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On naturally reductive left-invariant metrics of $\mathrm{SL}(2, \mathbb{R})$

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Abstract. On any real semisimple Lie group we consider a one-parameter family of left-invariant naturally reductive metrics. Their geodesic flow in terms of Killing curves, the Levi Civita connection and the main curvature properties are explicitly computed. Furthermore we present a group theoretical revisitation of a classical realization of all simply connected 3-dimensional manifolds with a transitive group of isometries due to L. Bianchi and É. Cartan. As a consequence one obtains a characterization of all naturally reductive left-invariant Riemannian metrics of SL(2, ℝ).

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1. Introduction

Naturally reductive homogeneous spaces may be regarded as a generalization of symmetric spaces and have revealed to be an important source of examples in Riemannian and pseudo-Riemannian geometry. Our interest in these spaces has its origin in the study of their *adapted complexifications*. Indeed in this situation it is possible to characterize and eventually describe the unique maximal adapted complexification associated to the Levi Civita connection (cf. [HI]).

For a real, noncompact semisimple Lie group G let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of its Lie algebra $\mathfrak{g}$ with respect to a maximal compact subalgebra $\mathfrak{k}$. Denote by B the Killing form of G and consider the distinguished one-parameter family of left-invariant metrics ν_m (degenerate for $m = 0$) uniquely defined by

$$\nu_m|_{\mathfrak{g}}(X, Y) = -mB(X_{\mathfrak{k}}, Y_{\mathfrak{k}}) + B(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$$

for any $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ and $Y = Y_{\mathfrak{k}} + Y_{\mathfrak{p}}$ in $\mathfrak{k} \oplus \mathfrak{p} \cong T_eG$. These arise as a natural continuation of the left-invariant naturally reductive Riemannian metrics ν_m, with $m > 0$, appearing in the classification given by C. Gordon in [G]. Note that they are all invariant with respect to the product group $L = G \times K$, with K

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the connected subgroup of G generated by $\mathfrak{k}$, whose factors act on G by left and right multiplication respectively.

By duality the analogous construction is carried out also in the case of G compact semisimple (see Section 3). Then one obtains a family of metrics whose Riemannian elements appear in the classification of all left-invariant naturally reductive Riemannian metrics due to D'Atri and Ziller ([DZ]). In fact one has

Every metric in the above families is naturally reductive.

Here this is shown by explicitly describing the geodesic flow in terms of Killing curves induced by "horizontal elements" in a reductive decomposition of Lie$(G \times K)$ (Proposition 3.1). Moreover the Levi Civita connection for left-invariant vector fields as well as basic properties of sectional and Ricci curvature are computed. Note that for every nondegenerate metric in the above families the projection to the associated symmetric space G/K is a pseudo-Riemannian submersion with totally geodesic fibers.

Let $\Delta \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ be the unit disk in $\mathbb{C}$, denote by θ the Kählerian metric on $\Delta \times \mathbb{C}^* \subset T\Delta$ whose potential is given by the Poincaré squared norm, fix a 3-dimensional orbit for the induced $\mathrm{SL}_2(\mathbb{R})$-action on $T\Delta$ and consider the pull-back $O^*\theta$ via the natural orbit map $O : \mathrm{SL}_2(\mathbb{R}) \to \Delta \times \mathbb{C}^*$. In the last section we introduce a one-parameter family of left-invariant metrics on $\mathrm{SL}_2(\mathbb{R})$ as linear combinations of $O^*\theta$ and the pull-back of the Poincaré form on the unit disk via the natural quotient map. For these the right $\mathrm{SO}_2(\mathbb{R})$-action turns out to be by isometries.

By recalling the classical realization of all simply connected 3-dimensional Riemannian manifolds with a transitive group of isometries due to É. Cartan ([C], cf. the earlier work of L. Bianchi [B]) one shows that, by acting with an automorphism if necessary, every left-invariant Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$ with 4-dimensional group of isometries can be obtained as above (Lemma 4.5). Moreover a simple computation implies that such metrics consist of those in the one-parameter family above indicated which are Riemannian. This closes the circle and as a consequence one has (see Theorem 4.7 for the precise statement)

A left-invariant Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$ *is naturally reductive if and only if its group of isometries is* 4*-dimensional.*

An analogous result holds for the universal covering of $\mathrm{SL}_2(\mathbb{R})$ and its quotients. We wish to point out a pleasant feature of the above description, namely that it makes explicit the Lie group structure of the underlined Riemannian manifolds appearing in the classical realization of É. Cartan.

2. Preliminaries

In this section basic definitions on naturally reductive metrics and some elementary consequences are given. Let M be a pseudo-Riemannian manifold. By definition every element of $\mathrm{Iso}(M) \setminus \{e\}$ acts effectively on M.

Definition 2.1 (cf. [O'N]). A pseudo-Riemannian metric ν on a manifold M is naturally reductive if there exist a connected Lie subgroup L of $\mathrm{Iso}(M)$ acting transitively on M and a decomposition $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{l}$, where $\mathfrak{h}$ is the Lie algebra of the isotropy group H at some point of M, such that $\mathrm{Ad}(H)\,\mathfrak{m} \subset \mathfrak{m}$ and

$$\tilde{\nu}([X, Y]_{\mathfrak{m}}, Z) = \tilde{\nu}(X, [Y, Z]_{\mathfrak{m}})$$

for all $X, Y, Z \in \mathfrak{m}$. Here $[\ ,\ ]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[\ ,\ ]$ and $\tilde{\nu}$ is the pull-back of ν to $\mathfrak{m}$ via the natural projection $L \to L/H \cong M$. In this setting we refer to $\mathfrak{h} \oplus \mathfrak{m}$ as a naturally reductive decomposition and to L/H as a naturally reductive realization of M.

For a naturally reductive realization L/H every geodesic through the base point eH is the orbit of a one-parameter subgroup of L generated by some $X \in \mathfrak{m}$ (see [O'N, page 313]). In fact for a Riemannian homogeneous manifold L/H with an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{h} \oplus \mathfrak{m}$ this property implies that L/H is a naturally reductive realization (see [BTV]).

As a direct consequence of the above definition and for later use we note the following

Lemma 2.2. *Let $M = L/H$ be a pseudo-Riemannian homogeneous space, Γ a discrete central subgroup of L acting properly discontinuously on M and endow $N := \Gamma \setminus M$ with the pushed-down metric. Then $\mathfrak{h} \oplus \mathfrak{m}$ is a naturally reductive decomposition for M if and only if so it is for N.*

Note that here L acts by isometries on $N = \Gamma \setminus L/H \cong L/\Gamma H$, the isotropy in ΓeH is ΓH and the ineffectivity, given by $\{g \in \Gamma H : lgl^{-1} \in \Gamma H \text{ for every } l \in L\}$, does not necessarily coincide with Γ (cf. Section 5). One also has

Lemma 2.3. *Let L be a noncompact semisimple Lie group endowed with a left-invariant Riemannian metric. Then the Lie group exponential and the Riemannian exponential in e do not coincide, i.e, L is not a naturally reductive realization of itself.*

Proof. First recall that for a left-invariant metric ν on L the two exponentials coincide if and only if ν is also right invariant (see, e.g, ex. A5, ch. II in [H]). But then Corollary 21.5 in [M1] would imply that L is compact, giving a contradiction. ☐

3. A family of left-invariant metrics

Let G be a noncompact connected semisimple Lie group and $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of its Lie algebra $\mathfrak{g}$ with respect to a maximal compact Lie subalgebra $\mathfrak{k}$. Denote by B the Killing form on $\mathfrak{g}$ and, for every real m, assign a left-invariant metric ν_m on G by defining its restriction on $\mathfrak{g} \cong T_eG$ as follows

$$\nu_m\big|_{\mathfrak{g}}(X, Y) = -mB(X_{\mathfrak{k}}, Y_{\mathfrak{k}}) + B(X_{\mathfrak{p}}, Y_{\mathfrak{p}}), \tag{3.1}$$

for any $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ and $Y = Y_{\mathfrak{k}} + Y_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p}$. Note that B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, thus such metrics are Riemannian, degenerate or pseudo-Riemannian depending on the sign of m.

Let K be the connected subgroup of G generated by $\mathfrak{k}$ (which is compact if G is a finite covering of a real form of a complex semisimple Lie group). Since $\mathfrak{k}$, $\mathfrak{p}$ and B are $\mathrm{Ad}(K)$-invariant, ν_m is also right K-invariant, *i.e.*, the action of $G \times K$ on G defined by $(g, k) \cdot l := glk^{-1}$ is by isometries. Here we allow discrete ineffectivity given by the diagonal in $Z(G) \times Z(G)$, where $Z(G) \subset K$ is the center of G. One has $G = (G \times K)/H$ with H the diagonal in $K \times K$.

Note that a different choice of a maximal compact connected subalgebra $\mathfrak{k}'$ induces an *equivalent* left-invariant Riemannian structure, *i.e.*, there exists an isometric isomorphism

$$(G, \nu_m) \to (G, \nu_m') \, .$$

This is given by the conjugation transforming $\mathfrak{k}$ in $\mathfrak{k}'$.

If G is compact semisimple, choose a noncompact real form G' of the universal complexification $G^{\mathbb{C}}$ of G, let $K = G \cap G'$ and consider the Cartan decomposition $\mathfrak{g}' = \mathfrak{k} + \mathfrak{p}'$ of $\mathfrak{g}'$ with respect to K. One has the dual $Ad(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p} := i\mathfrak{p}'$. The analogous argument as above shows that the left-invariant metric ν_m defined as in (3.1) is also right K-invariant. However since in this compact case $B|_{\mathfrak{p}}$ is negative definite, this is never Riemannian. Therefore when dealing with curvature properties we will prefer to consider $-\nu_m$. Of course multiplying by a scalar has no influence on the associated Levi Civita connection and geodesic flow which are computed below. Note that a different choice of the noncompact real form G' in $G^{\mathbb{C}}$ does not induce in general equivalent left-invariant metrics on G.

Let us show that these metrics are naturally reductive in both compact and noncompact case.

Proposition 3.1 (cf. [DZ], [G, proof of Th. 5.2). *Let G be a semisimple Lie group and ν_m be the left-invariant metric defined above. Then*

i) *the action of $G \times K$ by left and right multiplication is by isometries,*

ii) *the direct sum $\mathfrak{h} \oplus \mathfrak{m}$, with $\mathfrak{h}$ the isotropy Lie algebra and*

$$\mathfrak{m} := \{ \, (-mX_{\mathfrak{k}} + X_{\mathfrak{p}}, \, -(1+m)\, X_{\mathfrak{k}}) \in \mathfrak{g} \times \mathfrak{k} \, : \, X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p} \, \} \, ,$$

is a naturally reductive decomposition of $\mathfrak{g} \times \mathfrak{k} = Lie(G \times K)$. In particular for every $X = X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p} \cong T_eG$ the unique geodesic through e and tangent to X is given by $\gamma_X : \mathbb{R} \to G$,

$$t \;\longrightarrow\; \exp_G t \, (-mX_{\mathfrak{k}} + X_{\mathfrak{p}}) \, \exp_G t \, (1+m) X_{\mathfrak{k}} \, .$$

Proof. The first part follows from the above considerations. For $ii)$ let $\tilde{\nu}_m$ be the pull-back of ν_m to $\mathfrak{m}$ via the natural projection and consider the elements

$(-mX_{\mathfrak{k}}+X_{\mathfrak{p}}, -(1+m)X_{\mathfrak{k}}),\ (-mY_{\mathfrak{k}}+Y_{\mathfrak{p}}, -(1+m)Y_{\mathfrak{k}}),\ (-mZ_{\mathfrak{k}}+Z_{\mathfrak{p}}, -(1+m)Z_{\mathfrak{k}})$ of $\mathfrak{m}$. Then one has

$$\begin{aligned}
&\tilde{\nu}_m\big([(mX_{\mathfrak{k}}-X_{\mathfrak{p}}, (1+m)X_{\mathfrak{k}}),(mY_{\mathfrak{k}}-Y_{\mathfrak{p}}, (1+m)Y_{\mathfrak{k}})]_{\mathfrak{m}}, (-mZ_{\mathfrak{k}}+Z_{\mathfrak{p}},-(1+m)Z_{\mathfrak{k}})\big)\\
&\ = \nu_m\big(-(1+2m)[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] - m([X_{\mathfrak{k}}, Y_{\mathfrak{p}}]+[X_{\mathfrak{p}}, Y_{\mathfrak{k}}]) + [X_{\mathfrak{p}}, Y_{\mathfrak{p}}],\ Z_{\mathfrak{k}}+Z_{\mathfrak{p}}\big)\\
&\ = m(1+2m)B\big([X_{\mathfrak{k}}, Y_{\mathfrak{k}}],\ Z_{\mathfrak{k}}\big)\\
&\qquad - mB([X_{\mathfrak{p}}, Y_{\mathfrak{p}}],\ Z_{\mathfrak{k}}) - mB([X_{\mathfrak{k}}, Y_{\mathfrak{p}}],\ Z_{\mathfrak{p}}) - mB([X_{\mathfrak{p}}, Y_{\mathfrak{k}}],\ Z_{\mathfrak{p}})
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\nu}_m\big((-mX_{\mathfrak{k}}+X_{\mathfrak{p}},-(1+m)X_{\mathfrak{k}}), [(mY_{\mathfrak{k}}-Y_{\mathfrak{p}},(1+m)Y_{\mathfrak{k}}),(mZ_{\mathfrak{k}}-Z_{\mathfrak{p}},-(1+m)Z_{\mathfrak{k}})]_{\mathfrak{m}}\big)\\
&\ = \nu_m\big(X_{\mathfrak{k}}+X_{\mathfrak{p}},\ -(1+2m)[Y_{\mathfrak{k}}, Z_{\mathfrak{k}}] - m([Y_{\mathfrak{k}}, Z_{\mathfrak{p}}]+[Y_{\mathfrak{p}}, Z_{\mathfrak{k}}]) + [Y_{\mathfrak{p}}, Z_{\mathfrak{p}}]\big)\\
&\ = m(1+2m)B\big(X_{\mathfrak{k}}, [Y_{\mathfrak{k}}, Z_{\mathfrak{k}}]\big) - mB(X_{\mathfrak{k}}, [Y_{\mathfrak{p}}, Z_{\mathfrak{p}}])\\
&\qquad - mB(X_{\mathfrak{p}}, [Y_{\mathfrak{k}}, Z_{\mathfrak{p}}]) - mB(X_{\mathfrak{p}}, [Y_{\mathfrak{p}}, Z_{\mathfrak{k}}]).
\end{aligned}$$

Recalling that $B([X, Y], Z) = B(X, [Y, Z])$ this gives the same result, as was to be proved. □

Let us compute the Levi Civita connection and point out basic curvature properties in terms of left-invariant vector fields of G.

Proposition 3.2. *The Levi Civita connection of* ν_m *is given by*

$$\nabla_X Y = \frac{1}{2}\big([X, Y] + (1+m)\big([X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [Y_{\mathfrak{k}}, X_{\mathfrak{p}}]\big)\big) \tag{3.2}$$

for every $X = X_{\mathfrak{k}} + X_{\mathfrak{p}},\ Y = Y_{\mathfrak{k}} + Y_{\mathfrak{p}} \in \mathfrak{k} + \mathfrak{p}$.

Proof. Since ν_m is left-invariant, then the Levi Civita connection is given by

$$\nabla_X Y = \frac{1}{2}\left([X, Y] - \mathrm{ad}^t_X(Y) - \mathrm{ad}^t_Y(X)\right),$$

where ad^t_X denotes the transpose of ad_X with respect to ν_m (cf. [KN, page 201]). Now a straightforward computation yields

$$\mathrm{ad}^t_X(Y) = -[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] + m[X_{\mathfrak{p}}, Y_{\mathfrak{k}}] - [X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + \frac{1}{m}[X_{\mathfrak{p}}, Y_{\mathfrak{p}}],$$

and the statement follows from the above formula. □

Then one obtains the curvature tensor $R(X, Y)Z = \nabla_{[X,Y]}X - \nabla_X\nabla_Y X + \nabla_Y\nabla_X X$ and all sectional curvatures for nondegenerate planes given, for an orthogonal base $\{X,\ Y\}$, by (see [O'N])

$$K(X, Y) = \frac{\nu_m(R(X, Y)X,\ Y)}{\nu_m(X, X)\nu_m(Y, Y)}.$$

For planes generated by elements which are either in $\mathfrak{k}$ or in $\mathfrak{p}$ one has

Lemma 3.3. *Let G be noncompact semisimple endowed with the metric ν_m and X, Y orthogonal in $\mathfrak{g}$ with squared norm equal to 1 or -1. Then*

i) *if $X = X_{\mathfrak{k}}$ and $Y = Y_{\mathfrak{k}}$, then $K(X, Y) = -\frac{m}{4}B([X, Y], [X, Y])$,*

ii) *if $X = X_{\mathfrak{k}}$ and $Y = Y_{\mathfrak{p}}$, then $K(X, Y) = \frac{m^2}{4}B([X, Y], [X, Y])\nu_m(X, X)$,*

iii) *if $X = X_{\mathfrak{p}}$ and $Y = Y_{\mathfrak{p}}$, then*

$$K(X, Y) = \frac{4 + 3m}{4}B([X, Y], [X, Y]).$$

For G compact semisimple endowed with the metric $-\nu_m$ the above formulas hold with opposite sign.

Proof. First assume that $X = X_k$ and $Y = Y_k$ and note that $K = \exp(\mathfrak{k}) = \mathrm{Exp}_e(\mathfrak{k})$ is totally geodesic in G. Moreover, by construction, the restriction of ν_m on K is left and right K-invariant. Then $i)$ follows from the usual formula for bi-invariant metrics on Lie groups (see [O'N, page 305]).

For ii) let $X = X_{\mathfrak{k}}$ and $Y = Y_{\mathfrak{p}}$. Since $\nu_m(Y, Y) = 1$ and $\nu_m(X, X) = \pm 1$, one has

$$K(X, Y) = \nu_m(\,\nabla_{[X,Y]}X - \nabla_X\nabla_Y X + \nabla_Y\nabla_X X\,,\ Y\,)\nu_m(X, X).$$

By recalling (3.2), ν_m-orthogonality of $\mathfrak{k}$ with $\mathfrak{p}$ and the usual inclusions $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ one obtains

$$\begin{aligned}
&\nu_m\Big(\frac{1}{2}(\,[[X, Y], X] + (1 + m)[X, [X, Y]]\,) - \nabla_X\frac{m}{2}[X, Y]\,,\ Y\Big)\\
&= \nu_m\Big(\frac{m}{2}[X, [X, Y]] - \frac{m(2 + m)}{4}[X, [X, Y]],\ Y\Big)\\
&= -\frac{m^2}{4}\nu_m(\,[X, [X, Y]],\ Y\,) = \frac{m^2}{4}B([X, Y], [X, Y]),
\end{aligned}$$

where in the last line $B(\,[X, [X, Y]],\ Y\,) = -B([X, Y], [X, Y])$ is used. A similar computation yields iii).

The case of G compact follows by recalling that $K(\,\cdot\,,\,\cdot\,)$ is linear with respect to scalar multiplication of the metric. □

Then one has

Proposition 3.4 (cf. [DZ], [G]).

i) *Let G be a noncompact semisimple Lie group endowed with the metric ν_m. Then the sign of sectional curvatures is not definite for $m < -4/3$ and in the Riemannian case $m > 0$.*

ii) *Let G be a compact semisimple Lie group endowed with the metric $-\nu_m$. Then the sign of sectional curvatures is not definite for $m < -4/3$ and in the pseudo-Riemannian case $m > 0$. It is nonnegative for $-1 \le m < 0$*

Proof. For G noncompact $B|_{\mathfrak{k}}$ is negative definite and $B|_{\mathfrak{p}}$ is positive definite, thus i) is a direct consequence of the above lemma.

Similarly one obtains the compact case when $m < -4/3$ and $m > 0$. If G is compact and $-1 < m < 0$ endow $G \times K$ with the bi-invariant Riemannian metric $-\tilde{\nu}_m$ uniquely defined by

$$-\tilde{\nu}_m|_{\mathfrak{g}\times\mathfrak{k}}((X, Z), (Y, W)) = -B(X, Y) + \frac{m}{1+m} B(Z, W)$$

and consider the projection $F : (G \times K, -\tilde{\nu}_m) \to (G, -\nu_m)$, $(g, k) \to gk^{-1}$. Note that F is left G-equivariant, the induced horizontal space at the origin is $\mathfrak{m} := \{ (-mX_{\mathfrak{k}} + X_{\mathfrak{p}}, \, -(1+m)\, X_{\mathfrak{k}}) \in \mathfrak{g} \times \mathfrak{k} \, : \, X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p} \}$ and that the restriction $DF|_{\mathfrak{m}}$ is an isometry. Thus F is a Riemannian submersion.

Now $(G \times K, -\tilde{\nu}_m)$ is the product of two compact Riemannian symmetric spaces, thus all its sectional curvatures are nonnegative and by O'Neill's formula (see [O'N, page 213]) so are those of (G, ν_m). The case $m = -1$ follows by continuity. □

Note that in the compact Riemannian case, *i.e.*, G compact endowed with the metric $-\nu_m$ and $m < 0$, both situations with nonnegative or not definite sectional curvatures may appear.

Consider the Ricci tensor $\mathrm{Ric}(\cdot, \cdot)$ given, for every $X, Y \in \mathfrak{g}$, by the trace of the operator $Z \to R(X, Z)Y$. Since $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal, then one can choose a frame $E_1, \ldots, E_n$ with elements either in $\mathfrak{k}$ or $\mathfrak{p}$ and for both compact and noncompact cases one has (see [O'N, page 87])

$$\mathrm{Ric}(X, Y) = \sum_{j=1}^{n} \nu_m(E_j, \, E_j)\nu_m(\, R(X, E_j)Y, \, Ej \,).$$

Now any $X \in \mathfrak{k}$ with $|\nu_m(X, X)| = 1$ can be completed to a frame $X, E_2, \ldots, E_n$ and consequently one has

$$\mathrm{Ric}(X, X) = \sum_{j=2}^{n} \nu_m(E_j, \, E_j)\nu_m(\, R(X, E_j)X, \, Ej \,) = \nu_m(X, \, X) \sum_{j=2}^{n} K(X, \, E_j).$$

This along with Lemma 3.3 implies that the restriction $\mathrm{Ric}\,|_{\mathfrak{k}}$ is positive definite in both compact and noncompact cases. Moreover one has

Lemma 3.5. *For every $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$ one has* $\mathrm{Ric}(X, Y) = 0$, *i.e., $\mathfrak{k}$ and $\mathfrak{p}$ are Ricci orthogonal.*

Proof. Let $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$ and $E_j \in \mathfrak{k}$ chosen in a frame as above. Then

$$\begin{aligned}
&\nu_m\big(R(X, E_j)Y, Ej \big) = \nu_m\big(\nabla_{[X,E_j]}Y - \nabla_X\nabla_{E_j}Y + \nabla_{E_j}\nabla_X Y\ ,\ Ej \big) \\
&= \frac{1}{2}\nu_m\Big([[X, E_j], Y] + (1+m)[[X, E_j], Y] - \nabla_X\big([E_j, Y] + (1+m)[E_j, Y]\big) \\
&\qquad + \nabla_{E_j}\big([X, Y] + (1+m)[X, Y]\big),\ E_j\Big) \\
&= \frac{2+m}{2}\nu_m\left([[X, E_j], Y] - \frac{2+m}{2}[X, [E_j, Y]] + \frac{2+m}{2}[E_j, [X, Y]],\ E_j\right)
\end{aligned}$$

which, by recalling ν_m-orthogonality of $\mathfrak{k}$ and $\mathfrak{p}$, is zero.

A similar computation shows that $\nu_m(R(X, E_j)Y, Ej) = 0$ also when $E_j \in \mathfrak{p}$, implying that $\mathrm{Ric}(X, Y) = 0$. □

Therefore the signature of Ric is completely determined if one understands its restriction to $\mathfrak{p} \times \mathfrak{p}$. For this a root decomposition of $\mathfrak{g}$ will be the key tool. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, denote by Σ^+ a choice of positive roots for $\mathfrak{a}$ and by $C(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then one has the usual decomposition

$$\mathfrak{g} = C(\mathfrak{a}) \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{g}^{-\alpha},$$

involving the eigenspaces with respect to the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$. Moreover $\sum_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{g}^{-\alpha}$ is invariant with respect to the canonical involution s of $\mathfrak{g}$ which fixes $\mathfrak{k}$. By recalling that $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the 1 and -1 eigenspaces with respect to s, one obtains a decomposition $\delta^+ \oplus \delta^-$ of such direct sum with $\delta^+ \subset \mathfrak{k}$ and $\delta^- \subset \mathfrak{p}$.

Furthermore given a base $U_1, \dots U_r$ of $\sum_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$, with $U_j \in \mathfrak{g}^{\alpha_j}$ for some $\alpha_j \in \Sigma^+$, note that $Z_j := U_j + s(U_j)$ and $W_j := U_j - s(U_j)$, for $j = 1, \dots, r$, determine a base of δ^+ and δ^- respectively. One has:

Lemma 3.6. *Consider $H \in \mathfrak{p}$, choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ containing H and let Z_j, W_j, for $j = 1, \dots, r$, be defined as above. Then:*

i) $R(H, Y)H = 0$ *for every* $Y \in C(\mathfrak{a})$,

ii) $R(H, Z_j)H = \frac{m}{4}\alpha_j(H)^2 Z_j$,

iii) $R(H, W_j)H = -\frac{4+3m}{4}\alpha_j(H)^2 W_j$.

In particular

$$\mathrm{Ric}(H, H) = \sum_{j=1}^{r} -\frac{2+m}{2}\alpha_j(H)^2.$$

Proof. First note that

$$[H, Z_j] = \alpha_j(H)W_j \quad \text{and} \quad [H, W_j] = \alpha_j(H)Z_j.$$

Then

$$\begin{aligned} R(H, Z_j)H &= \nabla_{[H,Z_j]}H - \nabla_H \nabla_{Z_j} H + \nabla_{Z_j} \nabla_H H \\ &= \alpha_j(H)\nabla_{W_j} H - \frac{1}{2}\nabla_H\big([Z_j, H] + (1+m)[Z_j, H]\big) \\ &= \frac{1}{2}\big(\alpha_j(H)[W_j, H] + (2+m)\alpha_j(H)\nabla_H W_j\big) \\ &= \frac{1}{2}\left(-\alpha_j(H)^2 Z_j + \frac{2+m}{2}\alpha_j(H)^2 Z_j\right) = \frac{m}{4}\alpha_j(H)^2 Z_j \end{aligned}$$

which proves ii). A similar computation yields iii), while i) is trivial.

This shows that a base consisting of the Z_j, W_j and elements of $C(\mathfrak{a})$ diagonalizes the operator $X \to R(H, X)H$, thus $\mathrm{Ric}(H, H)$ is given by the sum of the relative eigenvalues, concluding the statement. □

As a consequence of the above lemmas one obtains:

Proposition 3.7. *Let G be a semisimple Lie group endowed with the metric ν_m if G is noncompact, or $-\nu_m$ if it is compact. Then $\mathfrak{k}$, $\mathfrak{p}$ are Ricci orthogonal and* $\mathrm{Ric}|_{\mathfrak{k}}$ *is positive definite. Moreover* $\mathrm{Ric}|_{\mathfrak{p}}$ *is negative definite, null or positive definite depending on the sign of* $2+m$.

4. The case of $\mathrm{SL}_2(\mathbb{R})$

Here different realizations for naturally reductive left-invariant Riemannian metrics on $\mathrm{SL}_2(\mathbb{R})$ are presented. First we show that, by acting with a Lie group automorphism if necessary, such metrics can all be obtained as linear combinations of the pull-back of the Poincaré form on $\Delta \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ and the pull-back from one $\mathrm{SL}_2(\mathbb{R})$-orbit in $\Delta \times \mathbb{C}^* \subset T\Delta$ of the Kählerian metric whose potential is given by the Poincaré squared norm.

By recalling a classical realization of all simply connected 3-dimensional Riemannian manifolds with a transitive group of isometries given by É. Cartan ([C], cf. [V] and the earlier work of L. Bianchi [B]) one characterizes the above metrics by having a 4-dimensional group of isometries L. The connected component of L is given by the product $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R})$ and acts on $\mathrm{SL}_2(\mathbb{R})$ by left and right multiplication. Up to rescaling by a constant, such metrics belong to the one-parameter family given in Proposition 3.1 and an analogous result holds for the universal covering of $\mathrm{SL}_2(\mathbb{R})$ and its quotients.

Let us consider the following realization of the real projective automorphism group

$$\mathrm{PSL}_2(\mathbb{R}) = \{\psi_{a,t} \;:\; a \in \Delta,\; t \in \mathbb{R}\}$$

where $\psi_{a,t} : \Delta \to \Delta$ are the holomorphic transformations of the unit disk in $\mathbb{C}$ defined by

$$\psi_{a,t}(z) := \frac{a + e^{-it}z}{1 + \bar{a}e^{-it}z}.$$

One sees that these consist of all Moebius transformations of Δ. In particular maps composition induces a Lie group structure on $\mathrm{PSL}_2(\mathbb{R})$ such that

$$P : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R}), \qquad g \mapsto C^{-1} \circ g \circ C\,, \tag{4.1}$$

is a surjective Lie group morphism with kernel $\{\pm e\}$, where

$$C : \Delta \to H^+ \quad \text{defined by} \quad z \mapsto i\,\frac{1+z}{1-z}\,,$$

is the Cayley transform from the unit disk to the upper half plane. Here $\mathrm{SL}_2(\mathbb{R})$ acts on $H^+ \subset \mathbb{P}^1(\mathbb{C})$ by linear projective transformations (see, e.g., [A, ch. 1.1]). Note that the restriction of P to $\mathrm{SO}_2(\mathbb{R})$, given by

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \longmapsto \psi_{0,2t}$$

is a two-to-one covering of the compact subgroup $\mathrm{PSO}_2(\mathbb{R}) = \{\psi_{0,t} \,:\, t \in \mathbb{R}\}$ of $\mathrm{PSL}_2(\mathbb{R})$ and one has $\mathrm{PSO}_2(\mathbb{R}) \cong \mathrm{SO}_2(\mathbb{R})/\{\pm e\}$.

Remark 4.1. We have chosen a realization of Moebius group of transformations which is slightly different from the usual one because in what follows it is convenient to have an easy to handle right $\mathrm{PSO}_2(\mathbb{R})$-action on $\mathrm{PSL}_2(\mathbb{R})$.

Note the unique group structure on $\Delta \times \mathbb{R}$ with neutral element $(0, 0)$ making of the universal covering

$$\Delta \times \mathbb{R} \to \mathrm{PSL}_2(\mathbb{R}), \qquad (a, t) \mapsto \psi_{a,t}$$

a Lie group homomorphism. Then one has $\mathrm{PSL}_2(\mathbb{R}) \cong (\Delta \times \mathbb{R}) / (\{0\} \times 2\pi\mathbb{Z})$ and from above it also follows that $\mathrm{SL}_2(\mathbb{R}) \cong (\Delta \times \mathbb{R}) / (\{0\} \times 4\pi\mathbb{Z})$.

Let us introduce a left-invariant metric on $\Delta \times \mathbb{R}$ as follows. On the trivial holomorphic tangent bundle $\Delta \times \mathbb{C} \cong T\Delta$ of the unit disc consider the squared norm function $\rho : \Delta \times \mathbb{C} \to \mathbb{R}^{\geq 0}$

$$\rho(z, w) = \frac{|w|^2}{(1-|z|^2)^2}$$

induced by the Poincaré metric $\omega = \frac{dz d\bar{z}}{(1-|z|^2)^2}$ on Δ. By direct computation one has

$$\partial\bar{\partial}\,\rho = \frac{1}{(1-|z|^2)^2}\left\{ \frac{2|w|^2(1+2|z|^2)}{(1-|z|^2)^2}\, dz \wedge d\bar{z} \right.$$
$$\left. + \frac{2}{(1-|z|^2)}\,(\bar{z}w\, dz \wedge d\bar{w} + z\bar{w}\, dw \wedge d\bar{z}) \;+\; dw \wedge d\bar{w} \right\}$$

and it is straightforward to check that ρ is strictly plurisubharmonic for $w \neq 0$. Therefore it induces a Kählerian metric on $\Delta \times \mathbb{C}^*$ defined by

$$\theta(\,\cdot\,,\,\cdot\,) \;=\; -\frac{i}{2}\,\partial\bar{\partial}\,\rho(J\,\cdot\,,\,\cdot\,), \tag{4.2}$$

with J the complex structure on $T(\Delta \times \mathbb{C}^*)$. For $\xi + i\eta := z$ and $\sigma + i\tau := w$ one has

$$\begin{aligned}\theta = \frac{1}{(1-|z|^2)^2}\Bigg\{ & \frac{2|w|^2(1+2|z|^2)}{(1-|z|^2)^2}\,(d\xi^2 + d\eta^2)\\ & + \frac{4}{(1-|z|^2)}\Big(\mathrm{Re}(\bar{z}w)(d\xi d\sigma + d\eta d\tau) + \mathrm{Im}\,(\bar{z}w)(d\xi d\tau - d\eta d\sigma)\Big)\\ & + (d\sigma^2 + d\tau^2)\Bigg\}\end{aligned}$$

and consequently

$$\begin{aligned}\theta = {} & \frac{2|w|^2(1+2|z|^2)}{(1-|z|^2)^4}\,(d\xi^2 + d\eta^2) \;+\; \frac{1}{(1-|z|^2)^2}\,(d\sigma^2 + d\tau^2)\\ & + \frac{4}{(1-|z|^2)^3}\,\big(\,(\xi\sigma + \eta\tau)\,(d\xi d\sigma + d\eta d\tau) \;+\; (\xi\tau - \eta\sigma)\,(d\xi d\tau - d\eta d\sigma)\,\big).\end{aligned}$$

Since ω is $\mathrm{PSL}_2(\mathbb{R})$-invariant, so are ρ and θ with respect to the induced action on $\Delta \times \mathbb{C}^*$ given by $\psi_{a,t}\cdot(z,w) \;=\; (\,\psi_{a,t}(z), (d\psi_{a,t})_z(w)\,)$. Furthermore, $\mathrm{PSL}_2(\mathbb{R})$ acts freely on $\Delta \times \mathbb{C}^*$, thus the pull-back of the restriction of θ on any $\mathrm{PSL}_2(\mathbb{R})$-orbit defines a left-invariant metric on $\mathrm{PSL}_2(\mathbb{R})$ and consequently on its universal covering $\Delta \times \mathbb{R}$. We wish to compute it for the $(\Delta \times \mathbb{R})$-orbit through $(0,1) \in \Delta \times \mathbb{C}^*$.

Lemma 4.2. *Consider the orbit map* $O : \Delta \times \mathbb{R} \to \Delta \times \mathbb{C}^*$,

$$(x+iy, t) \;\to\; \psi_{x+iy,t}\cdot(0,1) \;=\; (\,x+iy\,,\,(\,1-(x^2+y^2)\,)\,e^{-it}\,)\,.$$

The pull-back of θ *via* O *defines a left-invariant metric on* $\Delta \times \mathbb{R}$ *given by*

$$O^*\theta \;=\; 2\,\frac{d^2x \;+\; d^2y}{(1-(x^2+y^2))^2} \;+\; \left(dt \;+\; 2\,\frac{y\,dx \;-\; x\,dy}{1-(x^2+y^2)}\right)^2.$$

Proof. Let $a := x + iy$. By direct computation one has

$$\begin{aligned}O^*\theta = \frac{1}{(1-|a|^2)^2}\Big\{ & 2(1+2|a|^2)(dx^2+dy^2)\\ & + d((1-|a|^2)\cos t)^2 + d((1-|a|^2)\sin t)^2\\ & + 4\big[\,(x\cos t - y\sin t)\,(dx\,d((1-|a|^2)\cos t) \;-\; dy\,d((1-|a|^2)\sin t))\\ & + (x\sin t + y\cos t)(dx\,d((1-|a|^2)\sin t)) + dy\,d((1-|a|^2)\cos t)\,\big]\Big\}\,.\end{aligned}$$

That is,

$$\frac{1}{(1-|a|^2)^2}\Big\{2(1+2|a|^2)(dx^2+dy^2)+4x^2dx^2+4y^2dy^2$$
$$+(1+|a|^2)^2dt^2+8xydxdy$$
$$+4\big[-2x^2dx^2-2y^2dy^2-4xydxdy-x(1-|a|^2)dydt+y(1-|a|^2)dxdt\big]\Big\}$$

which becomes

$$2\frac{dx^2+dy^2}{(1-|a|^2)^2}+dt^2+4\frac{ydxdt-xdydt}{1-|a|^2}+4\frac{y^2dx^2-2xydxdy+x^2dy^2}{(1-|a|^2)^2},$$

implying the statement. □

Regarding the pull-back of θ on different orbits note that $\{0\}\times\mathbb{R}^{>0}$ is a global slice with respect to the $\Delta\times\mathbb{R}$-action on $\Delta\times\mathbb{C}^*$. That is, every orbit in $\Delta\times\mathbb{C}^*$ is of the form $(\Delta\times\mathbb{R})\cdot(0,l)$ for some positive l and an analogous computation as the one above shows that, up to rescaling by l^2, one obtains the same metric. Now note that

$$\frac{d^2x+d^2y}{(1-(x^2+y^2))^2}$$

is the the pull-back on $\Delta\times\mathbb{R}$ of the Poincaré form on Δ. In particular it is a left-invariant, semi definite symmetric form. Then any (positive definite) metric obtained as linear combination of it and $O^*\theta$ is left-invariant. As a simple consequence of Lemma 4.2 one has:

Lemma 4.3. *Up to rescaling, all Riemannian metrics on $\Delta\times\mathbb{R}$ obtained as linear combinations of the pull-back of the Poincaré form form on Δ and $O^*\theta$ are of the form*

$$\frac{dx^2+dy^2}{(1-(x^2+y^2))^2}+m\left(\frac{1}{2}dt+\frac{y\,dx-x\,dy}{1-(x^2+y^2)}\right)^2, \tag{4.3}$$

with $m\in\mathbb{R}^{>0}$.

Denote by K and G the Lie groups $S0_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ respectively and consider their universal coverings $\widetilde{K}=\{0\}\times\mathbb{R}\subset\Delta\times\mathbb{R}=\widetilde{G}$. Notice that the above introduced orbit map $O:\Delta\times\mathbb{R}\to\Delta\times\mathbb{C}^*$ is equivariant with respect to the right $\widetilde{K}$-action on $\widetilde{G}$ and rotations on the second component of $\Delta\times\mathbb{C}^*$. Moreover these leave the squared norm function ρ and $\theta(\cdot,\cdot)=-\frac{i}{2}\partial\bar\partial\rho(J\cdot,\cdot)$, invariant. It follows that the pull-back $O^*\theta$ on $\widetilde{G}$ is invariant with respect to the right $\widetilde{K}$-action.

Since the "Poincaré form" on $\widetilde{G}$ is also right $\widetilde{K}$-invariant, the left and right action of $\widetilde{G}\times\widetilde{K}$, given by $(h,k)\cdot g=hgk^{-1}$, is by isometries for every metric on $\widetilde{G}$ of the form (4.3). One also checks that the ineffectivity of this action is given by the discrete subgroup $\Lambda\cong\mathbb{Z}$ generated by $(\bar g,\bar g)\in\widetilde{G}\times\widetilde{K}$, where $\bar g=(0,2\pi)$ is the generator of the center in $\Delta\times\mathbb{R}=\widetilde{G}$.

Lemma 4.4. *For every Riemannian metric* μ *of the form* (4.3) *the Lie group* $\widetilde{G} \times \widetilde{K}/\Lambda$ *is the connected component of* $\mathrm{Iso}(\widetilde{G}, \mu)$ *and acts on* $\widetilde{G}$ *by left and right multiplication.*

Proof. From above it follows that $\widetilde{G} \times \widetilde{K}/\Lambda \subset \mathrm{Iso}(\widetilde{G}, \mu)$, therefore it is enough to show that $\mathrm{Iso}(\widetilde{G}, \mu)$ is 4-dimensional. For this note that the isotropy of $\mathrm{Iso}(\widetilde{G}, \mu)$ at any point of $\widetilde{G}$ is isomorphic to a subgroup of $O_3(\mathbb{R})$, thus it cannot have dimension two. As a consequence the only other possible dimension for $\mathrm{Iso}(\widetilde{G}, \mu)$ is six. However in this case one would have constant Ricci curvatures, contradicting Corollary 4.7 in [M2] and proving the statement. □

Lemma 4.5. *Let* $\widetilde{G}$ *be the universal covering of* G. *Then*

i) *Every left-invariant Riemannian metric on* $\widetilde{G}$ *with 4-dimensional group of isometries is, up to rescaling, the pull-back of a metric of the form* (4.3) *via a Lie group automorphism of* $\widetilde{G}$.
ii) *Let* Γ *be a discrete central subgroup of* $\widetilde{G}$. *Then the analogous statement holds for any left-invariant Riemannian metric on* $\Gamma \setminus \widetilde{G}$ *with 4-dimensional group of isometries and for push-downs to* $\Gamma \setminus \widetilde{G}$ *of metrics on* $\widetilde{G}$ *of the form* (4.3).

Proof. From [V, pages 350-357] it follows that every simply connected 3-dimensional Riemannian manifold with a transitive action of $\widetilde{G}$ and 4-dimensional group of isometries is isometrically equivalent to $\frac{1}{\sqrt{n}}\Delta \times \mathbb{R}$ endowed with a metric of the form

$$\frac{dx^2 + dy^2}{(1 - n(x^2 + y^2))^2} + \left(dt + \frac{l}{2}\,\frac{y\,dx - x\,dy}{1 - n(x^2 + y^2)}\right)^2$$

for some positive n and l. By applying the linear reparametrization $(x', y', t') = (\sqrt{n}\,x\,,\ \sqrt{n}\,y\,,\ 4\frac{n}{l}t\,)$ one sees that, up to rescaling, these correspond to the one-parameter family of metrics on $\Delta \times \mathbb{R}$ given in (4.3).

In particular given a left-invariant metric ν on $\widetilde{G}$ with 4-dimensional group of isometries one obtains, possibly after rescaling, an isometry $F : (\widetilde{G}, \nu) \to (\widetilde{G}, \mu)$, where μ is a metric of the form (4.3). Moreover by left-invariance we may assume that $F(e) = e$.

We need to show that this is a Lie group automorphism. For this note that F induces an injective Lie group morphism $\hat{F}$ from the group of left translations $\widetilde{G} \subset \mathrm{Iso}(\widetilde{G}, \nu)$ to the connected component $\widetilde{G} \times \widetilde{K}/\Lambda$ of $\mathrm{Iso}(G, \mu)$ given by $\hat{F}(h) \cdot x := F \circ h \circ F^{-1}(x)$.

A simple computation at the level of Lie algebras implies that $\hat{F}_*(X) = (\varphi_*(X), 0)$ for all $X \in \mathrm{Lie}(\widetilde{G}) = \mathfrak{sl}_2(\mathbb{R})$, where $\varphi_* : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$ is a Lie algebras morphism. Hence $\hat{F}(g) = [\varphi(g), e]$, where φ is the Lie group automorphism of $\widetilde{G}$ induced by φ_*. Finally from the definition of $\hat{F}$ it follows that the diagram

$$\begin{array}{ccc} \widetilde{G} & \stackrel{\hat{F}}{\to} & (\widetilde{G} \times \widetilde{K})/\Lambda \\ \downarrow & & \downarrow \\ (\widetilde{G},\ \nu) & \stackrel{F}{\to} & (\widetilde{G},\ \mu) \end{array}$$

is commutative, where the vertical arrows are the orbit maps through e. As a consequence $F = \varphi$, which proves i).

Now let Γ be a discrete central subgroup of $\widetilde{G}$ and ν be a left-invariant metric on $\Gamma \setminus \widetilde{G}$ with 4-dimensional group of isometries L. Then the pull-back $\tilde{\nu}$ on $\widetilde{G}$ of ν yields a left-invariant metric on $\widetilde{G}$ and the universal covering $\widetilde{L}$ of L acts by isometries on $\widetilde{G}$. Then the analogous argument as in the proof of Lemma 4.4 implies that $\dim \mathrm{Iso}(\widetilde{G}, \tilde{\nu}) = 4$.

From i) it follows that the there exists a metric $\tilde{\mu}$ of the form (4.3) and a Lie group automorphism $\widetilde{F}$ such that $\tilde{\nu} = \widetilde{F}^*(\tilde{\mu})$. Now note that the center $C \cong \{0\} \times 2\pi\mathbb{Z}$ of $\widetilde{G} \cong \Delta \times \mathbb{R}$ is isomorphic to $\mathbb{Z}$ and since $\widetilde{F}(C) = C$ and $\Gamma \subset C$, it follows that $\widetilde{F}(\Gamma) = \Gamma$. Thus $\widetilde{F}$ pushes down to a Lie group automorphism F of $\Gamma \setminus \widetilde{G}$ and one has a commutative diagram

$$\begin{array}{ccc} (\widetilde{G},\ \tilde{\nu}) & \stackrel{\widetilde{F}}{\to} & (\widetilde{G},\ \tilde{\mu}) \\ \downarrow & & \downarrow \\ (\Gamma \setminus \widetilde{G},\ \nu) & \stackrel{F}{\to} & (\Gamma \setminus \widetilde{G},\ \mu)\,, \end{array}$$

where the vertical arrows are the canonical quotients and μ is the push-down on $\Gamma \setminus \widetilde{G}$ of $\tilde{\mu}$. By construction $\widetilde{F}$ and the quotient maps are local isometries, thus $\nu = F^*\mu$, which proves ii). □

Identify as usual $T_eG \cong \mathfrak{g}$ with the 2×2 matrixes of zero trace and consider its basis $\{U,\ H,\ W\}$, with

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 4.6. *The push-down to G of a metric μ of the form (4.3) is the unique left-invariant metric whose restriction to $T_eG \cong \mathfrak{g}$ is represented in the base $\{U,\ H,\ W\}$ by the diagonal matrix*

$$\begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

i.e, up to rescaling $\mu|_{\mathfrak{g}} = -mB|_{\mathfrak{k}} + B|_{\mathfrak{p}}$, *with* $\mathfrak{k} = \mathfrak{so}_2(\mathbb{R})$. *In particular it is naturally reductive and the geodesic flow through* e *is given by*

$$Exp_e\,(uU + aH + bW) = \exp_G(-muU + aH + bW)\,\exp_K((1+m)uU),$$

with u, a, b *real.*

Proof. Let $P : G \to \mathrm{PSL}_2(R)$ be defined as in (4.1) and consider the following commutative diagram

$$\begin{array}{ccc} & \widetilde{G} \cong \Delta \times \mathbb{R} & \\ \swarrow & & \searrow \\ G & \xrightarrow{P} & \mathrm{PSL}_2(\mathbb{R})\,, \end{array}$$

where the diagonal arrows denote the universal coverings of G and of $\mathrm{PSL}_2(\mathbb{R})$. All maps are Lie group morphisms with discrete central kernel, thus any left-invariant metric on $\widetilde{G}$ pushes down to a unique left-invariant metric on G and $\mathrm{PSL}_2(\mathbb{R})$ making of P a local isometry. Consider local coordinates x, y and t in a neighborhood of the neutral element $\psi_{0,0}$ induced by the universal covering $\Delta \times \mathbb{R} \to \mathrm{PSL}_2(\mathbb{R})$, $(x+iy, t) \to \psi_{x+iy,t}$ and note that

$$P_*(U) = \frac{d}{ds}\bigg|_0 P(\exp(s\,U)) = \frac{d}{ds}\bigg|_0 \psi_{0,2s} = 2\frac{d}{dt}\bigg|_0,$$

$$P_*(H) = \frac{d}{ds}\bigg|_0 P(\exp(s\,H)) = \frac{d}{ds}\bigg|_0 \psi_{\frac{e^{2s}-1}{e^{2s}+1},0} = \frac{d}{dx}\bigg|_0$$

and

$$P_*(W) = \frac{d}{ds}\bigg|_0 P(\exp(s\,W)) = \frac{d}{ds}\bigg|_0 \psi_{i\frac{\sinh(s)}{\cosh(s)},0} = -\frac{d}{dy}\bigg|_0,$$

Then it follows that the restriction to T_eG of the push-down on G of a metric of the form (4.3) on $\widetilde{G}$ is as claimed. The last part is a direct consequence of Proposition 3.1. □

Let us collect the above information for $G = \mathrm{SL}_2(\mathbb{R})$ as follows.

Theorem 4.7. *Let* $\widetilde{G}$ *be the universal covering of* G (*and of* $\mathrm{PSL}_2(\mathbb{R})$) *and* $\widetilde{K}$ *that of* K (*and of* $\mathrm{PSO}_2(\mathbb{R})$).

i) *A left-invariant Riemannian metric on* $\widetilde{G}$ *is naturally reductive if and only if it has 4-dimensional group of isometries. By acting with a Lie group automorphism if necessary, such metrics can all be realized as linear combinations of the "Poincaré form" and the pull-back via an orbit map to* $\Delta \times \mathbb{C}^* \subset T\Delta$ *of* *the Kählerian metric* θ *given in* (4.2). *Up to rescaling, they consist of a one-parameter family explicitly given, in the coordinates* $(x+iy, t) \in \Delta \times \mathbb{R} \cong \widetilde{G}$, *by the expression*

$$\frac{dx^2 + dy^2}{(1-(x^2+y^2))^2} + m\left(\frac{1}{2}dt + \frac{ydx - xdy}{1-(x^2+y^2)^2}\right)^2,$$

with $m \in \mathbb{R}^{>0}$. *For all of them the connected component of the group of isometries is* $\widetilde{G} \times \widetilde{K}/\Lambda$ *acting on* $\widetilde{G}$ *by* $[h, k] \cdot g = hgk^{-1}$. *Here* $\Lambda \cong \mathbb{Z}$ *is the discrete ineffectivity.*

ii) *A left-invariant Riemannian metric on* G *or on* $\mathrm{PSL}_2(\mathbb{R})$ *is naturally reductive if and only if it has 4-dimensional group of isometries. By acting with a Lie group automorphism and up to rescaling it can be obtained as the push-down of an element of the family in* i). *For all of them the connected components of the groups of isometries are* $G \times K/\{\pm(e, e)\}$ *and* $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSO}_2(\mathbb{R})$ *acting, on* G *and* $\mathrm{PSL}_2(\mathbb{R})$ *respectively, by left and right multiplication. Furthermore their restriction to* $T_eG \cong \mathfrak{g}$ *is represented in the base* $\{U, H, W\}$ *by the diagonal matrix*

$$\begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Given a metric of the form (4.3) consider the realization L/H of $\widetilde{G}$ as a Riemannian homogeneous space, where $L := \widetilde{G} \times \widetilde{K}/\Lambda$ and $H := \{[k, k] \in L : k \in \widetilde{K}\}$ is the isotropy of L at e.

For any discrete subgroup Γ in the center $C \cong \mathbb{Z}$ of $\widetilde{G}$ let $\Gamma' := \{[g, e] \in L : g \in \Gamma\}$ and note that

$$\Gamma \setminus \widetilde{G} = \Gamma' \backslash L/H.$$

Then from Lemma 2.2 it follows that a metric on $\widetilde{G}$ of the form (4.3) is naturally reductive if and only if so is its push-down to $\Gamma \setminus \widetilde{G}$. Since by Lemma 4.6 this holds true for $G = (\{0\} \times 4\pi\mathbb{Z}) \setminus \widetilde{G}$, it also holds true for $\widetilde{G}$ and $\mathrm{PSL}_2(\mathbb{R}) = C \backslash \widetilde{G}$, where $\tilde{G} \cong \Delta \times \mathbb{R}$ as usual and $C = \{0\} \times 2\pi\mathbb{Z}$. Furthermore the action of L pushes down to an action of $(\Gamma \setminus \widetilde{G}) \times (\Gamma \setminus \widetilde{K})/E$, where the ineffectivity E is generated by $([\bar{g}], [\bar{g}])$, with $[\bar{g}]$ a generator of $\Gamma \setminus C$. In the cases of $\Gamma = \{0\} \times 4\pi\mathbb{Z}$ and $\Gamma = C$ this yields, by Lemma 4.4. the connected components of the group of isometries given in ii) for G and $\mathrm{PSL}_2(\mathbb{R})$ respectively.

Conversely given a left-invariant naturally reductive Riemannian metric ν on $\Gamma \setminus \widetilde{G}$ the same argument as in the proof of Lemma 4.4 implies that the dimension of the group of isometries cannot be neither five nor six. Dimension three is also to be excluded because of Lemma 2.3, thus $\dim \mathrm{Iso}(\Gamma \setminus \widetilde{G}, \nu) = 4$. In view of Lemmas 4.5 and 4.6 this concludes the proof. □

Remark 4.8. The same argument as above applies to prove a similar statement for all quotients of the form $\Gamma \setminus \widetilde{G}$, with Γ is a discrete subgroup in the center $C \cong \mathbb{Z}$ of $\widetilde{G}$. In this case the connected component of the group of isometries is given by $(\Gamma \setminus \widetilde{G}) \times (\Gamma \setminus \widetilde{K})/E$, where the ineffectivity E is as in the above proof.

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