# A Hörmander-Type Spectral Multiplier Theorem for Operators Without Heat Kernel 

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#### Abstract

Hörmander's famous Fourier multiplier theorem ensures the $L_{p}$-boundedness of $F\left(-\Delta_{\mathbb{R}} D\right)$ whenever $F \in \mathcal{H}(s)$ for some $s>\frac{D}{2}$, where we denote by $\mathcal{H}(s)$ the set of functions satisfying the Hörmander condition for $s$ derivatives. Spectral multiplier theorems are extensions of this result to more general operators $A \geq 0$ and yield the $L_{p}$-boundedness of $F(A)$ provided $F \in \mathcal{H}(s)$ for some $s$ sufficiently large. The harmonic oscillator $A=-\Delta_{\mathbb{R}}+x^{2}$ shows that in general $s>\frac{D}{2}$ is not sufficient even if $A$ has a heat kernel satisfying Gaussian estimates. In this paper, we prove the $L_{p}$-boundedness of $F(A)$ whenever $F \in \mathcal{H}(s)$ for some $s>\frac{D+1}{2}$, provided $A$ satisfies generalized Gaussian estimates. This assumption allows to treat even operators $A$ without heat kernel (e.g. operators of higher order and operators with complex or unbounded coefficients) which was impossible for all known spectral multiplier results.


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## 0. - Introducion

In this paper, we present a new spectral multiplier result motivated by Hörmander's famous Fourier multiplier theorem. In terms of the functional calculus $F \mapsto F(-\Delta)$ of the Laplace operator $\Delta$ on $\mathbb{R}^{D}$, Hörmander's theorem says the following:
$F \in \mathcal{H}(s)$ for some $\quad s>\frac{D}{2} \Longrightarrow F(-\Delta) \in \mathfrak{L}\left(L_{p}\left(\mathbb{R}^{D}\right)\right) \quad$ for all $\quad p \in(1, \infty)$.
Here we denote by $\mathcal{H}(s)$ the set of functions satisfying the Hörmander condition for $s$ derivatives:

$$
\mathcal{H}(s):=\left\{F: \mathbb{R}_{+} \rightarrow \mathbb{C} \text { bounded Borel funcion; } \sup _{t>0}\|\omega F(t \cdot)\|_{H^{s}\left(\mathbb{R}_{+}\right)}<\infty\right\}
$$

where $\omega \in \mathbb{C}_{c}^{\infty}\left(\mathbb{R}_{+}\right)$is a fixed 'partition of unity' function [i.e. $\sum_{l \in \mathbb{Z}} \omega\left(2^{l} t\right)=1$ for all $t \in \mathbb{R}_{+}$]. Christ [C] and Mauceri and Meda [MM] generalized this result to homogeneous Laplacians $\Delta$ on Lie groups $G$ of some homogeneous dimension $D$, i.e. $|B(x, r)| \sim r^{D}$ for all $x \in G, r>0$. Indeed, they obtained independently

$$
F \in \mathcal{H}(s) \quad \text { for some } \quad s>\frac{D}{2} \Longrightarrow F(-\Delta) \in \mathfrak{L}\left(L_{p}(G)\right) \quad \text { for all } \quad p \in(1, \infty) .
$$

In order to treat more general elliptic operators and irregular domains, Duong, Ouhabaz and Sikora [DOS] extended this result to arbitrary non-negative selfadjoint operators $A$ on (subsets of) metric measured spaces ( $\Omega, \mu, d$ ) of some dimension $D$, i.e. $|B(x, \lambda r)| \leq C \lambda^{D}|B(x, r)|$ for all $x \in \Omega, r>0, \lambda \geq 1$. They showed
(H) $\quad F \in \mathcal{H}(s)$ for some $s>\frac{D}{2} \Longrightarrow F(A) \in \mathfrak{L}\left(L_{p}(\Omega)\right)$ for all $p \in(1, \infty)$,
provided $A$ satisfies the so-called Plancherel estimate
(P) $\left\|F(t A)\left|B\left(\cdot, r_{t}\right)\right|^{1 / 2}\right\|_{1 \rightarrow 2} \leq C\|F\|_{L_{2}([0,1])}$ for all $F \in L_{\infty}([0,1]), t>0$
and $A$ satisfies Gaussian estimates, i.e. the $e^{-t A}$ have integral kernels $k_{t}(x, y)$ for which one has a pointwise upper bound of the following type:
(GE) $\quad\left|k_{t}(x, y)\right| \leq\left|B\left(x, r_{t}\right)\right|^{-1} g\left(\frac{d(x, y)}{r_{t}}\right) \quad$ for all $\quad x, y \in \Omega, t>0$.
Here the $r_{t}$ are suitable positive radii and $g: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is a suitable decay function. Note that (GE) without any additional assumption like (P) does not imply (H) since the harmonic oscillator $A=-\Delta+x^{2}$ on $\Omega=\mathbb{R}$ satisfies (GE) and has the following property [T]:
$F \in \mathcal{H}(s) \quad$ for some $\quad s \leq \frac{D}{2}+\frac{1}{6}-\varepsilon \nRightarrow F(A) \in \mathfrak{L}\left(L_{p}(\mathbb{R})\right) \quad$ for all $\quad p \in(1, \infty)$.
Furthermore, note that an elliptic operator $A$ of order $m$ on $\mathbb{R}^{D}$ with bounded measurable coefficients satisfies (GE) if $m \geq D$ [AT], [D1] or $m=2$ and the coefficients are real [A]. On the other hand, in general $A$ does not satisfy (GE) if $m<D$ [D3], [ACT] or the coefficients are unbounded [LSV]. But in many of these cases $A$ still satisfies so-called generalized Gaussian estimates [D1], [ ScV ]. By this we mean an estimate of the following type:
(GGE)

$$
\begin{gathered}
\left\|\chi_{B\left(x, r_{t}\right)} e^{-t A} \chi_{B\left(y, r_{t}\right)}\right\|_{p_{o} \rightarrow p_{o}^{\prime}} \leq\left|B\left(x, r_{t}\right)\right|^{\frac{1}{p_{o}^{\prime}}-\frac{1}{p_{o}}} g\left(\frac{d(x, y)}{r_{t}}\right) \\
\text { for all } \quad x, y \in \Omega, t>0
\end{gathered}
$$

and for some $p_{o} \in[1,2)$. Notice that (GGE) for $p_{o}=1$ is equivalent to (GE) [BK1]. The main result of the present paper is that (GGE) without any additional assumption implies the following adaptation of $(\mathrm{H})$ :
(î) $\quad F \in \mathcal{H}(s)$ for some $s>\frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}\left(L_{p}(\Omega)\right)$ for all $p \in\left(p_{o}, p_{o}^{\prime}\right)$.
We want to mention that, for the class of operators $A$ satisfying (GGE), the interval $\left[p_{o}, p_{o}^{\prime}\right]$ is optimal for the existence of the semigroup $\left(e^{-t A}\right)_{t \in \mathbb{R}_{+}}$on $L_{p}$ [D3]; this shows the optimality of our spectral multiplier theorem $(\tilde{\mathrm{H}})$.

Our main tool for the proof of $(\tilde{\mathrm{H}})$ is the singular integral theory developped in [BK2] which generalizes the classical singular integral theory based on Hörmander's well-known weak type $(1,1)$ condition for integral operators (in a weakened version due to Duong and McIntosh [DM]). This new singular integral theory based on (GGE) allows to extend other $L_{2}$-properties of $A$ (than the boundedness of $F(A)$ for $F \in \mathcal{H}(s)$ is considered in this paper) to $L_{p}$ for $p \in\left(p_{o}, p_{o}^{\prime}\right)$. We mention the properties of having maximal regularity [BK1], an $H^{\infty}$ functional calculus [BK2] or Riesz transforms [BK3], [HM].

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## 1. - Main result

We begin with some basic notation and assumptions. For the rest of this paper, $(\Omega, \mu, d)$ is a metric measure space of dimension $D$, i.e.

$$
|B(x, \lambda r)| \leq C \lambda^{D}|B(x, r)| \quad \text { for all } \quad x \in \Omega, r>0, \lambda \geq 1
$$

Here we denote by $B(x, r)$ the ball of center $x$ and radius $r$, and by $|B(x, r)|$ or $v_{r}(x)$ its volume; by $L_{p}^{\omega}(\Omega)$ we denote the weak $L_{p}(\Omega)$-spaces. Furthermore, we fix once and for all real numbers $p_{o} \in[1,2), m \in[2, \infty)$ and the following notation:

$$
r_{t}:=t^{1 / m} \quad \text { and } \quad g(t):=C e^{-b t \frac{m}{m-1}} \quad \text { for all } \quad t \in \mathbb{R}_{+} .
$$

Here $C$ and $b$ are positive constants whose values are of no interest and might change from one appearance of the function $g$ to the next without mentioning it. We denote by $\mathcal{H}(s)$ the set of functions satisfying the Hörmander condition for $s$ derivatives:

$$
\mathcal{H}(s):=\left\{F: \mathbb{R}_{+} \rightarrow \mathbb{C} \text { bounded Borel function; } \sup _{t>0}\|\omega F(t \cdot)\|_{H^{s}\left(\mathbb{R}_{+}\right)}<\infty\right\}
$$

where $\omega \in C_{c}^{\infty}\left(\mathbb{R}_{+}\right)$is a fixed 'partition of unity' function [i.e. $\sum_{l \in \mathbb{Z}} \omega\left(2^{l} t\right)=1$ for all $t \in \mathbb{R}_{+}$]. Now we can present the main result of this paper.

Theorem 1.1. Let $(\Omega, \mu, d)$ be a space of dimension $D$ and $A$ a non-negative self-adjoint operator on $L_{2}(\Omega)$ such that (GGE) holds. Then, for all $s>\frac{D+1}{2}$, there exists $C>0$ such that

$$
\|F(A)\|_{\mathfrak{L}\left(L_{p_{o}}(\Omega), L_{p_{o}}^{\omega}(\Omega)\right)} \leq C\left(\|F\|_{L_{\infty}\left(\mathbb{R}_{+}\right)}+\sup _{t>0}\|\omega F(t \cdot)\|_{H^{s}\left(\mathbb{R}_{+}\right)}\right)
$$

for all $F \in \mathcal{H}(s)$. In particular, the following implication holds:
$F \in \mathcal{H}(s)$ for some $\quad s>\frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}\left(L_{p}(\Omega)\right) \quad$ for all $\quad p \in\left(p_{o}, p_{o}^{\prime}\right)$.
Remark.
(a) An important example are the Riesz means $R_{\alpha}(A)$, where $R_{\alpha}(x):=(1-x)_{+}^{\alpha}$. Observe that $R_{\alpha} \in \mathcal{H}(s)$ if and only if $s<\alpha+\frac{1}{2}$. Hence, in the situation of Theorem 1.1 one has

$$
\left\|R_{\alpha}(t A)\right\|_{p \rightarrow p} \leq C_{p, \alpha} \quad \text { for all } \quad t>0, p \in\left(p_{o}, p_{o}^{\prime}\right), \alpha>\frac{D}{2}
$$

(b) Another important example are the imaginary powers $A^{i \tau}, \tau \in \mathbb{R}$. If we denote $P_{\tau}(x):=x^{i \tau}$ then $\left\|\omega P_{\tau}(t \cdot)\right\|_{H^{s}\left(\mathbb{R}_{+}\right)} \leq C_{s}(1+|\tau|)^{s}$ for all $\tau \in \mathbb{R}$, $s, t>0$. Hence, in the situation of Theorem 1.1 one has

$$
\left\|A^{i \tau}\right\|_{p \rightarrow p} \leq C_{p, s}(1+|\tau|)^{s} \quad \text { for all } \quad \tau \in \mathbb{R}, p \in\left(p_{o}, p_{o}^{\prime}\right), s>\frac{D+1}{2}
$$

(c) Theorem 1.1 is optimal with respect to $p$ since, for the class of operators $A$ satisfying (GGE), the interval $\left[p_{o}, p_{o}^{\prime}\right]$ is optimal for the existence for the semigroup $\left(e^{-t A}\right)_{t \in \mathbb{R}_{+}}$on $L_{p}$ [D3].
(d) Concerning optimality with respect to $s$ (the number of derivatives), we mention that our condition $s>\frac{D+1}{2}$ cannot be replaced by $s>\frac{D}{2}+\alpha$ with $\alpha<\frac{1}{6}$. Indeed, the Riesz means $R_{\alpha}(A)$ of the harmonic oscillator $A=-\Delta+x^{2}$ on $\mathbb{R}$ do not satisfy $R_{\alpha}(A) \in \mathfrak{L}\left(L_{p}(\mathbb{R})\right)$ for all $p \in(1, \infty)$ unless $\alpha \geq \frac{1}{6}$ [T, Theorem 2.1]. On the other hand, $A$ satisfies (GGE) for $p_{o}=1$, and $R_{\alpha} \in \mathcal{H}(s)$ for all $s<\alpha+\frac{1}{2}$.
Under the additional assumptions $(P)$ from above and $p_{o}=1$, our condition $s>\frac{D+1}{2}$ can be replaced by $s>\frac{D}{2}$ [DOS, Theorem 3.1].
(e) By standard methods [DM], [BK2], Theorem 1.1 can be extended to the case where $\Omega$ is only a subset of a space of dimension $D$. This allows to treat elliptic operators $A$ on irregular domains $\Omega \subset \mathbb{R}^{D}$; see Section 2.1 below.

## 2. - Examples

In this section, we give some examples of elliptic operators $A$ for which Theorem 1.1 applies, i.e. for which (GGE) holds.

## 2.1. - Higher order operators with bounded coefficients and Dirichlet boundary conditions on irregular domains

These operators $A$ are given by forms $\mathfrak{a}: V \times V \rightarrow \mathbb{C}$ of the type

$$
\mathfrak{a}(u, v)=\int_{\Omega} \sum_{|\alpha|=|\beta|=k} a_{\alpha \beta} \partial^{\alpha} u \overline{\partial^{\beta} v} d x
$$

where $V:=\stackrel{\circ}{H}^{k}(\Omega)$ for some arbitrary (irregular) domain $\Omega \subset \mathbb{R}^{D}$. We assume $a_{\alpha \beta}=\overline{a_{\beta \alpha}} \in L_{\infty}\left(\mathbb{R}^{D}\right)$ for all $\alpha, \beta$ and Garding's inequality

$$
\mathfrak{a}(u, u) \geq \delta\left\|\nabla^{k} u\right\|_{2}^{2} \quad \text { for all } \quad u \in V
$$

for some $\delta>0$ and $\left\|\nabla^{k} u\right\|_{2}^{2}:=\sum_{|\alpha|=k}\left\|\partial^{\alpha} u\right\|_{2}^{2}$. Then $\mathfrak{a}$ is a closed symmetric form, and the associated operator $A$ on $L_{2}(\Omega)$ is given by $u \in D(A)$ and $A u=g$ if and only if $u \in V$ and $\langle g, v\rangle=\mathfrak{a}(u, v)$ for all $v \in V$.

In this situation, we have for $p_{o}:=\frac{2 D}{m+D} \vee 1$ and $m:=2 k$ [D1], [AT, Section 1.7]:
$\left\|\chi_{B\left(x, r_{t}\right)} e^{-t A} \chi_{B\left(y, r_{t}\right)}\right\|_{p_{o} \rightarrow p_{o}^{\prime}} \leq r_{t}{ }^{D\left(\frac{1}{p_{o}^{\prime}}-\frac{1}{p_{o}}\right)} g\left(\frac{d(x, y)}{r_{t}}\right) \quad$ for all $x, y \in \Omega, t>0$.
Hence, by Remark (e) above, the conclusion of Theorem 1.1 holds:

$$
F \in \mathcal{H}(s) \text { for some } s>\frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}\left(L_{p}(\Omega)\right) \text { for all } p \in\left(p_{o}, p_{o}^{\prime}\right)
$$

## 2.2. - Schrödinger operators with singular potentials on $\mathbb{R}^{D}$

Now we study Schrödinger operators $A=-\Delta+V$ on $\mathbb{R}^{D}, D \geq 3$, where $V=V_{+}-V_{-}, V_{ \pm} \geq 0$ are locally integrable, and $V_{+}$is bounded for simplicity (for the general case, see e.g. [ ScV ]). We assume the following form bound:

$$
\left\langle V_{-} u, u\right\rangle \leq \gamma\left(\|\nabla u\|_{2}^{2}+\left\langle V_{+} u, u\right\rangle\right)+c(\gamma)\|u\|_{2}^{2} \quad \text { for all } \quad u \in H^{1}\left(\mathbb{R}^{D}\right)
$$

and some $\gamma \in(0,1)$. Then the form sum $A:=-\Delta+V=\left(-\Delta+V_{+}\right)-$ $V_{-}$is defined and the associated form is closed and symmetric with form domain $H^{1}\left(\mathbb{R}^{D}\right)$. By standard arguments using ellipticity and Sobolev inequality, (GGE) holds for $p_{o}=\frac{2 D}{D+2}$ and $m=2$ [after replacing $A$ by $A+c(\gamma)$ ]. Due to [LSV], $\left(e^{-t A}\right)_{t \in \mathbb{R}_{+}}$is bounded on $L_{q}\left(\mathbb{R}^{D}\right)$ for all $q \in\left(p_{\gamma}, p_{\gamma}^{\prime}\right)$ and
$p_{\gamma}:=\frac{2 D}{D(1+\sqrt{1-\gamma})+2(1-\sqrt{1-\gamma})}<\frac{2 D}{D+2}$. Hence, by interpolation, one obtains (GGE) even for all $p_{o} \in\left(p_{\gamma}, 2\right)$. Thus, Theorem 1.1 yelds
$F \in \mathcal{H}(s)$ for some $s>\frac{D+1}{2} \Longrightarrow F(A) \in \mathfrak{L}\left(L_{p}\left(\mathbb{R}^{D}\right)\right)$ for all $p \in\left(p_{\gamma}, p_{\gamma}^{\prime}\right)$.

## 2.3. - Elliptic operators on Riemannian manifolds

Let $A=-\Delta$ be the Laplacian on a Riemannian manifold $\Omega$. Let $d$ be the geodesic distance and $u$ the Riemannian measure. Assume that $\Omega$ satisfies the so-called volume doubling property and that the heat kernel $k_{t}(x, y)$ satisfies

$$
k_{t}(x, x) \leq C|B(x, \sqrt{t})|^{-1} \quad \text { for all } \quad x \in \Omega, t>0
$$

Then $\left(e^{t \Delta}\right)_{t \in \mathbb{R}_{+}}$satisfies (GE) [G] or, equivalently, (GGE) for $p_{o}=1$ and $m=2$. On Riemannian manifolds satisfying a local higher order Sobolev inequality, (GGE) holds even for suitable higher order elliptic operators $A$ [BC].

## 3. - Proof of the main result

The main tool for the proof of Theorem 1.1 is the following result [BK2, Theorem 1.1] which generalizes Hörmander's well-known weak type (1,1) condition for integral operators (in a weakened version due Duong and McIntosh [DM]) and provides a weak type ( $p_{o}, p_{o}$ ) condition for arbitrary operators.

Theorem 3.1. Let $(\Omega, d, \mu)$ be a space of homogeneous type and $A$ a nonnegative selfadjoint operator on $L_{2}(\Omega)$ such that (GGE) holds. Let $T \in \mathfrak{L}\left(L_{2}(\Omega)\right)$ satisfy

$$
\begin{equation*}
N_{p_{o}^{\prime}, r_{t} / 2}\left(\left(T D^{n} e^{-t A}\right)^{*} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y) \leq C\left(M_{2} f\right)(x) \tag{1}
\end{equation*}
$$

for all $t>0, f \in L_{p_{0}^{\prime}}(\Omega), x \in \Omega, y \in B\left(x, r_{t} / 2\right)$ and some $n \in \mathbb{N}$. Then we have $T \in \mathfrak{L}\left(L_{p_{o}}(\Omega), L_{p_{o}}^{\omega}(\Omega)\right)$.

Here we used the following notation:

$$
\begin{gathered}
M_{p} f(x):=\sup _{r>0} N_{p, r} f(x) \quad \text { [p-maximal operator] } \\
N_{p, r} f(x):=|B(x, r)|^{-1 / p}\|f\|_{L_{p}(B(x, r))}, D^{n} f(t):=\sum_{k=0}^{n}\binom{n}{k}(-1)^{k} f(k t) .
\end{gathered}
$$

Hence $I-D^{n} e^{-t A}$ can be seen as an approximation of the identity of order $n$ since we formally have $\frac{D^{n} f(t)}{t^{n}} \rightarrow(-1)^{n} f^{(n)}(0)$ for $t \rightarrow 0$.

Another central tool for the prof of Theorem 1.1 is the following result on the extension of generalized Gaussian estimates for real times $t \in \mathbb{R}_{+}$to complex times $z \in \mathbb{C}_{+}$; its proof is given in [B, Theorem 2.1].

Theorem 3.2. Let $(\Omega, \mu, d)$ be a space of dimension $D$ and $1 \leq p \leq 2 \leq q \leq$ $\infty$. Let A be a non-negative selfadjoint operator on $L_{2}(\Omega)$ such that

$$
\left\|\chi_{B\left(x, r_{t}\right)} e^{-t A} \chi_{B\left(y, r_{t}\right)}\right\|_{p \rightarrow q} \leq\left|B\left(x, r_{t}\right)\right|^{\frac{1}{q}-\frac{1}{p}} g\left(\frac{d(x, y)}{r_{t}}\right)
$$

for all $t \in \mathbb{R}_{+}, x, y \in \Omega$. Then we have

$$
\left\|\chi_{B\left(x, r_{z}\right)} e^{-z A} \chi_{B\left(y, r_{z}\right)}\right\|_{p \rightarrow q} \leq\left|B\left(x, r_{z}\right)\right|^{\frac{1}{q}-\frac{1}{p}} g\left(\frac{|z|}{\operatorname{Rez}}\right)^{D\left(\frac{1}{p}-\frac{1}{q}\right)} g\left(\frac{d(x, y)}{r_{z}}\right)
$$

for all $z \in \mathbb{C}_{+}, x, y \in \Omega$ and $r_{z}:=(\text { Rez })^{\frac{1}{m}-1}|z|$.
Some rather technical features of generalized Gaussian estimates are summarized in the following lemma; see [BK4, Proposition 2.1] and [BK2, Lemma 3.3(a)] for the proofs. We will denote by $A(x, r, k)$ the annular region $A(x, r, k):=B(x,(k+1) r) \backslash B(x, r)$.

Lemma 3.3. Let $(\Omega, \mu, d)$ be a space of dimension $D$ and $1 \leq p \leq q \leq \infty$. Let $R$ be a linear operator and $r>0$.
(i) The following are equivalent:
(a) We have for all $x, y \in \Omega$ :

$$
\left\|\chi_{B(x, r)} R \chi_{B(y, r)}\right\|_{p \rightarrow q} \leq v_{r}(x)^{\frac{1}{q}-\frac{1}{p}} g\left(\frac{d(x, y)}{r}\right)
$$

(b) We have for all $x, y \in \Omega$ and $u \in[p, q]$ :

$$
\left\|\chi_{B(x, r)} R \chi_{B(y, r)}\right\|_{u \rightarrow q} \leq v_{r}(x)^{\frac{1}{q}-\frac{1}{u}} g\left(\frac{d(x, y)}{r}\right)
$$

(c) We have for all $x \in \Omega$ and $k \in \mathbb{N}$ :

$$
\left\|\chi_{B(x, r)} R \chi_{A(x, r, k)}\right\|_{p \rightarrow q} \leq v_{r}(x)^{\frac{1}{q}-\frac{1}{p}} g(k) .
$$

(ii) If (a) holds then we have for all $s>0, f \in L_{q}(\Omega), x \in \Omega, y \in B(x, s)$ :

$$
N_{q, s}\left(R P_{B(y, 5 s)^{c}} f\right)(y) \leq g\left(r^{-1} s\right)\left(1+s^{-1} r\right)^{D / q} M_{p} f(x)
$$

In order to prove the assertion $F(A) \in \mathfrak{L}\left(L_{p_{o}}(\Omega), L_{p_{o}}^{\omega}(\Omega)\right)$ of Theorem 1.1 by means of our weak type $\left(p_{o}, p_{o}\right)$ criterion Theorem 3.1 , we have to check line (1) for $T=F(A)$. The main step is the following.

Proposition 3.4. Let $(\Omega, d, \mu)$ be a space of dimension $D$ and $A$ a non-negative selfadjoint operator on $L_{2}(\Omega)$ such that (GGE) holds. Then, for all $s>\frac{D+1}{2}$, there exist $\varepsilon_{1}, \varepsilon_{2}, C>0$ such that we have for all $F \in L_{\infty}\left(\mathbb{R}_{+}\right)$and $\sigma>0$ :
$N_{p_{o}^{\prime}, r_{t} / 2}\left(F(\sigma A)^{*} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y) \leq C\left(\left(\frac{t}{\sigma}\right)^{-\varepsilon_{1}} \vee\left(\frac{t}{\sigma}\right)^{-\varepsilon_{2}}\right)\|F \cdot \exp \|_{H^{s}\left(\mathbb{R}_{+}\right)}\left(M_{2} f\right)(x)$ for all $t>0, f \in L_{p_{o}^{\prime}}(\Omega), x \in \Omega, y \in B\left(x, r_{t} / 2\right)$.

Proof. By Lemma 3.3(i), the $L_{p_{o}} \rightarrow L_{p_{o}^{\prime}}$ estimate (GGE) in the hypothesis implies the following $L_{2} \rightarrow L_{p_{o}^{\prime}}$ estimate:

$$
\left\|\chi_{B\left(x, r_{t}\right)} e^{-t A} \chi_{B\left(y, r_{t}\right)}\right\|_{2 \rightarrow p_{o}^{\prime}} \leq\left|B\left(x, r_{t}\right)\right|^{\frac{1}{p_{o}^{\prime}}-\frac{1}{2}} g\left(\frac{d(x, y)}{r_{t}}\right) .
$$

By Theorem 3.2, the latter extends to complex times $z \in \mathbb{C}_{+}$as follows, denoting $r_{z}=(R e z)^{\frac{1}{m}-1}|z|$ and $\alpha=D\left(\frac{1}{2}-\frac{1}{p_{o}^{\prime}}\right)$ :

$$
\left\|\chi_{B\left(x, r_{z}\right)} e^{-z A} \chi_{B\left(y, r_{z}\right)}\right\|_{2 \rightarrow p_{o}^{\prime}} \leq\left|B\left(x, r_{z}\right)\right|^{\frac{1}{p_{o}^{\prime}}-\frac{1}{2}}\left(\frac{|z|}{R e z}\right)^{\alpha} g\left(\frac{d(x, y)}{r_{z}}\right) .
$$

This implies by Lemma 3.3(ii) for $R=\left(\frac{|z|}{R e z}\right)^{-\alpha} e^{-z A}$ :

$$
N_{p_{o}^{\prime}, r_{t} / 2}\left(e^{-z A} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y) \leq\left(\frac{|z|}{R e z}\right)^{\alpha}\left(1+\frac{r_{z}}{r_{t}}\right)^{D / p_{o}^{\prime}} g\left(\frac{r_{t}}{r_{z}}\right)\left(M_{2} f\right)(x)
$$

for all $t>0, f \in L_{p_{o}^{\prime}}(\Omega), x \in \Omega, y \in B\left(x, r_{t} / 2\right)$. The latter for $z=(1+i \tau) \sigma$ allows to estimate $N_{p_{o}^{\prime}, r_{t} / 2}\left(F(\sigma A)^{*} \chi_{\left.B\left(y, 4 r_{t}\right)^{c} f\right)(y) \text { by using the Fourier inversion }}\right.$ formula for $G:=F \cdot \exp$ (this approach is taken from [DOS, Lemma 4.3]):

$$
F(\sigma A)^{*}=\int_{\mathbb{R}} e^{-(1+i \tau) \sigma A} \overline{\widehat{G}(\tau)} d \tau
$$

Indeed, since $r_{(1+i \tau) \sigma}=\sqrt{1+\tau^{2}} \sigma^{1 / m}$, we can estimate as follows:

$$
\begin{aligned}
& N_{p_{o}^{\prime}, r_{t} / 2}\left(F(\sigma A)^{*} \chi_{\left.B\left(y, 4 r_{t}\right)^{c} f\right)(y)}\right. \\
& \quad \leq \int_{\mathbb{R}} N_{p_{o}^{\prime}, r_{t} / 2}\left(e^{-(1+i \tau) \sigma A} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y)|\widehat{G}(\tau)| d \tau \\
& \quad \leq \int_{\mathbb{R}}{\sqrt{1+\tau^{2}}}^{\alpha}\left(1+\sqrt{1+\tau^{2}} \frac{\sigma^{1 / m}}{t^{1 / m}}\right)^{D / p_{o}^{\prime}} g\left(\sqrt{1+\tau^{2}}-\frac{1}{\sigma^{1 / m}}\right)|\widehat{G}(\tau)| d \tau M_{2} f(x) \\
& \quad \leq\left(1+\frac{\sigma}{t}\right)^{D / p_{o}^{\prime m}} \int_{\mathbb{R}}{\sqrt{1+\tau^{2}}}^{D / 2} g\left({\sqrt{1+\tau^{2}}-1}_{\left.-\frac{t^{1 / m}}{\sigma^{1 / m}}\right)|\widehat{G}(\tau)| d \tau M_{2} f(x)} \quad \leq\left(1+\frac{\sigma}{t}\right)^{D / p_{o}^{\prime} m}\left(\int_{\mathbb{R}}\left(1+\tau^{2}\right)^{\frac{D}{2}-s} g\left({\sqrt{1+\tau^{2}}}^{-1} \frac{t^{1 / m}}{\sigma^{1 / m}}\right)^{2} d \tau\right)^{1 / 2}\|G\|_{H^{s}} M_{2} f(x) .\right.
\end{aligned}
$$

Hence the assertion is proved once we show for $\beta:=s-\frac{D}{2}>\frac{1}{2}$ :

$$
\int_{0}^{\infty}\left(1+\tau^{2}\right)^{-\beta} g\left({\sqrt{1+\tau^{2}}}^{-1} a\right) d \tau \leq C a^{1-2 \beta} \quad \text { for all } \quad a \geq 2
$$

First, the change of variables $u={\sqrt{1+\tau^{2}}}^{-1}$ yields

$$
\int_{0}^{\infty}\left(1+\tau^{2}\right)^{-\beta} g\left({\sqrt{1+\tau^{2}}}^{-1} a\right) d \tau=a^{2(1-\beta)} \int_{a^{-1}}^{\infty} g\left(u^{-1}\right) u^{1-2 \beta}\left(a^{2} u^{2}-1\right)^{-1 / 2} d u
$$

Since $\left(a^{2} u^{2}-1\right)^{1 / 2} \geq \frac{\sqrt{3}}{2} a u$ for all $u \in\left[2 a^{-1}, \infty\right)$, we have

$$
\int_{2 a^{-1}}^{\infty} g\left(u^{-1}\right) u^{1-2 \beta}\left(a^{2} u^{2}-1\right)^{-1 / 2} d u \leq \frac{2}{\sqrt{3}} a^{-1} \int_{0}^{\infty} g\left(u^{-1}\right) u^{-2 \beta} d u=C a^{-1}
$$

On the other hand, the remaining part of the integral can be estimated by

$$
\begin{aligned}
\int_{a^{-1}}^{2 a^{-1}} g\left(u^{-1}\right) u^{1-2 \beta}\left(a^{2} u^{2}-1\right)^{-1 / 2} d u & \leq g(a / 2) a^{2 \beta-1} \int_{a^{-1}}^{2 a^{-1}}(a u-1)^{-1 / 2} d u \\
& =g(a / 2) a^{2(\beta-1)} \int_{0}^{1} v^{-1 / 2} d v
\end{aligned}
$$

The last preparatory step for the proof of Theorem 1.1 is the following lemma.

Lemma 3.5. Let $n \in \mathbb{N}, \varepsilon>0$ and $E(u):=\sum_{k=0}^{n}\binom{n}{k}(-1)^{k} e^{-k u}, u \in \mathbb{R}_{+}$. Then

$$
\|E(\sigma \cdot)\|_{C^{n}\left(\left[\varepsilon, \varepsilon^{-1}\right]\right)} \leq C\left(1 \wedge \sigma^{n}\right) \quad \text { for all } \quad \sigma>0
$$

Proof. Fix $m \in\{0, \ldots, n\}$. First we treat the case of small $\sigma$. Since $\frac{E^{(m)}(t)}{t^{n-m}} \rightarrow \frac{n!}{(n-m)!}$ for $t \rightarrow 0$, we have $\left|E^{(m)}(t)\right| \leq C_{0} t^{n-m}$ for all $t \in[0,1]$. This implies for all $\sigma \in[0, \varepsilon]$ and $u \in\left[0, \varepsilon^{-1}\right]$ :

$$
\left|E(\sigma \cdot)^{(m)}(u)\right|=\sigma^{m}\left|E^{(m)}(\sigma u)\right| \leq \sigma^{m} C_{0}(\sigma u)^{n-m} \leq C_{0} \varepsilon^{n-m} \sigma^{n}
$$

Now we treat the case of large $\sigma$. Since $E^{(m)}(t)=\sum_{k=0}^{n} c_{k, m, n} e^{-k t}$ with $c_{0, m, n}=0$ for $m>0$, we deduce for all $\sigma, u \in[\varepsilon, \infty)$ :

$$
\left|E(\sigma \cdot)^{(m)}(u)\right|=\sigma^{m}\left|E^{(m)}(\sigma u)\right| \leq \sigma^{m} \sum_{k=0}^{n}\left|c_{k, m, n}\right| e^{-k \sigma \varepsilon} \leq C_{1}
$$

Finally, we come to the proof of Theorem 1.1. We use the symbol $\preceq$ to indicate domination up to constants independent of the relevant parameters.

Proof of Theorem 1.1. We want to apply our weak type ( $p_{o}, p_{o}$ ) criterion Theorem 3.1 for $T=F(A)$. Hence we have to show

$$
\begin{equation*}
N_{p_{o}^{\prime}, r_{t} / 2}\left(\left(F(A) D^{n} e^{-t A}\right)^{*} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y) \preceq \sup _{h>0}\left\|\omega F_{h}\right\|_{H^{s}}\left(M_{2} f\right)(x) \tag{2}
\end{equation*}
$$

for all $t>0, f \in L_{p_{o}^{\prime}}(\Omega), x \in \Omega, y \in B\left(x, r_{t} / 2\right)$ and some $n \in \mathbb{N}$. Choose $\varepsilon_{1} \geq \varepsilon_{2}>0$ as in Proposition 3.4 and $n \in \mathbb{N}$ such that $n>\varepsilon_{1} \vee s$. Denote $\delta:=\left(n-\varepsilon_{1}\right) \wedge \varepsilon_{2}>0$ and $\varphi(u):=u^{-\varepsilon_{1}} \vee u^{-\varepsilon_{2}}, E(u):=\sum_{k=o}^{n}\binom{n}{k}(-1)^{k} e^{-k u}$ for all $u \in \mathbb{R}_{+}$. Furthermore, for $\sigma>0$ we denote the dilations $F_{\sigma}:=F(\sigma \cdot)$, $\omega_{\sigma}:=\omega(\sigma \cdot)$ and $E_{\sigma}:=E(\sigma \cdot)$. Observe that $E_{\sigma}(A)=D^{n} e^{-\sigma A}$ and by Lemma 3.5

$$
\begin{equation*}
\varphi(\sigma)\left\|E_{\sigma}\right\|_{C^{n}(\operatorname{supp} \omega)} \preceq \varphi(\sigma)\left(1 \wedge \sigma^{n}\right) \leq \sigma^{-\delta} \wedge \sigma^{\delta} \quad \text { for all } \quad \sigma>0 \tag{3}
\end{equation*}
$$

Now (2) follows from Proposition 3.4, applied for $\omega F_{2^{-l}} E_{t 2^{-l}}$ instead of $F$ and $\sigma=2^{l}$, and then summation over $l \in \mathbb{Z}$ :

$$
\begin{array}{rlr}
N_{p_{o}^{\prime}, r_{t} / 2}\left(\left(F(A) D^{n} e^{-t A}\right)^{*} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y) & \\
& =N_{p_{o}^{\prime}, r_{t} / 2}\left(\sum_{l \in \mathbb{Z}}\left(\left(w_{2^{l}} F\right)(A) D^{n} e^{-t A}\right)^{*} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y) & {\left[\sum \omega_{2^{l}}=1\right]} \\
& \leq \sum_{l \in \mathbb{Z}} N_{p_{o}^{\prime}, r_{t} / 2}\left(\left(w F_{2^{-l}} E_{\left.\left.t 2^{-l}\right)\left(2^{l} A\right)^{*} \chi_{B\left(y, 4 r_{t}\right)^{c}} f\right)(y)}\right.\right. & {\left[E_{\sigma}(A)=D^{n} e^{-\sigma A}\right]} \\
& \leq \sum_{l \in \mathbb{Z}} \varphi\left(t 2^{-l}\right)\left\|w F_{2^{-l}} E_{t 2^{-l}} \cdot \exp \right\|_{H^{s}} M_{2} f(x) & \\
& \leq \sup _{h>0}\left\|\omega F_{h}\right\|_{H^{s}} M_{2} f(x) \sum_{l \in \mathbb{Z}} \varphi\left(t 2^{-l}\right)\left\|E_{t 2^{-l}} \cdot \exp \right\|_{C^{n}(\operatorname{supp} \omega)} & {[n \geq s]} \\
& \leq \sup _{h>0}\left\|\omega F_{h}\right\|_{H^{s}} M_{2} f(x) \sum_{l \in \mathbb{Z}}\left(t 2^{-l}\right)^{-\delta} \wedge\left(t 2^{-l}\right)^{\delta} & \\
\leq \sup _{h>0}\left\|\omega F_{h}\right\|_{H^{s}} M_{2} f(x) & &
\end{array}
$$

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