1} * (f * u)^{\wedge -1}$$

$$(f * u)^{\sim}(x, t) = (f * \mathcal{B}(t)u)(x)$$

$$(f * u)^{\wedge -1 \sim}(x,t) = \frac{(2\pi)^{-n/2}}{(2t)^{n/2}} e^{-x^2/4t} \left((f * \mathcal{B}(1/4t)u)(x/-2ti)\right),$$

$$[\hat{g}^{-1} * (f * u)^{\wedge -1}]^{\sim}(x, t) = \frac{(2\pi)^{-n/2}}{(2t)^{n/2}} \hat{g}^{-1} * [e^{-x^2/4t}((f * \mathcal{B}(1/4t)u)(x/-2ti))]$$

To this formula we apply the Fourier transform. To apply it conveniently we change in the formula the variables x into y and obtain after a few cancellations

$$(Au)^{\sim}(x,t) = e^{-x^2/4t}\{\hat{g}^{-1}(y) * [e^{-ty^2}((f * \mathcal{B}(t)u)(2tiy))]\}(x/2ti).$$

It follows that

$$(4) \qquad A_t v(x) = e^{-x^2/4t}\{\hat{g}^{-1}(y) * [e^{-ty^2}((f * v)(2tiy))]\}(x/2ti).$$

A further simplification happens when the multiplier g is a polynomial. It is enough to check it in case when $g(x) = x|^{\ell}$, $\ell = (\ell_1, \cdots, \ell_n)$, when $\hat{g}^{-1} = i^{-|\ell|}D^{\ell}$. It is immediately seen that

$$i^{-|\ell|}D_y^{\ell}[e^{-ty^2}w(y)] = e^{-ty^2}G'_y w(y)$$

where G'_y is a differential operator in y with coefficients which are polynomials in y and t with principal part $i^{-|\ell|}D_y^{\ell}$ and if the coefficients of G'_y are developed in monomials in y, $y|^m$, then each of these monomials will have for coefficient a polynomial in t divisible by $t^{|m|}$. Hence when we replace the variable y by x/2ti we obtain an operator $G_x^{(\ell)}$ again with polynomial coefficients in x and t whose principal part will be $(2t)^{|\ell|}D_x^{\ell}$ and we obtain the formula

$$(5) \qquad A_t v(x) = G_x^{(\ell)}((f * v)(x)), \text{ for } Au = x|^{\ell}(f * u).$$

If in addition, $f = D_x^k \delta_0$ then

$$(6) \qquad A_t v(x) = G_x^{(\ell)} D_x^k v(x) \text{ for } Au = x|^{\ell} D_x^k u.$$

In conclusion, for a differential operator with polynomial coefficients

$$(7) \qquad A = \sum_{|k| \leq m} P_k(x) D_x^k$$

we consider the order m_k of P_k. Hence, we can write

$$P_k(x) = \sum_{|\ell| \leq m_k} p_{k,\ell}\, x|^{\ell}.$$

We obtain for the operator (7) the formula:

$$(8) \qquad A_t v(x) = \sum_{|k| \leq m} \sum_{|\ell| \leq m_k} p_{k,\ell}\, G_x^{(\ell)} D_x^k v(x)$$

The maximum possible order of this operator is $N = \max_k (|k| + m_k)$. This maximum possible order will be achieved unless there are cancellations among the relevant coefficients $p_{k,\ell}$ and if it is achieved the principal part of A_t is given by

$$(9) \qquad \sum_{|k| \leq m} \sum_{|\ell| = N-|k|} p_{k,\ell}\, (2t)^{|\ell|} D_x^{k+\ell} \cdot 1.$$

In this case the coefficients of principal part are independent of x and this is a basic property of the operator A_t which allowed Baouendi to attain his results.

Remark 1. It should be noticed that for differential operators with polynomial coefficients, Baouendi constructs the operator A_t by another method (vide the text of his lecture).

Remark 2. A very plausible conjecture suggested by the results of Baouendi is that we have always solvability for the equation Au = v for a differential operator A with polynomial coefficients if the expression (9) doesn't vanish identically, i.e. there are no linear homogeneous relations between the relevant coefficients $p_{k,\ell}$. The solvability, however, would be available only in R-traces if R is the largest radius of a circle $B_R^2(R)$ which does not contain a common zero of the coefficients in the differential operator (9) (the coefficients being polynomials in t).

It is interesting to notice that by Hörmander's results, for solvability in distributions some relations between the coefficients are necessary, hence, as a general rule, there is no solvability in distributions. In contradistinction, if

1. Here $p_{k,\ell} = 0$ if $|\ell| > m_k$.

the above mentioned conjecture is valid, as a rule, except when certain relations hold, there is solvability in traces.

Remark 3. When the hypothesis of the conjecture of the preceding remark does not hold, there are counter-examples showing that there might be no solvability in traces. The simplest such example was suggested by R. D. Moyer and is the following: We take two space variables x_1 and x_2 and consider the operator $A = x_2\frac{\partial}{2\partial x_1} - x_1\frac{\partial}{1\partial x_2}$. One checks immediately that $A_t = A$, hence, there cannot be general solvability for the equation $(A_t\tilde{u})^{\sim}(x,t) = \tilde{v}(x,t)$ since it would require that $\tilde{v}(0,t) = 0$.

§9. Final remarks and problems.

The development of the theory of traces started quite recently and therefore there are plenty of basic problems which arise naturally in connection with this theory which were not settled as yet. Without a doubt there are many of these problems which could be easily solved but the solution has not been found yet because of lack of time. There are also some which seem to be rather difficult. One such was stated at the end of §4. In the present section we will give a few remarks concerning problems which are connected with further developments of the theory and its applications.

Remark 1. Symbolic Integrals. The symbolic integrals form an essential tool in the development of the theory of traces. For instance, in defining the scalar product between $\mathfrak{A}\mathcal{N}$ and $\mathfrak{A}\mathcal{N}'$ contained in $\mathfrak{A}\mathcal{N}$ or in the definition of convolutability and in several other instances. It is, however, to be noticed that the symbolic integral is constructed specifically for the purpose of studying analytic solutions of the heat equation $\frac{\partial u}{\partial t} = \Delta u$.

It is quite evident that by a linear change of variables x we could develop a completely similar and parallel theory pertaining to the equation $\frac{\partial u}{\partial t} = \Delta_1 u$ where Δ_1 is any homogeneous elliptic differential operator of second order with constant coefficients. However, for the application to such an equation we would need to introduce the symbolic integrals in a changed form, essentially by replacing the mean values over spheres (compare N.P., Ch. I, §5) by mean values over corresponding ellipsoids.

Another aspect of the S-integrals is the nature of the uniformity conditions which are rather restrictive. One may ask if there are no weaker conditions which would still assure the basic properties of the S-integrals which were used in our preceding developments. The weakening of uniformity conditions would lead to a larger class of convolutable traces. However, a heuristic argument to show that our definition is the right one lies in the fact that Baouendi who introduces the convolutors without the symbolic integrals[1.] obtains the same class of traces as we do.

Remark 2. Quasi-inversion of convolutors. In §7 we developped a method for constructing effectively a quasi-inverse f^{*-1} for a convolutor f. It required that the multiplier $g(x) = \hat{f}$ have the property that for some complex vector a, the function $\frac{1}{g(x+a)}$ be identifiable with a trace. The method, however, could be applied also if for some $a \in \mathbb{C}^n$, $\frac{1}{g(x+a)}$ is a distribution identifiable with a trace in elementary case (the same sort of argument as in §7, by using remark 2 of §5 instead of Theorem V).

1. It is done by considering the operator $u * \varphi$ with u a trace and $\varphi \in \mathcal{D}$ (which is an elementary convolution). Thus u is called a convolutor if this operator defined on $\mathcal{D} \subset \mathfrak{G}\mathcal{N}$ is continuous in the topology of $\mathfrak{G}\mathcal{N}$, hence extendable by continuity to the whole of $\mathfrak{G}\mathcal{N}$.

A simple example where this idea can be applied would be the case of $f = \frac{1}{i}\frac{\partial}{\partial x_n} + A$ where A is a differential operator with constant coefficients in all the variables excluding x_n. We wish to find the quasi-inverse of $f * f$. We note that $(f * f)^{\wedge} = g^2(x)$, $g(x) = \hat{f} = (x_n + \hat{A})$ where $\hat{A}$ is a polynomial independent of x_n. If for some $a \in \mathbb{C}^n$, the function $\frac{1}{g(x+a)}$ is a trace in elementary case then $\frac{1}{g^2(x+a)} = \frac{\partial}{\partial x_n} \frac{-1}{g(x+a)}$ which is a distribution identifiable with a trace in elementary case. We don't know how large is the class of differential operators with constant coefficients to which this procedure would apply.

Remark 3. Differential operators with general coefficients. In §8 we considered mostly differential operators with polynomial coefficients. What about more general coefficients? Let us write in general

$$A = \sum_{|k| \leq m} a_k D_x^k , \qquad a_k \in \mathfrak{a}\mathcal{K}.$$

By Theorem I, §6 each a_k is developable into a canonical series of Hermite polynomials, hence $\hat{a}_k^{-1}$ is developable into a canonical series of differential operators with constant coefficients and we can thus apply formulas 4 and 5 from §8 to obtain (at least formally) the operator A_t in form of a dfiferential operator of infinite order,with polynomial coefficients. It turns out, therefore, that to study differential operators with arbitrary coefficients in the theory of traces you have to consider differential problems of infinite order with polynomial coefficients in entire functions of Laplacian order 2 and finite type.

Remark 4. The non-solvability case. Assuming that the conjecture of Remark 2, §8, holds we know that the non-solvability for differential operators with polynomial coefficients happens only when certain relations between the coefficients of the operator hold. By a suitable linear change of variables x, however, we can obtain a new operator where the relation will not hold anymore so that there will be solvability in traces. However, if we go back by inverse transformations of the variables to the original operator we will realize that the solution, instead of being a trace relative to the heat equation, becomes a Δ_1-trace relative to the corresponding equation $\frac{\partial u}{\partial t} = \Delta_1 u$. It is clear, therefore, that we could expect general solvability for all operators A in question if we could form spaces of "super-traces" which will be sums of traces relative to different operators Δ_j.

Let us see, in case of two operators Δ_1 and Δ_2, what is involved in this idea. If we construct the trace-spaces $\mathfrak{A}\mathcal{X}^{(1)}$ and $\mathfrak{A}\mathcal{X}^{(2)}$ we notice that $\mathcal{D}$ belongs to both and is dense in both. This determines an identification mapping J between $\mathcal{D}$ as part of $\mathfrak{A}\mathcal{X}^{(1)}$ and $\mathcal{D}$ as part of $\mathfrak{A}\mathcal{X}^{(2)}$. We assume that this mapping is closable in topologies of the two spaces.[1] We proceed then as described at the beginning of §2 and obtain the space $(\mathfrak{A}\mathcal{X}^{(1)} \dot{+} \mathfrak{A}\mathcal{X}^{(2)})/G(-\bar{J})$ as the sum of the spaces $\mathfrak{A}\mathcal{X}^{(1)} + \mathfrak{A}\mathcal{X}^{(2)}$. This procedure can be extended to any number of operators Δ_j. It is easily shown that for every positive integer m there exists a positive integer N_m and operators Δ_j, $j=1, 2, \ldots, N_m$ such that for every differential operator of order $\leqq m$ with polynomial coefficients of order $\leqq m$ the equation $Au = v$ is solvable in $\Sigma \mathfrak{A}\mathcal{X}^{(j)}$ for every $v \in \cap \mathfrak{A}\mathcal{X}^{(j)}$. We can also choose an infinite sequence of operators Δ_j so that for every differential operator with polynomial coefficients, the equation is solvable when $v \in \bigcap_1^\infty \mathfrak{A}\mathcal{X}^{(j)}$ and with solution $u \in \sum_1^\infty \mathfrak{A}\mathcal{X}^{(j)}$.

<u>Remark 5. Uniqueness of solutions</u> The equations $Au = v$ we were considering before cannot have unique solutions since there exist non-zero solutions

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1. It is not proved as yet that this assumption holds.

of the homogeneous equations. In order to have uniqueness we have in the usual theory to impose on the solutions certain boundary conditions, i.e., conditions at infinity. We cannot directly define conditions at infinity for traces u. We may do it by imposing on each section $\tilde{u}(x, t)$ boundary conditions in the variables x, the conditions depending possibly on t so as to assure the uniqueness of the solution.

There is no clear way of trying to use traces on a bounded domain in $\mathbb{R}^n$ for two reasons: 1.° As mentioned in Remark 3, § 5, we cannot use for general traces the method of localization. 2.° If we considered traces of solutions in the space $\mathcal{ASHE}(\tilde{E} \times \mathbb{C}^1_+)$ where E is a domain in $\mathbb{R}^n$ and $\tilde{E}$ is the harmonicity cell of E, (see N.P., Ch. II, §1, Th.III, and Ch III, §1), we would notice that almost all the tools which we used in investigating the traces, in particular, the transformations (α) — $(\mathcal{E})$ of P.N., Ch. III, §1 will not work here since they will not preserve, in general, the domain $\tilde{E} \times \mathbb{C}^1_+$.

Therefore, it is a completely open question how to use traces in the treatment of boundary value problems on bounded domains where, in the usual theories, the most natural uniqueness theorems occur.