Twisted Alexander polynomials for irreducible $SL(2, \mathbb{C})$ -representations of torus knots

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Abstract. We prove that the twisted Alexander polynomial of a torus knot with an irreducible $SL(2, \mathbb{C})$ -representation is locally constant. In the case of a (2, q) torus knot, we can give an explicit formula for the twisted Alexander polynomial and deduce Hirasawa-Murasugi's formula for the total twisted Alexander polynomial. We also give examples which address a mis-statement in a paper of Silver and Williams.

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1. Introduction

Let *K* be a knot in the 3-sphere S^3 and $G(K) = \pi_1(S^3 - K)$ its knot group. In this paper, we consider the twisted Alexander polynomial $\Delta_{K,\rho}(t)$, which is defined as a rational expression over \mathbb{C} with one variable *t*, for a knot *K* associated with an irreducible representation $\rho : G(K) \to SL(2, \mathbb{C})$. The twisted Alexander polynomial for a knot with a linear representation was originally introduced by Lin in [9]. It was generalized and developed by Wada in [12] for finitely presentable groups which include link groups. If we put t = 1, it is known that $\Delta_{K,\rho}(1)$ equals the Reidemeister torsion of the exterior of a knot *K* for the same representation ρ , under the acyclicity condition [6].

When ρ is a nonabelian representation, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ becomes a Laurent polynomial over \mathbb{C} (see [7]). Since an irreducible representation is nonabelian, $\Delta_{K,\rho}(t)$ is a Laurent polynomial and all the coefficients of $\Delta_{K,\rho}(t)$ are complex valued functions on the space of irreducible representations in $SL(2, \mathbb{C})$. We then obtain the following.

Theorem 1.1. If K is a torus knot, then every coefficient of $\Delta_{K,\rho}(t)$ is a locally constant function, that is, a constant function on each connected component of the space of irreducible $SL(2, \mathbb{C})$ -representations.

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Remark 1.2. (1) Johnson [3] proved that the Reidemeister torsion of a torus knot is a locally constant function on the space of irreducible $SL(2, \mathbb{C})$ -representations. More generally, it is known that the Reidemeister torsion is locally constant for a Seifert fibered manifold [5].

(2) Kitayama observed in [8, Example 5.11] that every coefficient of the twisted Alexander polynomial of a torus knot is locally constant for SU(2)-representations.

This paper is organized as follows. In the next section, we review the definition for the twisted Alexander polynomial associated with $SL(2, \mathbb{C})$ -representations. In Section 3, we describe the representation space of a torus knot (Proposition 3.7) according to Johnson's lecture note [3]. In the last section, we give two kinds of proofs for Theorem 1.1 and an explicit formula for the twisted Alexander polynomial for (2, q) torus knots (Theorem 4.2). We also discuss the total twisted Alexander polynomial, due to Silver-Williams [11]. Hirasawa-Murasugi's formula [2] for the total twisted Alexander polynomial corresponding to parabolic representations of a (2, q) torus knot is shown very easily (Corollary 4.5). In particular, we present an example for the twisted Alexander polynomial which cannot be written as a product of cyclotomic polynomials (Example 4.7). The example addresses a mis-statement in a paper of Silver and Williams [11].

We shall give a self-contained description through the paper, so we determine the representation space of a torus knot in detail (although it seems to be known to experts).

2. Twisted Alexander polynomials

In this section, we review the definition of $\Delta_{K,\rho}(t)$ for an $SL(2, \mathbb{C})$ -representation ρ . There are several versions for the twisted Alexander polynomial, but in this paper we adopt the one due to Wada [12].

For a given knot K, we fix a presentation of its knot group G(K):

$$P = \langle x_1, \ldots, x_n \mid u_1, \ldots, u_{n-1} \rangle.$$

We may assume its deficiency is one, but it might not be a Wirtinger presentation. We take the abelianization homomorphism $\alpha : G(K) \to \mathbb{Z} = \langle t \rangle$.

Given representations $\rho : G(K) \to SL(2, \mathbb{C})$ and $\alpha : G(K) \to \langle t \rangle$, they naturally induce two ring homomorphisms $\tilde{\rho}$ and $\tilde{\alpha}$ from the group ring $\mathbb{Z}G(K)$ to $M(2, \mathbb{C})$ and $\mathbb{Z}[t, t^{-1}]$ respectively, where $M(2, \mathbb{C})$ is the matrix algebra of 2×2 matrices over \mathbb{C} . Then $\tilde{\rho} \otimes \tilde{\alpha}$ defines a ring homomorphism $\mathbb{Z}G(K) \to$ $M(2, \mathbb{C}[t, t^{-1}])$. Let F_n denote the free group on generators x_1, \ldots, x_n and

$$\Phi:\mathbb{Z}F_n\to M\left(2,\mathbb{C}[t,t^{-1}]\right)$$

the composite of the surjection $\mathbb{Z}F_n \to \mathbb{Z}G(K)$ induced by the presentation P and the ring homomorphism $\tilde{\rho} \otimes \tilde{\alpha}$.

Let us consider the $(n - 1) \times n$ matrix A whose (i, j) component is the 2×2 matrix

$$\Phi\left(\frac{\partial u_i}{\partial x_j}\right) \in M\left(2, \mathbb{C}[t, t^{-1}]\right),$$

where $\frac{\partial}{\partial x_j}$ (j = 1, ..., n) denotes the free differential calculus (see [1]). This matrix A is called the Alexander matrix of the presentation P associated with ρ .

For $1 \le j \le n$, let us denote by A_j the $(n-1) \times (n-1)$ matrix obtained from A by removing the *j*th column. We regard A_j as a $2(n-1) \times 2(n-1)$ matrix with coefficients in $\mathbb{C}[t, t^{-1}]$.

The following two lemmas are the foundations of the definition for the twisted Alexander polynomial (see [12] for the proof).

Lemma 2.1. det $\Phi(x_j - 1) \neq 0$ for some j.

Lemma 2.2. det A_j det $\Phi(x_k - 1) = \det A_k \det \Phi(x_j - 1)$ for $1 \le j < k \le n$.

From the above two lemmas, we can define the twisted Alexander polynomial of G(K) associated with the representation $\rho : G(K) \to SL(2, \mathbb{C})$ to be a rational expression

$$\Delta_{K,\rho}(t) = \frac{\det A_j}{\det \Phi(x_j - 1)}$$

provided det $\Phi(x_i - 1) \neq 0$.

Remark 2.3. Up to a factor of t^k ($k \in \mathbb{Z}$), this is an invariant of G(K) with ρ (see [12, Theorem 1]). Namely, it does not depend on the choices of a presentation P. Hence we can consider it as a knot invariant.

In general, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ depends on ρ . However the following proposition is known.

Proposition 2.4. If ρ and ρ' are conjugate as an $SL(2, \mathbb{C})$ -representation, then $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$.

Here a representation ρ is conjugate to a representation ρ' if there exists $S \in SL(2, \mathbb{C})$ such that $\rho(g) = S\rho'(g)S^{-1}$ for any $g \in G(K)$.

Under a generic assumption on ρ , the twisted Alexander polynomial becomes a Laurent polynomial (see [7, Theorem 3.1]).

Proposition 2.5. If $\rho : G(K) \to SL(2, \mathbb{C})$ is a nonabelian representation, then $\Delta_{K,\rho}(t)$ is a Laurent polynomial with coefficients in \mathbb{C} .

In this paper, we consider that each coefficient of $\Delta_{K,\rho}(t)$ is a complex valued function on the space of conjugacy classes of irreducible $SL(2, \mathbb{C})$ -representations.

3. Representation space of a (p,q) **torus knot**

In this section, we recall a parametrization of the space of conjugacy classes of irreducible $SL(2, \mathbb{C})$ -representations of a torus knot. This was demonstrated in the unpublished lecture notes [3] by Johnson.

Let (p, q) be a pair of coprime natural numbers. Hereafter let K = T(p, q) be the (p, q) torus knot and G(p, q) be its knot group. We take the following presentation of G(p, q):

$$G(p,q) = \langle x, y \, | \, x^p y^{-q} \rangle.$$

First we quickly review some terminologies of a linear representation in $SL(2, \mathbb{C})$. A representation $\rho : G(p,q) \to SL(2,\mathbb{C})$ is called irreducible if there does not exist a nontrivial proper invariant subspace of \mathbb{C}^2 under the natural action of $\rho(G(p,q))$. A representation $\rho : G(p,q) \to SL(2,\mathbb{C})$ is called reducible if ρ is not irreducible. That is, there is an invariant 1-dimensional subspace of \mathbb{C}^2 . A representation ρ is called abelian if $\rho(G(p,q))$ is an abelian subgroup of $SL(2,\mathbb{C})$. It is easy to see that an abelian representation is reducible.

Let *R* be the set of irreducible $SL(2, \mathbb{C})$ -representations of G(p, q). Fixing the generators *x* and *y*, *R* can be embedded into $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ by the map $R \ni \rho \mapsto (\rho(x), \rho(y)) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. From this embedding, the topology of *R* can be induced from $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Let \hat{R} be the space of conjugacy classes of irreducible representations, that is, the quotient space of *R* by conjugate action of $SL(2, \mathbb{C})$. In general \hat{R} has some connected components. For a given representation ρ , we write $\hat{\rho}$ for its conjugacy class.

From now on, we start to describe the structure of \hat{R} . Choosing a pair (r, s) of natural numbers satisfying ps - qr = 1, then $m = x^{-r}y^s \in G(p, q)$ represents a meridian of T(p, q). Let $\rho : G(p, q) \to SL(2, \mathbb{C})$ be an irreducible representation. For simplicity, we write a capital letter X for the image $\rho(x)$ of x, Y for $\rho(y)$ and so on.

Now we put $z = x^p = y^q \in G(p, q)$ which lies in the center of G(p, q). Recall that the center of $SL(2, \mathbb{C})$ is $\{\pm E\}$, where E is the identity matrix of degree 2.

Lemma 3.1. $Z = \pm E$.

Proof. Assume that $Z \neq \pm E$. We take an eigenvalue λ of Z and its eigenspace $V_{\lambda} \subset \mathbb{C}^2$. Because z is a center element of G(p, q), Z can be commuted with any matrix $S \in \rho(G(p, q))$. For any vector $v \in V_{\lambda}$,

$$Z(Sv) = S(Zv) = \lambda Sv.$$

Hence $Sv \in V_{\lambda}$ and it implies V_{λ} is an invariant subspace of ρ . By the irreducibility of ρ , V_{λ} is the full space \mathbb{C}^2 . Therefore $\lambda = \lambda^{-1} = \pm 1$. Here we may put $Z = \begin{pmatrix} \pm 1 & t \\ 0 & \pm 1 \end{pmatrix}$ up to conjugation. If $t \neq 0$, then the above eigenspace V_{λ} is not the full space \mathbb{C}^2 . This contradicts the irreducibility of ρ and then $Z = \pm E$.

Since $Z = X^p = Y^q = \pm E$, it holds that $X^{2p} = Y^{2q} = E$. On the other hand, we have the following.

Lemma 3.2. $X^r \neq \pm E, Y^s \neq \pm E$.

Proof. Assuming $X^r = \pm E$, we have $X^{2r} = E$. Since ps - qr = 1, 2ps = 2qr + 2 holds. Thus $X^{2ps} = X^{2qr+2}$. Hence we have $E = X^2$ and then $X = \pm E$. It means that the representation ρ is abelian, but this is a contradiction. It is similarly proved that $Y^s \neq \pm E$.

Here we let

$$\alpha^{\pm 1} = \exp(\pm \sqrt{-1\pi a/p})$$
 and $\beta^{\pm 1} = \exp(\pm \sqrt{-1\pi b/q})$

to be the eigenvalues of X and Y respectively, where we can assume that 0 < a < p and 0 < b < q. Since

$$X^{p} = (-E)^{a} = Y^{q} = (-E)^{b},$$

it holds that

$$a \equiv b \mod 2$$
.

From now on, let us fix tr X and tr Y. We consider a conjugacy class of the representation ρ , so that we may assume $X = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and Y is conjugate to $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ in $SL(2, \mathbb{C})$.

Remark 3.3. We remark that *a* is fixed but *b* is not. In fact, there are two choices of b (0 < b < q), namely b or $-b \mod q$. Both of them give the same trace tr $Y = 2\cos(\pi b/q)$.

If Y is an upper triangle matrix, then ρ is a reducible representation. In this case, the trace of the meridian image

$$M = X^{-r}Y^{s}$$

= $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{-r} \begin{pmatrix} \beta & * \\ 0 & \beta^{-1} \end{pmatrix}^{s}$ or $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{-r} \begin{pmatrix} \beta^{-1} & * \\ 0 & \beta \end{pmatrix}^{s}$

is given by

tr
$$M = \alpha^{-r} \beta^{\pm s} + \alpha^r \beta^{\mp s} = 2 \cos \pi (ra/p \pm sb/q).$$

Namely we obtain the following lemma.

Lemma 3.4. ρ is an irreducible representation if tr $M \neq \alpha^{-r} \beta^{\pm s} + \alpha^r \beta^{\mp s}$.

Now Y^s is conjugate to $\begin{pmatrix} \beta^s & 0\\ 0 & \beta^{-s} \end{pmatrix}$ in $SL(2, \mathbb{C})$, so that Y^s has the form of

$$\begin{pmatrix} \beta^{s} + \delta & * \\ * & \beta^{-s} - \delta \end{pmatrix} \text{ or } \begin{pmatrix} \beta^{-s} + \delta & * \\ * & \beta^{s} - \delta \end{pmatrix},$$

where δ is any complex number. Therefore

tr
$$M =$$
tr $(X^{-r}Y^s)$
= $\alpha^{-r}\beta^{\pm s} + \alpha^r\beta^{\mp s} + \delta(\alpha^{-r} - \alpha^r).$

Hence we can assume

tr
$$M = \alpha^{-r}\beta^s + \alpha^r\beta^{-s} + \delta(\alpha^{-r} - \alpha^r)$$

by replacing β if necessary. We note that b has been fixed. This value of tr M can be any complex number because δ can be so.

Lemma 3.5. If we put $U = X^{-r}$ and $V = Y^s$, then $X = Z^s U^q$ and $Y = Z^{-r} V^p$.

Proof. Direct calculations.

Lemma 3.6. For any irreducible representation $\rho : G(p,q) \to SL(2,\mathbb{C})$, if tr X, tr Y and tr M are fixed, then ρ is uniquely determined up to conjugation.

Proof. We fix tr X, tr Y and tr M. Then we prove that X and Y are uniquely determined in $SL(2, \mathbb{C})$ up to mutual conjugation. First the value of tr X determines tr Z and tr U, because $Z = X^p$ and $U = X^{-r}$. Hence Z can be determined since $Z = \pm E$. Similarly tr Y determines tr V. Here tr M = tr UV is fixed and U, V do not commute, so that U and V are determined in $SL(2, \mathbb{C})$ up to mutual conjugation. Therefore X and Y are uniquely determined up to conjugation by Lemma 3.5.

Proposition 3.7 (Johnson [3]). Each connected component $\hat{R}_{a,b}$ of \hat{R} is determined by the following data:

(1) 0 < a < p, 0 < b < q. (2) $a \equiv b \mod 2$. (3) $\operatorname{tr} X = 2\cos(\pi a/p)$, $\operatorname{tr} Y = 2\cos(\pi b/q)$ and $Z = (-E)^a$. (4) $\operatorname{tr} M \neq 2\cos\pi(ra/p \pm sb/q)$.

In particular, $\hat{R}_{a,b}$ is parametrized by tr M and has complex dimension one.

4. A formula for the (2, q) torus knot

We start to compute the twisted Alexander polynomial of the (p, q) torus knot from the presentation

$$G(p,q) = \langle x, y \, | \, x^p y^{-q} \rangle.$$

Let us denote the relator by $u = x^p y^{-q}$. In this case, we easily see that

$$\frac{\partial u}{\partial x} = 1 + x + \dots + x^{p-1}$$

holds. Then by definition we have

$$\begin{split} \Delta_{K,\rho}(t) &= \frac{\det \Phi\left(\frac{\partial u}{\partial x}\right)}{\det \Phi(y-1)} \\ &= \frac{\det(E+t^q X+t^{2q} X^2+\dots+t^{(p-1)q} X^{p-1})}{\det(t^p Y-E)} \\ &= \frac{\left(1+\alpha t^q+\dots+\alpha^{p-1} t^{(p-1)q}\right)\left(1+\alpha^{-1} t^q+\dots+\alpha^{-(p-1)} t^{(p-1)q}\right)}{1-\left(\beta+\beta^{-1}\right) t^p+t^{2p}}, \end{split}$$

where we have assumed that $X = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and Y is conjugate to $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ for $\alpha = \exp(\sqrt{-1\pi a/p})$, $\beta = \exp(\sqrt{-1\pi b/q})$. From the above description, it is easy to see that $\Delta_{K,\rho}(t)$ can be determined by the fixed a and b. This completes the proof of Theorem 1.1.

Remark 4.1. For a reducible nonabelian representation $\rho : G(p,q) \to SL(2,\mathbb{C})$, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is expressed via the classical Alexander polynomial. More precisely, the following holds:

$$\Delta_{K,\rho}(t) = \frac{\Delta_K(\mu t) \Delta_K(\mu^{-1}t)}{t^2 - (\operatorname{tr} M) t + 1}$$

= $\frac{(t^{pq} - \mu^{pq}) (t^{pq} - \mu^{-pq})}{(t^p - \mu^p) (t^p - \mu^{-p}) (t^q - \mu^q) (t^q - \mu^{-q})}$

where $\mu \in \mathbb{C}$ satisfies $\Delta_K(\mu^2) = 0$ and $\mu + \mu^{-1} = \operatorname{tr} M$ (see [7, Theorem 3.1]).

Theorem 1.1 also can be shown by the following argument. We now put K = T(p,q) on the standard torus T^2 in S^3 . Here S^3 cut along T^2 consists of two solid tori U_1 and U_2 . Let $\pi : (S^3 - K)_{\infty} \to S^3 - K$ be the infinite cyclic covering associated with $\alpha : G(p,q) \to \mathbb{Z} = \langle t \rangle$. For simplicity, we write U'_i to $U_i - K$, and set $\tilde{U}'_i = \pi^{-1}(U'_i)$ for i = 1, 2. Then we have $(S^3 - K)_{\infty} = \tilde{U}'_1 \cup \tilde{U}'_2$. For the union we obtain the Mayer-Vietoris exact sequence with twisted coefficients:

$$\to H_1(\tilde{U}_1'\cup\tilde{U}_2';\mathbb{C}^2_\rho)\to H_0(\tilde{U}_1'\cap\tilde{U}_2';\mathbb{C}^2_\rho)\to \oplus_i H_0(\tilde{U}_i';\mathbb{C}^2_\rho)\to H_0(\tilde{U}_1'\cup\tilde{U}_2';\mathbb{C}^2_\rho)\to 0,$$

where \mathbb{C}^2_{ρ} is $\mathbb{Z}G(p,q)$ -module defined by the representation $\rho: G(p,q) \to SL(2,\mathbb{C})$. The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is given by the ratio of the orders of

$$H_1(\tilde{U}'_1 \cup \tilde{U}'_2; \mathbb{C}^2_{\rho}) = H_1(S^3 - K; \mathbb{C}[t, t^{-1}]^2_{\rho \otimes \alpha})$$

and

$$H_0(\tilde{U}'_1 \cup \tilde{U}'_2; \mathbb{C}^2_{\rho}) = H_0(S^3 - K; \mathbb{C}[t, t^{-1}]^2_{\rho \otimes \alpha}).$$

so that it is determined by $H_0(\tilde{U}'_1 \cap \tilde{U}'_2; \mathbb{C}^2_\rho)$, $H_0(\tilde{U}'_i; \mathbb{C}^2_\rho)$ and $H_0(\tilde{U}'_1 \cup \tilde{U}'_2; \mathbb{C}^2_\rho)$. However these twisted homology groups depend only on the traces of X, Y and X^pY^q , because all the spaces U'_1 , U'_2 and $U'_1 \cap U'_2$ are homotopic to S^1 and the core curves are corresponding to x, y and x^py^q respectively. Namely the twisted Alexander polynomial is locally constant.

Now in the case of p = 2, we can give an explicit formula for the twisted Alexander polynomial. In this case, *a* must be 1 and then \hat{R} consists of $\frac{q-1}{2}$ components $\hat{R}_{1,b}$ (0 < b < q, *b* is odd).

Theorem 4.2. Let K be the (2, q) torus knot and ρ_b an irreducible representation with $\hat{\rho}_b \in \hat{R}_{1,b}$, Then the twisted Alexander polynomial is given by

$$\Delta_{K,\rho_b}(t) = \left(t^2 + 1\right) \prod_{0 < k < q, \ k: \text{odd}, \ k \neq b} \left(t^2 - \xi_k\right) \left(t^2 - \bar{\xi}_k\right),$$

where $\xi_k = \exp(\sqrt{-1\pi k/q})$.

Proof. Here we have $\alpha = \sqrt{-1}$. The numerator of $\Delta_{K,\rho_b}(t)$ is $1 + (\alpha + \alpha^{-1})t^q + t^{2q} = 1 + t^{2q}$. On the other hand, the denominator is $(t^2 - \xi_b)(t^2 - \overline{\xi}_b)$, because $\beta = \exp(\sqrt{-1\pi b/q})$. The polynomial $t^{2q} + 1$ has the factorization

$$t^{2q} + 1 = \left(t^2 + 1\right) \prod_{0 < k < q, \ k: \text{odd}} (t^2 - \xi_k)(t^2 - \bar{\xi}_k)$$

over $\mathbb{C}[t]$, so that we can obtain the desired formula.

Example 4.3. Let K = T(2, 3), the trefoil knot. In this case, there is just one connected component $\hat{R}_{1,1}$ and we see that

$$\Delta_{K,\rho}(t) = \frac{t^6 + 1}{t^4 - t^2 + 1} = t^2 + 1$$

holds for any ρ with $\hat{\rho} \in \hat{R}_{1,1}$ (see [10, Theorem 4.1]).

A representation $\rho : G(K) \to SL(2, \mathbb{C})$ is called parabolic if the image of any meridian is a matrix with trace 2. For a torus knot T(p, q), we can show the following.

Proposition 4.4. There exists uniquely a conjugacy class of a parabolic representation on any connected component $\hat{R}_{a,b}$.

Proof. This is a straightforward consequence of Proposition 3.7, namely 2 is a value allowed by Proposition 3.7 (4). In fact we can easily show that $2 \cos \pi (ra/p \pm sb/q)$ never coincides with 2.

The (2, q) torus knot is one of 2-bridge knots. For a parabolic representation of a 2-bridge knot, Silver-Williams introduced the total twisted Alexander polynomial, which is denoted by $D_{K,\rho}(t)$. It is defined by taking the product of $\Delta_{K,\rho}(t)$ over parabolic representations corresponding to the roots of the Riley polynomial (see [11] for details).

As an immediate corollary of Theorem 4.2 and Proposition 4.4, we have Hirasawa-Murasugi's formula of $D_{K,\rho}(t)$ for the (2, q) torus knot.

Corollary 4.5 (Hirasawa-Murasugi [2]). For the (2,q) torus knot, the total twisted Alexander polynomial $D_{K,\rho}(t)$ is given by

$$D_{K,\rho}(t) = \prod_{0 < b < q, \ b: \text{odd}} \Delta_{K,\rho_b}(t)$$
$$= \left(t^2 + 1\right) \left(t^{2q} + 1\right)^{\frac{q-3}{2}},$$

where $\hat{\rho}_b \in \hat{R}_{1,b}$.

Proof. Since each connected component $\hat{R}_{1,b}$ contains just one class of a parabolic representation, we can calculate the total twisted Alexander polynomial as follows.

$$\begin{aligned} D_{K,\rho}(t) &= \prod_{0 < b < q, \ b: \text{odd}} \Delta_{K,\rho_b}(t) \\ &= \frac{t^{2q} + 1}{(t^2 - \xi_1)(t^2 - \bar{\xi}_1)} \cdot \frac{t^{2q} + 1}{(t^2 - \xi_3)(t^2 - \bar{\xi}_3)} \cdots \frac{t^{2q} + 1}{(t^2 - \xi_{q-2})(t^2 - \bar{\xi}_{q-2})} \\ &= \frac{(t^{2q} + 1)^{\frac{q-1}{2}}}{\frac{t^{2q} + 1}{t^2 + 1}} = \left(t^2 + 1\right) \left(t^{2q} + 1\right)^{\frac{q-3}{2}}. \end{aligned}$$

This completes the proof.

Example 4.6. Let K = T(2, 5). Then there exist two connected components $\hat{R}_{1,1}$ and $\hat{R}_{1,3}$ in the irreducible $SL(2, \mathbb{C})$ -representation space of G(2, 5). A direct calculation shows that

$$\Delta_{K,\rho_{\pm}}(t) = t^{6} + \frac{1 \pm \sqrt{5}}{2}t^{4} + \frac{1 \pm \sqrt{5}}{2}t^{2} + 1$$

holds for any $\hat{\rho}_+ \in \hat{R}_{1,1}$ and $\hat{\rho}_- \in \hat{R}_{1,3}$. If we take the product of them, we obtain the total twisted Alexander polynomial

$$D_{K,\rho}(t) = \Delta_{K,\rho_{+}}(t) \cdot \Delta_{K,\rho_{-}}(t) = \left(t^{2} + 1\right) \left(t^{10} + 1\right).$$

The result reveals that $D_{K,\rho}(t)$ is a product of cyclotomic polynomials, although the twisted Alexander polynomial is not (see [2, Proposition 10.4] and [11, Theorem 6.1]).

Finally, let us consider the ρ -twisted Alexander polynomial Δ_1^{ρ} defined in [11, Section 3] (see also [4, Theorem 4.1]). It is related to our twisted Alexander polynomial $\Delta_{K,\rho}(t)$ as follows:

$$\Delta_1^{\rho} = \Delta_{K,\rho}(t) \cdot \Delta_0^{\rho},$$

where Δ_0^{ρ} is the order of the cokernel of ∂_1 for the chain complex

$$0 \longrightarrow \Lambda^2 \xrightarrow{\partial_2} \left(\Lambda^2\right)^2 \xrightarrow{\partial_1} \Lambda^2 \longrightarrow 0.$$

Here $\Lambda = \mathbb{C}[t, t^{-1}]$ and the differentials are given by

$$\partial_2 = \left(\Phi\left(\frac{\partial u}{\partial x}\right) \Phi\left(\frac{\partial u}{\partial y}\right)\right), \quad \partial_1 = \left(\Phi(x-1) \\ \Phi(y-1)\right).$$

Then Δ_0^{ρ} equals the greatest common divisor of the 2 × 2 subdeterminants of the matrix representing ∂_1 .

In [11, Corollary 6.3] Silver and Williams stated that the ρ -twisted Alexander polynomial corresponding to a parabolic representation of a torus knot is a product of cyclotomic polynomials. The next example shows that it is a false statement.

Example 4.7. Let K = T(4, 3). There are three connected components $\hat{R}_{1,1}$, $\hat{R}_{2,2}$ and $\hat{R}_{3,1}$. Let us focus on the component $\hat{R}_{1,1}$. In this case, $\alpha = \exp(\sqrt{-1\pi/4})$ and $\beta = \exp(\sqrt{-1\pi/3})$. Thus for any representation $\hat{\rho} \in \hat{R}_{1,1}$, we obtain

$$\Delta_{K,\rho}(t) = \frac{1 + \sqrt{2}t^3 + t^6 + t^{12} + \sqrt{2}t^{15} + t^{18}}{1 - t^4 + t^8}$$
$$= \left(1 + t^4\right) \left(1 + \sqrt{2}t^3 + t^6\right).$$

To get Δ_1^{ρ} for the knot K = T(4, 3), we calculate Δ_0^{ρ} for a representation ρ : $G(4, 3) \to SL(2, \mathbb{C})$ defined by

$$\rho(x) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ and } \rho(y) = \begin{pmatrix} \beta + \sqrt{-1} & -\gamma \\ \gamma & \beta^{-1} - \sqrt{-1} \end{pmatrix},$$

where $\gamma = \sqrt{-1 - \sqrt{3}}$. We then obtain

$$\partial_1 = \begin{pmatrix} \rho(x)t^3 - E\\ \rho(y)t^4 - E \end{pmatrix}$$

and $\Delta_0^{\rho} = 1$ by a direct calculation. In fact, there are two subdeterminants

$$f_{13}(t) = -\gamma t^4 \left(\alpha t^3 - 1 \right)$$
 and $f_{24}(t) = -\gamma t^4 \left(\alpha^{-1} t^3 - 1 \right)$

such that $gcd(f_{13}, f_{24}) = 1$, where $f_{ij}(t)$ is the determinant of the 2 × 2 matrix consisting of the *i*th and the *j*th rows of ∂_1 .

Therefore the ρ -twisted Alexander polynomial is given by

$$\Delta_1^{\rho} = \Delta_{K,\rho}(t) \cdot \Delta_0^{\rho}$$
$$= \left(1 + t^4\right) \left(1 + \sqrt{2}t^3 + t^6\right)$$

and not an integral polynomial. In particular, it is not a product of cyclotomic polynomials. Of course this formula is valid for a parabolic $SL(2, \mathbb{C})$ -representation in $\hat{R}_{1,1}$, because of Theorem 1.1 and Proposition 4.4.

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