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On a semilinear elliptic equation in $\mathbb{H}^n$

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Dedicated to Prof. Yadava

Abstract. We prove existence/nonexistence and uniqueness of positive entire solutions for some semilinear elliptic equations on the Hyperbolic space.

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1. Introduction

In this paper we discuss existence/nonexistence and uniqueness of positive entire (*i.e.* finite energy) solutions of the following semilinear elliptic equation on $\mathbb{H} = \mathbb{H}^n$, the n dimensional Hyperbolic space:

$$\Delta_{\mathbb{H}} u + \lambda u + u^p = 0. \qquad (Eq_\lambda)$$

Here $\Delta_{\mathbb{H}}$ denotes the Laplace-Beltrami operator on $\mathbb{H}$, λ is a real parameter and $p > 1$ if $n = 2$ while $1 < p \leq 2^* - 1$ if $n > 2$, where $2^* := \frac{2n}{n-2}$.

A basic information is that the bottom of the spectrum of $-\Delta_{\mathbb{H}}$ on $\mathbb{H}$ is

$$\lambda_1(-\Delta_{\mathbb{H}}) := \inf_{u \in H^1(\mathbb{H}) \setminus 0} \frac{\displaystyle\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^2 dV_{\mathbb{H}}}{\displaystyle\int_{\mathbb{H}} |u|^2 dV_{\mathbb{H}}} = \frac{(n-1)^2}{4}. \qquad (1.1)$$

A straightforward consequence of (1.1) is the following non existence result:

Theorem 1.1. *Let $n \geq 2$ and $\lambda > \frac{(n-1)^2}{4}$. Then (Eq_λ) has no positive solution. If $\lambda = \frac{(n-1)^2}{4}$ there are no positive solutions of (Eq_λ) in $H^1(\mathbb{H})$.*

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So, in the sequel, we will assume $\lambda \le \frac{(n-1)^2}{4}$. Another consequence of (1.1) is that if $\lambda < \frac{(n-1)^2}{4}$ then

$$\|u\|_\lambda := \left[\int_{\mathbb{H}} \left(|\nabla_{\mathbb{H}} u|^2 - \lambda u^2\right) dV\right]^{\frac{1}{2}} \ , \quad u \in C_c^\infty(\mathbb{H})$$

is a norm, equivalent to the $H^1(\mathbb{H})$ norm. This is no longer true if $\lambda = \frac{(n-1)^2}{4}$. A suitable definition of entire (or finite energy) solution in this case can be given taking into account an inequality which can be derived from Hardy-Sobolev-Maz'ya inequalities (see Appendix B).

A sharp Poincaré-Sobolev inequality and the space $\mathcal{H}$

For every $n \ge 3$ and every $p \in \left(1, \frac{n+2}{n-2}\right]$ there is $S_{n,p} > 0$ such that

$$S_{n,p}\left(\int_{\mathbb{H}} |u|^{p+1} dV_{\mathbb{H}}\right)^{\frac{2}{p+1}} \le \int_{\mathbb{H}} \left[|\nabla_{\mathbb{H}} u|^2 - \frac{(n-1)^2}{4} u^2\right] dV_{\mathbb{H}} \tag{1.2}$$

for every $u \in C_0^\infty(\mathbb{H})$. If $n = 2$ any $p > 1$ is allowed.

Inequality (1.2) implies that $\|u\|_{\frac{(n-1)^2}{4}}$ is a norm as well on $C_0^\infty(\mathbb{H})$ and we will denote by $\mathcal{H}$ the closure of $C_c^\infty(\mathbb{H})$ with respect to this norm.

Definition 1.2. We will say that a positive solution u of (Eq_λ) is an entire solution if it belongs to the closure of $C_c^\infty(\mathbb{H})$ with respect to the norm $\|.\|_\lambda$.

Before stating our main results, let us recall what it is known in the Euclidean case ($\mathbb{H}$ replaced by $\mathbb{R}^n$, $n \ge 3$): in the subcritical case, a positive entire solution exists iff $\lambda < 0$; for such $\lambda' s$ the solution is radially symmetric and unique (up to translations); in the critical case a positive entire solution exists iff $\lambda = 0$, it is unique (up to translations and dilations) and it is explicitely known.

At our best knowledge, not much is known about (Eq_λ). It naturally appears when dealing with the Euler-Lagrange equations associated to Hardy-Sobolev-Maz'ya inequalities (see Section 6). It is also related (see [9]) to the Yamabe equation on the Heisenberg group and hence to the Webster scalar curvature equation (see [14]-[19], and, for a more general setting, [7]).

A relation between differential operators with mixed homogeneity and hyperbolic geometry was earlier observed by Beckner [2] in dealing with sharp Grushin estimates. In connection with Beckner work, and trying to extend previous results by Garofalo-Vassilev on Yamabe-type equations on groups of Heisenberg type [15], Monti and Morbidelli [23] enlightened the role of hyperbolic symmetry built in Grushin-type equations.

Motivated by these papers, we started investigating, among other things, uniqueness for (Eq_λ) and the main result in this paper, already announced in [9], is the following

Theorem 1.3. *Let $\lambda \leq \frac{(n-1)^2}{4}$ if $n \geq 3$ and $\lambda \leq \frac{2(p+1)}{(p+3)^2}$ if $n = 2$. Then (Eq_λ) has at most one entire positive solution, up to hyperbolic isometries.*

We also establish sharp existence results:

Theorem 1.4. *Let $p > 1$ if $n = 2$ and $1 < p < 2^* - 1$ if $n \geq 3$. Then (Eq_λ) has a positive entire solution for any $\lambda \leq \frac{(n-1)^2}{4}$.*

In the critical case the situation is more complicated. We have the following

Theorem 1.5. *Let $n \geq 4$, $p = 2^* - 1$ and $\frac{n(n-2)}{4} < \lambda \leq \frac{(n-1)^2}{4}$. Then (Eq_λ) has a positive entire solution.*

The restriction on λ in Theorem 1.5 is in fact sharp:

Theorem 1.6. *Let $n \geq 3$, $p = 2^* - 1$ and $\lambda \leq \frac{n(n-2)}{4}$. Then (Eq_λ) does not have any entire positive solution.*

We also observed a surprising low dimension phenomena

Theorem 1.7. *Let $n = 3$ and $p = 5$. Then (Eq_λ) has no entire positive solution.*

The paper is organized as follows.

In Section 2 we improve a symmetry result given in [1] as a first step towards a sharp uniqueness analysis.

In Section 3 we establish decay estimates for positive entire solutions, a crucial step to prove in Section 4 our uniqueness result.

Section 5 and 6 are devoted to existence/nonexistence and applications.

In Appendix A we recall notations and basic facts on the half space model $\mathcal{R}^n_+$ and the ball model $\mathcal{B}^n$ for $\mathbb{H}^n$ and in Appendix B we derive the Poincaré-Sobolev inequality in $\mathbb{H}^n$.

Added in proof. After this paper was completed, we got to know from R. Musina of a paper by Benguria, Frank and Loss [3] concerning critical Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half space, where the equivalent formulation (1.2) is also given. Among other things, they prove that the best constant in (1.2) is given by the Sobolev constant and it is not achieved. Theorem 1.7 above improves this result.

In addition, Yan Yan Li informed us that, in a recent work with D. Cao [11], they obtained results on locally finite energy solutions for a related PDE, which lead to conjecture uniqueness of such solutions. Indeed our result applies and the conjecture turns out to be true.

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2. Hyperbolic symmetry

The main purpose in this section is to prove symmetry properties of solutions of (Eq_λ). The result we need, a refinement of a result in [1], is the following

Theorem 2.1. *Let $\lambda \leq \left(\frac{n-1}{2}\right)^2$, $n \geq 2$. Let u be a positive entire solution of (Eq_λ). Then u has Hyperbolic symmetry, i.e. there is $x_0 \in \mathbb{H}$ such that u is constant on hyperbolic spheres centered at x_0.*

The proof relies on moving planes techniques in connection with sharp Sobolev inequalities, and follows closely [1].

We start pointing out the main difference with respect to [1]. The proof therein relies on the classical sharp Sobolev inequality in $H^1(\mathbb{H}^n)$, $n \geq 3$ (see [17]):

$$\frac{n(n-2)\omega^{\frac{2}{n}}}{4}\left(\int_{\mathbb{H}} |u|^{\frac{2n}{n-2}} dV_{\mathbb{H}}\right)^{\frac{n-2}{n}} \leq \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^2 dV_{\mathbb{H}} - \frac{n(n-2)}{4}\int_{\mathbb{H}} u^2 dV_{\mathbb{H}}. \quad (2.1)$$

This allows to get symmetry for $H^1(\mathbb{H})$ positive solutions of (Eq_λ) in case $\lambda \leq \frac{n(n-2)}{4}$.

To prove symmetry for any $\lambda < \frac{(n-1)^2}{4}$ (and $n \geq 2$) it will be enough to replace Sobolev inequality with the sharp Poincaré-Sobolev inequality, which by density, holds true in $H^1(\mathbb{H})$, as well as in $\mathcal{H}$.

To prove our theorem up to the limiting case $\lambda = \left(\frac{n-1}{2}\right)^2$, we need first some facts about $\mathcal{H}$. Let $D^1_{\text{cyl}}(\mathbb{R}^k \times \mathbb{R}^{n-1})$ be the closure in $D^1(\mathbb{R}^{k+n-1})$ of C_0^∞ y-radial functions $u = u(|y|, z)$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{n-1}$. We have the following

Lemma 2.2 (An isometric model for $\mathcal{H}$). *Given $u \in C_0^\infty(\mathcal{R}^n_+)$, let*

$$(Tu)(y,z) := \frac{1}{\sqrt{2\pi}}\,|y|^{-\frac{n-1}{2}} u(|y|, z), \quad (y,z) \in \mathbb{R}^2 \times \mathbb{R}^{n-1}.$$

Then $\|Tu\|_{D^1(\mathbb{R}^{n+1})} = \|u\|_{\mathcal{H}}$ and hence T extends to an isometric isomorphism between $\mathcal{H}$ and $D^1_{\text{cyl}}(\mathbb{R}^2 \times \mathbb{R}^{n-1})$.

Moreover $(\mathcal{H}, \|u\|_{\mathcal{H}})$ is a Hilbert space with the inner product $\langle , \rangle_{\mathcal{H}}$ given by

$$\langle u_1, u_2\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{\mathbb{R}^{n+1}} \nabla Tu_1 \nabla Tu_2\, dx.$$

Proof. Let $v := Tu$. Then

$$2\pi|\nabla v|^2 = r^{-(n-1)}|\nabla_z u|^2 + r^{-(n-1)}u_r^2 + \left(\frac{n-1}{2}\right)^2 r^{-(n+1)}u^2 - \frac{n-1}{2} r^{-n}(u^2)_r.$$

Integrating in polar coordinates and then by parts

$$\int_{\mathbb{R}^{n+1}} |\nabla v|^2 dydz = \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} \left[\frac{|\nabla u|^2}{r^{n-2}} + \left(\frac{n-1}{2}\right)^2 \frac{u^2}{r^n} - \frac{n-1}{2} r^{-n+1}(u^2)_r\right] drdz$$
$$= \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} \left[\frac{|\nabla u|^2}{r^{n-2}} - \left(\frac{n-1}{2}\right)^2 \frac{u^2}{r^n}\right] drdz = \|u\|^2_{\mathcal{H}}.$$

Since $T(C_0^\infty(\mathbb{H}))$ contains all the cylindrically symmetric C^∞ functions with compact support in $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-1}$, and they are dense in $D^1_{\text{cyl}}(\mathbb{R}^2 \times \mathbb{R}^{n-1})$, by density T extends to an isometry from $\mathcal{H}$ onto $D^1_{\text{cyl}}(\mathbb{R}^2 \times \mathbb{R}^{n-1})$. The same arguments show that T preserves the scalar product. Finally, since convergence in $\mathcal{H}$ and in $D^1_{\text{cyl}}(\mathbb{R}^2 \times \mathbb{R}^{n-1})$ imply, up to subsequences, pointwise convergence, we can in particular conclude that any $u \in \mathcal{H}$ can be written as $r^{\frac{n-1}{2}} v(r, z)$ for some $v \in D^1_{\text{cyl}}(\mathbb{R}^2 \times \mathbb{R}^{n-1})$. □

Lemma 2.3. *Let $u \in \mathcal{H}$ be compactly supported in $\mathbb{H}$. Then $u \in H^1(\mathbb{H})$.*

Proof. Let $u_n \in C_c^\infty(\mathbb{H})$ such that $||u_n - u||_{\mathcal{H}} \to 0$. We may assume that support of u_n's are contained in a fixed compact subset K of $\mathbb{H}$. From the Poincaré-Sobolev inequality (1.2), we get u_n is Cauchy in $L^p(\mathbb{H})$ and hence converges to u in $L^p(\mathbb{H})$ for any $p > 2$. This implies $u_n \to u$ in $L^2(\mathbb{H})$ thanks to the assumption on the support of u_n's. Convergence in L^2 and convergence in $||.||_{\mathcal{H}}$ together implies ∇u_n is Cauchy in $L^2(\mathbb{H})$ and hence the convergence of u_n to u in $H^1(\mathbb{H})$. □

Lemma 2.4 (Poincaré-Sobolev inequality in $\mathcal{H}$). *Let $p > 1$ if $n = 2$ and $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$. There exists $S_{n,p} > 0$ such that*

$$S_{n,p}\left(\int_{\mathbb{H}} |u|^{p+1}\, dV_{\mathbb{H}}\right)^{\frac{2}{p+1}} \leq \|u\|^2_{\mathcal{H}} \qquad \forall\, u \in \mathcal{H}. \tag{2.2}$$

Equivalently, for every $v \in D^1_{\text{cyl}}(\mathbb{R}^2 \times \mathbb{R}^{n-1})$

$$(2\pi)^{\frac{p-2}{p}} S_{n,p}\left(\int_{\mathbb{R}^{n+1}} \frac{|v|^{p+1}}{|y|^{\frac{n+3-p(n-1)}{2}}}\right)^{\frac{2}{p+1}} dx \;\leq\; \|v\|^2_{D^1(\mathbb{R}^{n+1})}. \tag{2.3}$$

Remark 2.5. Notice that $n + 3 - p(n-1) < 0$ if $\frac{n+3}{n-1} < p \leq \frac{n+2}{n-2}$.

Proof. (2.2) follows from (1.2) and (2.3) follows from (2.2), Lemma 2.2 and

$$\int_{\mathbb{R}^2\times\mathbb{R}^{n-1}} \frac{\left(|y|^{-\frac{n-1}{2}} u(|y|, z)|\right)^{p+1}}{|y|^{\frac{n+3-p(n-1)}{2}}} dydz = 2\pi \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} \frac{|u|^{p+1}}{r^n} drdz$$
$$= 2\pi \int_{\mathbb{H}} |u|^{p+1}\, dV_{\mathbb{H}}.$$
□

Furthermore, a direct computation gives

Lemma 2.6. *Let $u \in \mathcal{H}$ be a positive solution of (Eq_λ), $\lambda = \left(\frac{n-1}{2}\right)^2$, $n \geq 2$. Let $v := \sqrt{2\pi}\, Tu$. Then v is a $D^1(\mathbb{R}^{n+1})$ solution of*

$$-\Delta v(y,z) = \frac{v^p}{|y|^{\frac{n+3-p(n-1)}{2}}} \qquad in \quad \mathbb{R}^2 \times \mathbb{R}^{n-1}. \tag{2.4}$$

Proof of Theorem 2.1. The proof is in the same lines as in [1]; so, we will only give an outline. The main difference is in the use of sharp Poincaré-Sobolev inequalities (2.2) and (1.2) instead of sharp Sobolev inequality (2.1).

Let A_t be a one parameter group of isometries of $\mathbb{H}$ which is $C^1(\mathbb{R}\times\mathbb{H}, \mathbb{H})$ and I be a reflection (*i.e.*, I is an isometry and $I^2 =$ Identity) satisfying the invariance condition $A_t I A_t = I$, $\forall t \in \mathbb{R}$. Define $I_t = A_t I A_{-t}$ and Let U_t be the hypersurface of $\mathbb{H}$ which is fixed by I_t. We also assume that $\cup_{t_1<t<t_2} U_t$ is open for all $t_1, t_2 \in \mathbb{R}$ and $\cup_{t\in\mathbb{R}} U_t = \mathbb{H}$. For $t \in \mathbb{R}$ define

$$Q_t = \cup_{-\infty<s<t} U_s, \qquad \text{and} \qquad Q^t = \cup_{t<s<\infty} U_s.$$

Then $I_t(Q_t) \subset Q^t$ and $I_t(Q^t) \subset Q_t$ for all $t \in \mathbb{R}$. Now define for $t \in \mathbb{R}$

$$u_t(x) = u(I_t(x)),\ x \in \mathbb{H}.$$

The theorem will be proved once we show the existence of a $t_0 \in \mathbb{R}$ such that $u_{t_0} = u$ in Q_{t_0} (see [1] for details.) As in [1] we will prove this in three steps. We will only give details for step 1; the other two steps follow as in [1]. Let

$$\Lambda = \{t \in \mathbb{R} \ :\ \forall s > t, u \geq u_s \text{ in } Q_s\}.$$

Step 1. Λ is nonempty.

Proof of Step 1. It is enough to show that for t large enough $u_t \geq u$ in Q^t. Since I_t is an isometry, $u_t \in \mathcal{H}$ and solves (Eq_λ). Write, as in Lemma 2.2, $v = \sqrt{2\pi}Tu$, $v_t = \sqrt{2\pi}Tu_t$. From Lemma 2.2, they are in $D^{1,2}(\mathbb{R}^n)$ and, furthermore, they solve (2.4). Now define

$$\Omega_t = \{(y,z) \in \mathbb{R}^2 \times \mathbb{R}^{n-1} \ :\ (|y|, z) \in Q^t\}.$$

Then Step 1 will be proved once we show that $v_t \geq v$ in Ω_t. Now, Ω_t is open in $\mathbb{R}^{n+1}$ and $v = v_t$ on $\partial\Omega_t$. We also have

$$-\Delta(v - v_t) = |y|^{-\tau}(v^p - v_t^p)$$

where $\tau = \frac{2(n+1)-p(n-1)}{2}$ as in (2.4). Now multiplying the above equation by $(v - v_t)^+$ integrating by parts and applying (2.2) to $(v - v_t)^+$ we get,

$$\begin{aligned}\int_{\Omega_t} |\nabla(v - v_t)^+|^2 \, dx &= \int_{\Omega_t} \frac{v^p - v_t^p}{|y|^\tau (v - v_t)} ((v - v_t)^+)^2 \, dx \\ &\le C \left(\int_{\Omega_t} \frac{v^{p+1}}{|y|^\tau} \right)^{\frac{p-1}{p+1}} \left(\int_{\Omega_t} \frac{((v - v_t)^+)^{p+1}}{|y|^\tau} \right)^{\frac{2}{p+1}} \\ &\le C \left(\int_{\Omega_t} \frac{v^{p+1}}{|y|^\tau} \right)^{\frac{p-1}{p+1}} \int_{\Omega_t} |\nabla(v - v_t)^+|^2 \, dx.\end{aligned}$$

Since $\int_{\mathbb{R}^{n+1}} |y|^{-\tau} v^{p+1} < \infty$, $\int_{\Omega_t} |y|^{-\tau} v^{p+1} \to 0$ as $t \to \infty$. Hence there exist a t_0 such that $\int_{\Omega_t} |\nabla(v - v_t)^+|^2 \, dx = 0$ for all $t > t_0$. This proves Step 1.

To complete the proof, one can show, following [1], that

Step 2. Λ is bounded below.

Step 3. $\tilde{u}_{t_0} = \tilde{u}$ in Q_{t_0} where $t_0 = \inf \Lambda$. □

3. Decay estimates

The proof of our uniqueness result relies on decay properties of entire positive solutions we are going to prove here. Let u be a positive symmetric solution of (Eq_λ). As a function on $\mathcal{B} := \mathcal{B}^n$, $u = u(|\xi|)$, $\xi \in \mathbb{R}^n$, $|\xi| < 1$ and

$$\left[\frac{1 - |\xi|^2}{2} \right]^2 \Delta u + (n-2) \left[\frac{1 - |\xi|^2}{2} \right] < \nabla u \, , \xi > + \lambda u + u^p = 0. \tag{3.1}$$

Setting $|\xi| := \tanh \frac{t}{2}$, $u(t) := u(\tanh \frac{t}{2})$, $q := (\sinh t)^{n-1}$, it is easy to see that $\int_{\mathcal{B}} |u|^p dV_{\mathcal{B}} = \omega_n \int_0^\infty q |u|^p$, $\int_{\mathcal{B}} |\nabla_{\mathcal{B}} u|^2 dV_{\mathcal{B}} = \omega_n \int_0^\infty q |u'|^2$ so that

$$u \in H^1(\mathcal{B}^n) \Leftrightarrow \omega_n \int_0^\infty \left[|u|^2 + |u'|^2 \right] q = \int_{\mathcal{B}} \left[|u|^2 + |\nabla_{\mathcal{B}} u|^2 \right] dV_{\mathcal{B}} < +\infty. \tag{3.2}$$

Poincaré-Sobolev inequality reads as follows: $\forall \, p \in (1, 2^* - 1]$, $\exists c(n, p) > 0$:

$$c(n,p) \left[\int_0^{+\infty} q |w|^{p+1} \right]^{\frac{2}{p+1}} \le \int_0^{+\infty} q |w'|^2 - \frac{(n-1)^2}{4} \int_0^{+\infty} q w^2 \tag{3.3}$$

$\forall \, w \in C_0^\infty([0, +\infty))$; if $n = 2$ any $p > 1$ is allowed. In addition, (3.1) rewrites

$$u'' + \frac{n-1}{\tanh t} u' + \lambda \, u + u^p = 0, \qquad u'(0) = 0 \tag{3.4}$$

as well as

$$(qu')' + \lambda q\, u + qu^p = 0, \qquad u'(0) = 0 \tag{3.5}$$

and if $u \in H^1(\mathcal{B})$ solves (3.1), then $\int_0^\infty q[|u'|^2 - \lambda u^2] = \int_0^\infty qu^{p+1}$. The above relations are no longer true if $u \notin H^1(\mathcal{B})$. However, we can derive an analogue of (3.2) and (3.3) for functions in $\mathcal{H}$ possessing hyperbolic symmetry. As above, and if there is no confusion, we will write $u(\tanh \frac{t}{2})$ as $u(t)$.

Lemma 3.1. *Let $u \in \mathcal{H}$ be a symmetric function. Then*

$$||u||^2_{\mathcal{H}} = \omega_n \int_0^\infty \left[\left(u' + \frac{n-1}{2} \tanh \frac{t}{2}\, u \right)^2 + \frac{(n-1)u^2}{\left(2\cosh \frac{t}{2}\right)^2} \right] q \; < \infty \tag{3.6}$$

and hence the Poincaré - Sobolev inequality for u rewrites as

$$c(n,p) \left[\int_0^\infty q|u|^{p+1} \right]^{\frac{2}{p+1}} \le \int_0^\infty \left[\left(u' + \frac{n-1}{2} \tanh \frac{t}{2} u \right)^2 + \frac{(n-1)u^2}{\left(2\cosh \frac{t}{2}\right)^2} \right] q. \tag{3.7}$$

If u satisfies (3.4) *and* (3.6) *then for every w satisfying* (3.6)*, we have*

$$\begin{aligned} &\int_{t_0}^\infty \left[\left(u' + \frac{n-1}{2} \tanh \frac{t}{2} u \right)\left(w' + \frac{n-1}{2} \tanh \frac{t}{2} w \right) + \frac{(n-1)uw}{\left(2\cosh \frac{t}{2}\right)^2} \right] q \\ &= \int_{t_0}^\infty qu^p w \quad \textit{where either} \quad t_0 = 0 \quad \textit{or} \quad w(t_0) = 0. \end{aligned} \tag{3.8}$$

Proof. Let $u \in C_0^\infty(\mathcal{B})$ and $v := \left(2\cosh^2(t/2)\right)^{\frac{n-1}{2}} u$. Integrating by parts,

$$\begin{aligned} \|u\|^2_{\mathcal{H}} :&= \omega_n \int_0^\infty \left[(u')^2 - \left(\frac{n-1}{2}\right)^2 u^2 \right] (\sinh t)^{n-1}\, dt \\ &= \omega_n \int_0^\infty \left[(v')^2 + \frac{n-1}{4} \left(\frac{v}{\cosh(t/2)} \right)^2 \right] (\tanh(t/2))^{n-1}\, dt. \end{aligned} \tag{3.9}$$

Now, given $u \in \mathcal{H}$, there is a sequence of radial functions $u_m \in C_c^\infty(\mathcal{B})$ such that $u_m \to_m u$ in $\mathcal{H}$ and a.e. Hence, if $v_m := \left(2\cosh^2(t/2)\right)^{\frac{n-1}{2}} u_m$, then $v_m \to_m v :=$

$\left(2\cosh^2(t/2)\right)^{\frac{n-1}{2}} u$ and

$$||u_m||^2_{\mathcal{H}} = \omega_n \int_0^\infty \left[(v_m')^2 + \frac{n-1}{4}\left(\frac{v_m}{\cosh(t/2)}\right)^2 \right] (\tanh(t/2))^{n-1}\, dt. \quad (3.10)$$

Since u_n is a Cauchy sequence in $\mathcal{H}$, we can then pass to the limit in (3.10) and see that (3.9) actually holds for every $u \in \mathcal{H}$. Now (3.6) follows just substituting $v = \left(2\cosh^2(t/2)\right)^{\frac{n-1}{2}} u$ in (3.9).

The Poincaré-Sobolev inequality (3.7) follows from (3.3) and (3.6).

As for (3.8), let w satisfy (3.6). Then there is a sequence $w_n \in C_0^\infty([0,\infty))$ with $w_n'(0) = 0$ such that $w_n \to w$ in the $\mathcal{H}$ norm given by (3.6). Since u satisfies (3.5), multiplying this relation by w_n and integrating by parts we see that (3.8) holds for w_n. Since w satisfies (3.6) we can pass to the limit proving (3.8). □

Now, let us notice that if u solves (3.4) and $E_u(t) := \frac{u'^2}{2} + \frac{\lambda}{2}u^2 + \frac{|u|^{p+1}}{p+1}$, then

$$\frac{d}{dt}E_u(t) = -\frac{n-1}{\tanh t}u'^2 \leq 0 \quad \forall t > 0.$$

In particular, u and u' remain bounded and hence u is defined for every t. Energy considerations also lead to monotonicity properties and exponential decay of positive solutions of (3.4) as can be seen below.

Lemma 3.2. *Let $n \geq 2$, $p > 1$. Let $v > 0$ be a solution of* (3.4). *If $\lambda < 0$ we also assume that v satisfies* (3.2). *Then $v'(t) < 0$ for every $t > 0$ and* $\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} v'(t) = 0$.

Proof. If $\lambda \geq 0$, equation (3.5) implies $(qv')'(t) < 0\ \ \forall t > 0$ and then $v'(t) < 0$ $\forall t > 0$ because $v'(0) = 0$. In particular, it exists $v(\infty) := \lim_{t\to+\infty} v(t)$ and, since E_v is decreasing, $v'(\infty) := \lim_{t\to+\infty} v'(t)$ exists as well and is zero because v is positive. Finally, (3.4) gives $v''(\infty) = -[\lambda v(\infty) + v^p(\infty)]$ with, necessarily, $v''(\infty) = 0$ and hence $v(\infty) = 0$.

If $\lambda < 0$, v is not decreasing and does not vanish at infinity, in general. However, assuming (3.2), we have that $\liminf_{t+\infty} q(t)[v^2(t) + v'^2(t)] = 0$. In particular, $0 = \lim_{t\to+\infty} E_v(t) < E_v(t)\ \forall t$ by monotonicity. Now, let by contradiction $v'(t_0) = 0$ for some $t_0 > 0$. Then $0 < E_v(t_0) = \frac{1}{2}\lambda v^2(t_0) + \frac{|v|^{p+1}}{p+1}(t_0)$ and hence $\lambda v(t_0) + v^p(t_0) > 0$ and hence, by equation (3.4), $v''(t_0) < 0$. Thus t_0 is the only zero of v' and $v' > 0$ in $(0, t_0)$. Since $E_v(0) > 0$ and $v'(0) = 0$, $\lambda v(t) + v^p(t) > 0$ for t small, and hence $v'' < 0$ for t small and hence v' is negative for t small, a contraddiction. This and (3.2) imply $v(\infty) = 0$. As above, $v'(\infty)$ exists and it is necessarily zero. □

Remark 3.3. In case $\lambda < 0$ and assuming $v \to_{t\to+\infty} 0$ instead of (3.2), the same argument as above implies again $v'(t) < 0$ for $t > 0$ and $v'(\infty) = 0$.

Lemma 3.4. *Let $n \geq 2$, $p > 1$, $\lambda < \frac{(n-1)^2}{4}$. Let $v > 0$ be a solution of* (3.4) *satisfying* (3.2). *Then*

$$\lim_{t\to+\infty}\frac{\log v^2}{t}=\lim_{t\to+\infty}\frac{\log v'^2}{t}=\lim_{t\to+\infty}\frac{\log[v^2+v'^2]}{t}=-\left[n-1+\sqrt{(n-1)^2-4\lambda}\right]$$

$$\frac{v'}{v}\to_{t\to+\infty}-\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}.$$

Proof. By Lemma 3.2, $v \to_{t\to+\infty} 0$. Then it exists $t_\epsilon > 0$ such that

$$\coth t \leq 1+\epsilon, \qquad v^p(t) \leq \epsilon v(t), \qquad \forall t \geq t_\epsilon.$$

Since, again by Lemma 3.2, v' is negative, we have, for $t \geq t_\epsilon$,

$$v'' + (n-1)(1+\epsilon)v' + \lambda v \leq v'' + (n-1)\coth t\, v' + \lambda\, v + v^p = 0 \tag{3.11}$$

$$v'' + (n-1)v' + (\lambda+\epsilon)v \geq v'' + (n-1)\coth t\, v' + \lambda\, v + v^p = 0. \tag{3.12}$$

Let $\mu^\pm(\epsilon) := \frac{-(n-1)(1+\epsilon)\pm\sqrt{(n-1)^2(1+\epsilon)^2-4\lambda}}{2}$, $\nu^\pm(\epsilon) := \frac{-(n-1)\pm\sqrt{(n-1)^2-4(\lambda+\epsilon)}}{2}$ be the characteristic roots of the differential polinomials in the left hand side of (3.11)-(3.12), respectively (notice that $\mu^\pm(\epsilon)$ are real and distincts for $\lambda \leq \frac{(n-1)^2}{4}$; to have $\nu^\pm(\epsilon)$ real and distinct we need to choose $\epsilon < \frac{(n-1)^2}{4} - \lambda$ and with this choice $\mu^-(\epsilon) < \nu^-(\epsilon)$). Let

$$w := v' - \mu^+(\epsilon)v, \qquad z := v' - \nu^+(\epsilon)v. \tag{3.13}$$

Then, from (3.11) and (3.12), we have, for $t \geq t_\epsilon$,

$$w' - \mu^-(\epsilon)w = v'' + (n-1)(1+\epsilon)v' + \lambda v \leq 0 \tag{3.14}$$

$$z' - \nu^-(\epsilon)z = v'' + (n-1)v' + (\lambda+\epsilon)v \geq 0. \tag{3.15}$$

From (3.14)-(3.15) we derive $\left(e^{-\mu^-(\epsilon)t}w\right)' \leq 0 \leq \left(e^{-\nu^-(\epsilon)t}z\right)'$ for every $t \geq t_\epsilon$ and hence, integrating such inequalities in $[\tau, t]$, $t_\epsilon \leq \tau \leq t$, we get, respectively,

$$v'(t) - \mu^+(\epsilon)v(t) = w(t) \leq \left(e^{-\mu^-(\epsilon)\tau}w(\tau)\right)e^{\mu^-(\epsilon)t} := c_\epsilon(\tau)e^{\mu^-(\epsilon)t} \tag{3.16}$$

$$v'(t) - \nu^+(\epsilon)v(t) = z(t) \geq \left(e^{-\nu^-(\epsilon)\tau}z(\tau)\right)e^{\nu^-(\epsilon)t} := d_\epsilon(\tau)e^{\nu^-(\epsilon)t} \tag{3.17}$$

for every $t \geq \tau \geq t_\epsilon$. Again by integration, we finally get, for every $t \geq \tau \geq t_\epsilon$,

$$\begin{aligned} v(t) &\leq \left(e^{-\mu^+(\epsilon)\tau}v(\tau)\right)e^{\mu^+(\epsilon)t} + c_\epsilon(\tau)\frac{e^{\mu^-(\epsilon)t} - e^{(\mu^-(\epsilon)-\mu^+(\epsilon))\tau+\mu^+(\epsilon)t}}{\mu^-(\epsilon)-\mu^+(\epsilon)} \\ &= \left[e^{-\mu^+(\epsilon)\tau}v(\tau) - \frac{c_\epsilon(\tau)}{\mu^-(\epsilon)-\mu^+(\epsilon)}e^{(\mu^-(\epsilon)-\mu^+(\epsilon))\tau}\right]e^{\mu^+(\epsilon)t} \qquad (3.18) \\ &\quad + \frac{c_\epsilon(\tau)e^{\mu^-(\epsilon)t}}{\mu^-(\epsilon)-\mu^+(\epsilon)} \end{aligned}$$

$$\begin{aligned} v(t) &\geq \left(e^{-\nu^+(\epsilon)\tau} v(\tau)\right) e^{\nu^+(\epsilon)t} + d_\epsilon(\tau) \frac{e^{\nu^-(\epsilon)t} - e^{(\nu^-(\epsilon)-\nu^+(\epsilon))\tau+\nu^+(\epsilon)t}}{\nu^-(\epsilon)-\nu^+(\epsilon)} \\ &= \left[e^{-\nu^+(\epsilon)\tau} v(\tau) - \frac{d_\epsilon(\tau)}{\nu^-(\epsilon)-\nu^+(\epsilon)} e^{(\nu^-(\epsilon)-\nu^+(\epsilon))\tau}\right] e^{\nu^+(\epsilon)t} \\ &\quad + \frac{d_\epsilon(\tau) e^{\nu^-(\epsilon)t}}{\nu^-(\epsilon)-\nu^+(\epsilon)}. \end{aligned} \tag{3.19}$$

Now, v positive, $\mu^-(\epsilon) < \mu^+(\epsilon)$ and (3.18) imply

$$e^{-\mu^+(\epsilon)\tau} v(\tau) - \frac{c_\epsilon(\tau)}{\mu^-(\epsilon)-\mu^+(\epsilon)} e^{(\mu^-(\epsilon)-\mu^+(\epsilon))\tau} \geq 0 \qquad \forall\, \tau \geq t_\epsilon$$

that is

$$v(\tau) \geq \frac{c_\epsilon(\tau) e^{\mu^-(\epsilon)\tau}}{\mu^-(\epsilon)-\mu^+(\epsilon)} = \frac{w(\tau)}{\mu^-(\epsilon)-\mu^+(\epsilon)} = \frac{v'(\tau)-\mu^+(\epsilon)v(\tau)}{\mu^-(\epsilon)-\mu^+(\epsilon)} \qquad \forall\, \tau \geq t_\epsilon.$$

Hence

$$v'(\tau) \geq \mu^-(\epsilon) v(\tau), \quad \forall\, \tau \geq t_\epsilon \tag{3.20}$$

and integrating on $[t_\epsilon, t]$ we find

$$v(t) \geq \left(v(t_\epsilon) e^{-\mu^-(\epsilon)t_\epsilon}\right) e^{\mu^-(\epsilon)t} \quad \forall\, t \geq t_\epsilon. \tag{3.21}$$

We notice, for future reference, that in case $\lambda \geq 0$ (3.20) and (3.21) hold true for any positive solution: assumption (3.2) is used, at this stage, to insure monotonicity and decay properties of v and hence it is required just in case $\lambda < 0$ (see Lemma 3.2; accordingly with Remark 3.3, we might have asked, alternatively, to v to vanish at infinity). We similarly infer from (3.19) that

$$\hat{d}_\epsilon(\tau) := e^{-\nu^+(\epsilon)\tau} v(\tau) - \frac{d_\epsilon(\tau)}{\nu^-(\epsilon)-\nu^+(\epsilon)} e^{(\nu^-(\epsilon)-\nu^+(\epsilon))\tau} \leq 0 \qquad \forall\, \tau \geq t_\epsilon.$$

Otherwise, since $\nu^-(\epsilon) < \nu^+(\epsilon)$, (3.19) gives $v(t) \geq \frac{1}{2}\hat{d}_\epsilon(\tau) e^{\nu^+(\epsilon)t}$. But, since $\lambda < \frac{(n-1)^2}{4}$ and hence $n-1+2\nu^+(\epsilon) > 0$, this inequality implies that $\int_0^\infty q v^2 = +\infty$, violating (3.2) (in case $\lambda < 0$, and hence $\nu^+(\epsilon) > 0$, it would even follow that v is unbounded). Thus, $\hat{d}_\epsilon(\tau) \leq 0 \quad \forall\, \tau \geq t_\epsilon$, that is

$$v(\tau) \leq \frac{d_\epsilon(\tau) e^{\nu^-(\epsilon)\tau}}{\nu^-(\epsilon)-\nu^+(\epsilon)} = \frac{z(\tau)}{\nu^-(\epsilon)-\nu^+(\epsilon)} = \frac{v'(\tau)-\nu^+(\epsilon)v(\tau)}{\nu^-(\epsilon)-\nu^+(\epsilon)} \qquad \forall\, \tau \geq t_\epsilon.$$

Hence

$$v'(\tau) \leq \nu^-(\epsilon) v(\tau), \quad \forall\, \tau \geq t_\epsilon \tag{3.22}$$

and integrating on $[t_\epsilon, t]$ we find

$$v(t) \le \left(v(t_\epsilon)e^{-\nu^-(\epsilon)t_\epsilon}\right)e^{\nu^-(\epsilon)t} \quad \forall\, t \ge t_\epsilon. \tag{3.23}$$

We see from (3.21) and (3.23) that

$$2\mu^-(\epsilon) \le \liminf_{t\to+\infty} \frac{\log v^2}{t} \le \limsup_{t\to+\infty} \frac{\log v^2}{t} \le 2\nu^-(\epsilon), \quad \forall \epsilon > 0$$

and hence $\lim_{t\to+\infty} \frac{\log v^2}{t} = -\left[n-1+\sqrt{(n-1)^2-4\lambda}\right]$.

From (3.20)-(3.22) and (3.21)-(3.23) we see that similar bounds hold true for v' as well and hence

$$\lim_{t\to+\infty} \frac{\log(v')^2}{t} = \lim_{t\to+\infty} \frac{\log(v^2+(v')^2)}{t} = -\left[n-1+\sqrt{(n-1)^2-4\lambda}\right].$$

Finally, taking lim sup and lim inf in (3.20)-(3.22) and then sending ϵ to zero, we see that $\frac{v'}{v} \to_{t\to+\infty} -\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}$. □

Remark 3.5. In case $\lambda < 0$ assumption (3.2) can be replaced by $\lim_{t\to+\infty} v = 0$. Thus, if $\lambda < 0$ a positive solution vanishes at infinity iff it is in H^1.

Let us now consider the limit case $\lambda = \frac{(n-1)^2}{4}$. Here we have more precise estimates, which will be actually needed later on.

Lemma 3.6. *Let $n \ge 2$, $\lambda = \frac{(n-1)^2}{4}$, $p > 1$. Let $v > 0$ be a solution of* (3.4). *Then*

$$\lim_{t\to+\infty} \frac{\log v^2}{t} = \lim_{t\to+\infty} \frac{\log(v')^2}{t} = \lim_{t\to+\infty} \frac{\log[v^2+(v')^2]}{t} = -(n-1). \tag{3.24}$$

Let, in addition, v satisfy (3.6). *Then there is $A > 0$ such that*

$$e^{\frac{n-1}{2}t}v(t) \to A \quad \textit{and} \quad e^{\frac{n-1}{2}t}v'(t) \to -\frac{n-1}{2}A \ \text{ as } t \to \infty. \tag{3.25}$$

Proof. Since, $\mu^\pm$ (see the proof of Lemma 3.4) are real and distinct, (3.14)-(3.18) and hence the bound from below $v(t) \ge c_\epsilon e^{\mu^-(\epsilon)t}$ given in (3.21), still holds true, because, at this stage, it was just required that v decreases to zero, a property satisfied here (Lemma 3.2). For the same reason, an upper bound for v and (3.20) will give a similar upper bound for $-v'$.

A bound from above for v can be obtained as follows. First observe that $v''(t)$ cannot be definitely negative and then it is definitely positive because, otherwise, it has a sequence $t_j \to_j +\infty$ of zeros, that is, denoted $w := v'$, then $w < 0$ and $w'(t_j) = 0$. Taking the derivative of (3.4), we get

$$w'' + (n-1)\coth t\, w' + \left(\lambda - \frac{n-1}{\sinh^2 t} + pv^{p-1}\right)w = 0$$

and we see that there is $\underline{t}$ such that $w''(t_j) > 0$ for $t_j \geq \underline{t}$, while w'' has to change sign at two consecutive zeros of w'. In turn, $v'' > 0$ for large t implies that v (and v') has exponential decay. In fact, if $v'' > 0$ for $t \geq \bar{t}$, then, using (3.4), we see that $0 > (n-1)\coth t v' + \lambda v$ for $t \geq \bar{t}$ and hence $\frac{v'}{v} \leq -\frac{\lambda \tanh \bar{t}}{n-1}$ and hence $v(t) \leq v(\bar{t}) e^{-\frac{\lambda \tanh \bar{t}}{n-1}(t-\bar{t})}$. As a consequence, as observed above, a similar upper bound holds true for $-v'$. Now, from (3.5) and (3.21), we get, for positive λ, $-(qv')(t) \geq -(qv')(t_\epsilon) + \lambda \int_{t_\epsilon}^t q e^{-(\frac{n-1}{2}+\epsilon)\tau} d\tau \geq c_\epsilon e^{(\frac{n-1}{2}-\epsilon)t}$ and then $-v'(t) \geq c_\epsilon e^{-(\frac{n-1}{2}+\epsilon)t}$. Here $\lambda = \frac{(n-1)^2}{4}$, so we got, for any given $\epsilon > 0$, positive numbers b_ϵ, c_ϵ such that

$$b_\epsilon e^{-(\frac{n-1}{2}+\epsilon)t} \leq v \leq c_\epsilon e^{-(\frac{n-1}{4}-\epsilon)t}, \quad b_\epsilon e^{-(\frac{n-1}{2}+\epsilon)t} \leq -v' \leq c_\epsilon e^{-(\frac{n-1}{4}-\epsilon)t}.$$

Now, let $V := (v, v')$, $F(V,t) := (0, (n-1)v'(1-\coth t) - v^p)$ and $\mathcal{A} := \begin{pmatrix} 0 & 1 \\ -\lambda & -(n-1) \end{pmatrix}$. We can write (3.5) as $V' = \mathcal{A}\, V + F(V,t)$.

The above asymptotic analysis gives $b := \limsup_{t\to+\infty} \frac{\log|V|}{t} < 0$. Since $|F(V,t)| \leq \epsilon |V|$ for t large and V small, b has to be a non positive characteristic root of $\mathcal{A}$ (see ([11, Theorem 4.3]), *i.e.* $b = \frac{1}{2}\left[-(n-1) \pm \sqrt{(n-1)^2 - 4\lambda}\right] = -\frac{n-1}{2}$. This proves (3.24).

To prove (3.25), note that using (3.24) we can rewrite (3.4) as

$$v'' + (n-1)v' + \frac{(n-1)^2}{4} v = f$$

where $|f(t)| \leq Ce^{-\alpha t}$, $t > 0$ for some $\alpha > \frac{n-1}{2}$ and $C > 0$. Let $w = v' + \frac{n-1}{2} v$. Then

$$\left(e^{\frac{n-1}{2}t} w\right)' = e^{\frac{n-1}{2}t}\left(w' + \frac{n-1}{2} w\right) = e^{\frac{n-1}{2}t} f(t). \tag{3.26}$$

Since v satisfy (3.6), we have $\liminf_{t\to\infty} q\left(v' + \frac{n-1}{2}\tanh\frac{t}{2}\ v\right)^2 = 0$. Thus, there exist a sequence $t_n \to \infty$ such that $e^{\frac{n-1}{2}t_n} w(t_n) \to 0$. Let $t < t_n$, integrating (3.26) from t to t_n and then taking the limit as $n \to \infty$ we get

$$e^{\frac{n-1}{2}t} w(t) = g(t) \tag{3.27}$$

where $|g(t)| \leq Ce^{(\frac{n-1}{2}-\alpha)t}$ for some $C > 0$. This implies

$$\left(e^{\frac{n-1}{2}t} v(t)\right)' = e^{\frac{n-1}{2}t} w(t) = g(t). \tag{3.28}$$

Integrating from 0 to t observing that g has exponential decay, we get

$$e^{\frac{n-1}{2}t} v(t) = v(0) + \int_0^t g(s)\, ds \to A \geq 0, \quad \text{as} \quad t \to \infty.$$

Suppose $A = 0$, then integrating (3.28) from t to ∞ we get

$$e^{\frac{n-1}{2}t}v(t) \le Ce^{(\frac{n-1}{2}-\alpha)t}$$

which contradicts (3.24) as $\alpha > \frac{n-1}{2}$. Now the estimate on $e^{\frac{n-1}{2}t}v'(t)$ follows from (3.27). This completes the proof of the lemma. □

Remark 3.7. Because of the asymptotic behaviour proved above, positive solutions of (Eq_λ) cannot be in H^1 if $\lambda = \frac{(n-1)^2}{4}$.

The final argument just relies on some exponential decay of v, v', which was shown to hold true also in case $\int_0^\infty q|v'|^2 = +\infty$. In this case, which is not covered by the above lemma, we can conclude anyway that

$$\limsup_{t\to+\infty} \frac{\log(v^2+v'^2)}{t} = -(n-1) + \sqrt{(n-1)^2 - 4\lambda}.$$

Remark 3.8. Exact asymptotic behaviour can be observed explicitely. Given $\lambda \le \frac{(n-1)^2}{4}$, let $p = 1 + \frac{2}{n-1\pm\sqrt{(n-1)^2-4\lambda}}$ or $p = 1 + \frac{4}{n-1\pm\sqrt{(n-1)^2-4\lambda}}$. Then

$$v^\pm(t) := \left[\frac{c}{1+\cosh t}\right]^{\frac{n-1\pm\sqrt{(n-1)^2-4\lambda}}{2}} \quad \text{and} \quad u^\pm := \left[\frac{c}{\cosh t}\right]^{\frac{n-1\pm\sqrt{(n-1)^2-4\lambda}}{2}}$$

are solutions of (3.4) for suitable constants c. Notice that while v^+, u^+ are entire solutions, v^- and u^- decay too slowly to have finite energy.

4. Proof of the uniqueness result

We prove here uniqueness of positive entire solutions u of (Eq_λ) when $n \ge 3$ and $\lambda \le \frac{(n-1)^2}{4}$ or $n = 2$ and $\lambda \le \frac{2(p+1)}{(p+3)^2}$. By Theorem 2.1 u is a solution of (3.4). We start proving a comparison lemma.

Lemma 4.1. *Let $\lambda \le \frac{(n-1)^2}{4}$. Let $p+1 \le 2^*$ if $n \ge 3$ and $p > 1$ if $n = 2$. Let u, v be distinct positive solutions of* (3.4). *Then*

(i) *Given $R, M > 0$, there is $\delta = \delta(u, R)$ such that if $v(0), u(0) \le M$ then*

$$u(t_i) = v(t_i), \quad 0 < t_1 < t_2 \le R, \quad \Rightarrow \quad t_2 - t_1 \ge \delta.$$

If, in addition, u satisfies (3.2) *or* (3.6) *depending on $\lambda < \frac{(n-1)^2}{4}$ or $\lambda = \frac{(n-1)^2}{4}$ respectively, and $v(0) < u(0)$, then*

(ii) *there is t_v such that $u(t_v) = v(t_v)$ and there is $\underline{t} = \underline{t}(u)$ such that*

$$v(t_1) = u(t_1), \quad t_1 \ge \underline{t} \quad \Rightarrow \quad v(t) > u(t) \quad \forall t > t_1.$$

Proof. We start proving that if $0 < t_1 < t_2 < +\infty$, $v(t_i) = u(t_i)$ then

$$\exists C(p,n) > 0 : 0 \le v \le u \quad \text{in} \quad [t_1, t_2] \Rightarrow \int_{t_1}^{t_2} q u^{p+1} \ge C(p,n). \tag{4.1}$$

If $t_2 = +\infty$, assumption $u(t_2) = v(t_2)$ can be replaced by assuming u satisfies (3.2) or (3.6) when $\lambda < \frac{(n-1)^2}{4}$ or $\lambda = \frac{(n-1)^2}{4}$ respectively.

To prove (4.1), let us write $w := u - v$. From (3.3) and (3.5) we get:

$$c(p,n)\left[\int_{t_1}^{t_2} q w^{p+1}\right]^{\frac{2}{p+1}} \le \int_{t_1}^{t_2} q|w'|^2 - \lambda \int_{t_1}^{t_2} q w^2$$
$$= \int_{t_1}^{t_2} q(u^p - v^p)w \le p\int_{t_1}^{t_2} q u^{p-1} w^2 \le p\left[\int_{t_1}^{t_2} q u^{p+1}\right]^{\frac{p-1}{p+1}} \times \left[\int_{t_1}^{t_2} q w^{p+1}\right]^{\frac{2}{p+1}}.$$

If $t_2 = +\infty$, and $\lambda < \frac{(n-1)^2}{4}$ the argument works unchanged, because $\int_0^\infty q|u'|^2 < +\infty \Rightarrow \int_0^\infty q|v'|^2 < +\infty$; in fact, from (3.2) and Lemma 3.2 it follows that $u' < 0$ and $u(t) \to_{t\to+\infty} 0$ and then $v(t) \to_{t\to+\infty} 0$ and, again by Lemma 3.2, $v' < 0$. Then, using (3.5), $\int_0^t q(v')^2 \le \int_0^t q(v')^2 - (qvv')(t) = -\int_0^t [qv']'v = \int_0^t \lambda q v^2 + q v^{p+1} \le \int_0^{t_1} \lambda q v^2 + q v^{p+1} + \int_0^\infty |\lambda| q u^2 + q u^{p+1} < +\infty$.

Let us consider the case $t_2 = +\infty$, and $\lambda = \frac{(n-1)^2}{4}$. The proof follows as above with the help of (3.7) and (3.8) once we show that v satisfies (3.6). To prove this, let $t_n > 0$. Then from (3.5) we have

$$\int_0^{t_n} q v^{p+1} = -q(t_n)v(t_n)v'(t_n) + \int_0^{t_n}\left[(v')^2 - \frac{(n-1)^2}{4}v^2\right]q$$
$$= -q(t_n)v(t_n)\left[v'(t_n) + \frac{n-1}{2}\tanh\frac{t_n}{2}\, v(t_n)\right]$$
$$+ \int_0^{t_n}\left[\left(v' + \frac{n-1}{2}\tanh\frac{t}{2}\, v\right)^2 + \frac{(n-1)v^2}{\left(2\cosh\frac{t}{2}\right)^2}\right]q.$$

Since $v(t) \le u(t), t \ge t_1$, and u satisfies (3.6) the left hand side is finite as $t_n \to \infty$. It remains to show that there exist a sequence $t_n \to \infty$ such that the first term on the right hand side is greater than or equal to zero in the limit as $n \to \infty$. Note that using (3.24) if $\liminf_{t\to\infty} e^{\frac{n-1}{2}t}(v'(t) + \frac{n-1}{2}\tanh\frac{t}{2}\, v(t)) > 0$ then $\liminf_{t\to\infty} e^{\frac{n-1}{2}t}(v'(t) + \frac{n-1}{2}v(t)) > 0$ and hence $e^{\frac{n-1}{2}t}u(t) \ge e^{\frac{n-1}{2}t}v(t) \to \infty$ which contradicts (3.25) as u satisfies (3.6). Therefore we can find a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} e^{\frac{n-1}{2}t_n}(v'(t_n) + \frac{n-1}{2}\tanh\frac{t_n}{2}\, v(t_n)) \le 0$ and hence $\lim_{n\to\infty} q(t_n)v(t_n)\left[v'(t_n) + \frac{n-1}{2}\tanh\frac{t_n}{2}\, v(t_n)\right] \le 0$.

Proof of (i). Assume, by contradiction, that there exist u_n, v_n with $u_n(0) \le M$ and $0 \le t_{1,n} < t_{2,n} \le R$ with $t_{2,n} - t_{1,n} \to_n 0$ such that $0 \le v_n \le u_n$ in $[t_{1,n}, t_{2,n}]$. We can also assume $u_n(0) \to_n \alpha$. Let u_α be the solution of (3.4) with $u_\alpha(0) = \alpha$. By continuous dependence, $\int_{t_{1,n}}^{t_{2,n}} q u_n^{p+1} \le \int_{t_{1,n}}^{t_{2,n}} q[1+u_\alpha^{p+1}] \to 0$, contraddicting (4.1).

Proof of (ii). Let first prove that there is t_0 such that $v(t_0) = u(t_0)$. Arguing by contraddition, we can assume $0 \le v \le u$ for every t.

Consider the case $\lambda < \frac{n-1^2}{4}$. Then $\int_0^\infty qv^2 \le \int_0^\infty qu^2 < +\infty$ and, as above, $v' \le 0$. In particular, $\liminf_{t\to+\infty} |qvu'|(t) = 0$ while, from (3.5), $0 < \int_0^t quv\left(u^{p-1} - v^{p-1}\right) = q\left[v'u - vu'\right](t) \le -(qvu')(t)$, a contradiction.

When $\lambda = \frac{n-1^2}{4}$ with the help of (3.8) we see that $0 = \int_0^\infty quv\left(u^{p-1} - v^{p-1}\right)$ which is possible only when $u = v$. As for the second statement, just observe that if $v(t) < u(t)$ for some $t > t_1$, then there is $t_2 > t_1$ with $v(t_2) = u(t_2)$ and either $v < u$ for $t > t_2$, or there is $t_3 > t_2$ with $v(t_3) = u(t_3)$ and $v < u$ in (t_2, t_3). In both cases, by (4.1), t_1 cannot be too large. □

An auxiliary energy

Let us introduce, following Kwong (see [18]),

$$\hat{v} := (\sinh^\alpha t)v, \quad \alpha := \frac{2(n-1)}{p+3}, \quad \beta := \alpha(p-1). \tag{4.2}$$

It results $(\sinh^\beta t)\hat{v}'' + \frac{1}{2}[\sinh^\beta t]'\hat{v}' + G(t)\hat{v} + \hat{v}^p = 0$ where

$$\begin{aligned} &G := A\sinh^\beta t + B\sinh^{\beta-2} t = (\sinh^{\beta-2} t)[A\sinh^2 t + B] \\ &A := \lambda - \frac{\alpha^2(p+1)}{2} = \lambda - \frac{2(n-1)^2(p+1)}{(p+3)^2}, \quad B := \frac{\alpha}{2}[2 - \alpha(p+1)]. \end{aligned} \tag{4.3}$$

Notice that $A = \lambda - \frac{n(n-2)}{4}$ if $n \ge 3,\ p = 2^* - 1$ and

$$\begin{aligned} &\alpha < \frac{n-1}{2} \quad \text{and} \quad 2(n-1) > \alpha(p+1) > n-1 \quad \forall\, p > 1 \\ &B > 0 \quad \text{if} \quad n = 2 \quad \text{and} \quad B < 0 \quad \text{if} \quad n \ge 3 \\ &\beta < 2 \quad \forall\, p < 2^* - 1 \quad \text{while} \quad \beta = 2 \quad \text{if} \quad n \ge 3 \quad \text{and} \quad p + 1 = 2^*. \end{aligned} \tag{4.4}$$

A crucial role will be played by the auxiliary energy

$$\begin{aligned} \mathcal{E}_{\hat{v}}(t) :&= \frac{1}{2}(\sinh^\beta t)\hat{v}'^2 + \frac{|\hat{v}|^{p+1}}{p+1} + \frac{1}{2}G\hat{v}^2 \\ &= \frac{\sinh^{\alpha(p+1)} t}{2} v^2\left[\left(\frac{\alpha}{\tanh t} + \frac{v'}{v}\right)^2 + \frac{2v^{p-1}}{p+1} + A + \frac{B}{\sinh^2 t}\right]. \end{aligned} \tag{4.5}$$

Notice that

$$\frac{d}{dt}\mathcal{E}_{\hat{v}}(t) = \frac{1}{2}G'\hat{v}^2 = \frac{1}{2}[A\beta \sinh^2 t + B(\beta - 2)]\hat{v}^2 \sinh^{\beta-3} t \cosh t. \tag{4.6}$$

We list below monotonicity properties of G.

Lemma 4.2.

(i) *Let $n = 2$. Then $\lambda \le \frac{2(p+1)}{(p+3)^2} \Rightarrow G'(t) < 0 \quad \forall t > 0$.*
(ii) *Let $n \ge 3$. If $p < 2^* - 1$, then*

$$\lambda < \frac{2(n-1)^2(p+1)}{(p+3)^2} \Rightarrow \exists \tilde{t} > 0 : G'(t)(\tilde{t} - t) > 0 \quad \forall t > 0, t \ne \tilde{t}$$

$$\lambda \ge \frac{2(n-1)^2(p+1)}{(p+3)^2} \Rightarrow G'(t) > 0 \quad \forall t.$$

If $p = 2^* - 1$, then

$$\lambda \ne \frac{n(n-2)}{4} \Rightarrow G'\left[\lambda - \frac{n(n-2)}{4}\right] > 0; \quad \lambda = \frac{n(n-2)}{4} \Rightarrow G' \equiv 0.$$

Proof.

(i) In fact, $B(\beta - 2) < 0 \quad \forall p > 1$ and $A \le 0$ if $\lambda \le \frac{2(p+1)}{(p+3)^2}$.
(ii) If $1 < p < \frac{n+2}{n-2}$ then $B(\beta - 2) > 0$ and $A < 0 \Leftrightarrow \lambda < \frac{2(n-1)^2(p+1)}{(p+3)^2}$ while if $p = \frac{n+2}{n-2}$ then $\beta = 2$ and hence G' has the sign of $A = \lambda - \frac{n(n-2)}{4}$. □

We now derive asymptotic properties of $\mathcal{E}_{\hat{v}}$.

Lemma 4.3.

(i) *If $n = 2$, $\mathcal{E}_{\hat{v}}(t) = \frac{\sinh^{\alpha(p+1)-2} t}{2} v^2[\alpha^2 + B] + \circ(1)$ at $t = 0$. If $n \ge 3$, $\mathcal{E}_{\hat{v}}(t) = \circ(1)$ at $t = 0$.*
(ii) *Let $n \ge 2$. Then $\lambda < \frac{2(n-1)^2(p+1)}{(p+3)^2} \Rightarrow \mathcal{E}_{\hat{v}}(t) \to_{t\to+\infty} 0$*

Proof.

(i) Since $\alpha(p+1) > 1$ we have $\mathcal{E}_{\hat{v}}(t) = \frac{\sinh^{\alpha(p+1)-2} t}{2} v^2[\alpha^2 + B] + \circ(1)$ for t close to zero. Now just recall that $\alpha(p+1) > 2$ iff $n \ge 3$.
(ii) Since $\lambda < \frac{(n-1)^2}{4}$, we derive from Lemma 3.4 that, as t goes to infinity,

$$\mathcal{E}_{\hat{v}}(t) = \frac{\sinh^{\alpha(p+1)} t}{2} v^2 \left[\left(\alpha - \frac{n-1+\sqrt{(n-1)^2 - 4\lambda}}{2}\right)^2 + A + \circ(1)\right]. \tag{4.7}$$

The result follows because, by Lemma 3.4, $v^2 \leq ce^{-(\epsilon+n-1+\sqrt{(n-1)^2-4\lambda})t}$ while $\alpha(p+1) < n-1+\sqrt{(n-1)^2-4\lambda} \Leftrightarrow \lambda < \frac{2(n-1)^2(p+1)}{(p+3)^2}$. □

We first prove a uniqueness result for a Dirichlet problems in $[0, T]$.

Proposition 4.4. *Let $\lambda \leq \frac{(n-1)^2}{4}$ and $p+1 \leq 2^*$ if $n \geq 3$. If $n = 2$ assume $\lambda \leq \frac{2(p+1)}{(p+3)^2}$. Then the Dirichlet problem*

$$\begin{aligned} &v'' + (n-1)\coth t\, v' + \lambda\, v + v^p = 0 \\ &v'(0) = 0, \quad v(T) = 0, \quad v(t) > 0 \quad \forall t \in [0, T) \end{aligned} \tag{4.8}$$

has at most one solution and no solution if $n \geq 3$, $p+1 = 2^*$, $\lambda = \frac{n(n-2)}{4}$.

Proof. We prove this proposition in two steps.

Step 1. Let u be a solution of (4.8) and let v be a solution of (3.4) with initial value $0 < v(0) \leq u(0)$. Then

$$\exists t_1 \in (0, T) : v(t_1) = u(t_1) \quad \text{and} \quad v(t) > 0 \quad \forall\, t \in [0, t_1].$$

Step 2. If, in addition, $v(T) = 0$, then $\exists t_2 \in (t_1, T) : u(t_2) = v(t_2)$.

Conclusion. Assume, by contraddiction, that there are two solutions u and v of (4.8) with, say, $v(0) < u(0)$. Let us denote by v_α the solution of (3.4) such that $v_\alpha(0) = \alpha \leq v(0)$ and set

$$\begin{aligned} A := \{\alpha \in (0, v(0)) :\ \exists t_{\alpha,1} < t_{\alpha,2} \quad \text{in} \quad (0, T) \quad \text{such that} \quad v_\alpha(t_{\alpha,i}) = u(t_{\alpha,i}) \\ \text{and} \quad v_\alpha(t) < u(t)\ \forall\, t \in [0, t_{\alpha,1}), \quad v_\alpha(t) > u(t)\ \forall\, t \in (t_{\alpha,1}, t_{\alpha,2})\}. \end{aligned}$$

In view of Step 2 and continuous dependence of v_α on α, A is a nonempty open set. In particular, $\inf A \notin A$ and hence $\inf A = 0$. In fact, $\inf A > 0 \Rightarrow \inf A \in A$, by continuous dependence and because, by Lemma (4.1)-(i), $t_{\alpha,2} - t_{\alpha,1} \geq \delta(u) > 0$ for every $\alpha \in A$. Now, if $\alpha_n \to 0$, by continuous dependence, $t_{\alpha_n,i} \to_n T$ for $i = 1, 2$, contraddicting Lemma 4.1-(i).

Proof of Step 1. It relies on a standard Sturm comparison argument: arguing by contraddiction, we find $t_0 \leq T$ with $v(t_0) = 0$ and $0 < v(t) < u(t)\ \forall t \in [0, t_0)$. From the equation for u and v, we get:

$$[q(v'u - vu')]' = quv(u^{p-1} - v^{p-1}) \tag{4.9}$$

and hence $0 \geq (qv'u)(t_0) = \int_0^{t_0} quv(u^{p-1} - v^{p-1})(\tau)d\tau > 0$.

Proof of Step 2. Let us assume, by contradiction, that $v(t) > u(t)$ in (t_1, T). Set $\gamma(t) := \frac{v(t)}{u(t)}$, $\gamma(T) := \lim_{t\to T} \frac{v(t)}{u(t)} = \frac{v'(T)}{u'(T)}$. We first claim that

$$\gamma(t) \qquad \text{is strictly increasing in} \quad [0, T] \tag{4.10}$$

and in fact $v'u - vu' > 0$ in $(0, T)$. To see this, notice first that, since $[u(t) - v(t)](t_1 - t) > 0 \,\forall\, t \neq t_1$, we have $[q(v'u - vu')]'(t_1 - t) > 0 \quad \forall\, t \neq t_1$ thanks to (4.9). If $(v'u - vu')(\hat{t}) = 0$ for some $\hat{t}$, necessarily bigger than t_1, then $(v'u - vu')(T) < 0$ while $u(T) = v(T) = 0$. This prove the claim.

Now, let, as above, $\hat{v} := (\sinh t)^\alpha v$, $\hat{u} := (\sinh t)^\alpha u$. Notice that $\mathcal{E}_{\hat{u}}(T) = \frac{1}{2}(\sinh T)^{2\alpha+\beta} u'^2(T)$ because $u(T) = 0$ and similarly for $\mathcal{E}_{\hat{v}}(T)$. Now, for any $t \in (0, T)$ and $\epsilon > 0$ it results

$$\begin{aligned}\frac{1}{2}(\sinh T)^{2\alpha+\beta}[v'^2(T) - \gamma^2(t)u'^2(T)] \;&=\; \mathcal{E}_{\hat{v}}(T) - \gamma^2(t)\mathcal{E}_{\hat{u}}(T)\\ = \mathcal{E}_{\hat{v}}(\epsilon) - \gamma^2(t)\mathcal{E}_{\hat{u}}(\epsilon) + \frac{1}{2}&\int_\epsilon^T G'(\tau)[\hat{v}^2 - \gamma^2(t)\hat{u}^2](\tau)d\tau.\end{aligned} \tag{4.11}$$

We first deal with the case $n \geq 3$ where, by Lemma 4.3-(i), $\mathcal{E}_{\hat{v}}(\epsilon)$, $\mathcal{E}_{\hat{u}}(\epsilon)$ go to zero as ϵ goes to zero. In addition, by Lemma 4.2-(ii) we know that if $p + 1 < 2^*$ there is $\tilde{t} \in (0, +\infty]$ such that $G' > 0$ in $(0, \tilde{t})$ and $G' < 0$ if $t > \tilde{t}$ while, if $p + 1 = 2^*$, then $G'(t) \neq 0$ for all $t > 0$ provided $\lambda \neq \frac{n(n-2)}{4}$.

Assume first $T \leq \tilde{t}$, and hence $\hat{v}^2(t) < \gamma^2(T)\hat{u}^2(t)$ for every $t \in [0, T]$ and $G' \neq 0$ in $[0, T]$. Taking $t = T$ in (4.11) and then sending ϵ to zero, we get

$$0 = \frac{1}{2}(\sinh T)^{2\alpha+\beta}[v'^2(T) - \gamma^2(T)u'^2(T)] = \frac{1}{2}\int_0^T G'[\hat{v}^2 - \gamma^2(T)\hat{u}^2] \neq 0.$$

If $\tilde{t} < T$, from $\gamma(\tilde{t}) < \gamma(T)$, we get, as above, but choosing $t = \tilde{t}$,

$$\begin{aligned}0 &< (\sinh T)^{2\alpha+\beta}[v'^2(T) - \gamma^2(\tilde{t})u'^2(T)]\\ &= \int_0^{\tilde{t}} G'[\hat{v}^2 - \gamma^2(\tilde{t})\hat{u}^2] + \int_{\tilde{t}}^T G'[\hat{v}^2 - \gamma^2(\tilde{t})\hat{u}^2] < 0\end{aligned}$$

Now, let $n = 2$ and $\lambda \leq \frac{2(p+1)}{(p+3)^2}$ so that $G'(t) < 0 \quad \forall\, t$ by Lemma 4.2-(i). Taking $t = \epsilon$ in (4.11) and using Lemma 4.3-(i), we see that

$$(\sinh T)^{2\alpha+\beta}[v'^2(T) - \gamma^2(t)u'^2(T)] = \circ(1) + \int_0^T G'[v^2(\tau) - \gamma^2(t)u^2(\tau)]\chi_{[t,T]}d\tau.$$

Since the integrand is non positive, sending t to zero we get (since $\gamma' > 0$),

$$0 < (\sinh T)^{2\alpha+\beta}[v'^2(T) - \gamma^2(0)u'^2(T)] \leq \int_0^T G'[v^2(\tau) - \gamma^2(0)u^2(\tau)]d\tau < 0.$$

Finally, in case $\lambda = \frac{n(n-2)}{4}$, $p+1 = 2^*$, we know that $G' \equiv 0$ and hence $\mathcal{E}_{\hat{v}} \equiv 0$. Since $A = 0$, this implies $\left[\left(\frac{\alpha}{\tanh t} + \frac{v'}{v}\right)^2 + \frac{2v^{p-1}}{p+1} + \frac{B}{\sinh^2 t}\right] = 0$ if $v(t) \neq 0$. Hence v cannot vanish. □

Remark 4.5. The last argument above also shows that (Eq_λ) has no postive H^1 solution if $\lambda = \frac{n(n-2)}{4}$ and $p+1 = 2^*$. In fact it also implies that if v is a solution of (3.4), with $v(0) > 0$, then $\frac{v'}{v} \to_{t\to+\infty} -\frac{n-2}{2}$. Hence v cannot satisfy (3.2), otherwise, in view of Lemma 3.4, it should result $\lim_{t\to+\infty} \frac{v'}{v} = -\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2} = -\frac{n}{2}$.

Corollary 4.6. *Let $n \geq 3$, $p+1 \leq 2^*$, $\lambda \leq \frac{(n-1)^2}{4}$ or $n = 2$ and $\lambda \leq \frac{2(p+1)}{(p+3)^2}$. Let u, v be solutions of* (3.4), *$u > 0$ and $0 < v(0) < u(0)$. Then $v(t) > 0 \quad \forall t$.*

In addition, if u satisfies (3.2) *or* (3.6) *depending on $\lambda < \frac{(n-1)^2}{4}$ or $\lambda = \frac{(n-1)^2}{4}$ respectively, then $u - v$ has exactly one zero.*

Proof. We argue by contraddiction, assuming v has a first zero t_0. Hence, denoted by v_α the solution of (3.4) such that $v_\alpha(0) = \alpha$, the set

$$A := \{\alpha \in (0, u(0)) : v_\alpha \quad \text{has a zero}\}$$

is, by continuous dependence, an open set containing $v(0)$. Let $(\underline{\alpha}, \overline{\alpha})$ be the largest open interval in A containing $v(0)$. If we denote by t_α the first zero of v_α, then $\alpha \to t_\alpha$ is continuous in $(\underline{\alpha}, \overline{\alpha})$. We claim that $t_\alpha \to +\infty$ as α goes to $\underline{\alpha}$. Assume, by contradiction, that $\alpha_n \to \underline{\alpha}^+$ and $t_{\alpha_n} \to \underline{t} < +\infty$. Since $v_{\alpha_n} \to v_{\underline{\alpha}}$ uniformly on bounded intervals, it results $v_{\underline{\alpha}}(\underline{t}) = 0$ and hence, by the definition of $\underline{\alpha}$, $\underline{\alpha} = 0$. Now, we notice that, exactly as in the proof of Step 1 in Proposition 4.4, we have that for every $\alpha \in A$ there is $\tau_\alpha \in (0, t_\alpha)$ such that $v_\alpha(\tau_\alpha) = u(\tau_\alpha)$. For a subsequence, $\tau_{\alpha_n} \to \tau_{\underline{\alpha}} \leq \underline{t}$ and hence $u(\tau_{\underline{\alpha}}) = \lim_n u(\tau_{\alpha_n}) = \lim_n v_{\alpha_n}(\tau_{\alpha_n}) = 0$, a contraddiction. Similarly, if $\alpha_n \to \overline{\alpha}^-$ then $t_{\alpha_n} \to +\infty$. This implies that, for large T, (4.8) has at least two solutions, contradicting Proposition 4.4.

Thus $v(t) > 0$ for every t. In particular, if $u \in H^1(\mathcal{B})$, it follows from Proposition 4.1-(ii) that there is t_v, a first zero of $u - v$ and that t_v is the unique zero if it is large enough, say $t_v \geq \underline{t}_u$. We first notice that t_v is large if $v(0)$ is small. In fact, since v_α goes to zero uniformly on bounded sets as α goes to zero, clearly t_v has to go to infinity as $v(0)$ goes to zero. Thus, for α small, $u - v_\alpha$ has exactly one zero. To prove that $u - v$ has exactly one zero if $v(0) < u(0)$, we argue by contraddiction, assuming

$$\overline{\alpha} := \sup\{\alpha \in (0, u(0)) : \ \forall \beta \in (0, \alpha)\ \exists! t_\beta > 0 \quad \text{such that} \quad u(t_\beta) = v_\beta(t_\beta)\} < u(0).$$

Then, there are $\alpha_n > \overline{\alpha}$ converging to $\overline{\alpha}$ such that $u - v_{\alpha_n}$ has at least two zeros, say $0 < t_{n,1} < t_{n,2}$, with $v_{\alpha_n}(t) > u(t)$ in $(t_{1,n}, t_{2,n})$. Since, by continuous dependence, $v_{\overline{\alpha}} - u$ has exactly one zero, say $\overline{t}$, then $t_{n,1} \to_n \overline{t}$ and $t_{n,2} \to_n +\infty$. But this contradicts Proposition 4.1-(ii). □

Proof of Theorem 1.3. By contradiction, assume u and v are two distinct entire positive solutions of (Eq_λ) with, say, $v(0) < u(0)$. By Corollary 4.6 $u - v$ has exactly one zero, say t_v and $v(t) > u(t)$ for $t \geq t_v$. As in Step 2, Proposition 4.4, $\gamma(t) := \frac{v(t)}{u(t)}$ is strictly increasing; here, this follows from

$$\lim_{t\to+\infty} q(v'u - vu')(t) = 0. \tag{4.12}$$

When $\lambda < \frac{(n-1)^2}{4}$, (4.12) follows from Lemma 3.4. When $\lambda = \frac{(n-1)^2}{4}$, we have from (3.25) that $|\gamma(t)| \leq M < \infty$ for all t. Thus

$$\liminf_{t\to\infty} u^{-2}(uv' - vu')(t) = \liminf_{t\to\infty} \gamma'(t) = 0.$$

Then (3.25) implies $\liminf_{t\to\infty} q(uv' - vu')(t) = 0$ and hence (4.12) follows as (4.9) shows that $q(uv' - vu')(t)$ is a decreasing function for $t > t_v$.

Now, write, as above, $\hat{v} := (\sinh t)^\alpha v$, $\hat{u} := (\sinh t)^\alpha u$ and let $\mathcal{E}_{\hat{v}}(t), \mathcal{E}_{\hat{u}}(t)$ be the related auxiliary energies. For any $T > 0$, $t \in (0, T)$ and $\epsilon > 0$ we have

$$\mathcal{E}_{\hat{v}}(T)-\gamma^2(t)\mathcal{E}_{\hat{u}}(T)=\mathcal{E}_{\hat{v}}(\epsilon)-\gamma^2(t)\mathcal{E}_{\hat{u}}(\epsilon)+\frac{1}{2}\int_\epsilon^T G'(\tau)[\hat{v}^2-\gamma^2(t)\hat{u}^2](\tau)d\tau \tag{4.13}$$

We first consider the case $\lambda < 2\frac{(n-1)^2(p+1)}{(p+3)^2}$ (in case $n \geq 3$, $p + 1 = 2^*$ this means $\lambda < \frac{n(n-2)}{4}$). Here the arguments are the same as in the proof of Step 2 in Proposition 4.4. If $n \geq 3$, by Lemma 4.3 $\mathcal{E}_{\hat{v}}(\epsilon), \mathcal{E}_{\hat{u}}(\epsilon)$ go to zero as ϵ goes to zero and as T goes to infinity and, by Lemma 4.2-(ii), $G'(t) < 0$ for all $t > 0$ if $p+1 = 2^*$ while, if $p+1 < 2^*$, there is $\tilde{t} \in (0, +\infty]$ such that $G' > 0$ in $(0, \tilde{t})$ and $G' < 0$ if $t > \tilde{t}$. Thus, taking in (4.13) $t = 0$ if $p + 1 = 2^*$ and $t = \tilde{t}$ if $p + 1 < 2^*$ and then sending ϵ to zero and T to infinity, we find

$$0 = \frac{1}{2}\int_0^\infty G'(\tau)[\hat{v}^2 - \gamma^2(t)\hat{u}^2](\tau)d\tau < 0$$

a contradiction. If $n = 2$, $G'(t) < 0 \quad \forall t$ by Lemma 4.2-(i) and, by Lemma 4.3-(ii), $\mathcal{E}_{\hat{v}}(\epsilon), \mathcal{E}_{\hat{u}}(\epsilon)$ go to zero and as T goes to infinity. Taking $t = \epsilon$ in (4.13) and using Lemma 4.3-(i), we see that

$$\mathcal{E}_{\hat{v}}(T) - \gamma^2(t)\mathcal{E}_{\hat{u}}(T) = \circ(1) + \frac{1}{2}\int_0^T G'[v^2(\tau) - \gamma^2(t)u^2(\tau)]\chi_{[t,T]}d\tau.$$

Since the integrand is non positive, sending t to zero and T to infinity we get

$$0 \leq \int_0^\infty G'[v^2(\tau) - \gamma^2(0)u^2(\tau)]d\tau < 0.$$

Let us now consider the case $n = 2$ and $\lambda = 2\frac{(n-1)^2(p+1)}{(p+3)^2}$. In this case $A = 0$ and $G' < 0$. Taking $t = \epsilon$ in (4.13), it rewrites, making use of (4.7), as

$$\frac{\sinh^{\alpha(p+1)} T}{2}\left\{v^2(T)\left[\left(\frac{p+1}{p+3}\right)^2 + \circ(1,T)\right] - \gamma^2(t)u^2(T)\left[\left(\frac{p+1}{p+3}\right)^2 + \tilde{\circ}(1,T)\right]\right\}$$

$$= \circ(1,t) + \frac{1}{2}\int_0^T G'[v^2(\tau) - \gamma^2(t)u^2(\tau)]\chi_{[t,T]}d\tau,\ \circ(1,T) \to_{T\to\infty} 0,\ \circ(1,t) \to_{t\to 0} 0.$$

Since by Corollary 4.6 $v(T) \geq u(T)$ for T large, taking the lim sup as t goes to zero, we get the contradiction

$$\frac{\sinh^{\alpha(p+1)} T}{2} v^2(T)\left[(1-\gamma^2(0))\left(\frac{p+1}{p+3}\right)^2 + \circ(1)\right] \leq \frac{1}{2}\int_0^T G'[v^2(\tau) - \gamma^2(0)u^2(\tau)]d\tau.$$

Finally, let $\lambda \in \left[\frac{2(n-1)^2(p+1)}{(p+3)^2}, \frac{(n-1)^2}{4}\right]$, $n \geq 3$. Thus $G' > 0$, and $\mathcal{E}_{\hat{v}}(0) = 0$. So

$$\mathcal{E}_{\hat{v}}(T) - \gamma^2(T)\mathcal{E}_{\hat{u}}(T) = \frac{1}{2}\int_0^T G'(\tau)[\hat{v}^2(\tau) - \gamma^2(T)\hat{u}^2(\tau)]d\tau \quad \forall\, T > 0 \tag{4.14}$$

where the right hand side is negative and decreasing in T; but this is impossible because

$$\mathcal{E}_{\hat{v}}(T) - \gamma^2(T)\mathcal{E}_{\hat{u}}(T) \to_{T\to+\infty} 0. \tag{4.15}$$

□

Proof of (4.15). Notice first that (4.5) and $\left|\frac{v'}{v}\right|$ bounded give

$$\begin{aligned}\mathcal{E}_{\hat{v}}(T) &= \frac{\sinh^{\alpha(p+1)} T}{2} v^2(T)\left[\alpha^2 + 2\alpha\frac{v'}{v} + \left(\frac{v'}{v}\right)^2 + O(e^{-2T}) + \frac{2v^{p-1}(T)}{p+1} + A\right] \\ &= \frac{\sinh^{\alpha(p+1)} T}{2} v^2(T)\left[\alpha^2 + A + 2\alpha\frac{v'}{v} + \left(\frac{v'}{v}\right)^2\right] + \circ(1) \quad \text{as } T \to +\infty.\end{aligned}$$

because $\alpha(p+1) - (n-1) < 2$ and $v^{p+1}(T) \leq c_\varepsilon e^{-[(\frac{(p+1)(n-1)}{2}) - \epsilon]T}$ and $\alpha < \frac{n-1}{2}$. Similarly for $\mathcal{E}_{\hat{u}}(T)$. From this, Lemma 3.4 and Lemma 3.6 we derive

$$\begin{aligned}\mathcal{E}_{\hat{v}}(T) - \gamma^2(T)\mathcal{E}_{\hat{u}}(T) &= \frac{\sinh^{\alpha(p+1)} T}{2}\frac{v(T)}{u(T)}(uv' - vu')\left[2\alpha + \left(\frac{v'(T)}{v(T)} + \frac{u'(T)}{u(T)}\right)\right] \\ &= O\left(e^{(\frac{\epsilon}{2} + \alpha(p+1))T}\right)(uv' - vu')(T)\left[2\alpha - (n - 1 + \sqrt{(n-1)^2 - 4\lambda}) + \circ(1)\right]\end{aligned}$$

Now, (4.15) follows from $0 < u^2(T)\gamma'(T) = (uv' - vu')(T) \leq c_\varepsilon e^{-[\frac{(n-1)(p+1)}{2} - \varepsilon]t}$ and $q(uv' - vu')(T) = \int_T^\infty quv(v^{p-1} - u^{p-1}) \leq \int_T^\infty e^{-[\frac{(n-1)(p-1)}{2} - \varepsilon]T}$ for T large. □

5. Existence and nonexistence

We start proving the non existence results stated in the Introduction.

Proof of Theorems 1.1 *and* 1.6. First, let $\lambda > \frac{(n-1)^2}{4}$ and assume, by contradiction, that (Eq_λ) has a positive solution. Recall that if $\Omega \subset \mathbb{H}$ is a smooth bounded domain, then $\lambda_1(\Omega)$, the first Dirichlet eigenvalue of $-\Delta_{\mathbb{H}}$ in Ω, has the property (see [4])

$$\lambda_1(\Omega) = \sup\{\lambda : \Delta_{\mathbb{H}}\phi + \lambda\phi \leq 0 \text{ for some } \phi > 0 \text{ in } \Omega\}.$$

Therefore $\lambda_1(\Omega) \geq \lambda$ for any smooth and bounded Ω and in particular for any geodesic ball. This is a contradiction because the first eigenvalue of the geodesic ball converges to $\frac{(n-1)^2}{4}$ as the radius goes to infinity (see [10, Chapter II, Section 5, Theorem 5]). Non existence of $H^1(\mathbb{H})$ positive solutions in case $\lambda = \frac{(n-1)^2}{4}$ has been discussed in Remark 3.7.

Next, let $n \geq 3$, $p = 2^* - 1$ and $\lambda < \frac{n(n-2)}{4}$ and assume, by contradiction, that (Eq_λ) has a positive entire solution. From Theorem 2.1 we know that the solution has hyperbolic symmetry and hence (3.4) has a solution. Let $\hat{u}$ and $\mathcal{E}_{\hat{u}}$ be as in Section 4. Then it follows from Lemma 4.2 and Lemma 4.3 that $\frac{d}{dt}\hat{E}_{\hat{u}}(t) < 0$ and $\hat{E}_{\hat{u}}(0) = 0 = \hat{E}_{\hat{u}}(\infty)$ which is a contradiction.

Finally, when $n \geq 3$, $p = 2^* - 1$ and $\lambda = \frac{n(n-2)}{4}$, non existence of positive entire solutions was derived in Remark 4.5 □

Proof of Theorem 1.7. Assume by contradiction that $u > 0$ is a solution of

$$u'' + \frac{2}{\tanh t}u' + \lambda u + u^5 = 0, \quad u > 0, \ u'(0) = 0 \tag{5.1}$$

satisfying the estimates in Lemma 3.4 and 3.6 when $\lambda < 1$ and $\lambda = 1$ respectively. In view of Theorems 1.1 and 1.6 we can assume that $\frac{3}{4} < \lambda \leq 1$.

We will arrive at a contradiction by means of a Pohozaev type identity (*cf.* [26] and [27]).

Let $f : [0, \infty) \to [0, \infty)$ be smooth, with $f(0) = 0$. Multipling (5.1) by $(\sinh^2 t) f u'$ and integrating on $[0, T]$, $T > 0$, we get

$$\begin{aligned} &\sinh^2 T \ f(T)\left[\frac{(u'(T))^2}{2} + \frac{\lambda u^2(T)}{2} + \frac{u^6(T)}{6}\right] \\ &= \int_0^T \left[h(t)(u'(t))^2 + g(t)\left(\frac{\lambda u^2(t)}{2} + \frac{u^6(t)}{6}\right)\right] \end{aligned} \tag{5.2}$$

where g and h are given by

$$g(t) = f \sinh 2t + f' \sinh^2 t, \quad \text{and} \quad h(t) = \frac{1}{2}\left[f' \sinh^2 t - f \ \sinh 2t\right]. \tag{5.3}$$

Multiplying (5.1) by hu and integrating by parts, we get

$$\begin{aligned}&\int_0^T \left[\frac{h''}{2} - \frac{h'}{\tanh t} + \frac{h}{\sinh^2 t} + \lambda h\right] u^2 + \int_0^T hu^6 - \int_0^T h(u')^2 \\ &= \frac{h'(T)u^2(T)}{2} - h(T)u(T)u'(T) - \frac{h(T)u^2(T)}{\tanh T}.\end{aligned} \tag{5.4}$$

By substituting (5.4) in (5.2) we get

$$\begin{aligned}&\sinh^2 T \;\; f(T)\left[\frac{(u'(T))^2}{2} + \frac{\lambda u^2(T)}{2} + \frac{u^6(T)}{6}\right] \\ &+ \left[\frac{h'(T)}{2} - \frac{h(T)}{\tanh T} - \frac{h(T)u'(T)}{u(T)}\right] u^2(T) = \int_0^T \left[A(t)u^2 + B(t)u^6\right]\end{aligned} \tag{5.5}$$

where A and B are as follows

$$A(t) = \frac{h''}{2} - \frac{h'}{\tanh t} + \frac{h}{\sinh^2 t} + \lambda h + \frac{\lambda}{2}(f \sinh 2t + f' \sinh^2 t) \tag{5.6}$$

$$B(t) = \frac{1}{6}\left(f \sinh 2t + f' \sinh^2 t + 6h\right). \tag{5.7}$$

Let us consider the two cases:

Case 1. $\lambda = 1$. In this case we choose $f(t) = t$. Then

$$\begin{aligned} h(t) &= \frac{1}{2}\left(\sinh^2 t - t \sinh 2t\right), \quad A(t) \equiv 0 \\ B(t) &= -\frac{2\sinh t}{3}\left[t \cosh t - \sinh t\right] < 0, \quad \forall\, t > 0.\end{aligned}$$

Now let us consider the left hand side of (5.5). We know from (3.25) that $u'(T) = -u(T) + \circ(1)$. In fact from the proof of (3.25) we observe that

$$u'(T) = -u(T) + O(e^{-\alpha t}), \;\; \alpha > 1 \;\text{ as } T \to \infty.$$

Using this fact and (3.25), the left hand side of (5.5) reduces to

$$\left[T \sinh^2 T \;-\; \frac{T}{2}\cosh 2T\right] u^2(T) + \circ(1) = \circ(1) \;\text{ as } T \to \infty.$$

Thus taking $T \to \infty$ in (5.5) we get the left hand side to be zero while the right hand side is negative, a contradiction. This completes the proof when $\lambda = 1$.

Case 2. $\frac{3}{4} < \lambda < 1$. Choose $f(t) = \sinh \omega t$ where $\omega = 2\sqrt{1-\lambda}$.

Then we have $f'(t) = \omega \cosh \omega t$, $f''(t) = \omega^2 f$ and $f'''(t) = \omega^3 \cosh \omega t = \omega^2 f'$. Direct calculation gives

$$h(t) = \frac{1}{2}\left(f' \sinh^2 t - f \sinh 2t\right), \quad A(t) \equiv 0$$
$$B(t) = -\frac{2\sinh t}{3}\left[(\sinh \omega t)\cosh t - \omega(\cosh \omega t)\sinh t\right] < 0.$$

Now we will estimate the left hand side of (5.5). Using Lemma 3.4 we can estimate

$$\sinh^2 T\ f(T)\left[\frac{(u'(T))^2}{2} + \frac{\lambda u^2(T)}{2} + \frac{u^6(T)}{6}\right]$$
$$= \frac{1}{8}\left[1 + \sqrt{1-\lambda} + \circ(1)\right] u^2(T) e^{2(1+\sqrt{1-\lambda})T} + \circ(1).$$

and

$$\left[\frac{h'(T)}{2} - \frac{h(T)}{\tanh T} - \frac{h(T)u'(T)}{u(T)}\right] u^2(T)$$
$$= \frac{1}{8}\left[1 - 2\lambda - \sqrt{1-\lambda} + \circ(1)\right] u^2(T) e^{2(1+\sqrt{1-\lambda})T} + \circ(1).$$

Thus the left hand side of (5.5) becomes $\frac{1}{4}[1-\lambda+\circ(1)]u^2(T)e^{2(1+\sqrt{1-\lambda})T} + \circ(1)$. Taking $T \to \infty$ in (5.5) we get the left hand side to be nonnegative while the right hand side is negative, contradiction. This completes the proof. $\square$

We end this section by proving the existence results.

We will prove in fact existence of ground state solutions. To this extent, given $\lambda \leq \frac{(n-1)^2}{4}$, let $\mathcal{H}_\lambda$ denote the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the norm $||u||_\lambda^2 = \int_{\mathbb{H}} \left[|\nabla_{\mathbb{H}} u|^2 - \lambda u^2\right] dV_{\mathbb{H}}$ (see Section 1). Denoted by $||\cdot||_\lambda$ and $\langle , \rangle_\lambda$ the norm and, respectively, the inner product in $\mathcal{H}_\lambda$, let us define

$$I(u) = \frac{||u||_\lambda^2}{\left(\int_{\mathbb{H}} |u|^{p+1} dV_{\mathbb{H}}\right)^{\frac{2}{p+1}}}, \quad u \in \mathcal{H}_\lambda, u \neq 0, \quad S_{\lambda,p} = \inf_{u \in \mathcal{H}_\lambda} I(u). \tag{5.8}$$

From (1.2) we have $S_{\lambda,p} > 0$ for $1 < p \leq 2^* - 1$ if $n \geq 3$ and $1 < p < \infty$ when $n = 2$. If $S_{\lambda,p}$ is achieved at some $u \in \mathcal{H}_\lambda$, then, up to a constant multiple, u will solve (Eq_λ).

We now restate and then prove Theorem 1.4 and Theorem 1.5.

Theorem 5.1. *Let $\lambda \leq \frac{(n-1)^2}{4}$. Let $p > 1$ if $n = 2$ and $1 < p < 2^* - 1$ if $n \geq 3$. Then $S_{\lambda,p}$ is achieved in $\mathcal{H}_\lambda$ and hence (Eq_λ) has a positive finite energy solution.*

Theorem 5.2. *Let $n \geq 4$, $p = 2^* - 1$ and $\frac{n(n-2)}{4} < \lambda \leq \frac{(n-1)^2}{4}$. Then $S_{\lambda,p}$ is achieved in $\mathcal{H}_\lambda$ and hence (Eq_λ) has a positive finite energy solution.*

Proofs of Theorem 5.1 *and Theorem*5.2. Let

$$N := \left\{ u \in \mathcal{H}_\lambda : u \neq 0,\ ||u||_\lambda^2 = \int_{\mathbb{H}} |u|^{p+1} dV_{\mathbb{H}} \right\}$$

be the Nehari manifold. We notice that $S_{\lambda,p} = \inf_{u\in N} I(u)$ and $I(u) = ||u||_\lambda^{\frac{2(p-1)}{p+1}} = \left[\int_{\mathbb{H}} |u|^{p+1} dV_{\mathbb{H}}\right]^{\frac{p-1}{p+1}}$, $\forall u \in N$.

In order to show that $S_{\lambda,p}$ is achieved, it is enough to exhibit a minimizing sequence $u_n \in N$ such that $u_n(x) \to u(x)$ for a.e. x for some $u \in N$ because then $\left[\int_{\mathbb{H}} |u|^{p+1} dV_{\mathbb{H}}\right]^{\frac{p-1}{p+1}} \leq S_{\lambda,p}$ by Fatou's lemma while the opposite inequality follows from the definition of $S_{\lambda,p}$ and the fact that $u \in N$.

So, let $u_n \in N$ be a minimizing sequence. Clearly, u_n is bounded in $\mathcal{H}_\lambda$. We will show that, up to dilations and translations, u_n converges weakly, and pointwise, to some $u \in N$. To begin with, given $z_0 \in \mathbb{R}^{n-1}$ and $R > 0$ let $B(z_0, R) := \{(r, z) \in \mathbb{H} : r^2 + |z - z_0|^2 < R\}$. Let $0 < \delta < (S_{\lambda,p})^{\frac{p+1}{p-1}}$. Using the concentration function $Q_n(R) = \sup_{z_0\in\mathbb{R}^{n-1}} \int_{B(z_0,R)} |u_n|^{p+1}\, dV_{\mathbb{H}}$, we can find $z_n \in \mathbb{R}^{n-1}$ and $R_n > 0$ such that

$$\delta = \int_{B(z_n,R_n)} |u_n|^{p+1} dV_{\mathbb{H}} = \sup_{z_0\in\mathbb{R}^{n-1}} \int_{B(z_0,R_n)} |u_n|^{p+1} dV_{\mathbb{H}}.$$

Define $v_n(r, z) = u_n((0, z_n) + R_n(r, z))$. Then $v_n \in N$ and is again minimizing, *i.e.* $||v_n||_\lambda^2 = \int_{\mathbb{H}} |v_n|^{p+1} dV_{\mathbb{H}} \to (S_{c,p})^{\frac{p+1}{p-1}}$, and, moreover,

$$\delta = \int_{B(0,1)} |v_n|^{p+1} dV_{\mathbb{H}} = \sup_{z_0\in\mathbb{R}^{n-1}} \int_{B(z_0,1)} |v_n|^{p+1} dV_{\mathbb{H}}. \tag{5.9}$$

By Ekeland principle we may assume v_n is a Palais-Smale sequence, *i.e.*

$$\langle v_n, u\rangle_\lambda = \int_{\mathbb{H}} |v_n|^{p-1} v_n u\, dV_{\mathbb{H}} + o(1) \tag{5.10}$$

uniformly for u in bounded sets of $\mathcal{H}_\lambda$. We can also assume $v_n \rightharpoonup v$ for some $v \in \mathcal{H}_\lambda$, pointwise and in $L^q_{\mathrm{loc}}(\mathbb{H}^n)$ for every $q < 2^*$. Choosing $u = v$ in (5.10), we get, passing to the limit, $||v||_\lambda^2 = \int_{\mathbb{H}} |v|^{p+1}\, dV_{\mathbb{H}}$.

Thus, in order to prove that $v \in N$, it remains to show that $v \neq 0$.

Assume, by contradiction, that $v = 0$. We claim that $v = 0$ implies that for every $z_0 \in \mathbb{R}^{n-1}$ and every $\phi \in C_c^\infty(B(z_0, 1))$, $0 \leq \phi \leq 1$ it results

$$\int_{\mathbb{H}} |\phi v_n|^{p+1} dV_{\mathbb{H}} \to 0 \quad \text{as} \quad n \to \infty \tag{5.11}$$

(note that ϕ need not vanish on $\{(0, z) : z \in \mathbb{R}^{n-1}\}$).

Let us prove this claim. Using $\phi^2 v_n$ as test function in (5.10) we get

$$\langle v_n, \phi^2 v_n\rangle_\lambda = \int_{\mathbb{H}} |v_n|^{p-1}(\phi v_n)^2 + o(1). \tag{5.12}$$

Now, if $\lambda < \frac{(n-1)^2}{4}$, then $\langle v_n, \phi^2 v_n\rangle_\lambda = \int_{\mathbb{H}} \left[\langle \nabla_{\mathbb{H}} v_n, \nabla_{\mathbb{H}}(\phi^2 v_n)\rangle - \lambda(\phi v_n)^2\right] dV_{\mathbb{H}} = \int_{\mathbb{H}} |\nabla_{\mathbb{H}}(\phi v_n)|^2 dV_{\mathbb{H}} - \int_{\mathbb{H}} r^2 |\nabla\phi|^2 v_n^2 dV_{\mathbb{H}} + \circ(1) = \int_{\mathbb{H}} |\nabla_{\mathbb{H}}(\phi v_n)|^2 dV_{\mathbb{H}} + \circ(1)$, *i.e.*

$$\langle v_n, \phi^2 v_n\rangle_\lambda = ||\phi v_n||_\lambda^2 + o(1). \tag{5.13}$$

The same holds true if $\lambda = \frac{(n-1)^2}{4}$, because, if $T v_n$ is as in Lemma 2.2, $\langle v_n, \phi^2 v_n\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{\mathbb{R}^{n+1}} \nabla T v_n \nabla(T(\phi^2 v_n))\, dx = \frac{1}{2\pi}\int_{\mathbb{R}^{n+1}} |\nabla(T(\phi v_n))|^2\, dx + o(1) = ||\phi v_n||_\lambda^2 + o(1)$.

Equations (5.13) and (5.12) give $||\phi v_n||_\lambda^2 = \int_{\mathbb{H}} |v_n|^{p-1}(\phi v_n)^2 dV_{\mathbb{H}} + o(1)$ which in turn implies, by Cauchy Schwartz and $S_{\lambda,p}^{\frac{1}{2}} ||\phi v_n||_{p+1} \le ||\phi v_n||_\lambda$,

$$S_{\lambda,p}\left(\int_{\mathbb{H}} |\phi v_n|^{p+1}\right)^{\frac{2}{p+1}} \le \left(\int_{\mathbb{H}} |\phi v_n|^{p+1}\right)^{\frac{2}{p+1}} \left(\int_{B(z_0,1)} |v_n|^{p+1}\right)^{\frac{p-1}{p+1}} + o(1)$$

and (5.11) follows, in view of the choice of δ.

Now, denoted by $B((1,0),R)$ the Euclidean ball of radius R and centre at $(1,0) \in \mathbb{R}\times\mathbb{R}^{n-1}$, equations (5.9) and (5.11) clearly imply that

$$\liminf_{n\to\infty} \int_{B((1,0),R)} |v_n|^{p+1} dV_{\mathbb{H}} > 0 \quad \forall R > 0. \tag{5.14}$$

This is impossible if $p+1 < 2^*$, because we assumed $v = 0$ and then Theorem 5.1 is proved.

We are left with the critical case $p+1 = 2^*$.

To rule out (5.14) also in the critical case we need to study the sequence inside $\mathbb{H}$. To do this, let $0 < R < 1$ and $\psi \in C_c^\infty(B((1,0),R))$ be such that $\psi = 1$ on $B\big((1,0), \frac{R}{2}\big)$ and $0 \le \psi \le 1$. Proceeding as before we will get

$$S_{\lambda,p}\left(\int_{\mathbb{H}} |\psi v_n|^{p+1}\right)^{\frac{2}{p+1}} \le \left(\int_{\mathbb{H}} |\psi v_n|^{p+1}\right)^{\frac{2}{p+1}} \left(\int_{B((1,0),R)} |v_n|^{p+1}\right)^{\frac{p-1}{p+1}} + o(1).$$

In view of (5.14), this implies

$$\liminf_{n\to\infty} \int_{B((1,0),R)} |v_n|^{p+1} dV_{\mathbb{H}} \ge (S_{\lambda,p})^{\frac{p+1}{p-1}} \quad \forall R > 0$$

and hence $\int_{\mathbb{H}\setminus B((1,0),R)} |v_n|^{p+1} \to 0$ for any $R > 0$. In turn, this implies, in view of (5.10),

$$\int_{\mathbb{H}} \left[|\nabla_{\mathbb{H}}(\psi v_n)|^2 - \lambda(\psi v_n)^2 \right] dV_{\mathbb{H}} = \int_{\mathbb{H}} |v_n|^{p-1}(\psi v_n)^2 dV_{\mathbb{H}} + o(1)$$
$$= \int_{\mathbb{H}} |\psi v_n|^{p+1} dV_{\mathbb{H}} + o(1)$$

where ψ is as above. Therefore $I(\psi v_n) \to S_{\lambda,p}$, ψv_n has compact support in $\mathbb{H}^n$ and hence $\psi v_n \in H^1(\mathbb{H}^n)$ (see Lemma 2.3) and $||\psi v_n||_2 \to 0$.

Let us observe that if $w \in C_c^\infty(\mathbb{R}^n_+)$ and we define $\tilde{w}(r,z) = r^{\frac{n-2}{2}} w(r,z)$, then

$$I(\tilde{w}) = \frac{\displaystyle\int_{\mathbb{R}^n_+} \left[|\nabla w|^2 - \left(\lambda - \frac{n(n-2)}{4}\right) \frac{w^2}{r^2} \right] drdz}{\left(\displaystyle\int_{\mathbb{R}^n_+} |w|^{2^*} \, drdz \right)^{\frac{2}{2^*}}}.$$

Since $\lambda > \frac{n(n-2)}{4}$, we obtain from [5]

$$S_{\lambda,p} = \inf_{w \in C_c^\infty(\mathbb{R}^n_+)} I(\tilde{w}) < S$$

where S is the best constant in the Euclidean Sobolev inequality in $\mathbb{R}^n$. But we have

$$S_{\lambda,p} = \lim_{n\to\infty} I(\psi v_n) = \lim_{n\to\infty} \frac{\displaystyle\int_{\mathbb{R}^n_+} \left[|\nabla w_n|^2 - \left(\lambda - \frac{n(n-2)}{4}\right) \frac{w_n^2}{r^2} \right] drdz}{\left(\displaystyle\int_{\mathbb{R}^n_+} |w_n|^{2^*} \, drdz \right)^{\frac{2}{2^*}}}$$

where $w_n(r,z) = r^{-\frac{n-2}{2}}(\psi v_n)(r,z)$. Since $||\psi v_n||_2 \to 0$, $||\psi v_n||_2^2 = \int_{\mathbb{R}^n_+} \frac{w_n^2}{r^2} \, drdz$ and $\int_{\mathbb{R}^n_+} |w_n|^{2^*} \, drdz \to (S_{\lambda,p})^{\frac{p+1}{p-1}}$, we get

$$S_{\lambda,p} = \lim_{n\to\infty} \left(\int_{\mathbb{R}^n_+} |\nabla w_n|^2 \, drdz \right) \left(\int_{\mathbb{R}^n_+} |w_n|^{2^*} \, drdz \right)^{-\frac{2}{2^*}} \geq S$$

which is a contradiction. This proves Theorem 5.2. $\square$

6. Some related problems

As mentioned in the Introduction, $(Eq)_\lambda$ appears while dealing with critical elliptic PDE with cylindrical symmetry. We start considering the Euler-Lagrange equation associated to Hardy-Sobolev-Maz'ya inequalities (B.5):

$$\Delta u(x) + \mu \frac{u(x)}{|y|^2} + \frac{|u(x)|^{p-2}u(x)}{|y|^t} = 0 \quad x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^h \tag{6.1}$$

where $p > 2$ and $p \le \frac{2N}{N-2}$, $N := k + h$, if $N \ge 3$ and $t := N - \frac{N-2}{2}p$, $\mu \in \mathbb{R}$. In case $k = 1$ we also require $u(0, z) \equiv 0$.

Existence of entire positive solutions is proved in [21] for $\mu < (\frac{k-2}{2})^2$ if $p \in (2, 2^*)$ and for $\mu \in (0, (\frac{k-2}{2})^2)$ and $N \ge 4$ if $p = 2^*$. An existence result in case $0 < \mu := (\frac{k-2}{2})^2$ and $p = 2^*$ was established in [28]. These results extend (in case $k \ne 2$) a result in [6], where $k > 1$ and $\mu = 0$.

Equation (6.1) does not carry radial symmetry in general, but has indeed cylindrical symmetry; this might imply cylindrical symmetry of solutions (actually, according to [16], ground state solutions are not cylindrically symmetric if $\mu \ll 0$). This was in fact proved in case $\mu = 0$: existence of cylindrically symmetric minimizers was obtained in [6] and [25] by means of rearrengement techniques and cylindrical symmetry of any minimizer was proved in [24]; finally, using moving plane techniques, cylindrical symmetry of any positive entire solution of (6.1) was proved in [13], as a first step towards the identification of positive entire solutions in the special case $t = 1$.

Now, if $w(y, z) = w(|y|, z)$ is a cylindrically symmetric solution of (6.1), then the transformation $u(r, z) := r^{\frac{N-2}{2}} w(r, z)$ (an isometry between Sobolev spaces, see Appendix B below) gives a solution of (Eq_λ) (in $\mathcal{R}^n_+$) with

$$n = h + 1, \quad \lambda = \mu + \frac{h^2 - (k-2)^2}{4}$$

and viceversa. So, we can derive from Theorems 1.4 and 1.5 existence of cylindrically symmetric positive entire solutions for (6.1):

- for every $\mu \le \left(\frac{k-2}{2}\right)^2$ if $k \ge 2$ and $p \le \frac{2N}{N-2}$ or $k = 1$ and $p < \frac{2N}{N-2}$
- for every $\mu \in \left(0, \frac{1}{4}\right]$ if $k = 1$, $p = \frac{2N}{N-2}$ and $h \ge 3$.

Our uniqueness/nonexistence results translate into uniqueness/nonexistence of cylindrically symmetric positive entire solutions. In particular, in case $k = 1$, there are no symmetric solutions if $h \ge 3$, $p = \frac{2N}{N-2}$ and $\mu \notin (0, \frac{1}{4}]$ or $h = 2$, $p = 5$ whatever is μ. However, in case $\mu = 0$, where we know positive entire solutions are symmetric, we can conclude that (6.1) has exactly one (up to dilations and translations) positive entire solution, and this solution has cylindrical symmetry. This uniqueness result was already proved in [13], but with a completely different method.

In some special cases we know explicit solutions for (Eq_λ) (see Remark 3.8) and hence for (6.1). We thus obtain in some cases a complete classification of (cylindrically symmetric) positive entire solutions. More precisely, given $\mu \leq \frac{(k-2)^2}{4}$, let $p = 2 + \frac{2}{h+\sqrt{(k-2)^2-4\mu}}$. We have the solution

$$u(y,z) = c(\mu,h,k)\frac{|y|^{\frac{\sqrt{(k-2)^2-4\mu}-(k-2)}{2}}}{\left[(1+|y|)^2+|z|^2\right]^{\frac{1}{p-2}}}.$$

If $\mu = 0$ and $k \geq 2$, then $p = 2\frac{N-1}{N-2}$ and u is as in [13]: we recover the classification result therein (in fact extended to the case $k = 1$).

If $\mu = \frac{(k-2)^2}{4}$, $k \geq 3$ and $h = k-2$ then $p = 2 + \frac{2}{h} = \frac{2N}{N-2}$; in this case

$$u(y,z) = \left[\frac{c(\mu,N)}{|y|\,[(1+|y|)^2+|z|^2]}\right]^{\frac{N-2}{4}}$$

is the extremal for a sharp Hardy-Sobolev-Maz'ya inequality (*cf.* [28]).

Another class of equations is given by critical Grushin-type equations

$$\Delta_y\varphi + (1+\alpha)^2|y|^{2\alpha}\Delta_z\varphi + \varphi^{\frac{Q+2}{Q-2}} = 0 \qquad (y,z) \in \mathbb{R}^k \times \mathbb{R}^h \tag{6.2}$$

where $\alpha > 0$, $Q = k + h(1+\alpha)$, $k, h \geq 1$ (see [23]). Solutions of (6.2) are extremals of the weighted Sobolev inequality

$$\tilde{S}\left(\int_{\mathbb{R}^N}|u|^{\frac{2Q}{Q-2}}dydz\right)^{\frac{Q-2}{Q}} \leq \int_{\mathbb{R}^N}\left(|\nabla_y u|^2 + (\alpha+1)^2|y|^{2\alpha}|\nabla_z u|^2\right)dydz. \tag{6.3}$$

A cylindrically symmetric solution φ of (6.2) gives, via the change of variables $\Phi(r,z) = r^{\frac{Q-2}{2(1+\alpha)}}\varphi(r^{\frac{1}{1+\alpha}}, z)$, $r = |y|$, a solution of (Eq_λ) (see [9]), with

$$n = h+1, \quad \lambda = \frac{1}{4}\left[h^2 - \left(\frac{k-2}{\alpha+1}\right)^2\right], \quad p = \frac{Q+2}{Q-2} < 2^* - 1.$$

Notice that, when $k = 2$, solutions of (6.2) correspond to solutions of (Eq_λ) with $\lambda = \frac{(n-1)^2}{4}$ (and similarly for (6.1)).

Again, if $k \neq 2$, extremals for (6.3) are in $H^1(\mathbb{H})$. This follows by

$$|||u|||^2 := \int_{\mathbb{R}^N}\left(|\nabla_y u|^2 + (\alpha+1)^2|y|^{2\alpha}|\nabla_z u|^2\right)dydz \geq \left(\frac{k-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{u^2}{|y|^2}$$

(a Hardy type inequality, see [12]) and the easy-to-check identity

$$\omega_k(\alpha+1)\int_{\mathbb{H}}|\nabla_{\mathbb{H}}\hat{u}|^2 + \left(\frac{k-2}{2(\alpha+1)}\right)^2\hat{u}^2dV_{\mathbb{H}} = |||u||| + \left(\frac{h(\alpha+1)}{2}\right)^2\int_{\mathbb{R}^N}\frac{u^2}{|y|^2}.$$

In particular, Theorem 1.3, with $n \geq 3$, yields uniqueness of cylindrically symmetric positive entire solutions of (6.2) if $k \geq 1$ and $h \geq 2$. In case $h = 1$, it results $\frac{2(p+1)}{(p+3)^2} = \frac{1}{4}\big(1 - \frac{1}{(Q-1)^2}\big) \geq \frac{1}{4}\big[1 - \big(\frac{k-2}{\alpha+1}\big)^2\big]$ unless $k = 2$, where $\lambda = \frac{1}{4}$ (a case which is not covered by Theorem 1.3) and we get uniqueness also in case $h = 1$, $k \neq 2$, thus improving the uniqueness result in [23].

A. Appendix

For convenience of the reader, we recall some basic facts on the models for $\mathbb{H}$ used in this paper.

The half space model $\mathcal{R}^n_+$

It is given by $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ endowed with the Riemannian metric $\frac{\delta_{ij}}{r^2}$. The associated distance is given by

$$d_{\mathcal{R}^n_+}((r,z),(r_0,z_0)) = 2\tanh^{-1}\left(\frac{|z-z_0|^2+(r-r_0)^2}{|z-z_0|^2+(r+r_0)^2}\right)^{\frac{1}{2}}, \quad (r,z) \in \mathbb{R}^+ \times \mathbb{R}^{n-1}.$$

Easily, the hyperbolic sphere $\mathbb{S}_R(r_0,z_0)$ centered in (r_0,z_0) and of radius R is the euclidean sphere centered at $(r_0\cosh R, z_0)$ and radius $r_0 \sinh R$. Also, it is given by $r_0\mathbb{S}_R + (0,z_0)$, a dilation and translation of the hyperbolic sphere of radius R and center at $(1,0)$, which writes as $\frac{(1+r)^2+|z|^2}{2r} = 1 + \cosh R$.

A function u on $\mathcal{R}^n_+$ has hyperbolic symmetry if its level sets are hyperbolic spheres centered at some (r_0,z_0), or, equivalently, if up to dilations and translations in z it is constant on hyperbolic spheres centered at $(1,0)$.

The hyperbolic gradient $\nabla_{\mathbb{H}}$ and the hyperbolic laplacian $\Delta_{\mathbb{H}}$ write, in $\mathcal{R}^n_+$, as

$$\nabla_{\mathbb{H}} = r\nabla, \qquad \Delta_{\mathbb{H}} = r^2\Delta - (n-2)r\partial_r.$$

In particular, since the volume form is given by $dV_{\mathbb{H}} = \frac{dr\,dz}{r^n}$, the $H^1(\mathbb{H})$ norm writes as

$$\|u\|^2_{\mathbb{H}} = \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} \left[\frac{|\nabla u|^2}{r^{n-2}} + \frac{u^2}{r^n}\right] dr\,dz.$$

Let us now write some explicit (hyperbolically symmetric) solutions of (Eq_λ). Given $\lambda \leq \frac{(n-1)^2}{4}$, let $p = 1 + \frac{2}{n-1+\sqrt{(n-1)^2-4\lambda}}$. By direct computations, one can see that for some constant $c = c(\lambda, n)$,

$$u(r,z) = \left[\frac{2cr}{(1+r)^2+|z|^2}\right]^{\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}} = \left[\frac{c}{1+\cosh R}\right]^{\frac{1}{p-1}}$$

is an entire solution of (Eq_λ), only depending on the hyperbolic distance R of (r, z) from $(1, 0)$. Notice that $u \in H^1(\mathbb{H})$ whenever $\lambda < \frac{(n-1)^2}{4}$. By Theorem (1.3), it is the unique entire solution, up to hyperbolic isometries. In the limiting case, *i.e.* $\lambda = \frac{(n-1)^2}{4}$, $u \in \mathcal{H}$ but is not in $H^1(\mathbb{H})$.

If, instead, $p = 1 + \frac{4}{n-1+\sqrt{(n-1)^2-4\lambda}}$, $\lambda \leq \frac{(n-1)^2}{4}$, we have, for a suitable constant c, the solution

$$\tilde{u}(r, z) = \left[\frac{2cr}{1 + r^2 + |z|^2}\right]^{\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}} = \left[\frac{c}{\cosh^2 R}\right]^{\frac{1}{p-1}}.$$

As above, in case $\lambda = \frac{(n-1)^2}{4}$, $u \in \mathcal{H}$ but is not in $H^1(\mathbb{H})$.

The ball model $\mathcal{B}^n$

It is given by $B^n := \{\xi \in \mathbb{R}^n : \quad |\xi| < 1\}$ endowed with the Riemannian metric $\frac{4\delta_{ij}}{(1-|\xi|^2)^2}$. The associated distance is given by

$$d_{\mathcal{B}^n}(\xi, \xi_0) = 2 \tanh^{-1}\left(\frac{|\xi - \xi_0|}{\sqrt{1 - 2 < \xi, \xi_0 > + |\xi|^2\, |\xi_0|^2}}\right)^{\frac{1}{2}}.$$

The hyperbolic gradient $\nabla_{\mathbb{H}}$ and the hyperbolic laplacian $\Delta_{\mathbb{H}}$ write as

$$\nabla_{\mathbb{H}} = \frac{1 - |\xi|^2}{2}\nabla, \qquad \Delta_{\mathbb{H}} = \left(\frac{1 - |\xi|^2}{2}\right)^2 \Delta + (n-2)\frac{1 - |\xi|^2}{2} < \xi, \nabla > .$$

The standard hyperbolic isometry M between the two models is given as follows. If e_0, e_j, $j = 1, \dots, n-1$ is the standard basis in $\mathbb{R} \times \mathbb{R}^{n-1}$, then

$$M(r, z) := \left(\frac{1 - r^2 - |z|^2}{(1+r)^2 + |z|^2}, \frac{2z}{(1+r)^2 + |z|^2}\right) = 2\frac{x + e_0}{|x + e_0|^2} - e_0, \quad x = (r, z).$$

M is a bijection of $\mathbb{R}^n \setminus \{-e_0\}$ onto itself, $M(\mathbb{R}^+ \times \mathbb{R}^{n-1}) = B^n$ and $M = M^{-1}$. Notice that, if $\xi := M(r, z)$, it results

$$|\xi|^2 = \frac{(1-r)^2 + |z|^2}{(1+r)^2 + |z|^2}, \quad \frac{1 - |\xi|^2}{2} = \frac{2r}{(1+r)^2 + |z|^2}, \quad \frac{1 - |\xi|^2}{1 + |\xi|^2} = \frac{2r}{1 + r^2 + |z|^2}.$$

So, we see that M sends a hyperbolic sphere in $\mathcal{R}^n_+$ centered at $(1, 0)$ into the hyperbolic sphere in $\mathcal{B}^n$ centered at 0 with (of course) the same radius. In particular, a function on $\mathcal{B}^n$ has hyperbolic symmetry iff, up to an hyperbolic isometry, is a radial function. Also, we see that the special solutions of (Eq_λ) described above, rewrite in the ball model as

$$u(\xi)=\left[c(\lambda,n)\frac{1-|\xi|^2}{2}\right]^{\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}},\ \tilde{u}(\xi)=\left[\sqrt{\tilde{c}(\lambda,n)}\frac{1-|\xi|^2}{1+|\xi|^2}\right]^{\frac{n-1+\sqrt{(n-1)^2-4\lambda}}{2}}.$$

B. Appendix

We derive here Poincaré-Sobolev inequalities (1.2) from Hardy-Sobolev-Maz'ya inequalities (B.5). We will use here $\mathcal{R}^n_+$, as a model for $\mathbb{H}^n$, $n \geq 2$.

B.1. Isometric Sobolev spaces and Poincaré inequality in $\mathbb{H}^n$

Let $u \in C_0^\infty(\mathbb{H}^n)$, $k \in \mathbb{N}$ and $N := k + (n-1)$. Define

$$v(y,z) = (T_k u)(y,z) := |y|^{-\frac{N-2}{2}} u(|y|, z), \qquad y \in \mathbb{R}^k,\ z \in \mathbb{R}^{n-1}.$$

Thus $T_k u \in C_0^\infty(\mathbb{R}^N)$, has cylindrical symmetry and is compactely supported by $\mathbb{R}^k \times \mathbb{R}^{n-1} \setminus \mathbb{R}^{n-1}$. Notice that, if $k \geq 2$, $T_k(C_0^\infty(\mathbb{H}^n))$ is dense in $D^1_{\mathrm{cyl}}(\mathbb{R}^N)$, the closure in $D^1(\mathbb{R}^N)$ of cylindrically symmetric $C_0^\infty(\mathbb{R}^N)$ functions.

We want to shaw isometric properties of T_k.

Let $p > 2$. In addition, let $p \leq \frac{2N}{N-2}$ if $N \geq 3$ and set $t := N - \frac{N-2}{2}p$. Then

$$\frac{1}{\omega_k}\int_{\mathbb{R}^N} \frac{v^p}{|y|^t} dydz = \int_{\mathbb{H}^n} u^p dV_{\mathbb{H}^n} \tag{B.1}$$

$$\frac{1}{\omega_k}\int_{\mathbb{R}^N} |\nabla v|^2 = \int_{\mathbb{H}^n}\left[|\nabla_{\mathbb{H}^n} u|^2 - \frac{(n-1)^2-(k-2)^2}{4}u^2\right]dV_{\mathbb{H}^n} \tag{B.2}$$

(ω_k is the volume of the k dimensional unite sphere, $\omega_1 := 2$). From (B.1) (with $t=2$), (B.2) and Hardy inequality ([20, 2.1.6 Corollary 3]) we derive

$$\begin{aligned}&\omega_k \int_{\mathbb{H}^n}\left[|\nabla_{\mathbb{H}^n} u|^2 - \frac{(n-1)^2}{4}u^2\right]dV_{\mathbb{H}^n}\\ &= \int_{\mathbb{R}^k\times\mathbb{R}^{n-1}} |\nabla v|^2 dydz - \frac{(k-2)^2}{4}\int_{\mathbb{R}^k\times\mathbb{R}^{n-1}} \frac{v^2}{|y|^2}dydz \geq 0\end{aligned} \tag{B.3}$$

for all $v \in C_0^\infty(\mathbb{R}^k \times \mathbb{R}^h)$, subject to the condition $u(0,z) \equiv 0$ in case $k=1$.

In particular, we get the well known Poincaré inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV_{\mathbb{H}^n} \geq \frac{(n-1)^2}{4}\int_{\mathbb{H}^n} u^2 dV_{\mathbb{H}^n} \qquad \forall u \in H^1(\mathbb{H}^n). \tag{B.4}$$

Notice that $\frac{(n-1)^2}{4}$ is the best constant in (B.4) (*i.e.* (1.1) holds true) because $\frac{(k-2)^2}{4}$ is the best constant in the Hardy ineqaulity.

As a consequence of (B.4), the right hand side in (B.2) defines, for $k \neq 2$, an equivalent norm on $H^1(\mathbb{H}^n)$. Thus T_k, $k \geq 3$ extends to an isometric isomorphism between $H^1(\mathbb{H}^n)$ and $D^1_{\mathrm{cyl}}(\mathbb{R}^N)$. Similarly if $k = 1$, provided $D^1_{\mathrm{cyl}}(\mathbb{R}^N)$ is replaced by the closure in $D^1(\mathbb{R}^N)$ of $C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^{n-1})$. Case $k = 2$ was considered in Lemma 2.2.

B.2. HSM inequality implies Poincaré-Sobolev inequality in $\mathbb{H}^n$, $n \geq 2$.

Let us first recall the HSM inequality.

Hardy-Sobolev-Maz'ya (HSM) inequality (*cf.* [20, 2.1.6 Corollary 3])

Let $k, h \in \mathbb{N}$, $N = k + h$. Let $p > 2$ and $p \leq \frac{2N}{N-2}$ if $N \geq 3$. Let $t = N - \frac{N-2}{2}p$ Then there is $c = c(N, p)$ such that

$$\left(\int_{\mathbb{R}^k \times \mathbb{R}^h} \frac{v^p}{|y|^t} dydz\right)^{\frac{2}{p}} \leq c \int_{\mathbb{R}^k \times \mathbb{R}^h} \left[|\nabla v|^2 - \left(\frac{k-2}{2}\right)^2 \frac{v^2}{|y|^2}\right] dydz \qquad \text{(B.5)}$$

for all $u \in C_0^\infty(\mathbb{R}^k \times \mathbb{R}^h)$, subject to the condition $u(0, z) \equiv 0$ in case $k = 1$.

Using the equality in (B.3) with $k = 1$, (B.5) and then (B.1), we readily get

$$S_{n,p} \left(\int_{\mathbb{H}} |u|^p dV_{\mathbb{H}}\right)^{\frac{2}{p}} \leq \int_{\mathbb{H}} \left[|\nabla_{\mathbb{H}} u|^2 - \frac{(n-1)^2}{4} u^2\right] dV_{\mathbb{H}} \quad \forall\, u \in C_0^\infty(\mathbb{H})$$

(p as in (HSM) inequality). This is Poincaré-Sobolev inequality (1.2).

Since in [20] inequality (B.5) is stated only in case $N > 2$, we indicate here the arguments to obtain (B.5). Actually, we follow closely [20].

Let us start with a basic inequality

Maz'ya inequality ([20, 2.1.6 Corollary 1])

Given $h, k \in \mathbb{N}$, let $N = k + h$, $\alpha > 1 - k$, $1 \leq q \leq \frac{N}{N-1}$, $\beta = \alpha - 1 + N\frac{q-1}{q}$. Then

$$\left(\int_{\mathbb{R}^k \times \mathbb{R}^h} |y|^{\beta q} |u|^q dydz\right)^{\frac{1}{q}} \leq c \int_{\mathbb{R}^k \times \mathbb{R}^h} |y|^\alpha |\nabla u| dydz \qquad \forall\, u \in C_0^\infty(\mathbb{R}^N).$$

A choice of α leads to the following inequality (*cf.* [20, 2.1.6 Corollary 2]):

$$\left(\int_{\mathbb{R}^k \times \mathbb{R}^h} |y|^{\frac{p-2}{2}(N-k)-k} |u|^p dydz\right)^{\frac{2}{p}} \leq c \int_{\mathbb{R}^k \times \mathbb{R}^h} |y|^{2-k} |\nabla u|^2 dydz \qquad \text{(B.6)}$$

for every $p > 2$ and $u \in C_0^\infty(\mathbb{R}^N)$. In fact, let us choose in Maz'ya inequality

$$q = \frac{2p}{p+2}, \qquad \alpha = \frac{p-2}{4}(N-k) - (k-1).$$

Notice that if $N > 2$ then $1 < q \leq \frac{N}{N-1}$ iff $2 < p \leq \frac{2N}{N-2}$ while if $N = 2$ then $1 < q < \frac{N}{N-1}$ for any $p > 2$. Correspondingly, let

$$\beta = \frac{p-2}{4}(N-k) - (k-1) - 1 + N\frac{p-2}{2p} = \frac{p+2}{2p}\left[\frac{p-2}{2}(N-k) - k\right].$$

With this choice, and writing Maz'ya inequality for $u^{\frac{p+2}{2}}$, we get

$$\begin{aligned}
&\left(\int_{\mathbb{R}^k\times\mathbb{R}^h} |y|^{\frac{p-2}{2}(N-k)-k}|u|^p dydz\right)^{\frac{p+2}{2p}} \\
&\leq c\int_{\mathbb{R}^k\times\mathbb{R}^h} |y|^{\frac{p-2}{4}(N-k)-\frac{k}{2}}|u|^{\frac{p}{2}}|y|^{-\frac{k-2}{2}}|\nabla u|dydz \\
&\leq c\left(\int_{\mathbb{R}^k\times\mathbb{R}^h} |y|^{\frac{p-2}{2}(N-k)-k}|u|^p\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^k\times\mathbb{R}^h} |y|^{2-k}|\nabla u|^2\right)^{\frac{1}{2}}
\end{aligned}$$

and hence (B.6). Now, if $u \in C_0^\infty(\mathbb{R}^N)$ and we plug $|y|^{\frac{k-2}{2}}u$ in (B.6) we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^k\times\mathbb{R}^h} |y|^{\frac{p-2}{2}(N-k)-k+p\frac{k-2}{2}}|u|^p dydz\right)^{\frac{2}{p}} \\
&\leq \int_{\mathbb{R}^k\times\mathbb{R}^h} |y|^{2-k}\Bigg[\left(\frac{k-2}{2}\right)^2|y|^{k-4}u^2 + |y|^{k-2}|\nabla u|^2. \\
&\quad + (k-2)|y|^{k-4}u < y, \nabla u > \Bigg]dydz \\
&= \int_{\mathbb{R}^k\times\mathbb{R}^h}\left[|\nabla u|^2 + \left(\frac{k-2}{2}\right)^2\frac{|u|^2}{|y|^2} + (k-2) < \frac{y}{|y|^2}, \nabla u >\right]dydz
\end{aligned}$$

provided $u(0,z) \equiv 0$ in case $k = 1$. Since, integrating by parts,

$$2\sum_{j=1}^k \int_{\mathbb{R}^k\times\mathbb{R}^h} u\frac{y_j}{|y|^2}\frac{\partial u}{\partial y_j} = -(k-2)\int_{\mathbb{R}^k\times\mathbb{R}^h}\frac{|u|^2}{|y|^2}$$

while $\frac{p-2}{2}(N-k) - k + p\frac{k-2}{2} = -N + \frac{p(N-2)}{2}$, (B.5) follows.

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