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The aim of this paper is the short description of some results obtained by the author in the last time on the following topics : (a) Structure theory and interpolation theory for function spaces of Sobolev - Besov type without and with weights, (b) Structure theory for nuclear spaces, (c) L_p - theory for a class of strong singular elliptic differential operators. In some sense, this paper is the continuation of the survey papers [20,22] . So we describe in the first line results obtained after finishing of these papers. But this paper is selfcontained, and we repeat the needed basis facts here.

1. Function spaces of Sobolev - Besov type without weights.-

1.1 Definitions.-

R_n denotes the n - dimensional real Euclidean space. $S'(R_n)$ is the set of tempered distributions , F denotes the Fourier - transformation in $S'(R_n)$, F^{-1} is the inverse Fourier - transformation. $L_p(R_n)$ are the usual L_p - spaces ; $1 < p < \infty$. For $0 \leq s < \infty$ and $1 < p < \infty$ we define the Banach spaces

$$(1)\quad H_p^s(R_n) = \{ f \mid f \in S'(R_n) ,\ \|f\|_{H_p^s(R_n)} = \| F^{-1}(1+|\xi|^2)^{\frac{s}{2}} Ff \|_{L_p(R_n)} < \infty \}$$

These are the well - known Lebesgue spaces (or Bessel - potential spaces, or Liouville spaces). Of course, (1) is also meaningful for $-\infty < s < 0$, but we restrict our attention to the case $s \geq 0$. It is $H^0_p(R_n) = L_p(R_n)$. For $0 \leq s =$ integer the spaces $H^s_p(R_n) = W^s_p(R_n)$ are the well - known Sobolev spaces.

(2) $$W^s_p(R_n) = \{f \mid f \in L_p(R_n),\ \|f\|_{W^s_p(R_n)} = \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L_p(R_n)} < \infty\} \ :$$

$\|f\|_{H^s_p}$ and $\|f\|_{W^s_p}$ are equivalent norms.

For $f \in L_p(R_n)$ we write

(3a) $$(\Delta_h f)(x) = f(x+h) - f(x) \ ; \quad h \in R_n \ :$$

and for an integer $\ell \geq 2$

(3b) $$\Delta^\ell_h f = \Delta_h (\Delta^{\ell-1}_h f).$$

Let $0 < s < \infty$; $1 < p < \infty$; $1 \leq q \leq \infty$. We choose integers j and ℓ with $0 \leq j < s$ and $\ell > s - j$ and define the Banach spaces

$$B^s_{p,q}(R_n) = \{ f \mid f \in W^j_p(R_n) \ ,$$

(4) $$\|f\|_{B^s_{p,q}(R_n)} = \|f\|_{W^j_p(R_n)} + \left[\int_{R_n} |h|^{-(s-j)q} \sum_{|\alpha| \leq j} \|\Delta^\ell_h D^\alpha f\|^q_{L_p} \frac{dh}{|h|^n}\right]^{\frac{1}{q}} < \infty \} .$$

(For $q = \infty$, we have to modify in the usual way). These are the well - known Besov - spaces, introduced by Besov [2] . Of course, the norm $\|f\|_{B^s_{p,q}}$ depends on the choice of j and ℓ ; but all these norms are equivalent to each other, and so we avoided additional indices. The spaces

(5) $$W^s_p(R_n) = B^s_{p,p}(R_n) \ ; \ 0 < s \neq \text{integer} \ ; \ 1 < p < \infty \ :$$

are the usual Slobodeckij spaces.

For the spaces $B^s_{p,q}(R_h)$ exist many equivalent norms, we refer to [13, 16, 21, 22] (also for the proof of the equivalence of the norms described in (4)).

Let Ω be a bounded domain in R_n with smooth boundary, $\partial\Omega \in C^\infty$. In the usual way, we define $H^s_p(\Omega)$ and $B^s_{p,q}(\Omega)$ (and so also $W^s_p(\Omega)$) as factor spaces. Let $0 \leq s < \infty$; $1 < p < \infty$. Then we define

(6) $H^s_p(\Omega) = H^s_p(R_n) / \{f(x) | f(x) = 0 \text{ a.e. in } \Omega\}$.

$$||f||_{H^s_p(\Omega)} = \inf ||g||_{H^s_p(R_n)},$$

where the infimum is taken over all $g(x) \in H^s_p(R_n)$ with $g(x) = f(x)$ a.e. in Ω. In a similar way, we define $B^s_{p,q}(\Omega)$; $0 < s < \infty$; $1 < p < \infty$; $1 \leq q \leq \infty$. It is possible to give a more explicite definition of the spaces $B^s_{p,q}(\Omega)$. For an integer $L > 0$ and $h \in R_n$ we write

$$\Omega_h^{\ell} = \bigcap_{j=0}^{\ell} \{x \,|\, x + jh \in \Omega\}.$$

With the aid of these sets one can prove, that holds

$$B^s_{p,q}(\Omega) = \{f | f \in L_p(\Omega) : ||f||^*_{B^s_{p,q}(\Omega)} = ||f||_{L_p(\Omega)} + \left[\int_{R_n} |h|^{-(s-j)q} \sum_{|\alpha| \leq j} || \Delta_h^{\ell} D^\alpha f||^q_{L_p(\Omega_h^{\ell})} \frac{dh}{|h|^n}\right]^{\frac{1}{q}} < \infty\}.$$

(with the usual modification for $q = \infty$).

j and L are integers ; $0 \leq j < s$; $L > j - s$. Δ^{ℓ}_h is defined in (3). $||f||^*_{B^s_{p,q}(\Omega)}$ is an equivalent norm (we avoided again the indices j and L). A proof is given in [12,23]. There are also many other equivalent norms. For $0 \leq s =$ integer the spaces $W^s_p(\Omega) = H^s_p(\Omega)$ are the usual Sobolev spaces : we must replace in (2) R_n by Ω. Further, we use the notation (5) with Ω instead of R_n (Slobodeckij spaces).

Finally, we denote with $\overset{\circ}{H}{}^s_p(\Omega)$ (or $\overset{\circ}{B}{}^s_{p,q}(\Omega)$) the completion of $C^\infty_0(\Omega)$ [the set of all complex infinitely differentiable functions with compact support in Ω] in $H^s_p(\Omega)$ (or $B^s_{p,q}(\Omega)$).

1.2 Structure theory.-

Let A be a Banach space and let $1 \leq q \leq \infty$. We consider the Banach space.

$$L_q(A) = \{a \mid a = \{a_j\}^\infty_{j=1};\ a_j \in A\ ,\ \|a\|_{\ell_q(A)} = (\sum_{j=1}^{\infty} \|a_j\|^q_A)^{\frac{1}{q}} < \infty\}$$

(with the usual modification for $q = \infty$).
With the aid of this notation, we are able to determine the structure of the spaces introduced in section 1.1.

Theorem 1.-

Let Ω be a bounded domain, $\partial\Omega \in C^\infty$.

(a) Let $s \geq 0$ and $1 < p < \infty$. The spaces $H^s_p(R_n)$, $H^s_p(\Omega)$, and $\overset{\circ}{H}{}^s_p(\Omega)$ are isomorphic to $L_p((0,1))$. All these spaces have a Schauder basis.

(b) Let $0 < s < \infty$; $1 < p < \infty$ and $1 \leq q < \infty$. The space $B^s_{p,q}(R_n)$ is isomorphic to $l_q(l_p)$. In all the spaces $B^s_{p,q}(R_n)$ and $B^s_{p,q}(\Omega)$ exist Schauder bases.

(c) Let $s > 0$ and $1 < p < \infty$. The spaces $B^s_{p,p}(R_n)$, $B^s_{p,p}(\Omega)$, and $\overset{\circ}{B}{}^s_{p,p}(\Omega)$ are isomorphic to l_p. They have a Schauder basis.

A proof of the theorem is given in [25]. There are further results in this direction. For instance : all spaces $B^s_{p,q}(\Omega) = (= \overset{\circ}{B}{}^s_{p,q}(\Omega))$ with $0 < s < \frac{1}{p}$; $1 < p < \infty$; $1 \leq q < \infty$ have a common Schauder basis, a system of functions of Haar type. A consequence of the theorem is, that the Sobolev spaces $W^s_p(\Omega)$; $0 \leq s =$ integer ; are isomorphic to $L_p((0,1))$, while the Slobodeckij spaces $W_p^s(\Omega)$; $0 < s \neq$ integer ; are isomorphic to l_p.

1.3 Interpolation theory.-

We use the real interpolation method $(A_o,A_1)_{\theta,q}$; $0 < \theta < 1$; $1 \leq q \leq \infty$; of Lions - Peetre, and the complex interpolation method $[A_o,A_1]_\theta$; $0 < \theta < 1$; of Lions - Calderon - S.G. Krejn. $\{A_o , A_1\}$ is an interpolation couple of Banach spaces. We do not repeat here the basis facts for these interpolation methods and refer for the real method to [3] ; chapter 3, and for the complex method to [4] , see also [7] or [9] .

Although we gave a detailed description of the interpolation theory for the spaces $H_p^s(\Omega)$ and $B_{p,q}^s(\Omega)$ in [23] , we repeat some of the results. This seems to be helpful for the understanding of further results.

Theorem 2.-

Let Ω be either the whole space R_n or a bounded domain in R_n with smooth boundary, $\partial\Omega \in C^\infty$. (The indices by the function spaces and the interpolation spaces are running in their natural domains of defintion, only additional restrictions are remarked).

(a) For $s_o \neq s_1$ holds

$$(B_{p,q_o}^{s_o}(\Omega) , B_{p,q_1}^{s_1}(\Omega))_{\theta,q} = B_{p,q}^{s_o(1-\theta)+s_1\theta}(\Omega). \tag{7}$$

The result is also true after replacing $B_{p,q_o}^{s_o}(\Omega)$ by $H_p^{s_o}(\Omega)$ or/and $B_{p,q_1}^{s_1}(\Omega)$ by $H_p^{s_1}(\Omega)$.

(b) For

$s_o \neq s_1$; $s = (1-\theta)s_o + \theta s_1$; $\frac{1}{p} = \frac{1-\theta}{p_o} + \frac{\theta}{p_1}$ holds

(8) $(B^{s_o}_{p_o,p_o}(\Omega)\ ,\ B^{s_1}_{p_1,p_1}(\Omega))_{\theta,p} = B^{s}_{p,p}(\Omega)$.

The result is also true after replacing $B^{s_o}_{p_o,p_o}(\Omega)$ by $H^{s_o}_{p_o}(\Omega)$ or/and $B^{s_1}_{p_1,p_1}(\Omega)$ by $H^{s_1}_{p_1}(\Omega)$.

(c) For $1 < q_o < \infty$; $1 < q_1 < \infty$:

$s = (1-\theta)s_o+\theta s_1$; $\frac{1}{p} = \frac{1-\theta}{p_o} + \frac{\theta}{p_1}$; $\frac{1}{q} = \frac{1-\theta}{q_o} + \frac{\theta}{q_1}$:

holds

(9) $[B^{s_o}_{p_o,q_o}(\Omega)\ ,\ B^{s_1}_{p_1,q_1}(\Omega)]_\theta = B^{s}_{p,q}(\Omega)$,

(10) $[H^{s_o}_{p_o}(\Omega)\ ,\ H^{s_1}_{p_1}(\Omega)]_\theta = H^{s}_{p}(\Omega)$.

A proof of this theorem is given in [21,23] . There are further results in this direction, see also [22] . Some of the described interpolation results also known, at least for $\Omega = R_n$. For formula (7), we refer to [10], for formula (8) to [6,14,15] and for formula (9) to [6,17] .

2. Function spaces of Sobolev - Slobodeckij type with weights.-

2.1 Definitions.-

In some papers, we considered function spaces with weights and the interpolation theory for these spaces [18, 19, 23] . The obtained results are summarized in [22]. There is also a description of applications to the theory of regular and singular elliptic differential operators. We do not repeat these results and consider a new class of function spaces with weights, which is a generalization of the spaces introduced in [18] and [19].

Let Ω be a (bounded or unbounded) arbitrary connected domain in R_n. (Assumptions for the smoothness of the boundary are not necessary). We consider a weight function $\rho(x)$, defined in Ω :

(11) $\quad 0 < \rho(x) \in C^\infty(\Omega)$;

(12) $\quad \forall\, \gamma \,, \exists\, c_\gamma > 0$ with $|D^\gamma \rho| \leq c_\gamma\, \rho^{1+|\gamma|}(x)$;

(13) $\quad \rho(x) \to \infty$ for $x \to \partial\,\Omega$ or $|x| \to \infty$:

(14) $\quad \exists\, a > 0$ with $\rho^{-a}(x) \in L_1(\Omega)$.

(13) means : for each positive number K exists a compact set $\omega_k \subset \Omega$ with $\rho(x) > K$ for $x \in \Omega - \omega_k$. For bounded domains (or domains with finite measure) (14) follows from (11) and (13).

Examples :

(a) Let Ω be an arbitrary connected bounded domain. With $d(x)$ we denote the distance of a point $x \in \Omega$ to the boundary $\partial\,\Omega$. It is not difficult to show the existence of two positive numbers c_1 and c_2 , and of a weight function $\rho(x)$ of the desired type with

$$c_1\, d(x) \leq \rho^{-1}(x) \leq c_2\, d(x).$$

$\rho^{-1}(x)$ is a " generalized distance - function".

(b) For $\Omega = R_n$ are

$\rho(x) = (1+|x|^2)^\eta$; $\rho(x) = e^{(1+|x|^2)^\eta}$: $\eta > 0$;

examples.

The considered class of weight functions is rather wide.

Now, we define Sobolev - Slobodeckij spaces with weights. Let $1 < p < \infty$; $s \geq 0$; ν and μ are real numbers with $\nu \geq \mu + sp$.

We write :

$$W^s_{p,\mu,\nu}(\Omega) = \{f \mid f \in D'(\Omega) \ ;$$

$$\|f\|_{W^s_{p,\mu,\nu}} = \Big[\int_\Omega \Big(\sum_{|\alpha|=s} \rho^{\mu}(x)|D^\alpha f(x)|^p + \rho^\nu(x)|f(x)|^p\Big)dx\Big]^{\frac{1}{p}} < \infty\}$$

for s = integer, and

$$W^s_{p,\mu,\nu}(\Omega) = \{f \mid f \in D'(\Omega) : \|f\|_{W^s_{p,\mu,\nu}} = \Big[\int_\Omega \rho^\nu(x)|f(x)|^p dx +$$

$$+ \int_{\Omega \times \Omega} \sum_{|\alpha|=[s]} \frac{|\rho^{\frac{\mu}{p}}(x)D^\alpha f(x) - \rho^{\frac{\mu}{p}}(y)D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}} \, dx\, dy\Big]^{\frac{1}{p}} < + \infty \}$$

for $s = [s] + \{s\}$; $[s]$ integer ; $0 < \{s\} < 1$. ($D'(\Omega)$ is the set of distributions over Ω). These spaces are introduced in [24]. In [24] , we considered also Besov spaces and Lebesgue spaces with weights of this type. It is also possible to weaken the assumptions for $\rho(x)$, or to consider the spaces $W^s_{p,\mu,\nu}(\Omega)$ with $\nu < \mu + sp$. [24]. But we restrict our attention to the above mentioned spaces.

2.2 Structure theory.-

The indices have the meaning of the last section, we notice only additional assumptions.

Theorem 3.-

(a) $W^s_{p,\mu,\nu}(\Omega)$ are Banach spaces. $C^\infty_o(\Omega)$ is a dense subset.

(b) Let $s \neq$ integer (Slobodeckij spaces with weights). Then, $W^s_{p,\mu,\nu}(\Omega)$ is isomorphic to l_p.

(c) Let s = integer (Sobolev spaces with weights). Then, $W^s_{p,\mu,\nu}(\Omega)$ is isomorphic to $L_p(0,1)$).

The parts (a) and (b) are proved in [24]. The part (c) is new, a partial result is proved in [26].

On the end of the last section we remarked the possibility of the consideration of the spaces $W^s_{p,\mu,\nu}(\Omega)$ with $\nu < \mu + sp$. In this case, $C_0^\infty(\Omega)$ is (generally) not dense in $W^s_{p,\mu,\nu}(\Omega)$. For the details, we refer to [24].

2.3 Interpolation theory.-

Theorem 4.-

Let m_1 and m_2 be integers ≥ 0 ; $1 < p_1, p_2 < \infty$; $\mu_1, \mu_2, \nu_1, \nu_2$ are real numbers with

$$\nu_1 \geq \mu_1 + p_1 m_1 \; ; \quad \nu_2 \geq \mu_2 + p_2 m_2 \; :$$

$$(\mu_1 - \nu_1) p_2 m_2 = (\mu_2 - \nu_2) p_1 m_1 .$$

(in the case $m_1 = m_2 = 0$, we assume $\mu_1 = \nu_1$ and $\mu_2 = \nu_2$).
Let $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \; ; \; s = (1-\theta) m_1 + \theta m_2 \; ;$$

$$\frac{\nu}{p} = (1-\theta)\frac{\nu_1}{p_1} + \theta\,\frac{\nu_2}{p_2} \quad ; \quad \mu = \nu + sp\,\frac{\mu_1 - \nu_1}{m_1 p_1} \quad .$$

(In the case $m_1 = 0$, $m_2 > 0$ we replace $\frac{\mu_1-\nu_1}{m_1 p_1}$ by $\frac{\mu_2-\nu_2}{m_2 p_2}$; in the case $m_1 = m_2 = 0$, we assume $\mu = \nu$).
Then holds

$$(15) \quad (W^{m_1}_{p_1,\mu_1,\nu_1}(\Omega),\, W^{m}_{p_2,\mu_2,\nu_2}(\Omega))_{\theta,p} = W^s_{p,\mu,\nu}(\Omega) \text{ for } s \neq \text{integer}$$

and

$$(16) \quad [W^{m_1}_{p_1,\mu_1,\nu_1}(\Omega),\, W^{m_2}_{p_2,\mu_2,\nu_2}(\Omega)]_\theta = W^s_{p,\mu,\nu}(\Omega) \text{ for } s = \text{integer}.$$

The theorem is proved in [24] . With the aid of the Besov spaces with weights and the Lebesgue spaces with weights it is possible to prove more general results ((15) without $s \neq$ integer ; (16) without s = integer). We refer to [24] . This interpolation theorem is the basis for the L_p - theory of a class of singular elliptic differential operators, described in the following sections.

3. Singular elliptic differential operators.-

3.1 Definitions.-

In [22] we summarized some results about regular and singular elliptic differential operators acting in function spaces with and without weights on the basis of the interpolation theory for these spaces. Now we consider new singular elliptic differential operators, which are generalizations of the operators considered in [8, 11, 18, 20] , see also [22].

Let Ω be an arbitrary (bounded or unbounded) connected domain in R_n. $\rho(x)$ denotes the weight function of section 2.1. Let m be an integer ; $m = 1,2, \ldots$; μ and ν are real numbers with

(17) $\quad \nu > \mu + 2m \quad ; \mu > 0.$

Let

(18) $\quad \varkappa_L = \frac{1}{2m} \left[\nu(2m-L)+\mu\right] \; ; \; L = 0, 1, \ldots, 2m.$

We consider the operator A.

$$(19)\qquad Au = \sum_{L=0}^{m} \sum_{|\alpha|=2L} \rho^{\chi_{2L}}(x) b_\alpha(x) D^\alpha u + \sum_{|\beta|<2m} a_\beta(x) D^\beta u$$

$b_\alpha(x)$ and $a_\beta(x)$ are real infinitely differentiable functions defined in Ω ;

$\forall\gamma$, $\exists\, c_\gamma > 0$ with $\sup_{\substack{x\in\Omega \\ |\alpha|=0,2,\dots,2m}} | D^\gamma b_\alpha(x) | \leq c_\gamma$;

$\exists\, c > 0$ with

(20)

$(-1)^m \sum_{|\alpha|=2m} b_\alpha(x)\, \xi^\alpha \geq c\, |\xi|^{2m}$ and $b_{(0,\dots,0)}(x) \geq c$;

$(-1)^L \sum_{|\alpha|=2L} b_\alpha(x)\, \xi^\alpha \geq 0$ for $L = 1,\dots,m-1$;

(we used $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ for $\xi = (\xi_1,\dots,\xi_n) \in R_n$) ;

(21) $\exists\, \varepsilon > 0$ with $D^\gamma a_\beta(x) = O(\rho^{\chi_{|\beta|}+|\gamma|-\varepsilon}(x))$; $0 \leq |\gamma| < \infty$

(20) is the ellipticity condition.

(21) shows that $\sum_{L=0} \sum_{|\alpha|=2L} \dots$ is the " main part " of A and $\sum_{|\beta|<2m} \dots$ is the " perturbation part ".

Examples.-

(a) Let Ω be a bounded domain and let $\rho^{-1}(x)$ be the generalized distance function described in the first example of section 2.1. Then is

$A\, u = \rho^{u}(x)\, (-\Delta)^m u + \rho^{\nu}(x)\, u$; (17) holds

an operator of the described type.

(b) Let $\Omega = R_n$. Then has

$A\, u = (1+|x|^2)^{n_1}\, (-\Delta)^m u + (1+|x|^2)^{n_2}\, u$; $n_2 > \max\,(n_1,0)$

the desired properties. (In both examples we have only a "main part"). It is easy to describe further examples. Special cases are also the operators considered in [18].

Further we define the (F) - space (complete locally convex space equiped with a countable set of semi - norms) $S_{\rho(x)}(\Omega)$,

$$S_{\rho(x)}(\Omega) = \{f \mid f \in C^{\infty}(\Omega),\ ||f||_{L,\alpha} = \sup_{x\in\Omega} \rho^{L}(x)|D^{\alpha}f(x)| < \infty$$

for L = 0,1,2,... and for all multiindices α }.
The spaces $W^{s}_{p,\mu,\nu}(\Omega)$ have the same meaning as in the sections 2.1 - 2.3.

3.2 L_p-theory.-

Firstly we consider A as a linear continuous map acting between function spaces.

Theorem 5.-

Let $\varkappa$ be an arbitrary real number ; $s \geq 0$; $1 < p < \infty$.
It exists a real number c, so that for all complex numbers λ with $\operatorname{Re}\lambda \leq c$ the operator $A - \lambda E$ (E is the identity) gives an isomorphic map from

$$W^{2m+s}_{p,\varkappa+p\mu(1+\frac{s}{2m}),\varkappa+p\nu(1+\frac{s}{2m})}(\Omega) \text{ onto } W^{s}_{p,\varkappa+p\mu\frac{s}{2m},\varkappa+p\nu\frac{s}{2m}}(\Omega)$$

and from $S_{\rho(x)}(\Omega)$ onto $S_{\rho(x)}(\Omega)$.

The theorem is proved in [26] , partial results are proved in [8, 18] . The basis for the proof is the interpolation theorem 4, together with some a - priori-estimates.
Secondly, we consider A as a linear unbounded operator acting in $L_{p,\varkappa}(\Omega)$ with the domain of definition

$$(22) \qquad D(A) = W^{2m}_{p,\varkappa+p\mu,\varkappa+p\nu}(\Omega).$$

$\varkappa$ is an arbitrary real number ; $1 < p < \infty$

Theorem 6.-

Let $A = A_{\varkappa,p}$ be the unbounded operator (19) , (22) acting in $L_{p,\varkappa}(\Omega)$.

(a) It holds

$$D(A^k) = W^{2mk}_{p,\varkappa+pk\mu,\varkappa+pk\nu}(\Omega) \; ; \; k = 1,2,\dots$$

(b) The spectrum of A ($= A_{\varkappa\, p}$) consists of a countable set of isolated eigenvalues. Each eigenvalue has a finite algebraic multiplicity. The eigenvalues and the generalized eigenvectors are same for all values of p and $\varkappa$ .The finite linear combinations of the generalized eigenvectors are dense in $S_{\rho(x)}(\Omega)$ and also dense in all spaces $W^s_{q,\sigma,\tau}(\Omega)$; $s \geq 0$; $1 < q < \infty$; σ and τ are real numbers ; $\tau \geq \sigma + qs$.

The part (a) is proved in [26] . The proof of (b) we shall publish in a later paper. (b) is analogous to the well known theory of Browder - Agmon for the density of the linear combinations of the generalized eigenvectors for regular elliptic differential operators [1], see also [5] .

3.3 L_2 - theory.-

Let A be the operator (19), (22) with $p = 2$ and $\varkappa = 0$. We assume additionnally

$$(Af,g)_{L_2} = (f,Ag)_{L_2} \; ; \; f \in C_0^\infty(\Omega) \; , \; g \in C_0^\infty(\Omega);$$

(symmetric property).

Theorem 7.-

(a) A is a self - adjoint semi - bounded operator,

$$(Af,f)_{L_2} \geq c \, ||f||^2_{L_2} \; ; \; f \in D(A) \; ,$$

with pure point spectrum. It exists four positive numbers c_1,

c_2, τ_1 and τ_2 with

(23) $\quad c_1\lambda^{\tau_1} \leq N(\lambda) \leq c_2\lambda^{\tau_2}$

($N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ is the eigenvalue - function, λ_j are the eigenvalues of A).

(b) For $0 < \theta < \infty$ holds

$$D(A^{\theta}) = W^{2m\theta}_{2,2\mu\theta,2\nu\theta}(\Omega).$$

This theorem follows from the theorem 6 and the considerations in [26] .

4. Nuclear function spaces.-

4.1 Isomorphic properties.-

In [20] we gave a systematic treatment about the connection between (singular) elliptic differential operators and nuclear function spaces. All nuclear spaces considered in [20] are isomorphic to the nuclear space s of rapidly decreasing sequences.

$$s = \{ \xi \mid \xi = (\xi_j)^{\infty}_{j=1} : \quad ||\xi||_L = \sup_j j^L |\xi_j| < \infty \quad \text{for } L = 0,1,2,\ldots \}$$

Now we formulate a general theorem in this direction.

We start with the description of the abstract background. Let A be a self - adjoint positive - definite operator with pure point spectrum acting in a separable complex Hilbert space, $D(A^{\infty}) = \bigcap_{j=0}^{\infty} D(A^j)$ equiped with the norms $||u||_j = ||A^j u||$; becomes an (F) - space. We denote the eigenvalues of A with λ_j ; $j = 1, 2, \ldots$ In [20] we remarked that $D(A^{\infty})$ is isomorphic to s if and only if there exist four positive numbers c_1, c_2, τ_1 and τ_2 with

$$c_1 j^{\tau_1} \leq \lambda_j \leq c_2 j^{\tau_2} .$$

Let Ω be an arbitrary (bounded or unbounded) domain in R_n ; smoothness assumptions for the boundary are not necessary. We consider

$$A_o \ f = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^m D^\alpha (a_{\alpha,\beta}(x) D^\beta f)$$

with the domain of definition $D(A_o) = C_o^\infty(\Omega)$, as an unbounded operator in $L_2(\Omega)$. We assume :

$a_{\alpha,\beta}(x)$ are real functions,

$a_{\alpha,\beta}(x) = a_{\beta,\alpha}(x) \in C^\infty(\Omega)$;

$\exists\, c > 0$ with

$(A_o f,f)_{L_2(\Omega)} \geq c \ ||f||^2_{L_2(\Omega)}$; $f \in C_o^\infty(\Omega)$.

A_o is a symmetric positive - definite operator.

Theorem 8.-

Let A be a positive - definite self - adjoint extension of A_o. Let $D(A^\infty)$ be a nuclear space. Then $D(A^\infty)$ is isomorphic to s.

We shall give the proof of the theorem in a later paper. Many interesting well - known nuclear function spaces are special cases of the last theorems, see [20] . We mention the Schwartz space $S(R_n)$, and the space $C^\infty(\overline{\Omega})$, where Ω is a bounded domain with $\partial\Omega \in C^\infty$, [20].

4.2 The space $S_{\rho(x)}(\Omega)$.

We assume that the weight function $\rho(x)$ and the space $S_{\rho(x)}(\Omega)$ have the same meaning as in section 3.1

Theorem 9.-

$S_{\rho(x)}(\Omega)$ is isomorphic to s.

The theorem is proved in [26] . It is a positive answer of the problem 2 of [20], p. 310. As a special case of the last theorem we mention that for an arbitrary bounded domain the space $C_o^{\infty}(\overline{\Omega}) = \{f \mid f \in C_o^{\infty}(R_n),\ \mathrm{supp}\ f \subset \overline{\Omega}\}$ $||f||_{\alpha} = \sup_{x \in \Omega} |D^{\alpha} f(x)|$, (α is an arbitrary multiindex), is isomorphic to s.

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