# Twisted Alexander polynomials for irreducible $S L(2, \mathbb{C})$-representations of torus knots 

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#### Abstract

We prove that the twisted Alexander polynomial of a torus knot with an irreducible $S L(2, \mathbb{C})$-representation is locally constant. In the case of a $(2, q)$ torus knot, we can give an explicit formula for the twisted Alexander polynomial and deduce Hirasawa-Murasugi's formula for the total twisted Alexander polynomial. We also give examples which address a mis-statement in a paper of Silver and Williams.


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## 1. Introduction

Let $K$ be a knot in the 3 -sphere $S^{3}$ and $G(K)=\pi_{1}\left(S^{3}-K\right)$ its knot group. In this paper, we consider the twisted Alexander polynomial $\Delta_{K, \rho}(t)$, which is defined as a rational expression over $\mathbb{C}$ with one variable $t$, for a knot $K$ associated with an irreducible representation $\rho: G(K) \rightarrow S L(2, \mathbb{C})$. The twisted Alexander polynomial for a knot with a linear representation was originally introduced by Lin in [9]. It was generalized and developed by Wada in [12] for finitely presentable groups which include link groups. If we put $t=1$, it is known that $\Delta_{K, \rho}(1)$ equals the Reidemeister torsion of the exterior of a knot $K$ for the same representation $\rho$, under the acyclicity condition [6].

When $\rho$ is a nonabelian representation, the twisted Alexander polynomial $\Delta_{K, \rho}(t)$ becomes a Laurent polynomial over $\mathbb{C}$ (see [7]). Since an irreducible representation is nonabelian, $\Delta_{K, \rho}(t)$ is a Laurent polynomial and all the coefficients of $\Delta_{K, \rho}(t)$ are complex valued functions on the space of irreducible representations in $S L(2, \mathbb{C})$. We then obtain the following.

Theorem 1.1. If $K$ is a torus knot, then every coefficient of $\Delta_{K, \rho}(t)$ is a locally constant function, that is, a constant function on each connected component of the space of irreducible $S L(2, \mathbb{C})$-representations.

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Remark 1.2. (1) Johnson [3] proved that the Reidemeister torsion of a torus knot is a locally constant function on the space of irreducible $S L(2, \mathbb{C})$-representations. More generally, it is known that the Reidemeister torsion is locally constant for a Seifert fibered manifold [5].
(2) Kitayama observed in [8, Example 5.11] that every coefficient of the twisted Alexander polynomial of a torus knot is locally constant for $S U(2)$-representations.

This paper is organized as follows. In the next section, we review the definition for the twisted Alexander polynomial associated with $S L(2, \mathbb{C})$-representations. In Section 3, we describe the representation space of a torus knot (Proposition 3.7) according to Johnson's lecture note [3]. In the last section, we give two kinds of proofs for Theorem 1.1 and an explicit formula for the twisted Alexander polynomial for $(2, q)$ torus knots (Theorem 4.2). We also discuss the total twisted Alexander polynomial, due to Silver-Williams [11]. Hirasawa-Murasugi's formula [2] for the total twisted Alexander polynomial corresponding to parabolic representations of a $(2, q)$ torus knot is shown very easily (Corollary 4.5). In particular, we present an example for the twisted Alexander polynomial which cannot be written as a product of cyclotomic polynomials (Example 4.7). The example addresses a mis-statement in a paper of Silver and Williams [11].

We shall give a self-contained description through the paper, so we determine the representation space of a torus knot in detail (although it seems to be known to experts).

## 2. Twisted Alexander polynomials

In this section, we review the definition of $\Delta_{K, \rho}(t)$ for an $S L(2, \mathbb{C})$-representation $\rho$. There are several versions for the twisted Alexander polynomial, but in this paper we adopt the one due to Wada [12].

For a given knot $K$, we fix a presentation of its knot group $G(K)$ :

$$
P=\left\langle x_{1}, \ldots, x_{n} \mid u_{1}, \ldots, u_{n-1}\right\rangle
$$

We may assume its deficiency is one, but it might not be a Wirtinger presentation. We take the abelianization homomorphism $\alpha: G(K) \rightarrow \mathbb{Z}=\langle t\rangle$.

Given representations $\rho: G(K) \rightarrow S L(2, \mathbb{C})$ and $\alpha: G(K) \rightarrow\langle t\rangle$, they naturally induce two ring homomorphisms $\widetilde{\rho}$ and $\widetilde{\alpha}$ from the group ring $\mathbb{Z} G(K)$ to $M(2, \mathbb{C})$ and $\mathbb{Z}\left[t, t^{-1}\right]$ respectively, where $M(2, \mathbb{C})$ is the matrix algebra of $2 \times 2$ matrices over $\mathbb{C}$. Then $\widetilde{\rho} \otimes \widetilde{\alpha}$ defines a ring homomorphism $\mathbb{Z} G(K) \rightarrow$ $M\left(2, \mathbb{C}\left[t, t^{-1}\right]\right)$. Let $F_{n}$ denote the free group on generators $x_{1}, \ldots, x_{n}$ and

$$
\Phi: \mathbb{Z} F_{n} \rightarrow M\left(2, \mathbb{C}\left[t, t^{-1}\right]\right)
$$

the composite of the surjection $\mathbb{Z} F_{n} \rightarrow \mathbb{Z} G(K)$ induced by the presentation $P$ and the ring homomorphism $\widetilde{\rho} \otimes \widetilde{\alpha}$.

Let us consider the $(n-1) \times n$ matrix $A$ whose $(i, j)$ component is the $2 \times 2$ matrix

$$
\Phi\left(\frac{\partial u_{i}}{\partial x_{j}}\right) \in M\left(2, \mathbb{C}\left[t, t^{-1}\right]\right)
$$

where $\frac{\partial}{\partial x_{j}}(j=1, \ldots, n)$ denotes the free differential calculus (see [1]). This matrix $A$ is called the Alexander matrix of the presentation $P$ associated with $\rho$.

For $1 \leq j \leq n$, let us denote by $A_{j}$ the $(n-1) \times(n-1)$ matrix obtained from $A$ by removing the $j$ th column. We regard $A_{j}$ as a $2(n-1) \times 2(n-1)$ matrix with coefficients in $\mathbb{C}\left[t, t^{-1}\right]$.

The following two lemmas are the foundations of the definition for the twisted Alexander polynomial (see [12] for the proof).

Lemma 2.1. $\operatorname{det} \Phi\left(x_{j}-1\right) \neq 0$ for some $j$.
Lemma 2.2. $\operatorname{det} A_{j} \operatorname{det} \Phi\left(x_{k}-1\right)=\operatorname{det} A_{k} \operatorname{det} \Phi\left(x_{j}-1\right)$ for $1 \leq j<k \leq n$.
From the above two lemmas, we can define the twisted Alexander polynomial of $G(K)$ associated with the representation $\rho: G(K) \rightarrow S L(2, \mathbb{C})$ to be a rational expression

$$
\Delta_{K, \rho}(t)=\frac{\operatorname{det} A_{j}}{\operatorname{det} \Phi\left(x_{j}-1\right)}
$$

provided $\operatorname{det} \Phi\left(x_{j}-1\right) \neq 0$.
Remark 2.3. Up to a factor of $t^{k}(k \in \mathbb{Z})$, this is an invariant of $G(K)$ with $\rho$ (see [12, Theorem 1]). Namely, it does not depend on the choices of a presentation $P$. Hence we can consider it as a knot invariant.

In general, the twisted Alexander polynomial $\Delta_{K, \rho}(t)$ depends on $\rho$. However the following proposition is known.

Proposition 2.4. If $\rho$ and $\rho^{\prime}$ are conjugate as an $S L(2, \mathbb{C})$-representation, then $\Delta_{K, \rho}(t)=\Delta_{K, \rho^{\prime}}(t)$.

Here a representation $\rho$ is conjugate to a representation $\rho^{\prime}$ if there exists $S \in$ $S L(2, \mathbb{C})$ such that $\rho(g)=S \rho^{\prime}(g) S^{-1}$ for any $g \in G(K)$.

Under a generic assumption on $\rho$, the twisted Alexander polynomial becomes a Laurent polynomial (see [7, Theorem 3.1]).

Proposition 2.5. If $\rho: G(K) \rightarrow S L(2, \mathbb{C})$ is a nonabelian representation, then $\Delta_{K, \rho}(t)$ is a Laurent polynomial with coefficients in $\mathbb{C}$.

In this paper, we consider that each coefficient of $\Delta_{K, \rho}(t)$ is a complex valued function on the space of conjugacy classes of irreducible $S L(2, \mathbb{C})$-representations.

## 3. Representation space of a $(p, q)$ torus knot

In this section, we recall a parametrization of the space of conjugacy classes of irreducible $S L(2, \mathbb{C})$-representations of a torus knot. This was demonstrated in the unpublished lecture notes [3] by Johnson.

Let $(p, q)$ be a pair of coprime natural numbers. Hereafter let $K=T(p, q)$ be the $(p, q)$ torus knot and $G(p, q)$ be its knot group. We take the following presentation of $G(p, q)$ :

$$
G(p, q)=\left\langle x, y \mid x^{p} y^{-q}\right\rangle .
$$

First we quickly review some terminologies of a linear representation in $S L(2, \mathbb{C})$. A representation $\rho: G(p, q) \rightarrow S L(2, \mathbb{C})$ is called irreducible if there does not exist a nontrivial proper invariant subspace of $\mathbb{C}^{2}$ under the natural action of $\rho(G(p, q))$. A representation $\rho: G(p, q) \rightarrow S L(2, \mathbb{C})$ is called reducible if $\rho$ is not irreducible. That is, there is an invariant 1 -dimensional subspace of $\mathbb{C}^{2}$. A representation $\rho$ is called abelian if $\rho(G(p, q))$ is an abelian subgroup of $S L(2, \mathbb{C})$. It is easy to see that an abelian representation is reducible.

Let $R$ be the set of irreducible $S L(2, \mathbb{C})$-representations of $G(p, q)$. Fixing the generators $x$ and $y, R$ can be embedded into $S L(2, \mathbb{C}) \times S L(2, \mathbb{C})$ by the map $R \ni$ $\rho \mapsto(\rho(x), \rho(y)) \in S L(2, \mathbb{C}) \times S L(2, \mathbb{C})$. From this embedding, the topology of $R$ can be induced from $S L(2, \mathbb{C}) \times S L(2, \mathbb{C})$. Let $\hat{R}$ be the space of conjugacy classes of irreducible representations, that is, the quotient space of $R$ by conjugate action of $S L(2, \mathbb{C})$. In general $\hat{R}$ has some connected components. For a given representation $\rho$, we write $\hat{\rho}$ for its conjugacy class.

From now on, we start to describe the structure of $\hat{R}$. Choosing a pair $(r, s)$ of natural numbers satisfying $p s-q r=1$, then $m=x^{-r} y^{s} \in G(p, q)$ represents a meridian of $T(p, q)$. Let $\rho: G(p, q) \rightarrow S L(2, \mathbb{C})$ be an irreducible representation. For simplicity, we write a capital letter $X$ for the image $\rho(x)$ of $x, Y$ for $\rho(y)$ and so on.

Now we put $z=x^{p}=y^{q} \in G(p, q)$ which lies in the center of $G(p, q)$. Recall that the center of $S L(2, \mathbb{C})$ is $\{ \pm E\}$, where $E$ is the identity matrix of degree 2 .

Lemma 3.1. $Z= \pm E$.
Proof. Assume that $Z \neq \pm E$. We take an eigenvalue $\lambda$ of $Z$ and its eigenspace $V_{\lambda} \subset \mathbb{C}^{2}$. Because $z$ is a center element of $G(p, q), Z$ can be commuted with any matrix $S \in \rho(G(p, q))$. For any vector $v \in V_{\lambda}$,

$$
Z(S v)=S(Z v)=\lambda S v
$$

Hence $S v \in V_{\lambda}$ and it implies $V_{\lambda}$ is an invariant subspace of $\rho$. By the irreducibility of $\rho, V_{\lambda}$ is the full space $\mathbb{C}^{2}$. Therefore $\lambda=\lambda^{-1}= \pm 1$. Here we may put $Z=$ $\left(\begin{array}{cc} \pm 1 & t \\ 0 & \pm 1\end{array}\right)$ up to conjugation. If $t \neq 0$, then the above eigenspace $V_{\lambda}$ is not the full space $\mathbb{C}^{2}$. This contradicts the irreducibility of $\rho$ and then $Z= \pm E$.

Since $Z=X^{p}=Y^{q}= \pm E$, it holds that $X^{2 p}=Y^{2 q}=E$. On the other hand, we have the following.

Lemma 3.2. $X^{r} \neq \pm E, Y^{s} \neq \pm E$.
Proof. Assuming $X^{r}= \pm E$, we have $X^{2 r}=E$. Since $p s-q r=1,2 p s=2 q r+2$ holds. Thus $X^{2 p s}=X^{2 q r+2}$. Hence we have $E=X^{2}$ and then $X= \pm E$. It means that the representation $\rho$ is abelian, but this is a contradiction. It is similarly proved that $Y^{s} \neq \pm E$.

Here we let

$$
\alpha^{ \pm 1}=\exp ( \pm \sqrt{-1} \pi a / p) \quad \text { and } \quad \beta^{ \pm 1}=\exp ( \pm \sqrt{-1} \pi b / q)
$$

to be the eigenvalues of $X$ and $Y$ respectively, where we can assume that $0<a<p$ and $0<b<q$. Since

$$
X^{p}=(-E)^{a}=Y^{q}=(-E)^{b}
$$

it holds that

$$
a \equiv b \bmod 2
$$

From now on, let us fix $\operatorname{tr} X$ and $\operatorname{tr} Y$. We consider a conjugacy class of the representation $\rho$, so that we may assume $X=\left(\begin{array}{cc}\alpha & 0 \\ 0 & \alpha^{-1}\end{array}\right)$ and $Y$ is conjugate to $\left(\begin{array}{cc}\beta & 0 \\ 0 & \beta^{-1}\end{array}\right)$ in $S L(2, \mathbb{C})$.

Remark 3.3. We remark that $a$ is fixed but $b$ is not. In fact, there are two choices of $b(0<b<q)$, namely $b$ or $-b \bmod q$. Both of them give the same trace $\operatorname{tr} Y=2 \cos (\pi b / q)$.

If $Y$ is an upper triangle matrix, then $\rho$ is a reducible representation. In this case, the trace of the meridian image

$$
\begin{aligned}
M & =X^{-r} Y^{s} \\
& =\left(\begin{array}{cc}
\alpha & 0 \\
0 & \alpha^{-1}
\end{array}\right)^{-r}\left(\begin{array}{cc}
\beta & * \\
0 & \beta^{-1}
\end{array}\right)^{s} \text { or }\left(\begin{array}{cc}
\alpha & 0 \\
0 & \alpha^{-1}
\end{array}\right)^{-r}\left(\begin{array}{cc}
\beta^{-1} & * \\
0 & \beta
\end{array}\right)^{s}
\end{aligned}
$$

is given by

$$
\operatorname{tr} M=\alpha^{-r} \beta^{ \pm s}+\alpha^{r} \beta^{\mp s}=2 \cos \pi(r a / p \pm s b / q)
$$

Namely we obtain the following lemma.

Lemma 3.4. $\rho$ is an irreducible representation if $\operatorname{tr} M \neq \alpha^{-r} \beta^{ \pm s}+\alpha^{r} \beta^{\mp s}$.
Now $Y^{s}$ is conjugate to $\left(\begin{array}{cc}\beta^{s} & 0 \\ 0 & \beta^{-s}\end{array}\right)$ in $S L(2, \mathbb{C})$, so that $Y^{s}$ has the form of

$$
\left(\begin{array}{cc}
\beta^{s}+\delta & * \\
* & \beta^{-s}-\delta
\end{array}\right) \text { or }\left(\begin{array}{cc}
\beta^{-s}+\delta & * \\
* & \beta^{s}-\delta
\end{array}\right),
$$

where $\delta$ is any complex number. Therefore

$$
\begin{aligned}
\operatorname{tr} M & =\operatorname{tr}\left(X^{-r} Y^{s}\right) \\
& =\alpha^{-r} \beta^{ \pm s}+\alpha^{r} \beta^{\mp s}+\delta\left(\alpha^{-r}-\alpha^{r}\right) .
\end{aligned}
$$

Hence we can assume

$$
\operatorname{tr} M=\alpha^{-r} \beta^{s}+\alpha^{r} \beta^{-s}+\delta\left(\alpha^{-r}-\alpha^{r}\right)
$$

by replacing $\beta$ if necessary. We note that $b$ has been fixed. This value of $\operatorname{tr} M$ can be any complex number because $\delta$ can be so.

Lemma 3.5. If we put $U=X^{-r}$ and $V=Y^{s}$, then $X=Z^{s} U^{q}$ and $Y=Z^{-r} V^{p}$.
Proof. Direct calculations.
Lemma 3.6. For any irreducible representation $\rho: G(p, q) \rightarrow S L(2, \mathbb{C})$, if $\operatorname{tr} X$, $\operatorname{tr} Y$ and $\operatorname{tr} M$ are fixed, then $\rho$ is uniquely determined up to conjugation.

Proof. We fix $\operatorname{tr} X, \operatorname{tr} Y$ and $\operatorname{tr} M$. Then we prove that $X$ and $Y$ are uniquely determined in $S L(2, \mathbb{C})$ up to mutual conjugation. First the value of $\operatorname{tr} X$ determines $\operatorname{tr} Z$ and $\operatorname{tr} U$, because $Z=X^{p}$ and $U=X^{-r}$. Hence $Z$ can be determined since $Z= \pm E$. Similarly $\operatorname{tr} Y$ determines $\operatorname{tr} V$. Here $\operatorname{tr} M=\operatorname{tr} U V$ is fixed and $U, V$ do not commute, so that $U$ and $V$ are determined in $S L(2, \mathbb{C})$ up to mutual conjugation. Therefore $X$ and $Y$ are uniquely determined up to conjugation by Lemma 3.5.

Proposition 3.7 (Johnson [3]). Each connected component $\hat{R}_{a, b}$ of $\hat{R}$ is determined by the following data:
(1) $0<a<p, 0<b<q$.
(2) $a \equiv b \bmod 2$.
(3) $\operatorname{tr} X=2 \cos (\pi a / p), \operatorname{tr} Y=2 \cos (\pi b / q)$ and $Z=(-E)^{a}$.
(4) $\operatorname{tr} M \neq 2 \cos \pi(r a / p \pm s b / q)$.

In particular, $\hat{R}_{a, b}$ is parametrized by $\operatorname{tr} M$ and has complex dimension one.

## 4. A formula for the $(2, q)$ torus knot

We start to compute the twisted Alexander polynomial of the $(p, q)$ torus knot from the presentation

$$
G(p, q)=\left\langle x, y \mid x^{p} y^{-q}\right\rangle .
$$

Let us denote the relator by $u=x^{p} y^{-q}$. In this case, we easily see that

$$
\frac{\partial u}{\partial x}=1+x+\cdots+x^{p-1}
$$

holds. Then by definition we have

$$
\begin{aligned}
\Delta_{K, \rho}(t) & =\frac{\operatorname{det} \Phi\left(\frac{\partial u}{\partial x}\right)}{\operatorname{det} \Phi(y-1)} \\
& =\frac{\operatorname{det}\left(E+t^{q} X+t^{2 q} X^{2}+\cdots+t^{(p-1) q} X^{p-1}\right)}{\operatorname{det}\left(t^{p} Y-E\right)} \\
& =\frac{\left(1+\alpha t^{q}+\cdots+\alpha^{p-1} t^{(p-1) q}\right)\left(1+\alpha^{-1} t^{q}+\cdots+\alpha^{-(p-1)} t^{(p-1) q}\right)}{1-\left(\beta+\beta^{-1}\right) t^{p}+t^{2 p}}
\end{aligned}
$$

where we have assumed that $X=\left(\begin{array}{ll}\alpha & 0 \\ 0 & \alpha^{-1}\end{array}\right)$ and $Y$ is conjugate to $\left(\begin{array}{ll}\beta & 0 \\ 0 & \beta^{-1}\end{array}\right)$ for $\alpha=\exp (\sqrt{-1} \pi a / p), \beta=\exp (\sqrt{-1} \pi b / q)$. From the above description, it is easy to see that $\Delta_{K, \rho}(t)$ can be determined by the fixed $a$ and $b$. This completes the proof of Theorem 1.1.
Remark 4.1. For a reducible nonabelian representation $\rho: G(p, q) \rightarrow S L(2, \mathbb{C})$, the twisted Alexander polynomial $\Delta_{K, \rho}(t)$ is expressed via the classical Alexander polynomial. More precisely, the following holds:

$$
\begin{aligned}
\Delta_{K, \rho}(t) & =\frac{\Delta_{K}(\mu t) \Delta_{K}\left(\mu^{-1} t\right)}{t^{2}-(\operatorname{tr} M) t+1} \\
& =\frac{\left(t^{p q}-\mu^{p q}\right)\left(t^{p q}-\mu^{-p q}\right)}{\left(t^{p}-\mu^{p}\right)\left(t^{p}-\mu^{-p}\right)\left(t^{q}-\mu^{q}\right)\left(t^{q}-\mu^{-q}\right)}
\end{aligned}
$$

where $\mu \in \mathbb{C}$ satisfies $\Delta_{K}\left(\mu^{2}\right)=0$ and $\mu+\mu^{-1}=\operatorname{tr} M$ (see [7, Theorem 3.1]).
Theorem 1.1 also can be shown by the following argument. We now put $K=$ $T(p, q)$ on the standard torus $T^{2}$ in $S^{3}$. Here $S^{3}$ cut along $T^{2}$ consists of two solid tori $U_{1}$ and $U_{2}$. Let $\pi:\left(S^{3}-K\right)_{\infty} \rightarrow S^{3}-K$ be the infinite cyclic covering associated with $\alpha: G(p, q) \rightarrow \mathbb{Z}=\langle t\rangle$. For simplicity, we write $U_{i}^{\prime}$ to $U_{i}-K$, and set $\tilde{U}_{i}^{\prime}=\pi^{-1}\left(U_{i}^{\prime}\right)$ for $i=1,2$. Then we have $\left(S^{3}-K\right)_{\infty}=\tilde{U}_{1}^{\prime} \cup \tilde{U}_{2}^{\prime}$. For the union we obtain the Mayer-Vietoris exact sequence with twisted coefficients:
$\rightarrow H_{1}\left(\tilde{U}_{1}^{\prime} \cup \tilde{U}_{2}^{\prime} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow H_{0}\left(\tilde{U}_{1}^{\prime} \cap \tilde{U}_{2}^{\prime} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow \oplus_{i} H_{0}\left(\tilde{U}_{i}^{\prime} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow H_{0}\left(\tilde{U}_{1}^{\prime} \cup \tilde{U}_{2}^{\prime} ; \mathbb{C}_{\rho}^{2}\right) \rightarrow 0$,
where $\mathbb{C}_{\rho}^{2}$ is $\mathbb{Z} G(p, q)$-module defined by the representation $\rho: G(p, q) \rightarrow S L(2, \mathbb{C})$. The twisted Alexander polynomial $\Delta_{K, \rho}(t)$ is given by the ratio of the orders of

$$
H_{1}\left(\tilde{U}_{1}^{\prime} \cup \tilde{U}_{2}^{\prime} ; \mathbb{C}_{\rho}^{2}\right)=H_{1}\left(S^{3}-K ; \mathbb{C}\left[t, t^{-1}\right]_{\rho \otimes \alpha}^{2}\right)
$$

and

$$
H_{0}\left(\tilde{U}_{1}^{\prime} \cup \tilde{U}_{2}^{\prime} ; \mathbb{C}_{\rho}^{2}\right)=H_{0}\left(S^{3}-K ; \mathbb{C}\left[t, t^{-1}\right]_{\rho \otimes \alpha}^{2}\right)
$$

so that it is determined by $H_{0}\left(\tilde{U}_{1}^{\prime} \cap \tilde{U}_{2}^{\prime} ; \mathbb{C}_{\rho}^{2}\right), H_{0}\left(\tilde{U}_{i}^{\prime} ; \mathbb{C}_{\rho}^{2}\right)$ and $H_{0}\left(\tilde{U}_{1}^{\prime} \cup \tilde{U}_{2}^{\prime} ; \mathbb{C}_{\rho}^{2}\right)$. However these twisted homology groups depend only on the traces of $X, Y$ and $X^{p} Y^{q}$, because all the spaces $U_{1}^{\prime}, U_{2}^{\prime}$ and $U_{1}^{\prime} \cap U_{2}^{\prime}$ are homotopic to $S^{1}$ and the core curves are corresponding to $x, y$ and $x^{p} y^{q}$ respectively. Namely the twisted Alexander polynomial is locally constant.

Now in the case of $p=2$, we can give an explicit formula for the twisted Alexander polynomial. In this case, $a$ must be 1 and then $\hat{R}$ consists of $\frac{q-1}{2}$ components $\hat{R}_{1, b}(0<b<q, b$ is odd $)$.

Theorem 4.2. Let $K$ be the $(2, q)$ torus knot and $\rho_{b}$ an irreducible representation with $\hat{\rho}_{b} \in \hat{R}_{1, b}$, Then the twisted Alexander polynomial is given by

$$
\Delta_{K, \rho_{b}}(t)=\left(t^{2}+1\right) \prod_{0<k<q, k: \text { odd, } k \neq b}\left(t^{2}-\xi_{k}\right)\left(t^{2}-\bar{\xi}_{k}\right),
$$

where $\xi_{k}=\exp (\sqrt{-1} \pi k / q)$.
Proof. Here we have $\alpha=\sqrt{-1}$. The numerator of $\Delta_{K, \rho_{b}}(t)$ is $1+\left(\alpha+\alpha^{-1}\right) t^{q}+$ $t^{2 q}=1+t^{2 q}$. On the other hand, the denominator is $\left(t^{2}-\xi_{b}\right)\left(t^{2}-\bar{\xi}_{b}\right)$, because $\beta=\exp (\sqrt{-1} \pi b / q)$. The polynomial $t^{2 q}+1$ has the factorization

$$
t^{2 q}+1=\left(t^{2}+1\right) \prod_{0<k<q, k: \text { odd }}\left(t^{2}-\xi_{k}\right)\left(t^{2}-\bar{\xi}_{k}\right)
$$

over $\mathbb{C}[t]$, so that we can obtain the desired formula.
Example 4.3. Let $K=T(2,3)$, the trefoil knot. In this case, there is just one connected component $\hat{R}_{1,1}$ and we see that

$$
\Delta_{K, \rho}(t)=\frac{t^{6}+1}{t^{4}-t^{2}+1}=t^{2}+1
$$

holds for any $\rho$ with $\hat{\rho} \in \hat{R}_{1,1}$ (see [10, Theorem 4.1]).
A representation $\rho: G(K) \rightarrow S L(2, \mathbb{C})$ is called parabolic if the image of any meridian is a matrix with trace 2 . For a torus knot $T(p, q)$, we can show the following.

Proposition 4.4. There exists uniquely a conjugacy class of a parabolic representation on any connected component $\hat{R}_{a, b}$.

Proof. This is a straightforward consequence of Proposition 3.7, namely 2 is a value allowed by Proposition 3.7 (4). In fact we can easily show that $2 \cos \pi(r a / p \pm s b / q)$ never coincides with 2.

The $(2, q)$ torus knot is one of 2-bridge knots. For a parabolic representation of a 2-bridge knot, Silver-Williams introduced the total twisted Alexander polynomial, which is denoted by $D_{K, \rho}(t)$. It is defined by taking the product of $\Delta_{K, \rho}(t)$ over parabolic representations corresponding to the roots of the Riley polynomial (see [11] for details).

As an immediate corollary of Theorem 4.2 and Proposition 4.4, we have Hira-sawa-Murasugi's formula of $D_{K, \rho}(t)$ for the $(2, q)$ torus knot.

Corollary 4.5 (Hirasawa-Murasugi [2]). For the $(2, q)$ torus knot, the total twisted Alexander polynomial $D_{K, \rho}(t)$ is given by

$$
\begin{aligned}
D_{K, \rho}(t) & =\prod_{0<b<q, b: \text { odd }} \Delta_{K, \rho_{b}}(t) \\
& =\left(t^{2}+1\right)\left(t^{2 q}+1\right)^{\frac{q-3}{2}}
\end{aligned}
$$

where $\hat{\rho}_{b} \in \hat{R}_{1, b}$.
Proof. Since each connected component $\hat{R}_{1, b}$ contains just one class of a parabolic representation, we can calculate the total twisted Alexander polynomial as follows.

$$
\begin{aligned}
D_{K, \rho}(t) & =\prod_{0<b<q, b: \text { odd }} \Delta_{K, \rho_{b}}(t) \\
& =\frac{t^{2 q}+1}{\left(t^{2}-\xi_{1}\right)\left(t^{2}-\bar{\xi}_{1}\right)} \cdot \frac{t^{2 q}+1}{\left(t^{2}-\xi_{3}\right)\left(t^{2}-\bar{\xi}_{3}\right)} \cdots \frac{t^{2 q}+1}{\left(t^{2}-\xi_{q-2}\right)\left(t^{2}-\bar{\xi}_{q-2}\right)} \\
& =\frac{\left(t^{2 q}+1\right)^{\frac{q-1}{2}}}{\frac{t^{2 q}+1}{t^{2}+1}}=\left(t^{2}+1\right)\left(t^{2 q}+1\right)^{\frac{q-3}{2}}
\end{aligned}
$$

This completes the proof.
Example 4.6. Let $K=T(2,5)$. Then there exist two connected components $\hat{R}_{1,1}$ and $\hat{R}_{1,3}$ in the irreducible $S L(2, \mathbb{C})$-representation space of $G(2,5)$. A direct calculation shows that

$$
\Delta_{K, \rho_{ \pm}}(t)=t^{6}+\frac{1 \pm \sqrt{5}}{2} t^{4}+\frac{1 \pm \sqrt{5}}{2} t^{2}+1
$$

holds for any $\hat{\rho}_{+} \in \hat{R}_{1,1}$ and $\hat{\rho}_{-} \in \hat{R}_{1,3}$. If we take the product of them, we obtain the total twisted Alexander polynomial

$$
D_{K, \rho}(t)=\Delta_{K, \rho_{+}}(t) \cdot \Delta_{K, \rho_{-}}(t)=\left(t^{2}+1\right)\left(t^{10}+1\right)
$$

The result reveals that $D_{K, \rho}(t)$ is a product of cyclotomic polynomials, although the twisted Alexander polynomial is not (see [2, Proposition 10.4] and [11, Theorem 6.1]).

Finally, let us consider the $\rho$-twisted Alexander polynomial $\Delta_{1}^{\rho}$ defined in [11, Section 3] (see also [4, Theorem 4.1]). It is related to our twisted Alexander polynomial $\Delta_{K, \rho}(t)$ as follows:

$$
\Delta_{1}^{\rho}=\Delta_{K, \rho}(t) \cdot \Delta_{0}^{\rho}
$$

where $\Delta_{0}^{\rho}$ is the order of the cokernel of $\partial_{1}$ for the chain complex

$$
0 \longrightarrow \Lambda^{2} \xrightarrow{\partial_{2}}\left(\Lambda^{2}\right)^{2} \xrightarrow{\partial_{1}} \Lambda^{2} \longrightarrow 0
$$

Here $\Lambda=\mathbb{C}\left[t, t^{-1}\right]$ and the differentials are given by

$$
\partial_{2}=\left(\Phi\left(\frac{\partial u}{\partial x}\right) \Phi\left(\frac{\partial u}{\partial y}\right)\right), \quad \partial_{1}=\binom{\Phi(x-1)}{\Phi(y-1)} .
$$

Then $\Delta_{0}^{\rho}$ equals the greatest common divisor of the $2 \times 2$ subdeterminants of the matrix representing $\partial_{1}$.

In [11, Corollary 6.3] Silver and Williams stated that the $\rho$-twisted Alexander polynomial corresponding to a parabolic representation of a torus knot is a product of cyclotomic polynomials. The next example shows that it is a false statement.

Example 4.7. Let $K=T(4,3)$. There are three connected components $\hat{R}_{1,1}, \hat{R}_{2,2}$ and $\hat{R}_{3,1}$. Let us focus on the component $\hat{R}_{1,1}$. In this case, $\alpha=\exp (\sqrt{-1} \pi / 4)$ and $\beta=\exp (\sqrt{-1} \pi / 3)$. Thus for any representation $\hat{\rho} \in \hat{R}_{1,1}$, we obtain

$$
\begin{aligned}
\Delta_{K, \rho}(t) & =\frac{1+\sqrt{2} t^{3}+t^{6}+t^{12}+\sqrt{2} t^{15}+t^{18}}{1-t^{4}+t^{8}} \\
& =\left(1+t^{4}\right)\left(1+\sqrt{2} t^{3}+t^{6}\right)
\end{aligned}
$$

To get $\Delta_{1}^{\rho}$ for the knot $K=T(4,3)$, we calculate $\Delta_{0}^{\rho}$ for a representation $\rho$ : $G(4,3) \rightarrow S L(2, \mathbb{C})$ defined by

$$
\rho(x)=\left(\begin{array}{cc}
\alpha & 0 \\
0 & \alpha^{-1}
\end{array}\right) \quad \text { and } \quad \rho(y)=\left(\begin{array}{cc}
\beta+\sqrt{-1} & -\gamma \\
\gamma & \beta^{-1}-\sqrt{-1}
\end{array}\right),
$$

where $\gamma=\sqrt{-1-\sqrt{3}}$. We then obtain

$$
\partial_{1}=\binom{\rho(x) t^{3}-E}{\rho(y) t^{4}-E}
$$

and $\Delta_{0}^{\rho}=1$ by a direct calculation. In fact, there are two subdeterminants

$$
f_{13}(t)=-\gamma t^{4}\left(\alpha t^{3}-1\right) \quad \text { and } \quad f_{24}(t)=-\gamma t^{4}\left(\alpha^{-1} t^{3}-1\right)
$$

such that $\operatorname{gcd}\left(f_{13}, f_{24}\right)=1$, where $f_{i j}(t)$ is the determinant of the $2 \times 2$ matrix consisting of the $i$ th and the $j$ th rows of $\partial_{1}$.

Therefore the $\rho$-twisted Alexander polynomial is given by

$$
\begin{aligned}
\Delta_{1}^{\rho} & =\Delta_{K, \rho}(t) \cdot \Delta_{0}^{\rho} \\
& =\left(1+t^{4}\right)\left(1+\sqrt{2} t^{3}+t^{6}\right)
\end{aligned}
$$

and not an integral polynomial. In particular, it is not a product of cyclotomic polynomials. Of course this formula is valid for a parabolic $S L(2, \mathbb{C})$-representation in $\hat{R}_{1,1}$, because of Theorem 1.1 and Proposition 4.4.

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