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p-Harmonic measure is not additive on null sets

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Dedicated to the memory of Tom Wolff.
Without his work this note would not have been possible.

Abstract. When $1 < p < \infty$ and $p \neq 2$ the p-harmonic measure on the boundary of the half plane $\mathbb{R}^2_+$ is not additive on null sets. In fact, there are finitely many sets E_1, E_2,...,E_κ in $\mathbb{R}$, of p-harmonic measure zero, such that $E_1 \cup E_2 \cup ... \cup E_\kappa = \mathbb{R}$.

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1. Introduction

We consider the p-harmonic measure associated to the operator

$$L_p(u) = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),$$

the p-Laplacian of a function u, for $1 < p < \infty$. A p-harmonic function in a domain $\Omega \subseteq \mathbb{R}^n (n \geq 2)$ is a weak solution of $L_p u = 0$; that is, $u \in W^{1,p}_{\text{loc}}(\Omega)$ and

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle\, dx = 0$$

whenever $\varphi \in C_0^\infty(\Omega)$. Weak solutions of $L_p(u) = 0$ are indeed in the class $C^{1,\alpha}_{\text{loc}}$, where α depends only on p and n ([DB], [L1].) A lower semicontinuous $v : \Omega \to \mathbb{R} \cup \{\infty\}$ is p-superharmonic provided that $v \not\equiv \infty$, and for each open $D \subset \overline{D} \subset \Omega$ and each u continuous on $\overline{D}$ and p-harmonic in D, the inequality $v \geq u$ on ∂D implies $v \geq u$ in D.

Let E be a subset of $\partial\Omega$. Consider the class $\mathcal{C}(E, \Omega)$ of nonnegative p-superharmonic functions v in Ω such that

$$\liminf_{X \in \Omega, X \to \zeta} v(X) \geq \chi_E(\zeta)$$

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for all $\zeta \in \partial\Omega$. The *p-harmonic measure* of the set E relative to the domain Ω is the function $\omega_p(., E, \Omega)$ whose value at any $X \in \Omega$ is given by

$$\omega_p(X, E, \Omega) = \inf\{v(X) : v \in \mathcal{C}(E, \Omega)\} .$$

We often omit the variable X and the domain Ω and write $\omega_p(E, \Omega)$ or just $\omega_p(E)$. The function $\omega_p(E, \Omega)$ is p-harmonic in Ω, satisfies

$$0 \le \omega_p(E, \Omega) \le 1,$$

and $\omega_p(E, \Omega)$ has boundary values 1 at all regular points interior to E and boundary values 0 at all regular points interior to $\partial\Omega \setminus E$. For these and additional potential-theoretic properties of the p-Laplacian see [GLM] and the book [HKM].

When $p = 2$ harmonic functions have the mean value property. Suppose Ω is a Dirichlet regular domain, then $\omega_2(X, \cdot, \Omega)$ is a probability measure on $\partial\Omega$ and the integral

$$\int_{\partial\Omega} f(\zeta)\, d\omega_2(X, \zeta, \Omega)$$

gives the solution to the Dirichlet problem for a given boundary data function f.

When $p \ne 2$, due to the nonlinearity of the p-Laplacian, p-harmonic functions need not satisfy the mean value property and the sum of two p-harmonic functions need not be p-harmonic. Consequently $\omega_p(X, \cdot, \Omega)$ is not additive on $\partial\Omega$, hence not a measure.

Very little is known about measure-theoretic properties of p-harmonic measure when $p \ne 2$. Assume that Ω is Dirichlet regular. Then for all compact subsets E of the boundary $\partial\Omega$ we have

$$\omega_p(E, \Omega) + \omega_p(\partial\Omega \backslash E, \Omega) = 1; \tag{1.1}$$

and if E and F are both compact, disjoint, and $\omega_p(E, \Omega) = \omega_p(F, \Omega) = 0$ then

$$\omega_p(E \cup F, \Omega) = 0. \tag{1.2}$$

These results can be found in [GLM] and also in [HKM].

Some conditions on the smallness of a compact set F in terms of Hausdorff dimension or capacity that imply $\omega_p(E \cup F, \Omega) = \omega_p(E, \Omega)$ can be found in [AM], [K] and [BBS].

Martio asked in [M1] whether p-harmonic measure defines an outer measure on the zero level; i.e., whether (1.2) remains true when E and F are allowed to intersect and to be noncompact.

In this note we answer Martio's question negatively by showing that ω_p is not additive on null sets when $p \ne 2$. We construct an example when $\Omega = \mathbb{R}^2_+$ is the upper half-space and $\partial\Omega = \mathbb{R}$. We may consider the point at infinity as a part of the boundary but it is not difficult to see that $\omega_p(\{\infty\}, \mathbb{R}^2_+) = 0$. Points in $\mathbb{R}^2_+$ will be denoted by (x, y) or X interchangeably.

Theorem 1.1. *Let* $1 < p < \infty$ *and* $p \neq 2$. *Then there exist finitely many sets* $E_1, E_2, \ldots, E_\kappa$ *on* $\mathbb{R}$ *such that*

$$\omega_p(E_k, \mathbb{R}^2_+) = 0, \quad \omega_p\left(\mathbb{R} \setminus E_k, \mathbb{R}^2_+\right) = 1, \quad \text{and} \quad \bigcup_{k=1}^{\kappa} E_k = \mathbb{R}.$$

Furthermore, the sets E_k *verify* $|\mathbb{R} \setminus E_k| = 0$.

Here $|.|$ stands for Lebesgue measure on the real line.

Corollary 1.2. *There exist A and* $B \subseteq \mathbb{R}$ *such that*

$$\omega_p(A, \mathbb{R}^2_+) = \omega_p(B, \mathbb{R}^2_+) = 0 \quad \text{and} \quad \omega_p(A \cup B, \mathbb{R}^2_+) > 0.$$

Thus $\omega_p(\cdot, \mathbb{R}^2_+)$ *is not additive on null sets.*

Corollary 1.3. *Let* $1 < p < \infty$ *and* $p \neq 2$. *Then* $\omega_p(X, \cdot, \mathbb{R}^2_+)$ *is not a Choquet capacity for each* $X \in \Omega$. *In fact there exists an increasing sequence of sets* $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_j \subseteq \cdots \subseteq \mathbb{R}$ *so that*

$$\lim_{j\to\infty} \omega_p(B_j) < \omega_p\left(\bigcup_{j=1}^{\infty} B_j\right).$$

To prove Corollary 1.2, choose $k_0 = \min\{k : \omega_p(E_1 \cup E_2 \cup \ldots E_k) > 0\}$ and let $A = E_1 \cup E_2 \cup \ldots E_{k_0-1}$, $B = E_{k_0}$.

Corollary 1.3 follows from Theorem 1.1 as in the tree case given in [KLW]. The definition of Choquet capacity can be found in [HKM].

Both the Theorem and its corollaries can be extended to $\mathbb{R}^n_+ (n \geq 3)$ by adding $n - 2$ dummy variables.

Until recently, there has been no ground for conjecturing the answer to Martio's and some other questions about p-harmonic measures. A sequence of papers [CFPR], [KW], [ARY] and [KLW], is devoted to studying p-harmonic measure and Fatou theorem for bounded p-harmonic functions in an overly simplified model – forward directed regular κ-branching trees. On such trees, Theorem 1 is proved and for each fixed p the exact value of the minimum of Hausdorff dimension of Fatou sets over all bounded p-harmonic functions is given in [KW] and [KLW].

In [KLW] the construction of the sets in Theorem 1 for trees starts with a basic p-harmonic function u that does not satisfy the mean value property, follows with a Riesz product and then a stopping time argument. It is really quite simple. In $\mathbb{R}^2_+$ we follow the same procedures. The basic p-harmonic function is infinitely more complicated and is provided by remarkable examples of Wolff for $2 < p < \infty$, and of Lewis for $1 < p < 2$ ([Wo1], [Wo2] and [L2]). On a tree there is a perfect independence among branches and the Riesz product includes all generations; in $\mathbb{R}^2_+$ we obtain an approximate independence by introducing large gaps in the Riesz product. Finally, instead of a stopping time argument, we use an ingenious lemma of Wolff [Wo1] on gap series of p-harmonic functions, to estimate the p-harmonic function whose boundary values are given by an infinite product.

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2. Preliminaries

In this section we recall several properties of p-harmonic functions which are needed in the proofs.

If $u(X)$ is p-harmonic and $c \in \mathbb{R}$, then $c + u(X)$, $cu(X)$ and $u(cX)$ are p-harmonic. If u is a nonnegative p-harmonic function in Ω and B is a ball such that $2B \subseteq \Omega$, then $\sup_B u \leq C \inf_B u$ for some $C = C(n, p) > 0$ (Harnack inequality). A nonconstant p-harmonic function in a domain cannot attain its supremum or infimum (Strong Maximum Principle). If a sequence of p-harmonic functions converges uniformly then the limit is also p-harmonic.

We list now some basic properties of p-harmonic measure.

1. If $\omega_p(X, E, \Omega) = 0$ at some $X \in \Omega$ then $\omega_p(Y, E, \Omega) = 0$ for any other $Y \in \Omega$ by Harnack inequality.
2. If $E_1 \subseteq E_2 \subseteq \partial\Omega$ then $\omega_p(E_1, \Omega) \leq \omega_p(E_2, \Omega)$ (monotonicity).
3. If $\Omega_1 \subseteq \Omega_2$ and $E \subseteq \partial\Omega_1 \cap \partial\Omega_2$ then $\omega_p(E, \Omega_1) \leq \omega_p(E, \Omega_2)$ (Carleman's principle).
4. If $E_1 \supseteq E_2 \supseteq, \ldots, \supseteq E_j \supseteq \ldots$ are closed sets on $\partial\Omega$, then

$$\omega_p \left(\bigcap_{j=1}^{\infty} E_j, \Omega \right) = \lim_{j \to \infty} \omega_p(E_j)$$

(upper semicontinuity on closed sets).

See chapter 11 in [HKM] for these properties.

We follow [Wo1] and let $W^{p|\lambda}$ be the class of all functions $f : \mathbb{R}^2_+ \to \mathbb{R}$ which are λ-periodic in the x variable ($f(x + \lambda, y) = f(x, y)$) and satisfy

$$\|f\|^p_{p|\lambda} = \int_{[0,\lambda)\times(0,\infty)} |\nabla f(x, y)|^p \, dx\, dy < \infty\,,$$

where the gradient is taken in the sense of distributions. If $f \in W^{p|\lambda}$ then the function f has a well-defined trace on $\mathbb{R}$; and among the functions g such that $g - f \in W^{p|\lambda}$ has trace 0 on $\mathbb{R}$, there is a unique g, denoted by $\hat{f}$, which minimizes $\|g\|_{p|\lambda}$. The function $\hat{f}$ is the unique p-harmonic function in $\mathbb{R}^2_+$ with boundary values f on $\mathbb{R}$. Moreover, there exists $\xi \in \mathbb{R}$ so that

$$|\hat{f}(x, y) - \xi| \leq 2e^{-\dfrac{\gamma y}{\lambda}} \|f\|_\infty$$

for some $\gamma = \gamma(p) > 0$, [Wo1]. Extend then $\hat{f}$ to $\mathbb{R}$ by its boundary values. The comparison principle holds in this setting: let $f, g \in W^{p|\lambda}$ satisfy $f \leq g$ in the Sobolev sense on $\mathbb{R}$, then $\hat{f} \leq \hat{g}$ in $\mathbb{R}^2_+$ ([Ma], [Wo1]).

The following lemma of Wolff ([Wo1]) is a substitute for a "local comparison principle" (unknown for $p \neq 2$) for p-harmonic functions. It is not difficult to prove (2.1) below for $y < A\nu^{-1}$ and (2.3) below for $y > 1$. However, a much deeper analysis is needed to obtain (2.1) and (2.3) below on the two sides of the line $y = A\nu^{-\alpha}$ for some $0 < \alpha < 1$. We shall need the full force of Wolff's lemma.

Wolff's Lemma [Wo1]. *Let $1 < p < \infty$. Define $\alpha = 1 - 2/p$ if $p \geq 2$ and $\alpha = 1 - p/2$ if $p < 2$. Let $\epsilon > 0$ and $0 < M < \infty$. Then there are small $A = A(p, \epsilon, M) > 0$ and large $\nu_0 = \nu_0(p, \epsilon, M) < \infty$ so that the following are true:*

If $\nu > \nu_0$ is an integer, $f, g, q \in Lip_1(\mathbb{R})$ are periodic with periods $1, 1, \nu^{-1}$ respectively, and

$$\max(\|f\|_\infty, \|g\|_\infty, \|q\|_\infty, \|f\|_{Lip_1}, \|g\|_{Lip_1}, \nu^{-1}\|q\|_{Lip_1}) \leq M,$$

then for $(x, y) \in \mathbb{R}^2_+$ we have

$$|\widehat{(qf + g)}(x, y) - (\hat{q}(x, y)f(x) + g(x))| < \epsilon \quad \textit{if} \quad y < A\nu^{-\alpha}. \tag{2.1}$$

If, in addition to the above, $\hat{q}(x, y) \to 0$ as $y \to \infty$, then

$$|\widehat{(qf + g)}(x, A\nu^{-\alpha}) - g(x)| < \epsilon \tag{2.2}$$

and

$$|\widehat{(qf + g)}(x, y) - \hat{g}(x, y)| < \epsilon \quad \textit{if} \quad y > A\nu^{-\alpha}. \tag{2.3}$$

The key to [Wo1] and [L2] is the existence of a basic function Φ which shows the failure of the mean value property for periodic p-harmonic functions in the class $W^{p|\lambda}(\mathbb{R}^2_+)$ when $p \neq 2$. The mean of $\Phi(x, 0)$ on $[0, 1]$ equals the limit of Φ at ∞ when $p = 2$.

Theorem 2.1. (Wolff and Lewis [Wo1], [L2]) *For $1 < p < \infty$ and $p \neq 2$ there exists a Lipschitz function $\Phi\colon \overline{\mathbb{R}^2_+} \to \mathbb{R}$ such that $L_p\Phi = 0$, Φ has period 1 in the x variable $\Phi(x + 1, y) = \Phi(x, y)$,*

$$\int_{[0,1)\times(0,\infty)} |\nabla\Phi|^p dxdy < +\infty,$$

$$\int_0^1 \Phi(x, 0)dx > 0, \quad \textit{but} \quad \Phi(x, y) \to 0 \quad \textit{as} \quad y \to \infty.$$

Note that when $p \neq 2$, the p-harmonic function $|X|^{\frac{p-n}{p-1}}$ if $p \neq n$, or $\log|X|$ if $p = n$, fails to have the mean value property on any sphere or half plane in $\mathbb{R}^n \setminus \{0\}$ ($n \geq 2$.) But these functions are not periodic.

3. Proofs

Proof of Theorem 1.1. Fix $p \neq 2,\ 1 < p < \infty$. Let Φ be the basic function of Wolff and Lewis. Note that $\Phi(x,0)$ must take both positive and negative values by the comparison principle. Replacing Φ by $c\Phi$ ($c > 0$ small constant), if necessary, we may assume

$$\|\Phi\|_\infty < \frac{1}{2} \tag{3.1}$$

and

$$\int_0^1 \log(1+\Phi(x,0))dx > 0.$$

Fix a positive integer κ such that

$$\sum_{k=1}^{\kappa} a_k > 0 \quad \text{and} \quad \prod_{k=1}^{\kappa}(1+a_k) > 1,$$

where

$$a_k = \min\left\{\Phi(x,0) : x \in \left[\frac{k-1}{\kappa}, \frac{k}{\kappa}\right]\right\} \tag{3.2}$$

Let

$$L = \|\Phi\|_{Lip_1},$$

and fix $\Lambda > 1$ and an integer $n_0 > 5$ so that

$$1 < \Lambda < \prod_{k=1}^{\kappa}(1+a_k)^{\frac{1}{\kappa}} \tag{3.3}$$

and

$$3^{-n_0} < \min\left\{1+\max\{a_k\} - \Lambda, \frac{L}{\kappa}\right\}. \tag{3.4}$$

For convenience we write $f(x)$ for $f(x,0)$ and $\omega_p(E)$ for $\omega_p(E, \mathbb{R}^2_+)$ from now on.

We shall choose *inductively* an increasing sequence of positive powers of the integer κ

$$1 < \nu_1 < \nu_2 < \dots$$

and shall define for each $k \in [1,\kappa]$ two sequences of functions on $\mathbb{R}$

$$q_1^k(x) = \Phi\left(x + \frac{k-1}{\kappa}\right), \quad f_1^k(x) = 1 + q_1^k(x) \tag{3.5}$$

and

$$q_j^k(x) = \Phi\left(\nu_j x + \frac{k-1}{\kappa}\right), \quad f_j^k(x) = f_{j-1}^k(x)(1+q_j^k(x)). \tag{3.6}$$

After these are defined, we observe from (3.2), (3.3) and the periodicity of $\Phi(x)$ that

$$\prod_{k=1}^{\kappa} f_j^k(x) = \prod_{i=1}^{j}\prod_{k=1}^{\kappa}\left(1+\Phi\left(\nu_i x + \frac{k-1}{\kappa}\right)\right) > \Lambda^{\kappa j} \quad \text{for all} \quad x. \tag{3.7}$$

Next, it follows from (3.1) that for $j \geq 1$

$$\|q_j^k\| < \frac{1}{2}, \tag{3.8}$$

$$2^{-j} < f_j^k < \left(\frac{3}{2}\right)^j, \tag{3.9}$$

$$\|q_j^k\|_{Lip_1} \leq L\nu_j, \tag{3.10}$$

and

$$\|f_j^k\|_{Lip_1} \leq L\nu_j 2^j\,. \tag{3.11}$$

We then define for each $k \in [1,\kappa]$ a set

$$E_k = \{x \in \mathbb{R} \colon f_j^k(x) > \Lambda^j \quad \text{for infinitely many} \quad j's\}\,.$$

Observe that (3.7) implies

$$\bigcup_{k=1}^{\kappa} E_k = \mathbb{R}.$$

To finish the proof we need to establish

$$\omega_p(E_k) = 0, \quad \omega_p\left(\mathbb{R}\setminus E_k, \mathbb{R}_2^+\right) = 1, \quad \text{and} \quad |\mathbb{R}\setminus E_k| = 0$$

for each k.

We start by discussing the choice of $\{\nu_j\}$ and two other sequences $\{r_j\}$ and $\{t_j\}$; we always assume $\{\nu_j\}$ are positive powers of κ, and $\{r_j\}$ and $\{t_j\}$ are negative powers of κ.

Set $r_0 = t_0 = 1$ and $\nu_1 = 1$. After $\{\nu_1, \nu_2, \ldots, \nu_j\}$, $\{r_0, r_1, \ldots, r_{j-1}\}$ and $\{t_0, t_1, \ldots, t_{j-1}\}$ are chosen, the functions

$$\{q_1^k, q_2^k, \ldots, q_j^k\}$$

and

$$\{f_1^k, f_2^k, \ldots, f_j^k\}$$

are then defined by (3.5) and (3.6) for each $k \in [1, \kappa]$. We then choose $r_j > 0$ so that

$$r_j < \min\{t_{j-1},\ (L\nu_j 6^{j+1})^{-1}\} \tag{3.12}$$

and that

$$|\widehat{f_j^k}(x, y) - f_j^k(x)| < 3^{-j-1} \quad \text{if} \quad 0 \le y \le r_j \tag{3.13}$$

for all $k \in [1, \kappa]$.

Let $f = g = f_j^k, q = q_{j+1}^k,\ M = L\nu_j 2^j$ and $\epsilon = 3^{-j-1}$ in Wolff's lemma; then ν_{j+1} and t_j can be chosen from (2.1) and (2.3) so that

$$\nu_{j+1}^{-1} < t_j < r_j \tag{3.14}$$

$$|\widehat{f_{j+1}^k}(x, y) - f_j^k(x)(1 + \widehat{q_{j+1}^k}(x, y))| < 3^{-j-1} \quad \text{if} \quad 0 < y \le t_j \tag{3.15}$$

and

$$|\widehat{f_{j+1}^k}(x, y) - \widehat{f_j^k}(x, y)| < 3^{-j-1} \quad \text{if} \quad y \ge t_j \tag{3.16}$$

for all $k \in [1, \kappa]$. The fact that $0 < \alpha < 1$ in Wolff's lemma is needed here to ensure that we can always find a t_j such that $\nu_{j+1}^{-1} < t_j < r_j$. We also need the fact that $\widehat{q_{j+1}^k}(x, y) \to 0$ as $y \to \infty$ to obtain (3.16). This ends the induction procedure.

For each $k \in [1, \kappa]$ the sequence $\{\widehat{f_j^k}\}$ converges to a p-harmonic function f^k on $\mathbb{R}_+^2$ uniformly on compact subsets. Since $\{t_j\}$ is decreasing, it follows from (3.16) that

$$|\widehat{f_N^k}(x, y) - \widehat{f_j^k}(x, y)| < 3^{-j} \quad \text{if} \quad y \ge t_j \tag{3.17}$$

for all $N \ge j$ and $k \in [1, \kappa]$; and from (3.15) and (3.17) that

$$\widehat{f_N^k}(x, y) > \frac{1}{2} f_j^k(x) - 3^{-j} \quad \text{if} \quad t_{j+1} \le y \le t_j \tag{3.18}$$

for all $N \ge j + 1$ and $k \in [1, \kappa]$. To see (3.18), observe that, since $y \ge t_{j+1}$, we get by (3.17),

$$|\widehat{f_N^k}(x, y) - \widehat{f_{j+1}^k}(x, y)| < 3^{-j-1}.$$

On the other hand, since $y \le t_j$, by (3.15) and (3.1) we have

$$\widehat{f_{j+1}^k}(x, y) > \frac{1}{2} f_j^k(x) - 3^{-j-1}.$$

We are now ready to prove $\omega_p(E_k) = 0$ and $\omega_p(\mathbb{R} \setminus E_k) = 1$ for all $k \in [1, \kappa]$. In view of the Harnack inequality and the strong maximum principle, it is enough to prove $\omega_p(X_0, E_k, \mathbb{R}^2_+) = 0$ and $\omega_p\left(X_0, \mathbb{R} \setminus E_k, \mathbb{R}^2_+\right) = 1$ for a fixed point $X_0 \in \mathbb{R}^2_+$. We take $X_0 = (0, 1)$. We fix k and from now on, we omit k in the subscripts and superscripts of E_k, q_j^k and f_j^k. Let $G_j = \{x : f_j(x) > \Lambda^j\}$, so that we have

$$E = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} G_j.$$

By monotonicity we get $\omega_p(E) \le \omega_p\left(\bigcup_{j=n}^{\infty} G_j\right)$. Therefore to show $\omega_p(E) = 0$ it suffices to prove that for some $C > 0$,

$$\omega_p\left(X_0, \bigcup_{j=n}^{\infty} G_j\right) \le C\Lambda^{-n} \quad \text{for all} \quad n > n_0. \tag{3.19}$$

In fact it is enough to show that for some $C > 0$,

$$\omega_p\left(X_0, \bigcup_{j=n}^{N} G_j\right) < C\Lambda^{-n} \quad \text{for all} \quad N > n > n_0 \tag{3.20}$$

Let us see how (3.20) implies (3.19). Observe that $\mathbb{R} \setminus \bigcup_{j=n}^{N} G_j$, $N \ge n$ is a decreasing sequence of closed sets on $\mathbb{R}$. Since the characteristic function of an open set is bounded and lower semicontinous, it is resolutive. Thus, we have

$$\omega_p\left(\bigcup_{j=n}^{N} G_j\right) = 1 - \omega_p\left(\mathbb{R} \setminus \bigcup_{j=n}^{N} G_j\right)$$

and

$$\omega_p\left(\bigcup_{j=n}^{\infty} G_j\right) = 1 - \omega_p\left(\mathbb{R} \setminus \bigcup_{j=n}^{\infty} G_j\right)$$

(See (9.31) and (11.4) of [HKM].) By the upper semicontinuity of p-harmonic measure on closed sets, we can let N go to ∞ to get

$$\lim_{N\to\infty} \omega_p\left(\bigcup_{j=n}^{N} G_j\right) = 1 - \omega_p\left(\mathbb{R} \setminus \bigcup_{j=n}^{\infty} G_j\right).$$

Therefore we conclude

$$\lim_{N\to\infty} \omega_p\left(\bigcup_{j=n}^{N} G_j\right) = \omega_p\left(\bigcup_{j=n}^{\infty} G_j\right).$$

By monotonicity we have $\omega_p(\mathbb{R} \setminus E) \geq \omega_p\left(\mathbb{R} \setminus \bigcup_{j=n}^{\infty} G_j\right)$; the equality $\omega_p(\mathbb{R} \setminus E) = 1$ follows again from (3.20).

We need to establish (3.20). Define for each $j > n_0$ a set

$$H_j = \bigcup \left\{ I : \kappa\text{-adic closed interval of length } t_j,\ \max_{x \in I} f_j(x) \geq \Lambda^j - 3^{-j-1} \right\}$$

and let

$$T_j = H_j \times [0, t_j].$$

Observe that from the definition of H_j we have

$$f_j(x) < \Lambda^j - 3^{-j-1} \quad \text{on} \quad H_j \setminus \overset{o}{H}_j \tag{3.21}$$

where $\overset{o}{H}_j$ is the relative interior of H_j. Hence, it follows that

$$G_j \subseteq \overline{G_j} \subseteq \overset{o}{H}_j \subseteq H_j.$$

Note from (3.8), (3.9), (3.10), (3.11), (3.12), and (3.14) that we have

$$|f_j(x) - f_j(x')| \leq L\nu_j 2^j t_j < 3^{-j} 6^{-1} \quad \text{if} \quad |x - x'| \leq t_j. \tag{3.22}$$

Therefore the inequality

$$\min_{H_j} f_j \geq \Lambda^j - 3^{-j} 2^{-1} \tag{3.23}$$

holds. Finally, from (3.13) and (3.14) we deduce

$$\widehat{f_j}(x, y) > \Lambda^j - 3^{-j} \quad \text{on} \quad T_j \tag{3.24}$$

We pause for a remark. If the statement

$$\widehat{f_N}(x, y) > C\Lambda^j \quad \text{on} \quad \partial T_j \setminus \overset{o}{H}_j \quad \text{for all} \quad N \geq j > n_0 \tag{3.25}$$

were true, then it would follow from the comparison principle applied on the domain $\mathbb{R}^2_+ \setminus \cup_{j=1}^N T_j$ and the convergence of $\{\widehat{f_j}\}$ that

$$\omega_p\left(X_0, \bigcup_{j=n}^{N} G_j\right) \leq \omega_p\left(X_0, \bigcup_{j=n}^{N} \partial T_j \setminus \overset{o}{H}_j\right) \leq C^{-1} \Lambda^{-n} \widehat{f_N}(X_0) < C(X_0)\Lambda^{-n}.$$

This would give (3.20) and thus $\omega_p(E) = 0$. Since (3.25) need not be true on vertical edges in ∂T_j, we need to modify the sets T_j.

The connected components of T_j are mutually disjoint rectangles Q of height t_j and of widths integer multiples of t_j. This class of rectangles is mapped to itself by the family of mappings $(x, y) \mapsto (m\nu_j^{-1} + x, y)$, $m \in \mathbb{Z}$.

Suppose $Q = [a, b] \times [0, t_j]$ is such a component. Then

$$f_j(a),\ \ f_j(b) < \Lambda^j - 3^{-j-1} \tag{3.26}$$

by (3.21). There are two possibilities.

Case I: $\max\limits_{[a,b]} f_j \le \Lambda^j$.
In this case define Q^* to be the empty set $\emptyset$, and note from (3.26) and the definition of G_j that

$$\overline{G_j} \cap [a, b] = \emptyset\,. \tag{3.27}$$

Case II: $\max\limits_{[a,b]} f_j > \Lambda^j$.
In this case let $I_j^Q = [a, a + t_j]$ and $J_j^Q = [b - t_j, b]$, and note from (3.22), (3.23), and (3.26) that

$$\Lambda^j - 3^{-j} < f_j(x) < \Lambda^j - 3^{-j-2} \quad \text{on} \quad I_j^Q \cup J_j^Q,$$

so that we have

$$\overline{G_j} \cap (I_j^Q \cup J_j^Q) = \emptyset\,. \tag{3.28}$$

To modify Q in Case II, we need the following fact.

Fact. If I is a κ-adic closed interval of length t_ℓ ($\ell > n_0$) on which $f_\ell(x) \ge \Lambda^\ell - 3^{-\ell}$, then I contains a κ-adic closed subinterval of length $t_{\ell+1}$ on which $f_{\ell+1}(x) > \Lambda^{\ell+1}$.

To see this, we write $f_{\ell+1} = (1 + q_{\ell+1}) f_\ell$ and note that I contains $t_\ell \nu_{\ell+1}$ periods of $q_{\ell+1}$. So from (3.2), the interval I has at least $t_\ell \nu_{\ell+1}$ κ-adic subintervals of length $\kappa^{-1}\nu_{\ell+1}^{-1}$ on which $q_{\ell+1} \ge \max\{a_k\}$. Let I'' be any one of such subintervals and let I' be any κ-adic subinterval of I'' of length $t_{\ell+1}$. Then

$$f_{\ell+1} \ge (\Lambda^\ell - 3^{-\ell})(1 + \max\{a_k\}) > \Lambda^{\ell+1} \quad \text{on} \quad I'$$

by (3.4).

Therefore, we may choose two sequences of κ-adic closed intervals:

$$I_j^Q \supseteq I_{j+1}^Q \supseteq I_{j+2}^Q \supseteq \dots$$

and

$$J_j^Q \supseteq J_{j+1}^Q \supseteq J_{j+2}^Q \supseteq \dots$$

such that $|I_\ell^Q| = |J_\ell^Q| = t_\ell$ and

$$f_\ell(x) > \Lambda^\ell - 3^{-\ell} \quad \text{on} \quad I_\ell^Q \cup J_\ell^Q \tag{3.29}$$

for all $\ell \geq j$. Let

$$a^* = \bigcap_{\ell=j}^{\infty} I_\ell^Q \quad \text{and} \quad b^* = \bigcap_{\ell=j}^{\infty} J_\ell^Q . \tag{3.30}$$

Clearly we have the inclusioin $[a+t_j, b-t_j] \subseteq [a^*, b^*] \subseteq [a, b]$. Finally replace Q by

$$Q^* = [a^*, b^*] \times [0, t_j]$$

in Case II.

Set

$$T_j^* = \bigcup \{Q^* : Q \quad \text{a component of} \quad T_j\},$$

and

$$H_j^* = T_j^* \cap \{y = 0\}.$$

Then it follows from (3.27) and (3.28) that

$$G_j \subseteq \overline{G_j} \subseteq \overset{o}{H_j^*} \subseteq H_j^* \subseteq T_j^* \subseteq T_j.$$

Claim. $\widehat{f_N}(x, y) > \Lambda^j/3 \quad \text{on } \partial T_j^* \setminus \overset{o}{H_j^*} \quad \text{for all } N \geq j.$

To establish the claim, note first that $\partial T_j^* \setminus \overset{o}{H_j^*} \subseteq T_j$, so that (3.24) implies

$$\widehat{f_j}(x, y) > \Lambda^j - 3^{-j} > \frac{\Lambda^j}{3} \quad \text{on} \quad \partial T_j^* \setminus \overset{o}{H_j^*} .$$

Next assume $N \geq j+1$. On $T_j^* \cap \{t_{j+1} \leq y \leq t_j\}$, it follows from (3.18) and (3.23) that

$$\widehat{f_N}(x, y) > \frac{1}{2} f_j(x) - 3^{-j} > \frac{1}{2}(\Lambda^j - 3^{-j}2^{-1}) - 3^{-j} > \frac{\Lambda^j}{3}.$$

The portion $V = (\partial T_j^* \setminus \overset{o}{H_j^*}) \cap \{0 \leq y \leq t_{j+1}\}$ consists of vertical line segments only. Suppose $(x, y) \in V$, then $x = a^*$ or b^*, associated with some component $[a, b] \times [0, t_j]$ of T_j, as defined in (3.30). If $(x, y) \in V \cap \{t_{\ell+1} \leq y \leq t_\ell\}$ for some $\ell \in [j+1, N-1]$, then

$$\widehat{f_N}(x, y) > \frac{1}{2} f_\ell(x) - 3^{-\ell} > \frac{1}{2}(\Lambda^\ell - 3^{-\ell}) - 3^{-\ell} > \frac{\Lambda^j}{3}$$

by (3.18) and (3.29). Finally, if $(x, y) \in V \cap \{0 \leq y \leq t_N\}$, then

$$\widehat{f_N}(x, y) > f_N(x) - 3^{-N-1} > \Lambda^N - 3^{-N} - 3^{-N-1} > \frac{\Lambda^j}{3}$$

by (3.13), (3.14) and (3.29). This proves the claim.

From the claim we deduce that the function $u(x, y) = 3\Lambda^{-n}\widehat{f_N}(x, y)$ has values $u(x, y) > 1$ on

$$\overline{\bigcup_{j=n}^{N} \partial T_j^* \cap \{y > 0\}} = \bigcup_{j=n}^{N} (\partial T_j^* \setminus H_j^{*o}).$$

We can now pass to a subset to conclude

$$u(x, y) > 1 \quad \text{on} \quad \overline{\partial\Bigg(\bigcup_{j=n}^{N} T_j^*\Bigg) \cap \{y > 0\}},$$

for $N \geq n > n_0$.

Repeat now the argument after (3.25).The statement (3.20) follows by applying the comparison principle to the functions u and $\omega_p\left(\cup_{j=n}^{N} G_j\right)$ on the domain $\mathbb{R}^2_+ \setminus \cup_{j=n}^{N} T_j^*$. This completes the proof of $\omega_p(E_k, \mathbb{R}^2_+) = 0$ and $\omega_p(\mathbb{R} \setminus E_k, \mathbb{R}^2_+) = 1$.

It remains to prove $|\mathbb{R} \setminus E_k| = 0$ for all $k \in [1, \kappa]$. Define Ψ on $[0, 1)$ so that

$$\Psi(x) = \log(1 + a_\ell) \quad \text{on} \quad \left[\frac{\ell - 1}{\kappa}, \frac{\ell}{\kappa}\right),\ 1 \leq \ell \leq \kappa,$$

and extend Ψ periodically to $\mathbb{R}$ so that $\Psi(x + 1) = \Psi(x)$ for all x. Recall that $a_\ell = \min\left\{\Phi(x) : x \in [\frac{\ell-1}{\kappa}, \frac{\ell}{\kappa}]\right\}$. Define for each $k \in [1, \kappa]$ a sequence of functions h_1^k, $h_2^k, h_3^k, \ldots$ so that

$$h_j^k(x) = \Psi\left(\nu_j x + \frac{k - 1}{\kappa}\right) - m,$$

where $m = \frac{1}{\kappa} \sum\limits_{k=1}^{\kappa} \log(1 + a_\ell)$.

Fix k in $[1, \kappa]$. Note that h_j^k is constant on each interval $\left[\frac{i-1}{\kappa\nu_j}, \frac{i}{\kappa\nu_j}\right)$, i an integer, and has average zero with respect to the Lebesgue measure μ on each interval

$$\left[\frac{i - 1}{\kappa\nu_{j-1}}, \frac{i}{\kappa\nu_{j-1}}\right).$$

Here we have set $\nu_{-1} = \kappa^{-1}$. Therefore the functions $h_1^k, h_2^k, h_3^k, \ldots$ are orthogonal in L^2. Since the sequence is uniformly bounded, it has partial sums

$$h_1^k + h_2^k + \cdots + h_j^k = o(j^{3/4}) \quad \mu - a.e.$$

Since

$$\log f_j^k \geq \sum_{\ell=1}^{j} \Psi\left(\nu_\ell x + \frac{k - 1}{\kappa}\right) = mj + \sum_{1}^{j} h_\ell^k(x)$$

and $1 < \Lambda < e^m$, therefore for μ-almost every x there exist an integer $j(x) > 0$ so that

$$f_j^k(x) > \Lambda^j \quad \text{for all} \quad j > j(x).$$

This says that $|\mathbb{R}^1 \setminus E_k| = 0$.

4. Questions and Comments

Many questions concerning p-harmonic measure and p-harmonic functions remain unanswered.

4.1. Are there *compact* sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ so that we have

$$\omega_p(A, \mathbb{R}^2_+) = \omega_p(B, \mathbb{R}^2_+) = 0,$$

but $\omega_p(A \cup B, \mathbb{R}^2_+) > 0$?

4.2. Can the number κ of sets in Theorem 1.1 be as small as 2?

Based on a theorem of Baernstein [B], we conjecture that when p is close to 2 and $p \neq 2$, $\kappa = 5$ suffices. In the tree case, κ must be and can be any integer ≥ 3 [KLW].

Theorem 4.1. (Baernstein [B]) *Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2$. For a set $S \subseteq \partial\mathbb{D}$ let S^* be the closed arc on $\partial\mathbb{D}$ centered at 1 of length $|S|$. Suppose that $E \subseteq \partial\mathbb{D}$ is the union of two disjoint closed arcs of equal positive length, and that the two components of $\partial\mathbb{D} \setminus E$ have unequal length, then there exist p_1 and p_2 (depending on E) with $1 < p_1 < 2 < p_2 < \infty$ such that*

$$\omega_p(0, E, \mathbb{D}) > \omega_p(0, E^*, \mathbb{D}) \quad \textit{for} \quad p_1 < p < 2 \tag{4.1}$$

and

$$\omega_p(0, E, \mathbb{D}) < \omega_p(0, E^*, \mathbb{D}) \quad \textit{for} \quad 2 < p < p_2\,. \tag{4.2}$$

If $E \subseteq \partial\mathbb{D}$ is the union of two disjoint closed arcs of unequal positive length for which the components of $\partial\mathbb{D} \setminus E$ do have equal length, then inequalities opposite to (4.1) and (4.2) are true.

According to Baernstein's theorem, there exist $1 < p_1 < 2 < p_2 < \infty$ so that for each $p \in (p_1, 2) \cup (2, p_2)$, there is one set J among the four $\{e^{i\theta} : \theta \in [0, \frac{4\pi}{5}]\}$, $\{e^{i\theta} : \theta \in [0, \frac{2\pi}{5}] \cup [\frac{4\pi}{4}, \frac{6\pi}{5}]\}$, $\{e^{i\theta} : \theta \in [0, \frac{6\pi}{5}]\}$ and $\{e^{i\theta} : \theta \in [0, \frac{4\pi}{5}] \cup [\frac{6\pi}{5}, \frac{8\pi}{5}]\}$, which satisfies

$$\omega_p(0, J, \mathbb{D}) < |J|/2\pi. \tag{4.3}$$

From this, a p-harmonic function $\hat{\Psi}$ on $\mathbb{D}$ having Lipschitz continuous boundary values Ψ may be constructed so that $\hat{\Psi}(0) = 0$ and

$$\sum_{k=1}^{5} \Psi(e^{i(\theta + k2\pi/5)}) > c > 0 \quad \text{for every} \quad \theta \in [0, 2\pi]; \tag{4.4}$$

consequently,

$$\frac{1}{2\pi}\int_0^{2\pi}\Psi(e^{i\theta})d\theta > c > 0.$$

On the other hand, using p-capacity estimates we can show that if $1 < p < \frac{3}{2}$ and J is an arc of the unit circle then (4.3) holds provided $|J| < \delta_0(p)$. This implies that for $1 < p < \frac{3}{2}$, there exists $\hat{\Psi}$ for which $\hat{\Psi}(0) = 0$ and (4.4) holds with 5 replaced by some $\kappa = \kappa(p)$.

Let $\Psi_n(e^{i\theta}) = \Phi(e^{in\theta})$ for integers $n \geq 1$. It is not clear, and probably false, whether $\widehat{\Psi}_n(0) = 0$. Therefore it is unclear how to adapt Wolff's lemma to disks. Unlike in the half plane, shortening the period of the boundary function on $\partial\mathbb{D}$ complicates the p-harmonic solution in $\mathbb{D}$.

4.3. Given any Lipschitz function Ψ on $\partial\mathbb{D}$, let $\widehat{\Psi}$ be the p-harmonic function in $\mathbb{D}$ with boundary values Ψ, and let $\Psi_n(e^{i\theta}) = \Psi(e^{in\theta})$ shortening the period. Suppose $\widehat{\Psi}(0) \leq \frac{1}{2\pi}\int_0^{2\pi}\Psi(e^{i\theta})d\theta$. We ask whether

$$\widehat{\Psi(0)} \leq \widehat{\Psi}_n(0) \leq \frac{1}{2\pi}\int_0^{2\pi}\Psi(e^{i\theta})d\theta \quad \text{for} \quad n \geq 2;$$

and what the value $\lim\limits_{n\to\infty}\widehat{\Psi}_n(0)$ might be.

4.4. Not much is known about the structure of the sets having p-harmonic measure zero. Sets $E \subseteq \mathbb{R}^n$ of absolute p-harmonic measure zero, $\omega_p(E \cap \partial\Omega, \Omega) = 0$ for all bounded domains Ω, are exactly those of p-capacity zero. There exist sets on $\partial\mathbb{R}^n_+$ of Hausdorff dimension $n-1$ that have zero p-harmonic measure with respect to $\mathbb{R}^n_+$ when $p \neq 2$. There are also sufficient conditions on sets $E \subseteq \partial\mathbb{R}^n_+$ in terms of porosity, that imply $\omega_p(E, \mathbb{R}^n_+) = 0$. For these and more, see [HM], [M2] and [W].

Further questions and discussions on p-harmonic measures can be found in [B] and [HKM].

4.5. Given a function u in $\mathbb{R}^n_+$, denote by $\mathcal{F}(u)$ the Fatou set

$$\left\{x \in \mathbb{R}^{n-1}\colon \lim_{y\to 0} u(x,y) \text{ exists and it is finite}\right\}.$$

Fatou's Theorem states that $\mathbb{R}^{n-1} \setminus \mathcal{F}(u)$ has zero $(n-1)$-dimensional measure for any bounded 2-harmonic function u in $\mathbb{R}^n_+$. When $1 < p < \infty$ and $p \neq 2$, the Hausdorff dimension of the Fatou set of any bounded p-harmonic function in $\mathbb{R}^n_+$ is bounded below by a positive number $c(n, p)$ independent of the function [FGMS], [MW].

Deep and unexpected examples in [Wo1], [Wo2] and [L2] show that Fatou Theorem relative to the Lebesgue measure fails when $p \neq 2$.

Theorem 4.2. (Wolff and Lewis [Wo1], [L2]) *For $1 < p < \infty$ and $p \neq 2$, there exists a bounded p-harmonic function u on $\mathbb{R}^2_+$ such that the Fatou set $\mathcal{F}(u)$ has zero length, and there exists a bounded positive p-harmonic function v on $\mathbb{R}^2_+$ such that the set*

$$\left\{x \in \mathbb{R} : \limsup_{y \to 0} v(x, y) > 0\right\}$$

has zero length.

Define the infimum of the dimensions of Fatou sets to be

$$\dim_{\mathcal{F}}(p) = \inf\left\{\dim \mathcal{F}(u) : u \text{ bounded p-harmonic in } \mathbb{R}^2_+\right\},$$

and the dimension of the p-harmonic measure to be

$$\dim \omega_p = \inf\left\{\dim E : E \subseteq \mathbb{R}^1,\ \omega_p(E, \mathbb{R}^2_+) = 1\right\}.$$

We ask what the values of $\dim_{\mathcal{F}}(p)$ and $\dim \omega_p$ are, and conjecture that $\dim \omega_p = \dim_{\mathcal{F}}(p) < 1$ when $p \neq 2$.

The question and the conjecture are based on results in [KW]. In the case of forward directed regular κ-branching trees ($\kappa > 1$) whose boundary is normalized to have dimension 1, the infimum of the dimensions of Fatou sets $\dim_{\mathcal{F}}(\kappa, p)$ is attained and is given by

$$\dim_{\mathcal{F}}(\kappa, p) = \min\left\{\frac{\log \sum_1^{\kappa} e^{x_j}}{\log \kappa} : \sum_1^{\kappa} x_j |x_j|^{p-2} = 0\right\};$$

furthermore $0 < \dim_{\mathcal{F}}(\kappa, p) < 1$ except when $p = 2$ or $\kappa = 2$, and in the exceptional case $\dim_{\mathcal{F}}(\kappa, p) = 1$.

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