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The Cauchy problem for hyperbolic systems with Hölder continuous coefficients with respect to the time variable

KUNIHIKO KAJITANI AND YASUO YUZAWA

Abstract. We discuss the local existence and uniqueness of solutions of certain nonstrictly hyperbolic systems, with Hölder continuous coefficients with respect to time variable. We reduce the nonstrictly hyperbolic systems to the parabolic ones and by use of the *Tanabe-Sobolevski's method* and the Banach scale method we construct a semi-group which gives a representation of the solution to the Cauchy problem.

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1. Introduction

We consider the following Cauchy problem:

$$\begin{cases} \partial_t u(t,x) = \displaystyle\sum_{j=1}^{d} A_j(t,x)\partial_j u(t,x) \\ \qquad\qquad +B(t,x)u(t,x) + f(t,x), \quad \text{in } [0,T]\times\mathbb{R}^d, \\ u(0,x) = u_0(x), \qquad x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where each A_j and B are $N\times N$ matrix functions, f, u and u_0 are N component vector functions and $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$. We assume that this system has weak hyperbolicity, that is,

(A.I) All eigenvalues of $\displaystyle\sum_{j=1}^{d} A_j(t,x)\xi_j$ are real valued in $[0,T]\times\mathbb{R}^d_x\times\{\mathbb{R}^d_\xi\setminus\{0\}\}$ and their multiplicity does not exceed ν.

Many papers are devoted to the study of wellposedness in the Gevrey classes for the Cauchy problem (1.1). When all A_j are smooth enough with respect to t, then this property was proved for the order $1\le s<1+1/(\nu-1)$ by M. D. Bronstein in [1] in

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the higher order scalar case and by K. Kajitani in [4] in the system case, respectively. Moreover they have shown it in the case that the coefficient also depend on x. When each A_j has only μ-Hölder continuity in t for some $0 < \mu \le 1$, the Cauchy problem is also wellposed in the Gevrey classes but the Gevrey order must be lower than the smooth case. The first result in the Hölder continuous case was derived by F. Colombini, E. Jannelli and S. Spagnolo in [2]. They proved that the Cauchy problem to the second order equation $u_{tt} = a(t)u_{xx}$ was Gevrey wellposed for the order $1 \le s < 1 + \mu/2$ and, it is important, this order is optimal. T. Nishitani in [8] extended to the second order equations with coefficients also depending on x, and then Y. Ohya and S. Tarama in [9] extended that the higher order scalar equation was Gevrey wellposed for $1 \le s < 1 + \mu/\nu$. The system case was investigated by Kajitani in [5], and he showed that the weakly hyperbolic systems were wellposed in the Gevrey classes for $1 \le s < 1 + \mu/(\nu + 1)$.

When the coefficients depend only on t, D'Ancona, T. Kinoshita and Spagnolo in [3] proved the Gevrey wellposedness for $1 \le s < 1 + \mu/\nu$ to 3×3 weakly hyperbolic systems with coefficients depending on t. To prove it, they derived the energy estimates for the approximate symbols and moreover Yuzawa in [11] has treated the general systems of which coefficients depend only on time variable.

In this paper, we shall extend their result to any $N \times N$ system whose coefficients depend also on the space variables by using the other approach, semi-group method called *Tanabe-Sobolevski method* (cf. [6, 10]) and consequently obtain the energy estimates.

To state our results we shall introduce the Gevrey classes and their properties.

Definition 1.1. Let $s \ge 1$, then we denote by $\gamma^{(s)}(\mathbb{R}^d)$ the set of all functions satisfying the following condition: for any compact subset K of $\mathbb{R}^d$, there exist constants $C_K > 0$ and $A_K > 0$ such that

$$|\partial_x^\alpha u(x)| \le C_K A_K^{|\alpha|} |\alpha|!^s$$

for any $x \in K$ and $\alpha \in \mathbb{N}^d$ and we define $\gamma_0^{(s)}(\mathbb{R}^d) = \gamma^{(s)}(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$.

Definition 1.2. Let k be an integer and $0 < \mu \le 1$. For a Banach space Y, we denote by $C^{k,\mu}([0, T]; Y)$ the set of functions $u(t)$ which are k times differentiable in Y with respect to t and $(\partial/\partial t)^k u(t)$ are μ-Hölder continuous in Y: there exists a constant $C > 0$ such that

$$\|\partial_t^l u(t)\|_Y \le C \quad (0 \le l \le k), \qquad ||\partial_t^k u(t) - \partial_t^k u(t')||_Y \le C|t - t'|^\mu$$

for $t, t' \in [0, T]$. We write $C^{k,0}([0, T]; Y)$ as $C^k([0, T]; Y)$ in brief.

Definition 1.3. For $\rho \ge 0$, $s > 1$, $h > 0$ and $l \in \mathbb{R}$, we define

$$H^l_{\Lambda(\rho)}(\mathbb{R}^d) = \left\{ u \in L^2_x(\mathbb{R}^d);\ \langle\xi\rangle_h^l e^{\Lambda(\rho)} \hat{u}(\xi) \in L^2_\xi(\mathbb{R}^d) \right\},$$

where $\langle\xi\rangle_h = \sqrt{h^2+|\xi|^2}$, $\Lambda(\rho) = \Lambda(\rho,\xi;s,h) = \rho\,\langle\xi\rangle_h^{1/s}$ and $\hat{u}(\xi)$ stands for a Fourier transform of $u(x)$:

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}u(x)dx,$$

and for $\rho < 0$ we define $H^l_{\Lambda(\rho)}(\mathbb{R}^d)$ as the dual space of $H^{-l}_{\Lambda(-\rho)}(\mathbb{R}^d)$.

When $\rho = 0$, $H^l_{\Lambda(0)} = H^l_0$ is a usual Sobolev space and we write them as H^l in brief. $H^l_{\Lambda(\rho)}$ is a Hilbert space with inner product

$$(u,v)_{H^l_{\Lambda(\rho)}} = (\langle\xi\rangle^l\, e^{\rho\langle\xi\rangle_h^{1/s}}\,u,\ \langle\xi\rangle^l\, e^{\rho\langle\xi\rangle_h^{1/s}}\,v)_{L^2}$$

and we define the norm of $H^l_{\Lambda(\rho)}$ by $||u||_{H^l_{\Lambda(\rho)}} = \|\,\langle\xi\rangle^l\, e^{\rho\langle\xi\rangle_h^{1/s}}\,\hat{u}(\xi)\|_{L^2}$.

Definition 1.4. We define $e^{\Lambda(\rho)} = e^{\rho\langle D\rangle_h^{1/s}}$ a pseudo differential operator of order infinity such as

$$e^{\Lambda(\rho)}u(x) = e^{\rho\langle D_x\rangle_h^{1/s}}u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi+\Lambda(\rho,\xi)}\hat{u}(\xi)d\xi$$

for $u(x)$ in $H^l_{\Lambda(\rho)}$.

Definition 1.5. Let $\rho(t)$ be a positive definite function in $[0,T]$, k an integer ≥ 0, $0 < \mu \leq 1$ and $l \in \mathbb{R}$. Then, we denote by $e^{-\Lambda(\rho(t))}C^{k,\mu}([0,T];H^l)$ the class of functions $f(t,x)$ for which to every $t \in [0,T]$,

$$e^{\Lambda(\rho(t))}f(t,x) = e^{\rho(t)\langle D_x\rangle^{1/s}}f(t,x) \in C^{k,\mu}([0,T];H^l).$$

We note the relations between $\gamma_0^{(s)}(\mathbb{R}^d)$ and $H^l_{\Lambda(\rho)}(\mathbb{R}^d)$.

Proposition 1.6 (cf. Lemma 1.2 in [4]). *For any $u(x)$ in $\gamma_0^{(s)}(\mathbb{R}^d)$ and $l \in \mathbb{R}$, there exists a constant $\rho_u > 0$ such that $u(x)$ in $H^l_{\Lambda(\rho_u)}(\mathbb{R}^d)$.*

Conversely, if $u(x)$ belongs to $H^l_{\Lambda(\rho)}(\mathbb{R}^d)$ for some $\rho > 0$, then u belongs to $\gamma^{(s)}(\mathbb{R}^d)$.

Now, we shall state the main theorems.

Theorem 1.7. *Let $1 \leq s < 1+\mu/\nu$, $\sigma = (\nu-1)(1-1/s)$ and $0 < \mu \leq 1$ and take $\delta \geq 0$ such that $s(\delta+\sigma+1) > 1+\mu$. Assume that* (A.I) *and the following condition* (A.II) *are valid*;

(A.II) *each $A_j(t,x)$ belongs to $C^{0,\mu}([0,T];\gamma^{(s)}(\mathbb{R}^d))$ for $j = 1,\ldots,d$ and $B(t,x) \in C^0([0,T];\gamma^{(s)}(\mathbb{R}^d))$.*

Then for every $u_0(x)$ *in* $H^l_{\Lambda(T+\rho_0)}(\mathbb{R}^d)$ *and for every* $f(t,x)$ *in* $e^{-\Lambda(T-t+\rho_0)}C([0,T];H^l(\mathbb{R}^d))$*, there exists a unique solution* $u(t,x)$ *of the Cauchy problem* (1.1)*, which is in* $e^{-\Lambda(T-t)}C([0,T];H^{l+\delta}(\mathbb{R}^d))\cap e^{-\Lambda(T-t)}C^1([0,T];H^{l-1+\delta}(\mathbb{R}^d))$ *and satisfies*

$$||u(t,\cdot)||_{H^{l+\delta}_{\Lambda(T-t)}} \leq C\left(||u_0||_{H^l_{\Lambda(T+\rho_0)}} + \int_0^t ||f(r,\cdot)||_{H^l_{\Lambda(T-r+\rho_0)}} dr\right), \tag{1.2}$$

for any $l\in\mathbb{R}$ *and* $0\leq t\leq T$.

Considering the property of the finite propagation of the solution for the weakly hyperbolic system and Proposition 1.6, the following theorem is concluded by Theorem 1.7.

Theorem 1.8. *Assume that* (A.I) *and* (A.II)*. If* $1\leq s<1+\mu/\nu$ *and* $0<\mu\leq 1$*, then for any* $f(t,x)$ *in* $C([0,T];\gamma^{(s)}(\mathbb{R}^d))$ *and* $u_0(x)$ *in* $\gamma^{(s)}(\mathbb{R}^d)$*, there is a unique solution* $u(t,x)$ *in* $C^1([0,T];\gamma^{(s)}(\mathbb{R}^d))$ *of the equation* (1.1).

2. Preliminaries

In this section we shall introduce some notation and fundamental propositions on the pseudo differential operator theory.

We denote by $S^m_{\rho,\delta,l}$ $(0\leq\delta<\rho\leq 1)$ a class of symbols $p(x,\xi)$ satisfying

$$|p|_{S^m_{\rho,\delta,l}} = \sup_{(x,\xi)\in\mathbb{R}^{2d},|\alpha+\beta|\leq l} \frac{|p^{(\alpha)}_{(\beta)}(x,\xi)|}{\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}} < \infty,$$

where $p^{(\alpha)}_{(\beta)}(x,\xi)=D^\beta_x\partial^\alpha_\xi p(x,\xi)$. We write $S^m_{1,0,l}$ as S^m_l in brief.

We denote by $(S^m_{\rho,\delta,l})^{N\times N}$ a class of matrix symbols $P(x,\xi)=(p_{ij}(x,\xi))_{1\leq i,j\leq N}$ such that all p_{ij} are in $S^m_{\rho,\delta,l}$ and we define

$$|P|_{S^m_{\rho,\delta,l}} = \max_{1\leq i,j\leq N} |p_{ij}|_{S^m_{\rho,\delta,l}}.$$

We often write $(S^m_{\rho,\delta,l})^{N\times N}$ as $S^m_{\rho,\delta,l}$ in brief and $S^m_{\rho,\delta}=\cap_l S^m_{\rho,\delta,l}$.

For $p(x,\xi)\in S^m_{\rho,\delta}$, we define the pseudo differential operator $p(x,D_x)$:

$$p(x,D_x)u(x)=\frac{1}{(2\pi)^d}\int e^{ix\xi}p(x,\xi)\hat{u}(\xi)d\xi.$$

The following proposition is well known as the boudedness in H^q of pseudo differential operators.

Proposition 2.1. *Let* $0 \le \delta < \rho \le 1$ *and*

$$M_0(q) = \left[\frac{2(|q| + l_0 + d + 1)}{2 - \delta}\right] + l_0, \qquad l_0 = \left[\frac{\delta(d+1)}{\rho - \delta}\right] + 1.$$

Assume that $l \ge M_0(q)$ *and* $p(x,\xi)$ *be in* $S^m_{\rho,\delta,l}$. *Then, we have*

$$||p(x, D_x)u||_{H^q} \le C_q |p|_{S^m_{\rho,\delta,M_0(q)}} \|u\|_{H^{q+m}}, \tag{2.1}$$

for $u \in H^{q+m}$.

Next, we denote by $\Gamma^m_{s,r,l}$ a class of symbols $p(x,\xi)$ satisfying

$$|p|_{\Gamma^m_{s,r,l}} = \sup_{(x,\xi)\in\mathbb{R}^{2d}, |\alpha+\beta|\le l, \gamma\in\mathbb{Z}^d_+} \frac{|p^{(\alpha)}_{(\beta+\gamma)}(x,\xi)|}{r^{|\gamma|}|\gamma|!^s \langle\xi\rangle^{m-|\alpha|}} < \infty. \tag{2.2}$$

For a symbol $p(x,\xi) \in \Gamma^m_{s,r,l}$, we define an operator $p_{\Lambda(\rho)}(x, D_x)$:

$$p_{\Lambda(\rho)}(x, D_x) = e^{\Lambda(\rho)} p(x, D_x) e^{-\Lambda(\rho)},$$

where $\Lambda(\rho) = \rho\langle D_x\rangle_h^{1/s}$. It follows from Kumano-go's formula that the symbol of $p_{\Lambda(\rho)}(x, D_x)$ give by

$$\sigma(p_{\Lambda(\rho)})(x,\xi) = \frac{1}{(2\pi)^d}\text{Os-}\iint_{\mathbb{R}^{2d}} e^{-iy\eta+\Lambda(\xi+\eta)-\Lambda(\xi)} p(x+y,\xi)dyd\eta, \tag{2.3}$$

where Os-$\iint$ means an oscillatory integral. We denote by $p_\Lambda(x,\xi)$ the symbol of $p_{\Lambda(\rho)}(x, D_x)$ in brief.

From (2.3) and Taylor's formula,

$$p_\Lambda(x,\xi) = p(x,\xi) + \sum_{0<|\gamma|<N} p_{(\gamma)}(x,\xi)\omega_\gamma(\rho,\xi) + r_N(p)(x,\xi), \tag{2.4}$$

where

$$\omega_\gamma(\rho,\xi) = \frac{1}{\gamma!} e^{-\rho\langle\xi\rangle_h^\kappa} \partial_\xi^\gamma (e^{\rho\langle\xi\rangle_h^\kappa})$$

and

$$r_N(p)(x,\xi) = \frac{N}{(2\pi)^d}$$

$$\times \sum_{|\gamma|=N} \text{Os-}\iint_{\mathbb{R}^{2d}} \int_0^1 (1-\theta)^{N-1} e^{-iy\eta+\Lambda(\xi+\eta)-\Lambda(\xi)} p_{(\gamma)}(x+\theta y,\xi)\omega_\gamma(\rho,\xi+\eta)d\theta dy d\eta.$$

Proposition 2.2 ([5]). *Let $p(x,\xi)$ be in $\Gamma^m_{s,r,l}$, $\Lambda(\rho)=\rho\langle\xi\rangle_h^{1/s}$ and $s>1$.*

(i) *ω_γ and $r_N(p)$ satisfy the following inequalities:*

$$|\partial_\xi^\alpha \omega_\gamma(\rho,\xi)| \le C_{\alpha\gamma}(|\rho|+h^{-\kappa})^{|\gamma|}\langle\xi\rangle_h^{-(1-\kappa)|\gamma|-|\alpha|}, \tag{2.5}$$

$$|\partial_\xi^\alpha D_x^\beta r_N(p)(x,\xi)| \le C_{\alpha\beta N}(|\rho|+h^{-\kappa})^{N}\langle\xi\rangle_h^{m-N(1-\kappa)-|\alpha|}, \tag{2.6}$$

for $(x,\xi)\in\mathbb{R}^{2d}$, $\alpha,\beta\in\mathbb{Z}^d_+$, where $\kappa=\frac{1}{s}$.

(ii) *If ρ satisfies*

$$|\rho|\le(24^s dr)^{-1/s}, \tag{2.7}$$

and l satisfies

$$l \ge \left[\frac{sl'}{s-1}\right]+l'+\left[\frac{d}{2}\right]+1, \tag{2.8}$$

then the symbol $p_\Lambda(x,\xi)$ belongs to $S^m_{l'}$. Moreover, there exists a constant $C_{l'}$ such that

$$|p_\Lambda|_{S^m_{l'}} \le C_{l'}|p|_{\Gamma^m_{s,r,l}}.$$

The following proposition is the fundamental property on the hyperbolic polynomial:

Proposition 2.3 ([1]). *Let $p(t,x,\lambda,\xi)$ be a hyperbolic polynomial of order N, that is, p can be factorized by real roots:*

$$p(t,x,\lambda,\xi)=\sum_{|\alpha|+j=N} a_{j,\alpha}(t,x)\lambda^j\xi^\alpha=\prod_{j=1}^N(\lambda-\lambda_j(t,x,\xi)),$$

where all $\lambda_j(t,x,\xi)$ are real valued for $(t,x,\xi)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. Assume that the multiplicity of $\lambda_j(t,x,\xi)$ is at most $\nu(\le N)$ and all the coefficients $a_{j,\alpha}(t,x)$ and $D_x^\beta a_{j,\alpha}(t,x)$ ($|\beta|\le\nu$) are bounded in $[0,T]\times\mathbb{R}^d$, then $p(t,x,\lambda,\xi)$ satisfies the following estimates:

$$\frac{1}{\left|p(t,x,\lambda-i\langle\xi\rangle_h^\kappa,\xi)\right|} \le C(|\lambda|+\langle\xi\rangle_h)^{-N+\nu}\langle\xi\rangle_h^{-\kappa\nu}, \tag{2.9}$$

$$\frac{|p^{(\alpha)}_{(\beta)}(t,x,\lambda-i\langle\xi\rangle_h^\kappa,\xi)|}{\left|p(t,x,\lambda-i\langle\xi\rangle_h^\kappa,\xi)\right|} \le C\langle\xi\rangle_h^{-\kappa|\alpha|+(1-\kappa)|\beta|} \qquad (|\alpha+\beta|\le\nu), \tag{2.10}$$

for $\lambda\in\mathbb{C}$ such that $\mathrm{Im}\lambda<0$ *and for (x,ξ) in $[0,T]\times\mathbb{R}^d$.*

3. The Cauchy problem in Sobolev spaces

In this section we assume that $1 \leq s < 1 + \mu/\nu$ and put $\kappa = 1/s$.

We reintroduce the equation (1.1):

$$\begin{cases} P(t,x,D)u(t,x) = -if(t,x), & (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

where

$$P(t,x,D) = D_t - \sum_{j=1}^{d} A_j(t,x)D_j + iB(t,x),$$

and $D = (D_t, D_x) = (D_t, D_1, D_2, \ldots, D_d)$, $D_t = -i\partial_t$ and $D_i = -i\partial_i$.

For some non negative and continuously differentiable function $\rho(t)$, we set $v(t,x) = e^{\Lambda(\rho(t))}u(t,x) = e^{\rho(t)\langle D_x\rangle_h^\kappa}u(t,x)$. Then we can reduce the problem (1.1) to

$$\begin{cases} P_\Lambda(t,x,D_t,D_x)v(t,x) = g(t,x), \\ v(0,x) = v_0(x), \end{cases} \tag{3.1}$$

where

$$D_{t,\Lambda} = D_{t,\Lambda(\rho(t))} = D_t + i\rho'(t)\langle D_x\rangle_h^\kappa,$$

$$P_\Lambda(t,x,D_t,D_x) = D_{t,\Lambda}I - \sum_{j=1}^{d} A_{j,\Lambda}(t,x,D_x)D_j + iB_\Lambda(t,x,D_x),$$

for $(t,x) \in \mathbb{R}^{1+d}$, $v_0(x) = e^{\Lambda(\rho(0))}u_0(x)$, $g(t,x) = -ie^{\Lambda(\rho(t))}f(t,x)$, and each $A_{j,\Lambda}(t,x,D_x)$ and B_Λ are pseudo differential operators such as

$$A_{j,\Lambda}(t,x,D_x) = A_{j,\Lambda(\rho(t))}(t,x,D_x) = e^{\rho(t)\langle D_x\rangle_h^\kappa}A_j(t,x)e^{-\rho(t)\langle D_x\rangle_h^\kappa},$$

$$B_\Lambda(t,x,D_x) = B_{\Lambda(\rho(t))}(t,x,D_x) = e^{\rho(t)\langle D_x\rangle_h^\kappa}B(t,x)e^{-\rho(t)\langle D_x\rangle_h^\kappa}.$$

We shall solve the equation (3.1) in Sobolev space H^l by using the semi group for $i\rho'(t)\langle D_x\rangle_h^\kappa)I - A_\Lambda(t))$.

4. Construction of $((\lambda + i\rho'(\tau)\langle D_x\rangle_h^\kappa))I - A_\Lambda(\tau))^{-1}$

Let τ be fixed in $[0,T]$. In this section, we shall construct the inverse of $((\lambda + i\rho'(\tau)\langle D_x\rangle_h^\kappa)I - A_\Lambda(\tau))$ for a complex number λ with $\mathrm{Im}\lambda \leq -h_0$. We consider the following resolvent equation :

$$((\lambda + i\rho'(\tau)\langle D_x\rangle_h^\kappa)I - A_\Lambda(\tau))v = g, \tag{4.1}$$

for $g \in H^q$, where

$$A(\tau) = A(\tau, x, D_x) = \sum_{j=1}^{d} A_j(\tau, x) D_j, \tag{4.2}$$

and

$$A_\Lambda(\tau) = A_{\Lambda(\rho(\tau))}(t, x, D_x) = \sum_{j=1}^{d} A_{j,\Lambda(\rho(\tau))}(\tau, x, D_x) D_j. \tag{4.3}$$

Let us define several symbols as follows:

$$\begin{aligned}
A(t, x, \xi) &= \sigma(A)(t, x, \xi) = \sum_{j=1}^{d} A_j(t, x)\xi_j, \\
A_\Lambda(t, x, \xi) &= \sigma(A_{\Lambda(\rho(t))})(t, x, \xi) = \sum_{j=1}^{d} A_{j,\Lambda(\rho(t))}(t, x, \xi)\xi_j, \\
P(t, x, \lambda, \xi) &= \sigma(P)(t, x, \lambda, \xi) = \lambda I - \sum_{j=1}^{d} A_j(t, x, \xi)\xi_j + iB(t, x), \\
P_\Lambda(t, x, \lambda, \xi) &= \sigma(P_\Lambda)(t, x, \lambda, \xi) \\
&= \lambda_\Lambda I - \sum_{j=1}^{d} A_{j,\Lambda(\rho(t))}(t, x, \xi)\xi_j + iB_{\Lambda(\rho(t))}(t, x, \xi), \\
H(t, x, \lambda, \xi) &= {}^{co}(\lambda_\Lambda I - A_\Lambda(t, x, \xi)), \\
p(t, x, \lambda, \xi) &= \det(\lambda I - A(t, x, \xi)), \\
p_L(t, x, \lambda, \xi) &= \det(\lambda_\Lambda I - A_\Lambda(t, x, \xi)), \\
M(t, x, \lambda, \xi) &= \frac{1}{p(t, x, \lambda_\Lambda, \xi)} I, \\
\lambda_\Lambda &= \lambda + i\rho'(t)\langle\xi\rangle_h^\kappa.
\end{aligned}$$

We shall consider an operator $\tilde{P}(t, x, \lambda, D_x) = (\lambda I - A_\Lambda(t, x, D_x)) \circ H(t, x, \lambda, D_x)$, where $H(t, x, \lambda, D_x)$ is the operator with the symbol $H(t, x, \lambda, \xi)$ and $\circ$ means an operator product. By Proposition 2.2, if $\rho(t)$ satisfies

$$0 \le \rho(t) \le (24^s dr)^{-\kappa} \qquad \text{in} \quad [0, T], \tag{4.4}$$

then $A_\Lambda(t, x, \xi)$ is in $S^1_{l'}$, so $H(t, \lambda)$ is in $S^{N-1}_{l'}$, where l' satisfies (2.8). Since we can take l in (2.8) arbitarly, $H(t, \lambda) \in \cap_{l'>0} S^{N-1}_{l'} = S^{N-1}$ that is,

$$\left| H^{(\alpha)}_{(\beta)}(t, x, \lambda, \xi) \right| \le C(|\lambda| + \langle\xi\rangle_h)^{N-1-|\alpha|}, \tag{4.5}$$

for $(t, x, \lambda, \xi) \in [0, T] \times \mathbb{R}^d \times \mathbb{C} \times \mathbb{R}^d$, $\alpha, \beta \in \mathbb{Z}^d_+$.

Denote by $\tilde{S}^m_{\rho,\delta}$ a set of symbols $a(t,x,\lambda,\xi)$ which are holomorphic in $\lambda \in \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \le 0\}$ and satisfying

$$|\partial_\xi^\alpha D_x^\beta a(t,x,\lambda,\xi)| \le C_{\alpha\beta}\langle\xi\rangle_h^{m-\rho|\alpha|+\delta|\beta|}, x,\xi \in \mathbb{R}^d, \lambda \in \mathbb{C}, \mathrm{Im}\lambda \le 0, \alpha,\beta \in \mathbb{Z}^d_+.$$

Next we shall compare $p_L(t,x,\lambda,\xi)$ and
$p_\Lambda(t,x,\lambda,\xi) = \sigma(e^\Lambda p(t,x,\lambda + i\rho(t)\langle D_x\rangle_h^\kappa), D_x)e^{-\Lambda})(t,x,\lambda,\xi)$. Since

$$p(t,x,\lambda,\xi) = \sum_{\pi\in S_N} \mathrm{sgn}\pi \cdot a_{1\pi(1)}a_{2\pi(2)}\cdots a_{N\pi(N)},$$

where $a_{ij} = a_{ij}(t,x,\lambda,\xi)$ is the (i,j)-component of $(\lambda I - A(t,x,\xi))$ and S_N stands for the set of permutations of $\{1,2,\cdots,N\}$, we can write

$$\begin{aligned} p(t,x,\lambda,D_x) = \sum_{\pi\in S_N} &\mathrm{sgn}\pi \\ &\times a_{1\pi(1)}(x,\lambda,D_x)\circ a_{2\pi(2)}(x,\lambda,D_x)\circ\cdots\circ a_{N\pi(N)}(x,\lambda,D_x) + q_1 \end{aligned}$$

and consequently

$$\begin{aligned} p_\Lambda(t,x,\lambda,D_x) = \sum_{\pi\in S_N} &\mathrm{sgn}\pi\, a_{1\pi(1),\Lambda}(x,D_x)\circ a_{2\pi(2),\Lambda}(x,D_x)\circ\cdots \\ &\cdots\circ a_{N\pi(N),\Lambda}(x,D_x) + q_{1\Lambda}, \end{aligned}$$

where $a_{jk,\Lambda}(x,D) = e^\Lambda a_{jk}e^{-\Lambda}$ and $q_1(t,x,\lambda,\xi) \in \tilde{S}^{N-1}_{1,0}$, because

$$\begin{aligned} &\sigma(a_{1\pi(1),\Lambda}(x,D_x)\circ a_{2\pi(2),\Lambda}(x,D_x)\circ\cdots\circ a_{N\pi(N),\Lambda}(x,D_x))(x,\xi) \\ &\quad = a_{1\pi(1),\Lambda}(x,\xi)a_{2\pi(2),\Lambda}(x,\xi)\cdots a_{N\pi(N),\Lambda}(x,\xi) + q_\pi(x,\xi) \end{aligned}$$

for any $\pi \in S_N$, where $q_\pi(t,x,\lambda,\xi) \in \tilde{S}^{N-1}_{1,0}$. Hence we have

$$p_\Lambda(t,x,\lambda,\xi) = p_L(t,x,\lambda,\xi) + q_2(t,x,\lambda,\xi), \tag{4.6}$$

where $q_2 \in \tilde{S}^{N-1}_{1,0}$ that is,

$$\left|q_2{}^{(\alpha)}_{(\beta)}(t,x,\lambda,\xi)\right| \le C_{\alpha\beta}(|\lambda| + \langle\xi\rangle)^{N-1-|\alpha|}, \tag{4.7}$$

for $(t,x,\lambda,\xi) \in [0,T]\times\mathbb{R}^d\times\mathbb{C}\times\mathbb{R}^d, \Im\lambda \le 0$ and $\alpha,\beta \in \mathbb{N}^d$. Therefore it follows from (4.5), (4.6) and (4.7) that we have

$$\sigma(\tilde{P})(t,x,\lambda,\xi) = p_\Lambda(t,x,\lambda,\xi)I + Q_1(t,x,\lambda,\xi), \tag{4.8}$$

where Q_1 satisfies

$$|Q_1{}^{(\alpha)}_{(\beta)}(t,x,\lambda,\xi)| \le C_{\alpha\beta}(|\lambda| + \langle\xi\rangle_h^\kappa)^{N-1-|\alpha|}, \tag{4.9}$$

for $(t,x,\lambda,\xi) \in [0,T]\times\mathbb{R}^d\times\mathbb{C}\times\mathbb{R}^d$, $\mathrm{Im}\lambda \le 0$ and $\alpha,\beta \in \mathbb{Z}^d_+$.

We shall construct the inverse of $((\lambda + i\rho'(t)\langle D_x\rangle_h^\kappa)I - A_\Lambda(\tau))$ of form

$$((\lambda+i\rho'(t)\langle D_x\rangle_h^\kappa)I-A_\Lambda(\tau))^{-1}=H(\tau,\lambda+i\rho'(t)\langle D_x\rangle_h^\kappa)\circ(\tilde{P})^{-1}(\tau,\lambda+i\rho'(t)\langle D_x\rangle_h^\kappa),$$

where $(\tilde{P})^{-1}$ is the inverse operator of $\tilde{P}$. Let us show that $(\tilde{P})^{-1}$ exists. Define a operator $S(t,\lambda) = S(t,x,\lambda,D_x)$ as

$$S(t,x,\lambda,D_x) = I - \tilde{P}(t,x,\lambda,D_x)\circ M(t,x,\lambda,D_x). \tag{4.10}$$

where $M(t,x,\lambda,\xi) = p(t,x,\lambda + i\rho'(t)\langle D_x\rangle_h^\kappa,\xi)^{-1}$. Noting that $p(\tau,x,\lambda,\xi)$ is elliptic for $|\lambda| \geq M\langle\xi\rangle_h$, we can prove the following proposition by use of Proposition 2.3.

Proposition 4.1. *Assume that* $1 < s < \nu/(\nu-1)$ *and* q *satisfies* $M_0(q) < l$, *where* $M_0(q)$ *is given in Proposition 2.1 with* $\rho = \kappa$, $\delta = 1-\kappa$. *Then, there exists a positive constant* $h > 0$ *such that for any* $\tau \in [0,T]$ *and* $\lambda \in \mathbb{C}$ $\quad(\mathrm{Im}\lambda \leq 0)$.

$$|\partial_\xi^\alpha D_x^\beta S(t,x,\lambda,\xi)| \leq C_{\alpha\beta}(|\lambda| + \langle\xi\rangle_h)^{\nu(1-\kappa)-1-\kappa|\alpha|+(1-\kappa)|\beta|}. \tag{4.11}$$

Proof. If $|\lambda| \geq M\langle\xi\rangle_h$, then $p(t,x,\lambda+i\rho'(t)\langle\xi\rangle_h^\kappa,\xi)$ is elliptic for $|\lambda| \geq M\langle\xi\rangle_h (M \gg 1)$, so we have

$$|p(t,x,\lambda+i\rho'(t)\langle\xi\rangle_h^\kappa,\xi)| \geq c_0(|\lambda| + \langle\xi\rangle_h)^m,$$

for $|\lambda| \geq M\langle\xi\rangle_h$, which implies (4.11) evidently for $|\lambda| \geq M\langle\xi\rangle_h$. From (4.8), the symbol of $\tilde{P}(t,x,\lambda,D_x)\circ M(t,x,\lambda,D_x)$ is

$$\begin{aligned}
&\sigma\left(\tilde{P}(t,\lambda)\circ M(t,\lambda)\right)(x,\xi)\\
&= I + \sum_{0<|\gamma|<n}\frac{1}{\gamma!}(p_\Lambda(t,x,\lambda,\xi))^{(\gamma)}\left(\frac{1}{p(t,x,\lambda+i\rho'(t)\langle\xi\rangle_h^\kappa,\xi)}\right)_{(\gamma)} I\\
&\quad + \sum_{0\leq|\gamma|<n}\frac{1}{\gamma!}Q^{(\gamma)}(t,x,\lambda,\xi)\left(\frac{1}{p(t,x,\lambda+i\rho'(t)\langle\xi\rangle_h^\kappa,\xi)}\right)_{(\gamma)} I + R_n(t,x,\lambda,\xi)
\end{aligned}$$

where R_n is in $\tilde{S}_{\kappa,1-\kappa}^{(1-\kappa)\nu-\kappa n}$. Moreover, by virtue of Proposition 2.2 and Proposition 2.3

$$\left(p_\Lambda(t,x,\lambda+i\rho'(t)\langle\xi\rangle_h^\kappa,\xi)\right)^{(\gamma)}\left(\frac{1}{p(t,x,\lambda+i\rho'(t)\langle\xi\rangle_h^\kappa,\xi)}\right)_{(\gamma)} \in \tilde{S}_{\kappa,1-\kappa}^{(1-2\kappa)|\gamma|} \subset \tilde{S}_{\kappa,1-\kappa}^{-\delta_1}$$

for $1 \leq |\gamma| \leq n$, and

$$Q_1^{(\gamma)}(t,x,\lambda,\xi)\left(\frac{1}{p_\Lambda}\right)_{(\gamma)}(t,x,\lambda+i\rho'(t)\langle\xi\rangle_h^\kappa,\xi)I \in \tilde{S}_{\kappa,1-\kappa}^{(1-\kappa)\nu-1-\kappa|\gamma|} \subset \tilde{S}_{\kappa,1-\kappa}^{-\delta_1}$$

for $0 \leq |\gamma| \leq n$, where $\delta_1 = \min\{2\kappa-1, 1-\nu(1-\kappa)\} = 1-\nu(1-\kappa) > 0$. Therefore we can see $S(t,x,\lambda,\xi)$ is in $\tilde{S}_{\kappa,1-\kappa}^{-\delta_1}$. This implies (4.11). □

From Proposition 4.1 it follows that if we take the parameter $h > 0$ sufficiently large, there exists a $(I - S(\tau, \lambda))^{-1}$ as Neumann series

$$\begin{aligned}(I - S(\tau, \lambda))^{-1} &= \sum_{j=0}^{\infty} S(\tau, \lambda)^j \\ &= \left\{ \tilde{P}(\tau, x, \lambda, D_x) \circ M(\tau, x, \lambda, D_x) \right\}^{-1}.\end{aligned} \tag{4.12}$$

Therefore we obtain the inverse operator of $(\lambda + i\rho'(\tau)\langle D_x\rangle_h^{\kappa})I - A_{\Lambda(\rho(\tau))}(\tau))$ as

$$\begin{aligned}&((\lambda + i\rho'(\tau)\langle D_x\rangle_h^{\kappa})I - A_{\Lambda(\rho(\tau))}(\tau, x, D_x))^{-1} \\ &= H(\tau, x, \lambda, D_x) \circ M(\tau, x, \lambda, D_x) \circ \sum_{j=0}^{\infty} S(\tau, x, \lambda, D_x)^j \\ &= H(\tau) \circ M(\tau) + H(\tau) \circ M(\tau) \circ \sum_{j=1}^{\infty} S(\tau)^j.\end{aligned} \tag{4.13}$$

Proposition 2.2 and Proposition 2.3 yield,

$$\sigma(H(\tau, \lambda) \circ M(\tau, \lambda))(x, \xi) = H(\tau, \lambda, \xi)M(\tau, \lambda, \xi) + Q_2(\tau, x, \lambda, \xi),$$

where $Q_2 \in \tilde{S}_{\kappa,1-\kappa}^{\nu(1-\kappa)-1-\kappa}$. Moreover, noting that $\sigma(H \circ M \circ S) \in \tilde{S}_{\kappa.1-\kappa}^{2\nu(1-\kappa)-2}$, we get

$$\begin{aligned}&\sigma(((\lambda + i\rho'(\tau)\langle D_x\rangle_h^{\kappa})I - A_{\Lambda(\rho(\tau))}(\tau, x, D_x))^{-1})(\tau, x, \lambda, \xi) \\ &\quad = H(\tau, \lambda, \xi)M(\tau, \lambda, \xi) + Q_3(\tau, x, \lambda, \xi),\end{aligned} \tag{4.14}$$

where $Q_3 \in \tilde{S}_{\kappa,1-\kappa}^{-\delta}$, where $\delta = \min\{1 + \kappa - \nu(1 - \kappa), 2(1 - \nu(1 - \kappa))\} = 2(1 - \nu(1 - \kappa)) > 0$.

5. The equation with time-independent coefficients

In this section, we shall solve the following equation: for fixed $\tau \in [0, T)$,

$$\begin{cases} P_{\Lambda(\rho(\tau))}(\tau, x, D_{t,\Lambda(\rho(\tau))}, D_x)v(t, x), = g(t, x), \quad T \geq t > \tau, x \in \mathbb{R}^d \\ v(\tau, x) = v_0(x). \end{cases} \tag{5.1}$$

We define

$$\begin{aligned}V_0(t, \tau) &= V_0(t, \tau, x, D_x) \\ &= \frac{1}{2\pi i} \int_{\mathrm{Im}\lambda=0} e^{i\lambda(t-\tau)}((\lambda + i\rho'(\tau)\langle D_x\rangle_h^{\kappa})I - A_{\Lambda(\rho(\tau))}(\tau, x, D_x))^{-1} d\lambda\end{aligned} \tag{5.2}$$

for $0 \leq \tau \leq t \leq T$.

Proposition 5.1. *Let* $1 < s < 1 + \frac{\mu}{\nu}$, $\mu \leq 1$, $\kappa = 1/s$, *and* $\sigma = (\nu - 1)(1 - \kappa)$. *Then there is* $\kappa' > 0$ *such that* $\kappa < \kappa' < 1$, $1 - \kappa' < \kappa'$ *and* $V_0(t, \tau)$ *is a pseudo differential operator of which symbol is in* $S^{\sigma}_{\kappa', 1-\kappa'}$. *Moreover there is a constant* $C > 0$ *such that*

$$\|V_0(t, \tau)u\|_{H^q_{\Lambda(\rho)}} \leq C\|u\|_{H^{q+\sigma}_{\Lambda(\rho)}}, \quad T \geq t > \tau, \tag{5.3}$$

for $u \in H^q_{\Lambda(\rho)}$.

Proof. It is sufficient to prove that there is $\kappa' > 0$ such that $1 > \kappa' > 1 - \kappa'$ and

$$|\partial_\xi D^\alpha_x V_0(t, \tau, x, \xi)| \leq C_{\alpha\beta}\langle\xi\rangle_h^{\sigma - \kappa'|\alpha| + (1-\kappa')|\beta|}, \quad T \geq t \geq \tau, x, \xi \in \mathbb{R}^n. \tag{5.4}$$

for $\alpha, \beta \in \mathbb{N}^d$. Since it is known from Proposition 3.3 in [6] that $V_0(t, \tau)(x, \xi)$ is in $S^{\nu(1-\kappa)}_{\kappa, 1-\kappa}$, it suffices to

$$|V_0(t, \tau)(x, \xi)| \leq C\langle\xi\rangle^{\sigma}_h, \qquad T \geq t \geq \tau, x, \xi \in \mathbb{R}^n. \tag{5.5}$$

In fact, we can choose κ' such that $\kappa > \kappa'$ and $\kappa' > 1 - \kappa'$ because of $\kappa > 1 - \kappa$. $V_0(t, \tau)(x, \xi)$ is in $S^{\nu(1-\kappa)}_{\kappa, 1-\kappa}$ implies

$$|\partial_\xi D^\alpha_x V_0(t, \tau, x, \xi)| \leq C_{\alpha\beta}\langle\xi\rangle_h^{\sigma - \kappa'|\alpha| + (1-\kappa')|\beta|}, \quad T \geq t \geq \tau, x, \xi \in \mathbb{R}^n. \tag{5.6}$$

for $|\alpha + \beta| \geq \frac{1-\kappa}{\kappa - \kappa'}$. Using the interpolation theorem we can get (5.4) from (5.5) and (5.6). Now we shall show (5.5). (5.2) yields

$$V_0(t, \tau)(x, \xi) = \frac{1}{2\pi i}\int_{\mathrm{Im}\lambda = 0} e^{i\lambda(t-\tau)}((\lambda + i\rho'(\tau)\langle D_x\rangle^\kappa_h)I - A_{\Lambda(\rho(\tau))}{}^{-1}(\tau, x, \xi))d\lambda$$

$$= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\mathrm{Im}\lambda = 0, |\mathrm{Re}\lambda| \leq R} e^{i\lambda(t-\tau)}((\lambda + i\rho'(\tau)\langle D_x\rangle^\kappa_h)I - A_{\Lambda(\rho(\tau))}{}^{-1}(\tau, x, \xi))d\lambda.$$

Besides,

$$\int_{\mathrm{Im}\lambda = 0, |\mathrm{Re}\lambda| \leq R} e^{i\lambda(t-\tau)}((\lambda + i\rho'(\tau)\langle D_x\rangle^\kappa_h)I - A_{\Lambda(\rho(\tau))})^{-1}(\tau, x, \xi))d\lambda$$

$$= \int_{C_R} e^{i\lambda(t-\tau)}((\lambda + i\rho'(\tau)\langle D_x\rangle^\kappa_h)I - A_{\Lambda(\rho(\tau))})^{-1}(\tau, x, \xi))d\lambda$$

$$- \int_{\gamma_R} e^{i\lambda(t-\tau)}((\lambda + i\rho'(\tau)\langle D_x\rangle^\kappa_h)I - A_{\Lambda(\rho(\tau)))})^{-1}(\tau, x, \xi))d\lambda,$$

where $C_R = \{\lambda \in \mathbb{C}; \mathrm{Im}\lambda = 0, |\mathrm{Re}\lambda| \leq R\} \cup \gamma_R$, $\gamma_R = \{|\lambda| = R, \mathrm{Im}\lambda \leq 0\}$. Noting

$$\lim_{R\to\infty}\int_{\gamma_R} e^{i\lambda(t-\tau)}((\lambda + i\rho'(\tau)\langle D_x\rangle^\kappa_h)I - A_{\Lambda(\rho(\tau))})^{-1}(\tau, x, \xi))d\lambda = 0,$$

we get

$$V_0(t,\tau)(x,\xi) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\mathrm{Im}\lambda=0,|\mathrm{Re}\lambda|\le R} e^{i\lambda(t-\tau)}\Big\{H(\tau,x,\lambda,\xi)M(\tau,x,\lambda,\xi) + Q_3(\tau,x,\lambda,\xi)\Big\}d\lambda$$
$$=: \lim_{R\to\infty}\{I_{1R}(x,\xi)+I_{2R}(x,\xi)\},$$

where

$$I_{1R}(x,\xi) = \frac{1}{2\pi i}\int_{C_R} e^{i\lambda(t-\tau)}\frac{H(\tau,x,\lambda,\xi)}{p(\tau,x,\lambda+i\rho'(\tau)\langle\xi\rangle_h^\kappa,\xi)}d\lambda$$

and

$$I_{2R}(x,\xi) = \frac{1}{2\pi i}\int_{C_R} e^{i\lambda(t-\tau)}Q_3(\tau,x,\lambda,\xi)d\lambda$$

Noting that the roots of $p(t,x,\lambda+i\rho'(\tau)\langle\xi\rangle_h^\kappa,\xi)=0$ are contained in the domaim C_R for large R, we can prove similarly as Proposition 2.1 in [11] that $I_{1R}(x,\xi)$ satisfies (5.5). Next we prove that $I_{2R}(x,\xi)$ satisfies (5.5). Since Q_3 satisfies from (4.14)

$$|Q_3(\tau,x,\lambda,\xi)| \le C(|\lambda|+\langle\xi\rangle_h)^{2(\nu(1-\kappa)-1)}$$

we can estimate

$$\begin{aligned}|I_{2R}(x,\xi)| &= \left|\frac{1}{2\pi i}\int_{C_R} e^{i\lambda(t-\tau)}Q_3(\tau,x,\lambda,\xi)d\lambda\right|\\ &\le C\int_{|\mathrm{Im}\lambda|\le R}(|\lambda|+\langle\xi\rangle_h)^{2(\nu(1-\kappa)-1)}d\mathrm{Im}\lambda\\ &\le C\langle\xi\rangle_h^{2(\nu(1-\kappa)-1)+1} \le C\langle\xi\rangle_h^{(\nu-1)(1-\kappa)}.\end{aligned}$$

Here we used the inequality $2(\nu(1-\kappa)-1)+1 \le (\nu-1)(1-\kappa)$ which is implied by the assumption $\kappa > \frac{\nu}{\nu+\mu} \ge \frac{\nu}{\nu+1}$. Thus we compleled the proof of Proposition 5.1. □

It is easily seen that $V_0(t,\tau)$ satisfies

$$\begin{cases} D_t V_0(t,\tau) = \big\{-i\rho'(\tau)\langle D_x\rangle_h^\kappa + A_{\Lambda(\rho(\tau))}(\tau,x,D_x)\big\}V_0(t,\tau) \quad (\tau < t \le T)\\ V_0(\tau,\tau) = \tilde{I}\end{cases}, \tag{5.7}$$

where $\tilde{I}$ means the identity operator. Thus we have obtained a solution $v(t)$ of (5.1) as follows:

$$v(t) = V_0(t,0)v_0 + \int_0^t V_0(t,r)g(r)dr.$$

6. Construction of the semi-group

We shall construct a semi-group $V(t,\tau) = V(t,\tau,x,D_x)$ for the generator $\rho'(t)\langle D_x\rangle_h^\kappa - iA_{\Lambda(\rho(t))}(t,x,D_x) + B(t,x)$;

$$\begin{cases} D_t V(t,x;\tau) = \{-i\rho'(t)\langle D_x\rangle_h^\kappa + A_{\Lambda(\rho(t))}(t,x,D_x) + iB(t,x)\}V(t,x;\tau), \\ \qquad\qquad t > \tau \geq 0, \\ V(\tau,\tau) = I. \end{cases} \tag{6.1}$$

We note that $V(t,\tau)$ has a semi group property, that is, $V(t,\tau) = V(t,r)V(r,\tau)$ for $T \geq t \geq r \geq \tau \geq 0$. In order to construct $V(t,\tau)$, we shall use *Tanabe-Sobolevski's method* and Banach scale method (cf. [6, 10]).

We shall seek the pseudo differential operator $V(t,\tau) = V(t,\tau,x,D_x)$ which satisfies the following equation:

$$V(t,\tau,x,D_x) = V_0(t,\tau,x,D_x) + \int_\tau^t V_0(t,r,x,D_x)\Phi(r,\tau,x,D_x)dr. \tag{6.2}$$

If $V(t,\tau)$ satisfies (6.1), $\Phi(t,x,D;\tau)$ must satisfy the following equation:

$$\Phi(t,\tau,x,D_x) = R(t,\tau,x,D_x) + \int_\tau^t R(t,r,x,D_x)\Phi(r,\tau,x,D_x)dr, \tag{6.3}$$

where

$$\begin{aligned} R(t,\tau,x,D_x) = \{&-i\rho'(t)\langle D_x\rangle_h^\kappa + A_{\Lambda(\rho(t))}(t,x,D_x) + i\rho'(\tau)\langle D_x\rangle_h^\kappa \\ &- A_{\Lambda(\rho(\tau))}(\tau,x,D_x) + iB(t,x)\} \circ V_0(t,\tau,x,D_x). \end{aligned} \tag{6.4}$$

Conversely, if $\Phi(t,\tau,x,D_x)$ satisfies (6.3), then $V(t,\tau,x,D_x)$ satisfies (6.2).

We shall construct a solution $\Phi(t,\tau,x,D_x)$ of the equation (6.3) as follows:

$$\Phi(t,\tau,x,D_x) = \sum_{j=0}^\infty \Phi_j(t,\tau,x,D_x), \tag{6.5}$$

where

$$\begin{cases} \Phi_0(t,\tau,x,D_x) = R(t,\tau,x,D_x), \\ \Phi_j(t,\tau,x,D_x) = \displaystyle\int_\tau^t R(t,r,x,D_x) \circ \Phi_{j-1}(r,\tau,x,D_x)dr \quad (j \geq 1). \end{cases} \tag{6.6}$$

For simplicity of notation denote $H^q_{\Lambda(\rho)}$ by H^q_ρ. We recall

$$||u||_{H^q_\rho} = ||\langle D_x\rangle_h^q e^{\rho\langle D_x\rangle_h^\kappa} u||_{L^2} \quad \kappa = \frac{1}{s}.$$

We know that for $l \geq 0$ and for $\rho_0 \geq \rho' > \rho \geq 0$

$$||\langle D_x \rangle_h^l u||_{H^q_\rho} \leq C(\rho' - \rho)^{-sl} ||u||_{H^q_{\rho'}}. \tag{6.7}$$

Let denote $\sigma = (\nu - 1)(1 - \kappa)$ and take $\delta \geq 0$ such that $1 > s(\sigma + \delta)$, $1 + \mu > s(1+\sigma+\delta)$. It is possible, because of the assumption $s\sigma < 1$ and $1+\mu > s(1+\sigma)$. Denote $s_1 = s(\sigma + \delta + 1)$, $s_0 = s(\sigma + \delta)$ and $\epsilon = \min\{1 - s_0, 1 + \mu - s_1\} = 1 + \mu - s_1 > 0$.

Proposition 6.1. *There is $C_R > 0$ such that for $T \geq t \geq \tau \geq 0$ and for $\rho_0 \geq \rho' > \rho \geq 0$*

$$\|R(t, \tau)u\|_{H^{q+\delta}_\rho} \leq C_R\{|t - \tau|^\mu(\rho' - \rho)^{-s_1} + (\rho' - \rho)^{-s_0}\}\|u\|_{H^q_{\rho'}} \tag{6.8}$$

Proof. It follows from (6.4), (5.3) and (6.7) that

$$\begin{aligned}\|R(t,\tau)u\|_{H^{q+\delta}_\rho} &\leq C_R(|t-\tau|^\mu||\langle D_x\rangle_h^{1+\delta+\sigma}u||_{H^q_\rho} + ||\langle D_x\rangle_h^{\delta+\sigma}u||_{H^q_\rho}) \\ &\leq C_R\{|t - \tau|^\mu(\rho' - \rho)^{-s_1} + (\rho' - \rho)^{-s_0}\}||u||_{H^q_{\rho'}}\end{aligned}$$

for $\rho_0 \geq \rho' > \rho \geq 0$, which proves proposition. □

Now we shall prove that (6.5) is convergent.

Proposition 6.2. *There are $C_1 > 0$, $C_2 > 0$ such that for $u \in H^q_{\rho'}$, $T \geq t \geq \tau \geq 0$, and for $\rho_0 \geq \rho' > \rho \geq 0$*

$$\|\Phi_j(t,\tau)u\|_{H^{q+\delta}_\rho} \leq C_j(t-\tau)^j \left\{\frac{(t-\tau)^{\mu(j+1)}}{(\rho'-\rho)^{s_1(j+1)}} + \frac{1}{(\rho'-\rho)^{s_0(j+1)}}\right\} \|u\|_{H^{q+\delta}_{\rho'}}, \quad j = 0, 1, \cdots, \tag{6.9}$$

where $s_1 = \nu(s - 1) + s$ and $s_0 = (\nu - 1)(s - 1)$ and

$$C_j = C_1 C_2^j (j!)^{-\epsilon}, \quad \epsilon = \min\{1 - s_0, 1 + \mu - s_1\} = 1 + \mu - s_1 > 0. \tag{6.10}$$

Proof. We shall prove this augument (6.9) by induction. Proposition 6.1 assures (6.9) for $j = 0$. Assume that (6.9) holds for $j - 1$. Then from (6.6) and (6.8)

$$\begin{aligned}||\Phi_j(t,\tau,x,D_x)u||_{H^{q+\delta}_\rho} &\leq \int_\tau^t ||R(t,r,x,D_x) \circ \Phi_{j-1}(r,\tau,x,D_x)u||_{H^{q+\delta}_\rho} dr \\ &\leq C_R \int_\tau^t \left\{\frac{|t - r|^\mu}{(\rho'' - \rho)^{s_1}} + \frac{1}{(\rho'' - \rho)^{s_0}}\right\} \\ &\quad \times C_{j-1}(r - \tau)^{j-1} \left\{\frac{(r - \tau)^{\mu j}}{(\rho' - \rho'')^{s_1 j}} + \frac{1}{(\rho' - \rho'')^{s_0 j}}\right\} dr\|u\|_{H^q_{\rho'}}\end{aligned} \tag{6.11}$$

for $\rho_0 \geq \rho' > \rho'' > \rho \geq 0$. Here we choose $\rho'' = \frac{\rho'+j\rho}{j+1}$. Taking account of the following equalities and inequalities

$$\frac{1}{(\rho''-\rho)^p(\rho''-\rho)^q} = \frac{(a+b)^{p+q}}{a^p b^q}(\rho'-\rho)^{-p-q},$$

where $\rho'' = \frac{a\rho+b\rho'}{a+b}$, $0 < \rho < \rho'$ and $a > 0, b > 0, p > 0, q > 0$,

$$\int_\tau^t (t-r)^a(r-\tau)^b dr = (t-\tau)^{a+b+1}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)},$$

where $a > -1, b > -1$,

$$(A+B)(A^k+B^k) \leq 4(A^{k+1}+B^{k+1}),$$

where $A, B > 0$ and k a positive integer and

$$\left(1+\frac{1}{j}\right)^j \leq e, \quad j = 1, 2, \cdots$$

we can get from (6.11)

$$4e^{s_1}C_{j-1}C_R \max\{\frac{(j+1)^{s_1}\Gamma(j(\mu+1))}{\Gamma((j+1)(\mu+1)}, \frac{(j+1)^{s_1}\Gamma(j)}{\Gamma(j+\mu+1)}, (j+1)^{s_0-1}\} \leq C_j. \quad (6.12)$$

By virture of the assumption $s < 1+\frac{\mu}{\nu}$ we can take $\epsilon = \min\{1+\mu-s_1, 1-s_0\} = 1+\mu-s_1 > 0$ such that

$$\max\left\{\frac{(j+1)^{s_1}\Gamma(j(\mu+1))}{\Gamma((j+1)(\mu+1))}, \frac{(j+1)^{s_1}\Gamma(j)}{\Gamma(j+\mu+1)}, (j+1)^{s_0-1}\right\} \leq Cj^{-\epsilon}.$$

Therefore we can find C_1, C_2 such that C_j given by (6.10) satisfies (6.12). □

From Proposition 6.2, we can observe that $\Phi(t,\tau,x,D_x)$ is well-defined by (6.5), so $V(t,\tau,x,D_x)$ can be also defined by (6.2). Precisely,

Proposition 6.3. *Let $q \in \mathbb{R}$, $u \in H^q_{\rho'}$, $T \geq t \geq \tau \geq 0$ and $\rho_0 \geq \rho' > \rho \geq 0$. Denote $s_1 = s(\sigma+\delta+1)$, $s_0 = s(\sigma+\delta)$ and $\epsilon = \min\{1-s_0, 1+\mu-s_1\} = 1+\mu-s_1$.*
(i) *There exists a constant $C_2 > 0$ such that*

$$||\Phi(t,x;\tau)u||_{H^{q+\delta}_\rho}$$

$$\leq C_2\left(\frac{(t-\tau)^\mu}{(\rho'-\rho)^{s_1}}+\frac{1}{(\rho'-\rho)^{s_0}}\right)\exp\left\{\left[C_2\frac{(t-\tau)^{\mu+1}}{(\rho'-\rho)^{s_1}}+\frac{t-\tau}{(\rho'-\rho)^{s_0}}\right]^{1/\epsilon}\right\}||u||_{H^q_{\rho'}}. \quad (6.13)$$

(ii) *There exists a constant $C_3 > 0$ such that*

$$||V(t,x;\tau)u||_{H^{q+\delta}_\rho} \leq C_3 \Bigg\{ (\rho' - \rho)^{-s_0} + \left(\frac{(t-\tau)^{1+\mu}}{(\rho' - v\rho)^{s_1+s\sigma}} + \frac{t-\tau}{(\rho'-\rho)^{s_0+s\sigma}} \right) \\ \times \exp C_3 \left(\frac{(t-\tau)^{1+\mu}}{(\rho'-\rho)^{s_1}} + \frac{(t-\tau)}{(\rho'-\rho)^{s_0}} \right)^{1/\epsilon} \Bigg\} ||u||_{H^q_{\rho'}}. \tag{6.14}$$

Proof. (i): We note that for a positive integer j and $x > 0$

$$\sup_{x \geq 0} x^j e^{-x} = \frac{x^j}{\sum_{n=0}^{\infty} x^n n!^{-1}} < j!,$$

implies that $\frac{1}{j!^\epsilon} \leq \frac{e^\epsilon}{x^{\epsilon j}}$ for $x > 0$, $\epsilon > 0$ and if we take $x = (2A)^{\frac{1}{\epsilon}}$

$$\sum_{j=0}^{\infty} \frac{A^j}{j!^\epsilon} \leq 2e^{\epsilon (2A)^{\frac{1}{\epsilon}}}$$

for $A > 0$, $\epsilon > 0$. If we take A like

$$A = C_2 \left\{ \frac{(t-\tau)^{1+\mu}}{(\rho' - \rho)^{s_1}} + \frac{(t-\tau)}{(\rho' - \rho)^{s_0}} \right\},$$

we can see that (6.13) holds.
(ii): (6.14) is a direct result of (5.3), (6.2) and of (6.13). □

Put

$$v(t) = V(t,0)v_0 + \int_0^t V(t,r)g(r)dr$$

then, it is obvious that $v(t)$ is a solution of the equation (3.1), moreover, it follows from (6.14) that if we take $\rho' = \rho_0$, $\rho = 0$, there is a constant $C > 0$ such that for $T \geq t > 0$

$$||v(t,\cdot)||_{H^{q+\delta}} \leq C \left(||v_0||_{H^q_{\rho_0}} + \int_0^t ||g(r,\cdot)||_{H^q_{\rho_0}} dr \right). \tag{6.15}$$

Since $u(t,x) = e^{-\Lambda(T-t)} v(t,x)$ satisfies (1.1) and (6.15) implies (1.2), we have completed the proof of Theorem 1.7.

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Institute of Mathematics
Universitiy of Tsukuba
1-1-1 Tennoudai, Tsukuba
Ibaraki 305-8571, Japan
kajitani@math.tsukuba.ac.jp

Nippon Institute of Technology
4-1 Gakuen-dai Miyashiro-cho Minami-Saitama
Saitama, Japan