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PROPAGATION OF SINGULARITIES FOR $\bar{\partial}$.

by

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Let M be a relatively compact domain in $\mathbb{C}^n$ with a smooth boundary, which we denote by bM.

We will assume that there is a real-valued C^∞ function r defined in a neighborhood of bM, such that $dr \neq 0$, $r > 0$ outside of $\bar{M}$, $r = 0$ on bM and $r < 0$ in M.

Furthermore, we will assume that M is pseudo-convex ; that is, if $P \in bM$ and $(a^1, \ldots, a^n) \in \mathbb{C}^n$ with :

$$\sum_{i=1}^{n} r_{z_i}(P)\, a^i = 0,$$

then

$$\sum_{i,j} r_{z_i \bar{z}_j}(P)\, a^i \bar{a}^j \geqslant 0.$$

Here we wish to discuss the following problem :

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given a (0,1)-form $\alpha = \Sigma\, \alpha_j\, d\bar z_j$ on $\bar M$, which satisfies the compatibility conditions $\bar\partial\alpha = 0$ (i.e. $\alpha_{j\bar z_k} = \alpha_{k\bar z_j}$), to find a function u on $\bar M$ such that :

$$\begin{cases} \bar\partial u = \alpha \\ \text{sing supp } (u) \subset \text{sing supp } (\alpha) = \bigcup \text{sing supp } (\alpha_j). \end{cases}$$

Since $\bar\partial$ is elliptic on functions, we know that :

$$M \cap \text{sing supp}(u) \subset M \cap \text{sing supp } (\alpha).$$

Hence the problem is concerned with behaviour at the boundary.

First consider the so called "global" problem in which sing supp $(\alpha) = \emptyset$

We have the following result :

THEOREM. If $M \subset\subset \mathbb{C}^n$ is a pseudo-convex domain with a C^∞ boundary and if α is a (0,1)-form on $\bar M$ whose components are all in $C^\infty(\bar M)$ and which satisfies the compatibility conditions $\bar\partial\alpha = 0$; then, for each integer $m \geqslant 0$ there exists a function $u_m \in C^m(\bar M)$ such that $\bar\partial u_m = \alpha$.

The proof of this theorem is given in [9];it is based on the weighted L_2-estimates of HÖRMANDER (see [5]). More precisely, for every $\tau \geqslant 0$ consider the norm $\| \ \|_{(\tau)}$ defined by :

$$\| u \|^2_{(\tau)} = \int_M |u|^2\, e^{-\tau|z|^2}\, dV.$$

Then there exists $C \geqslant 0$ such that

$$\text{(H)} \qquad \tau\|\varphi\|^2_\tau \leqslant C(\|\bar\partial\varphi\|^2_{(\tau)} + \|\bar\partial^*_\tau\varphi\|^2_{(\tau)}) \quad ,$$

for all (0,1)-forms φ in $C^\infty(\bar M) \cap \mathcal{D}\text{om}(\bar\partial^*)$ and all $\tau \geqslant 0$. Here $\bar\partial^*_\tau$ denotes the adjoint with respect to $\| \|_{(\tau)}$.

We set :

$$Q_\tau(\varphi,\psi) = (\bar\partial\varphi, \bar\partial\psi)_{(\tau)} + (\bar\partial^*_\tau\varphi, \bar\partial^*_\tau\psi)_{(\tau)} .$$

Given a (0,1)-form α we consider the following variational problem :

find $\varphi_\tau \in \mathrm{Dom}(\bar\partial^*)$ such that

$$Q_\tau(\varphi_\tau, \psi) = (\alpha, \psi)_{(\tau)}$$

for all $\psi \in \mathrm{Dom}(\bar\partial^*) \cap C^\infty(\bar M)$. Using (H) we can show that there exists a unique solution φ_τ and that for any m, $\varphi_\tau \in C^m(\bar M)$ if τ is large enough.
If $\bar\partial\alpha = 0$, we set $u_m = \bar\partial^*_\tau \varphi_\tau$, where τ is chosen so large that $\varphi_\tau \in C^{m+1}(\bar M)$.

More generally, if M is a complex manifold with a smooth boundary, we must assume that there exists a strongly pluri-subharmonic function in a neighborhood of bM ; this function is then used to replace $|z|^2$ in the above proof and, modulo some technicalities, the analogue of the above result follows. Examples of GRAUERT, see [2] , show that for general pseudo-convex manifolds the result does not hold.

The following example shows that "local" regularity does not hold in general.

Let $M \subset\subset \mathbb{C}^2$ and suppose that in a neighborhood of $P \in bM$ the boundary is "flat" (i.e. the Levi form vanishes identically). Thus we can introduce holomorphic coordinates (z,w) in a neighbordood U of P , such that P is at the origin and

$$U \cap bM = \{ (z,w) \mid \mathrm{Re}(w) = 0 \} .$$

Let $P \in V \subset\subset W \subset\subset U$ and let ρ be a C^∞ function which equals 1 in V and equals 0 outside of W .

Consider $\alpha = \bar\partial \left(\frac{\rho}{w}\right) = \frac{\bar\partial\rho}{w}$. I claim that there does not exist a function u with $\bar\partial u = \alpha$ and sing supp $u \subset$ sing supp α .

Suppose there is such a u, then u is bounded on compact subsets of $\bar{M} - (\bar{V} - W)$. Let $h = \frac{\rho}{w} - u$, then h is holomorphic in M, it is very large near P and bounded outside of V. Now consider the restriction of h to the set $Re(w) = \delta$, $Im(w) = 0$. Choosing δ sufficiently small we obtain a holomorphic function of z which is arbitrarily large at the origin but bounded on a contour enclosing the origin. This gives a contradiction.

Several examples of the same nature lead to the following conjecture.

CONJECTURE A.

Suppose M is a pseudo-convex complex manifold with a smooth boundary and suppose $P \in bM$. Then the following are equivalent.

a) There exists a neighborhood U of P such that, whenever $\alpha = \bar{\partial} v$ on $\bar{M}$, then there exists u on $\bar{M}$ with $\alpha = \bar{\partial} u$ and $U \cap \text{sing supp } u \subset U \cap \text{sing supp } \alpha$.

b) There do not exist any connected non-trivial analytic varieties, whose intersection with U lies in bM.

At this point we can prove the following result which gives a "finite" version of the conjecture.

For $P \in bM$ let $T_P^{1,0}(bM)$ denote the space of vectors of type $(1,0)$ which are tangent to bM at P, i.e. if $L \in T_P^{1,0}(bM)$ then in terms of holomorphic coordinates, we have :

$$L = \sum a^j \frac{\partial}{\partial z_j} ,$$

where

$$\sum a^j r_{z_j}(P) = 0.$$

For $L \in T_P^{1,0}(bM)$ let L' be a C^∞ vector field on a neighborhood U of P such that $L'_P = L$ and $L'_Q \in T_Q^{1,0}(bM)$ for $Q \in U \cap bM$. We denote by $T_P^{0,1}(bM)$ the conjugate of $T_P^{1,0}(bM)$.

Let $\mathcal{L}(L', \bar{L}')$ denote the Lie algebra defined by the vectors L' and $\bar{L}'$.

We set :

$$\mathscr{L}(L', \bar{L}') = \bigcup_{j=0}^{\infty} \mathscr{L}^j(L', \bar{L}') ,$$

where $\mathscr{L}^0(L', \bar{L}')$ denotes the linear combinations of L' and $\bar{L}'$ and :

$$\mathscr{L}^j(L', \bar{L}') = \mathscr{L}^{j-1}(L', \bar{L}') + [\mathscr{L}^{j-1}(L', \bar{L}'), \mathscr{L}^0(L', \bar{L}')] .$$

Finally for each $L \in T_P^{1,0}(bM)$ we denote by $\mathscr{L}_P(L, \bar{L})$ and $\mathscr{L}_P^j(L, \bar{L})$ those subspaces of the complexified tangent space $\mathbb{C}\, T_P(bM)$ which are restrictions of the spaces $\mathscr{L}(L', \bar{L}')$ and $\mathscr{L}^j(L', \bar{L}')$ respectively. Modulo the subspace $T_P^{1,0}(bM) + T_P^{0,1}(bM)$, this definition does not depend on the particular extension L'.

THEOREM .

If M is pseudo-convex and if $P \in bM$ has the following property :

(*) If $L \in T_P^{1,0}(bM)$, $L \neq 0$ then for some j we have :

$$\mathscr{L}_P^j(L, \bar{L}) \not\subset T_P^{1,0}(bM) + T_P^{0,1}(bM) .$$

Then $\bar{\partial}$ is locally regular at P, in the sense of (a) in conjecture A.

At present, the proof of this theorem is much too complicated. In fact, we believe that * is equivalent to subellipticity at P which then implies the local regularity (a).

DEFINITION.

$P \in bM$; we say that $\bar{\partial}$ is subelliptic at P if there exists a neighborhood U of P and constants $\varepsilon > 0$, $C > 0$ such that

$$\Sigma \, |||D_i \varphi_j|||^2_{\varepsilon-1} \leq C(\|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2 + \|\varphi\|^2)$$

for (0,1)-forms φ in $C_0^\infty(U \cap \bar{M}) \cap \mathrm{Dom}(\bar{\partial}^*)$.

Here $|||u|||_s$ is defined by :

$$|||u|||_s^2 = \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi'|^2)^{\frac{s}{2}} \; \tilde{u}(\xi', r) \; d\xi' \, dr$$

where $(X_1, \ldots, X_{2n-1}, r)$ denote the boundary coordinates (bM is given by $\{ r = 0 \}$), $\xi' = (\xi_1, \ldots, \xi_{2n-1})$ and $\tilde{u}(\xi', r)$ is the Fourier transform of u in $X_1, \ldots, X_{2n-1}$ for fixed r.

By the results in [10] subellipticity at P implies local regularity at P. We then make the following conjecture :

<u>CONJECTURE B</u>.

If M is pseudo-convex, $P \in bM$, then $\bar{\partial}$ is subelliptic at P if and only if * holds.

If for each $L \in T_p^{1,0}(bM)$ the lowest number j given in * is one then the Levi-form is positive definite; subellipticity holds with $\varepsilon = \frac{1}{2}$ (see [7]).

Let $L_1, \ldots, L_{n-1}$ be a basis of the vector fields of type $(1,0)$ tangent to bM in a neighborhood of P . Then $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}$, are independent and we can get such a purely imaginary vector field N so that $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}$, N are basis for all tangential vector fields in a neighborhood of P .

Let

$$[L_i, \bar{L}_j] = c_{ij} \; N + \Sigma a_{ij}^k \; L_k + \Sigma b_{ij}^k \; \bar{L}_k ,$$

then c_{ij} represents the Levi-form.

Conjecture B has been proved in case the matrix (c_{ij}) is diagonalizable (see [8]).

In particular if c_{ij} has at most one non zero eigen-value. The conjecture has been proven in many other cases also and we hope to give a proof in all generality soon.

Please observe that the type of result described above implies existence of global solutions with certain regularity properties. For example, an easy consequence of the regularity that we discuss is that orthogonal projection of L_2 on holomorphic functions preserves regularity at the boundary. For an exposition of this type of results see [1] .

Recently there have been many results on regularity properties of $\bar{\partial}$ in the sense of sup and Hölder norms (see [3], [4] and [6]). These results have been obtained on strongly pseudo-convex domains, an important element in proofis the Levi polynomial ; that is, for each point $P \in bM$ at which the Levi form is strongly pseudo-convex, it is easy to construct a second order polynomial which vanishes at P and such that in a neighborhood U of P all its zeros lie in $(U - \bar{M}) \cup \{P\}$.

L. NIRENBERG and the Author have discovered the following example (see [11]) which shows that such functions do not exist, in general even if (*) holds.

Let (z,w) denote the coordinates on $\mathbb{C}^2$ and define r by :

$$r = \mathrm{Re}(w) + |z|^8 + \frac{15}{7}|z|^2 \mathrm{Re}(z^6)$$

the domain defined by $r \leqslant 0$ is pseudo-convex ,
the boundary is strongly pseudo-convex outside of the line $z = 0$, $\mathrm{Re}(w) = 0$ and on this line, condition * holds with $j = 4$.

THEOREM.

If h is a holomorphic function defined in a neighborhood U of $(0,0)$ and if $h(0,0) = 0$; then there exist points (z_1,w_1) and (z_2,w_2) in U, such that $h(z_i,w_i) = 0$, $r(z_1,w_1) > 0$ and $r(z_2,w_2) < 0$.

In particular, this shows that there is no way of introducing local holomorphic coordinates relative to which this surface is convex in a neighborhood of $(0,0)$.

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REFERENCES

[1] FOLLAND G.B. and KOHN J.J, "The Neumann problem for the Cauchy-Riemann complex" , Annals of Mathematical Studies, Princeton University Press (1972).

[2] GRAUERT H., "Bemerkenswerte pseudokonvexe mannifaltigkeiten", Math. Z. 81 (1963), 377-391.

[3] GRAUERT H. and LIEB I., "Das Ramirezsche Integral und die Gleichung $\bar{\partial}f = \alpha$ in Bereich der beschrenkten Formen", to appear in Rice University Studies.

[4] HENKIN G. , "Integral representations of functions on strongly pseudo-convex domains and certain applications", Mat. Sbornik 78(1969), 611-632.

[5] HÖRMANDER L., "L^2 estimates and existence theorems for the $\bar{\partial}$ operator" , Acta Math. 113 (1965), 89-152.

[6] KERZMAN N., "Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudo-convex domains", Comm. Pure Appl. Math. 24 (1971), 301-379.

[7] KOHN J.J., "Harmonic integrals on strongly pseudo-convex manifolds I and II", Ann. of Math. 78 (1963), 112-148 ; ibid. 79 (1964), 450-472.

[8] KOHN J.J., "Boundary behaviour of $\bar{\partial}$ on weakly pseudo-convex manifolds of dimension two", Journal of Diff. Geometry (to appear).

[9] KOHN J.J., "Global regularity of $\bar{\partial}$ on weakly pseudo-convex manifolds" , Trans. A.M.S. (to appear).

[10] KOHN, J.J. and NIRENBERG L., "Non-coercive boundary value problems" , Comm. Pure Appl. Math. 18 (1965), 443-492.

[11] KOHN J.J. and NIRENBERG L., "An example of a pseudo-convex surface which does not admit a holomorphic separating function", (to appear).