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A Spherical Harnack Inequality for Singular Solutions of Nonlinear Elliptic Equations

CHIUN-CHUAN CHEN – CHANG-SHOU LIN

Abstract. We consider a positive singular solution of

$$(0.1) \quad \Delta u + g(u) = 0 \text{ in } B_2 \setminus \Gamma,$$

where $g(t)$ is locally bounded and positive for $t > 0$, Γ is a closed subset of \bar{B}_1 with vanishing Newton capacity, B_R is the open ball of radius R and center 0 in \mathbb{R}^n , and $n \geq 3$. By employing the method of moving planes and the localization method of R. Schoen, we prove the following inequality,

$$\frac{g(u(x))}{u(x)} \leq c d(x)^{-2} \text{ for } |x| \leq 1,$$

where c is a positive constant and $d(x)$ is the distance from x to Γ , provided that

$$g(t)t^{-\frac{n+2}{n-2}} \text{ is nonincreasing in } t \text{ for } t \text{ large.}$$

This inequality is new even when $u(x)$ is radially symmetric.

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1. – Introduction

In this paper we continue our previous work [5] to investigate the local behavior of positive singular solutions to the partial differential equation

$$(1.1) \quad \Delta u + g(u) = 0 \text{ in } B_2 \setminus \Gamma,$$

where B_R is the n -dimensional ball of center 0 and radius R , $n \geq 3$ and Γ is a closed subset of the closed ball \bar{B}_2 . The nonlinear term $g(t)$ is a positive

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and locally bounded function in $(0, \infty)$. A function u is said to be a solution of (1.1) if $u \in W_{loc}^{2,p}(B_2 \setminus \Gamma)$ for any $p > 1$ and u satisfies equation (1.1) in the distribution sense. Equation (1.1) often arises in many branches of applied science, as well as in many different disciplines of mathematics. When $g(t) = t^{\frac{n+2}{n-2}}$, equation (1.1) is the so-called constant scalar curvature equation in conformal geometry. Recently, there have been many works devoted to studying singular solutions of (1.1) when the singular set Γ consists of a single point only. We refer the interested readers to [3], [5], [6], [7], [8], [10], [11], [14], [15], [17], [19] and the references therein.

In this paper we are mainly concerned with apriori estimates on singular solutions and their applications to asymptotic symmetry. Unlike many other works, the singular set Γ is not limited to a discrete set. In fact, Γ can be as big as a submanifold with codimension 2. To state our result, we recall that the Newton capacity of a closed set Γ is defined by

$$\text{Cap}(\Gamma) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla v|^2 : v \in C_0^\infty(\mathbb{R}^n) \text{ and } v \geq 1 \text{ on } \Gamma \right\}.$$

Note that for a closed set Γ of Hausdorff dimension $\leq n - 2$, the Newton capacity $\text{Cap}(\Gamma)$ always vanishes. Throughout the paper, we always assume $\text{Cap}(\Gamma) = 0$ and $g(t)$ is positive for $t > 0$ and satisfies

$$(1.2) \quad \begin{cases} g(x) & \text{is locally bounded for } t \in [0, \infty) \text{ and} \\ t^{-\frac{n+2}{n-2}}g(t) & \text{is nonincreasing in } t \text{ for } t \text{ large.} \end{cases}$$

The main result of this paper is the following *spherical* Harnack inequality.

THEOREM 1.1. *Let Γ be a closed set in \bar{B}_1 with $\text{Cap}(\Gamma) = 0$, and let $g(t)$ satisfy (1.2). Assume that $u(x)$ is a positive solution of (1.1) for $x \in B_2 \setminus \Gamma$. Then there exists a constant $c > 0$ such that*

$$(1.3) \quad \frac{g(u(x))}{u(x)} \leq c d^{-2}(x) \text{ for } |x| \leq 1,$$

where $d(x) = d(x, \Gamma)$ denotes the distance from x to Γ .

We remark that inequality (1.3) is new even when $\Gamma = \{0\}$ and $u(x) = u(|x|)$ is radially symmetric. When $g(t)$ satisfies a power rule at infinity, that is, there is a positive number $p \in (1, \frac{n+2}{n-2})$ such that $g(t)t^{-p}$ is bounded from below and above by two positive constants, inequality (1.3) is reduced to

$$(1.4) \quad u(x) \leq c d^{-\frac{2}{p-1}}(x) \text{ for } |x| \leq 1.$$

Inequality (1.4) was previously known only when the limit $\lim_{t \rightarrow +\infty} \frac{g(t)}{t^p}$ is a positive number. See [5] and [16].

Inequality (1.3) is called a *spherical* Harnack inequality when $\Gamma = \{0\}$ because it implies that the Harnack inequality holds for u on each *sphere*

$S_r = \{x : |x| = r\}$. For each $r \in (0, 1]$, we set $v_r = u(rx)$ and rewrite (1.1) for v_r as a linear elliptic equation

$$\Delta v_r(x) + C_r(x)v_r(x) = 0 \text{ for } \frac{1}{2} \leq |x| \leq 2,$$

where $C_r(x) = r^2 \frac{g(v_r(x))}{v_r(x)}$. By (1.3), $C_r(x)$ satisfies $C_r(x) \leq c |x|^{-2}$ for $\frac{1}{2} \leq |x| \leq 2$. Since $C_r(x)$ is uniformly bounded in the annulus $\{x : \frac{1}{2} \leq |x| \leq 2\}$, the classical Harnack inequality and the gradient estimate give us that

$$\max_{|x|=1} v_r(x) \leq c \min_{|x|=1} v_r(x), \quad \text{and} \quad |\nabla v_r(x)| \leq c v_r(x) \text{ for } |x| = 1.$$

Scaling back to u , we have

$$(1.5) \quad \max_{|x|=r} u(x) \leq c \min_{|x|=r} u(x), \quad \text{and} \quad |\nabla u(x)| \leq c |x|^{-1} u(x) \text{ for } 0 < |x| \leq 1.$$

These spherical Harnack inequalities (1.5) give us the conclusion of the first part of the following corollary.

COROLLARY 1.2. *Suppose $g(t)$ satisfies (1.2) and u is a positive solution of (1.1) with $\Gamma = \{0\}$. If 0 is a nonremovable singularity, then*

$$(1.6) \quad \lim_{x \rightarrow 0} u(x) = +\infty.$$

Furthermore, $u(x)$ is asymptotically symmetric as x tends to 0 , that is,

$$(1.7) \quad u(x) = \bar{u}(r)(1 + o(1))$$

for $r = |x|$, where $\bar{u}(r)$ is the spherical average of u over the sphere $S_r = \{x : |x| = r\}$ and $o(1)$ tends to zero as $r \rightarrow 0$.

The asymptotic symmetry of singular solutions of (1.1) with $\Gamma = \{0\}$ has been proved by Caffarelli-Gidas-Spruck [3] and C. Li [11] under the condition that $g(t)$ is Lipschitz and

$$(1.8) \quad g(t)t^{-\frac{n+2}{n-2}} \text{ is nonincreasing for all } t > 0.$$

To prove (1.7), Caffarelli, Gidas and Spruck introduced a “measure-theoretic” version of the method of moving planes (MMP). The idea of their approach is quite simple, nevertheless the technique to carry out the idea is very complicated. The difficulty of their method arises from the fact that the solution u is defined only *locally*. In this case, they showed that the method of moving planes could be applied only along certain admissible directions. Later on, C. Li simplified their argument and proved (1.7) assuming g to be Lipschitz and to satisfy (1.8) only. Note that (1.6) of Corollary 1.2 is a consequence of the asymptotic symmetry in their approach. This is the reason why the monotonicity

condition (1.8) has to be assumed for *all* $t > 0$ in Li's argument. In this paper, we do not use the new version of MMP. Instead, we employ a *blow-up* argument of R. Schoen to accomplish our goal. Together with the standard MMP, we are able to use it to establish the spherical Harnack inequality (1.3). The MMP was first invented by Alexandroff and was later used to prove radial symmetry of positive solutions of semilinear elliptic equations by Ni, Gidas, Nirenberg [9] and Serrin [18]. Recently, it was greatly improved to be able to study the asymptotic behavior of positive singular solutions as well as the blow-up behavior of a sequence of smooth solutions to the scalar curvature equation or mean field type equations. See [3], [5], [7], [11], [12], [13], [14], [15] and the references therein. In this paper, we continue to apply refined arguments of the MMP, developed in [5], [7], [14]. However, due to the lack of regularity of g and the assumption that (1.3) holds only for large t , a new ingredient of ideas is required.

The blow-up technique mentioned above was first introduced by Schoen [16], [17] in his works on the constant scalar curvature equation. This method allows us to localize certain points where solutions can be blown-up to be an entire solution in \mathbb{R}^n . When $g(t)$ satisfies a power law at ∞ , the limiting function is well-understood. In our case, without any further assumption on $g(t)$ at infinity, we could only prove the existence of the limiting function which might be a constant at the worst case. To avoid it, we have to choose a second set of blowing-up points. Thus, the technique itself has its own interests. We believe it should be helpful in other problems. Our work is the first example to apply the localization method of Schoen to such generality of nonlinear terms.

For the proof of the asymptotic symmetry (1.7), we will use the standard blow-up argument. Thus, it is important to classify entire positive solutions of the limiting equation. The Liouville-type theorem below plays an essential role for this.

THEOREM 1.3. *Let g be a nonnegative, locally bounded function satisfying (1.8). Suppose that u is a nonconstant positive solution of*

$$\Delta u + g(u) = 0 \quad \text{in } \mathbb{R}^n.$$

Then u is radially symmetric with respect to some point of \mathbb{R}^n and $g(t) = c t^{\frac{n+2}{n-2}}$ for $0 < t \leq \max_{\mathbb{R}^n} u$, where c is a positive constant.

Theorem 1.3 was proved in [5] and [1] where, in addition to (1.8), g is also assumed to be Lipschitz. (In [5], the assumption that $g(t)t^{-\frac{n+2}{n-2}}$ tends to a positive number as $t \rightarrow +\infty$ was also required). Without the smoothness assumption on g , we need to refine some arguments in the process of moving planes.

This paper is organized as follows. The proof of Theorem 1.1 will be given in Section 2 and Section 3. In Section 2, we assume (1.3) does not hold and use the localization method of Schoen to blow-up solutions. The key step is to guarantee that the limiting function is *not* a constant function in the

whole \mathbb{R}^n . On the other hand, in Section 3, we employ the MMP to show that the limiting function is actually a constant. Thus, we obtain a contradiction and complete the proof of Theorem 1.1. In Section 4, we prove the Liouville theorem, Theorem 1.3 and more general, the symmetry of singular solutions of (1.1) in $\mathbb{R}^n \setminus \mathbb{R}^k$. The latter is important when we come to investigate the asymptotic symmetry for the case when the singular set Γ is a k -dimensional submanifold of \mathbb{R}^n for $0 \leq k \leq n - 2$. This asymptotic result will be established in Section 5, where Corollary 1.2 is a special case of the result presented there.

2. – Proof of Theorem 1.1, Part I

In this section, we are going to give a proof of Theorem 1.1. We prove (1.3) by contradiction. When inequality (1.3) fails, we first employ a blow-up method of Schoen to blow-up solutions into a positive entire function in \mathbb{R}^n . The key point is to make sure that the positive entire function is not identical to a positive constant in the whole \mathbb{R}^n . This is not obvious due to the generality of the nonlinear term $g(t)$. On the other hand, to obtain a contradiction, we will use the method of moving planes to show that the limiting function is actually a constant. This part of proof is long and technically complicated. The reason is partly due to the lack of regularity of $g(t)$ and condition (1.2) which holds only for *large* t . If an additional condition $g'(t) \geq 0$ is assumed, then the proof could be made considerably simpler. Since both blow-up argument and the MMP necessarily use a lot of notations, in the step 1 of proof of Theorem 1.1, we assume $\Gamma = \{0\}$ in order to make the presentation more readable. See Remark 2.2 for the general case. For the application of the method of moving planes, this assumption of Γ does not make any difference because of the result of removable singularity of bounded superharmonic functions. The removability result is stated in Lemma 2.1 below. Since its proof is quite elementary, we present the proof here for the sake of completeness.

LEMMA 2.1. *Suppose that h is a positive C^1 superharmonic function in $\bar{\Omega} \setminus \Gamma$, where $\Gamma \subseteq \bar{\Omega}$ is closed and $Cap(\Gamma) = 0$. Then*

$$h(x) \geq \inf_{y \in \partial\Omega \setminus \Gamma} h(y) \text{ for } x \in \Omega \setminus \Gamma .$$

PROOF. Set

$$h_\varepsilon^- = \min(h(x), m - \varepsilon) - m + \varepsilon \text{ for } x \in \Omega \setminus \Gamma ,$$

where $\varepsilon > 0$ and $m = \inf_{y \in \partial\Omega \setminus \Gamma} h(y)$. Clearly, $h_\varepsilon^-(x)$ is a bounded superharmonic function and $h_\varepsilon^-(x) \leq 0$. Moreover, $h_\varepsilon^-(x) = 0$ in a neighborhood of $\partial\Omega \setminus \Gamma$. By setting $h_\varepsilon^-(x) = 0$ outside of Ω , $h_\varepsilon^-(x)$ is a continuous nonpositive superharmonic function of $\mathbb{R}^n \setminus \Gamma$. Since Γ has zero capacity and $h_\varepsilon^-(x)$

is bounded in \mathbb{R}^n , we conclude that $h_\varepsilon^-(x)$ is a superharmonic function in \mathbb{R}^n by the classical theorem of removable singularity of bounded superharmonic functions. By the standard smoothing processes, we then have

$$h_\varepsilon^-(x) \geq \inf_{|y|=R} h_\varepsilon^-(y) = 0 \quad \text{for } |x| \leq R$$

with R large. Thus, $h_\varepsilon^-(x) \equiv 0$ and $h(x) \geq m - \varepsilon$ for $x \in \Omega \setminus \Gamma$ as claimed. Now let $\varepsilon \rightarrow 0^+$, we obtain $h(x) \geq m$ on $\Omega \setminus \Gamma$. \square

PROOF OF THEOREM 1.1. As mentioned above, we assume $\Gamma = \{0\}$. For simplicity, we may also assume that $g(t)$ is continuous. For the general case, it is easy to make a necessary modification of the following arguments. First, we recall that there is a $t_0 \geq 0$ such that

$$g(t)t^{-\frac{n+2}{n-2}} \text{ is nonincreasing for } t \geq t_0.$$

Suppose (1.3) fails. Then there exists a sequence $\{x_i\}$ such that

$$(2.1) \quad g(u(x_i))/u(x_i) = \sup_{|x| \geq |x_i|} g(u(x))/u(x)$$

and

$$(2.2) \quad \lim_{i \rightarrow +\infty} (g(u(x_i))/u(x_i))|x_i|^2 = +\infty.$$

Set

$$(2.3) \quad v_i(y) = M_i^{-1}u(x_i + \tilde{M}_i^{-1}y), \quad \text{where}$$

$$(2.4) \quad M_i = u(x_i) \text{ and } \tilde{M}_i = (g(u(x_i))/u(x_i))^{\frac{1}{2}}.$$

Using the localization method of R. Schoen, we claim:

STEP 1. There is sequence of $\{z_i\}$ so that if we replace x_i by z_i , then (2.1) and (2.2) hold, and v_i of (2.3) converges to a nonconstant function $v(x)$ in $C_{loc}^{1,\alpha}(\mathbb{R}^n)$.

To prove Step 1, we set $r_i = |x_i|$ and

$$(2.5) \quad S_i(z, x) \equiv (g(u(z))/u(z))^{\frac{1}{2}} \left(\frac{r_i}{4} - |z - x| \right)$$

for $z \in B(x, r_i/4)$. Let z_i and \bar{x}_i satisfy

$$(2.6) \quad S_i(z_i, \bar{x}_i) = \sup_{\substack{z \in \bar{B}(x, r_i/4) \\ |x|=r_i}} S_i(z, x).$$

In particular, we have

$$(2.7) \quad S_i(z_i, \bar{x}_i) \geq S_i(x_i, x_i) = \frac{r_i}{4} \left[\frac{g(u(x_i))}{u(x_i)} \right]^{\frac{1}{2}} \rightarrow +\infty$$

as $i \rightarrow +\infty$. We note that by (2.1), z_i satisfies

$$(2.8) \quad \begin{cases} |z_i| \geq \frac{3}{4}r_i & \text{and} \\ g(u(z_i))/u(z_i) = \sup_{|x| \geq |z_i|} (g(u(x))/u(x)). \end{cases}$$

Set $v_i(y) = M_i^{-1}u(z_i + \tilde{M}_i^{-1}y)$, where

$$M_i = u(z_i) \quad \text{and} \quad \tilde{M}_i = (g(u(z_i))/u(z_i))^{\frac{1}{2}}.$$

Obviously, $v_i(y)$ is smooth in the ball $\{y : |y| \leq R_i\}$, where

$$R_i := \tilde{M}_i |z_i| \geq S(z_i, \bar{x}_i) \rightarrow +\infty$$

as $i \rightarrow +\infty$, and v_i satisfies

$$(2.9) \quad \Delta v_i + h_i(v_i) = 0,$$

where

$$(2.10) \quad h_i(t) = \frac{g(M_i t)}{g(M_i)}.$$

For any $R > 0$ and $|y| \leq R$, let $z = z_i + \tilde{M}_i^{-1}y$. Then for large i

$$\begin{aligned} \frac{r_i}{4} - |z - \bar{x}_i| &\geq \frac{r_i}{4} - |z_i - \bar{x}_i| - \tilde{M}_i^{-1}|y| = \tilde{M}_i^{-1}(S_i(z_i, \bar{x}_i) - |y|) \\ &\geq \frac{1}{2}\tilde{M}_i^{-1}S_i(z_i, \bar{x}_i) = \frac{1}{2} \left(\frac{r_i}{4} - |z_i - \bar{x}_i| \right). \end{aligned}$$

Thus

$$(2.11) \quad \begin{aligned} \frac{h_i(v_i(y))}{v_i(y)} &= \frac{u(z_i)}{g(u(z_i))} \frac{g(u(z))}{u(z)} \\ &= \left(\frac{S(z, \bar{x}_i)}{S(z_i, \bar{x}_i)} \right)^2 \left(\frac{\frac{r_i}{4} - |z_i - \bar{x}_i|}{\frac{r_i}{4} - |z - \bar{x}_i|} \right)^2 \leq 4. \end{aligned}$$

By (2.11), the Harnack inequality can be applied to v_i since (2.9) can be written as a linear equation

$$\Delta v_i + \left(\frac{h(v_i)}{v_i} \right) v_i = 0.$$

Thus, $v_i(y)$ is uniformly bounded in any compact set of \mathbb{R}^n . Applying the elliptic estimates, $v_i(y)$ is bounded in $W_{\text{loc}}^{2,p}(\mathbb{R}^n)$ for any $p > 1$. Therefore by passing to a subsequence (still denoted by v_i), v_i converges to v in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$ for any $0 < \alpha < 1$.

Before continuing our proof further, we should stop to give some explanation in order to make the subsequent arguments more clear. First, in the most pathological situation, $g(M_i t_i)/g(M_i)$ might drop to zero for a sequence of $t_i \downarrow 1$ and meanwhile, for any $R > 0$, the measure $|\{y \in B_R : v_i(y) < t_i\}|$ tends to zero. In this case, $v(y) \equiv 1$ in \mathbb{R}^n . On the other hand, if we assume additionally that g' exists and $g'(t) \geq 0$ for $t \geq t_0$, then by (1.2), $g'(t)$ satisfies

$$0 \leq t g'(t) \leq \frac{n+2}{n-2} g(t) \quad \text{for } t \geq t_0.$$

From it, h_i satisfies for $|y| \leq R$

$$\begin{aligned} |\nabla h_i(y)| &\leq \frac{M_i g'(M_i v_i)}{g(M_i)} |\nabla v_i| \leq \frac{n+2}{n-2} \frac{g(M_i v_i)}{v_i g(M_i)} |\nabla v_i| \\ &= \frac{n+2}{n-2} \frac{h_i(v_i)}{v_i} |\nabla v_i| \leq c(R). \end{aligned}$$

By elliptic estimates, v_i converges to v in C^2 . Since $v_i(0) = 1$, $\Delta v_i(0) = -h_i(v_i(0)) = -1$. Thus $\Delta v(0) = -1$ which implies $v \not\equiv 1$ in \mathbb{R}^n .

Now we return to the proof of step 1. If $v \not\equiv 1$ in \mathbb{R}^n , then step 1 is proved. Assume $v \equiv 1$. We will choose another v_i and z_i as follows. For any $\sigma_0 \in (0, 1/2)$, there exists a $z_i^* \in \{t z_i \mid t \geq 1\}$ such that $u_i(z_i^*) = (1 - \sigma_0)u_i(z_i)$ when i is large. Since $\sigma_0 > 0$, by the assumption $v \equiv 1$, we have

$$(2.12) \quad \begin{cases} |z_i^* - z_i| (g(u(z_i))/u(z_i))^{\frac{1}{2}} \longrightarrow +\infty, \quad \text{and} \\ \frac{g(u(z_i^*))}{u(z_i^*)} \leq \frac{g(u(z_i))}{u(z_i)}, \end{cases}$$

where the second inequality is due to (2.8).

Let $v_i^*(y) = (M_i^*)^{-1}u(z_i^* + (\tilde{M}_i^*)^{-1}y)$, where

$$M_i^* = u(z_i^*) \quad \text{and} \quad \tilde{M}_i^* = (g(u(z_i^*))/u(z_i^*))^{\frac{1}{2}}.$$

As before, v_i^* satisfies

$$(2.13) \quad \Delta v_i^* + h_i^*(v_i^*) = 0$$

with $h_i^*(t) = g(M_i^* t)/g(M_i^*)$.

Since $g(t)t^{-\frac{n+2}{n-2}}$ is nonincreasing for $t > t_0$, we have for large i

$$(2.14) \quad \frac{g(u(z_i^*))}{u(z_i^*)} \geq \frac{g(u(z_i))}{u(z_i)} \left(\frac{u(z_i^*)}{u(z_i)} \right)^{\frac{4}{n-2}} = (1 - \sigma_0)^{\frac{4}{n-2}} \frac{g(u(z_i))}{u(z_i)}.$$

Hence by $|z_i^*| \geq |z_i|$,

$$(2.15) \quad \left(\frac{g(z_i^*)}{u(z_i^*)} \right) |z_i^*|^2 \rightarrow +\infty.$$

For any $R > 0$ and $|y| \leq R$, let $x = z_i^* + (\tilde{M}_i^*)^{-1}y$. Recall $|z_i^* - z_i| = |z_i^*| - |z_i|$. Then

$$(2.16) \quad \begin{aligned} |x| - |z_i| &\geq |z_i - z_i^*| - (\tilde{M}_i^*)^{-1}|y| \\ &= (\tilde{M}_i^*)^{-1}[\tilde{M}_i^*|z_i - z_i^*| - |y|] \geq 0, \end{aligned}$$

by (2.12) and (2.14). Thus by (2.8),

$$(2.17) \quad \frac{g(u(x))}{u(x)} \leq \frac{g(u(z_i))}{u(z_i)} \leq (1 - \sigma_0)^{\frac{-4}{n-2}} \frac{g(u(z_i^*))}{u(z_i^*)},$$

which implies that $h_i^*(v_i^*)/v_i^*$ is uniformly bounded in any compact set of \mathbb{R}^n . Thus, there exists a subsequence of v_i^* which converges to v^* in $C_{loc}^{1,\alpha}(\mathbb{R}^n)$. We claim that $v^* \not\equiv 1$ in \mathbb{R}^n .

Suppose $v^* \equiv 1$. Then $u(z_i) \geq u(x) \geq (1 - 2\sigma_0)u(z_i)$ for large i and $|y| \leq 1$, where $x = z_i^* + (\tilde{M}_i^*)^{-1}y$. Thus, by (1.2) and (2.12), we have for large i

$$\begin{aligned} h_i^*(v_i^*(y)) &= \frac{g(u(x))}{g(u(z_i^*))} \geq \frac{g(u(z_i))}{g(u(z_i^*))} \left(\frac{u(x)}{u(z_i)} \right)^{\frac{n+2}{n-2}} \\ &\geq \frac{u(z_i)}{u(z_i^*)} \left(\frac{u(x)}{u(z_i)} \right)^{\frac{n+2}{n-2}} \geq (1 - \sigma_0)^{-1} (1 - 2\sigma_0)^{\frac{n+2}{n-2}} \equiv c_0 > 0. \end{aligned}$$

Integrating (2.13), we have

$$- \int_{\partial B_1} \frac{\partial v_i^*}{\partial r} d\sigma = \int_{B_1} h_i^*(v_i^*) dx \geq c_0 \text{ vol}(B_1),$$

which yields a contradiction, since the left hand side tends to 0 as $i \rightarrow +\infty$. This proves Step 1.

REMARK 2.2. When the singular set Γ is not a discrete set, z_i^* can be chosen as follows. Note that since $\text{Cap}(\Gamma) = 0$ and $r_i = d(x_i)$ is small for

large i , the set $\{x \in \bar{B}_1 \mid d(x) \geq r_i\}$ is connected. For any $\sigma_0 \in (0, 1)$, z_i^* can be chosen so that

$$u(z_i^*) = (1 - \sigma_0)u(z_i) \quad \text{and} \quad d(z_i^*) \geq r_i.$$

Thus,

$$\frac{g(u(z_i^*))}{u(z_i^*)} d(z_i^*)^2 \geq (1 - \sigma_0)^{\frac{4}{n-2}} \frac{g(u(z_i))}{u(z_i)} r_i^2 \rightarrow +\infty$$

as $i \rightarrow +\infty$. Set $v_i^*(y)$ as before. For any $R > 0$, and $|y| \leq R$, let $x = z_i^* + (\tilde{M}_i^*)^{-1}y$. Then

$$\begin{aligned} d(x) &\geq d(z_i^*) - (\tilde{M}_i^*)^{-1}|y| = (\tilde{M}_i^*)^{-1} \left(\sqrt{\frac{g(u(z_i^*))}{u(z_i^*)}} d(z_i^*) - |y| \right) \\ &\geq \frac{3}{4}d(z_i^*) - |y| \geq \frac{3}{4}d(z_i^*) \geq \frac{3}{4}r_i. \end{aligned}$$

By (2.8), we have

$$\frac{g(u(x))}{u(x)} \leq \frac{g(u(z_i))}{u(z_i)} \leq (1 - \sigma_0)^{-\frac{4}{n-2}} \frac{g(u(z_i^*))}{u(z_i^*)},$$

which implies that $h_i^*(v_i^*)/v_i^*$ is uniformly bounded in any compact set of \mathbb{R}^n . The rest of the argument for Step 1 is the same as in the case when $\Gamma = \{0\}$.

The following lemma was proved in [3] (Lemma 2.1 in [3]) when $\Gamma = \{0\}$. We follow the same argument to prove the general case when $\text{Cap}(\Gamma) = 0$. Note that in Lemma 2.3 below, we assume $\Gamma \subseteq \bar{B}_1 \subset B_2$. When $\Gamma \cap \partial B_2 \neq \emptyset$, we suspect that the same conclusion should hold. However, we can establish the same result by requiring some additional condition for g for the case $\Gamma \cap \partial B_2 \neq \emptyset$. See Lemma 2.4 below.

LEMMA 2.3. *Let u be a positive singular solution of (1.1). Suppose that Γ and g satisfy the hypothesis of Theorem 1.1. Then $g(u) \in L^1(\bar{B}_1)$.*

PROOF. Since the capacity of Γ vanishes, there exists a nonnegative harmonic function η_j in $\mathbb{R}^n \setminus \Gamma$ such that

$$\begin{aligned} \eta_j(x) &= 0 \quad \text{for } x \in \Gamma, \quad \eta_j(x) \rightarrow 1 \text{ uniformly in } C_{loc}(\mathbb{R}^n \setminus \Gamma) \text{ and} \\ \int_{\mathbb{R}^n} |\nabla \eta_j|^2 dx &\rightarrow 0 \quad \text{as } j \rightarrow +\infty. \end{aligned}$$

For any $k > 0$, let $\varphi(t)$ be a smooth nonincreasing function satisfying

$$\varphi(t) = \begin{cases} 1 & \text{if } t < k \\ 0 & \text{if } t \geq 2k. \end{cases}$$

Then (1.1) implies

$$\begin{aligned}
 (2.18) \quad \int_{B_2} \varphi(u)\eta_j g(u)dx &= \int \nabla u \nabla (\varphi(u) \cdot \eta_j)dx - \int_{\partial B_2} \varphi(u)\eta_j \frac{\partial u}{\partial \nu} d\sigma \\
 &= \int \varphi'(u)|\nabla u|^2 \eta_j dx + \int_{B_2} \nabla(\Phi(u)) \nabla \eta_j dx - \int_{\partial B_2} \varphi(u)\eta_j \frac{\partial u}{\partial \nu} d\sigma
 \end{aligned}$$

where $\Phi(t) = \int_0^t \varphi(s)ds$. Let k be fixed and let $j \rightarrow +\infty$. Since $\varphi'(t) \leq 0$, (2.18) implies

$$\int_{B_2} \varphi(u)g(u)dx \leq \int_{\partial B_2} \left| \frac{\partial u}{\partial \nu} \right| d\sigma.$$

By passing $k \rightarrow +\infty$, $g(u) \in L^1(\bar{B}_1)$ is proved. □

When Γ is not a compact set in B_2 , we need a stronger assumption than (1.2) to ensure $g(u) \in L^1(\bar{B}_1)$.

LEMMA 2.4. *Let Γ be a closed set of \bar{B}_2 with $\text{Cap}(\Gamma) = 0$. Assume that in addition to (1.2), g satisfies*

- (i) $g(t)/t$ is increasing in t for $t \geq t_0$, and
- (ii) $g(t) \geq t^{1+p_0}$ for some $p_0 > 0$.

Then $g(u) \in L^1(\bar{B}_1)$ for any positive singular solution u to (1.1)

PROOF. For any $l = 0, 1, \dots, l_0$, let $\varphi_l(t)$ be a nonincreasing function in t such that

$$\varphi_l(t) = \begin{cases} t^{1-lp_0}g(t)^{-1} & \text{if } t < k \\ 0 & \text{if } t > 2k, \end{cases}$$

where l_0 is an integer so that $t^{1-l_0p_0}/g(t) \in L^1(\mathbb{R}^+)$ and $t^{1-(l_0-1)p_0}/g(t) \notin L^1(\mathbb{R}^+)$. (If $t/g(t) \in L^1(\mathbb{R}^+)$, we let $l_0 = 0$). By (ii), we have $p_0l_0 < 1$. Set

$$\Phi_l(t) = \int_0^t \varphi_l(s)ds \quad \text{for } l = 0, 1, \dots, l_0.$$

Thus, for $0 \leq l \leq l_0 - 1$, we have for t large,

$$\Phi_l(t) \leq c \int_{t_0}^t s^{-(l+1)p_0} ds \leq c_1 t^{1-(l+1)p_0},$$

and $\Phi_{l_0}(t)$ is bounded in \mathbb{R}^+ . For each $l = 0, 1, \dots, l_0$, choose $\xi_l(x) \in C_0^\infty(B_2)$ such that $\xi_l(x) = 1$ for $|x| \leq 1 + 2^{-(l_0+1-l)}$ and $\xi_l(x) = 0$ for $|x| \geq 1 + 2^{-(l_0-l)}$. Multiplying $\eta_j \varphi_l(u)\xi_l(x)$ on (1.1) and by passing $j \rightarrow +\infty$ and $k \rightarrow +\infty$ as in (2.18), we have

$$\begin{aligned}
 \int_{B_2} u^{1-lp_0}(x)\xi_l(x)dx &= \int_{B_2} \varphi_l(u)g(u)\xi_l(x)dx \\
 &\leq \int_{B_2} \Phi_l(u)|\Delta \xi_l(u)|dx \\
 &\leq c_1 \int_{B_2} u^{1-(l+1)p_0}(x)|\Delta \xi_l(x)|dx,
 \end{aligned}$$

and

$$\int_{B_2} u(x)^{1-l_0 p_0} \xi_{l_0}(x) dx \leq c_2 \int_{B_2} |\Delta \xi_{l_0}(x)| dx .$$

Starting from l_0 and ending at $l = 0$, we have

$$\int_{B_{3/2}} u(x) dx \leq c_3 .$$

Similar to (2.18), we then have $g(u) \in L^1(\bar{B}_1)$. □

Now we come back to the proof of Theorem 1.1.

STEP 2. Let $z_i, M_i = u(z_i), \tilde{M}_i = (g(u(z_i))/u(z_i))^{\frac{1}{2}}$ be defined as in Step 1. We claim

$$(2.19) \quad \lim_{i \rightarrow +\infty} g(M_i) M_i^{-\frac{n}{n-2}} = +\infty .$$

To prove (2.19), we note that since $v(y) \not\equiv 1$ in \mathbb{R}^n , there exists a positive constant c_0 independent of i such that

$$\int_{B_1} h_i(v_i(y)) dy \geq c_0 > 0 .$$

Let $B_i = B(z_i, \tilde{M}_i^{-1})$. By scaling, we have

$$\begin{aligned} \int_{B_i} g(u(x)) dx &= \left(\frac{M_i^{\frac{n}{n-2}}}{g(M_i)} \right)^{\frac{n-2}{2}} \int_{B_1(0)} h_i(v_i(y)) dy \\ &\geq c_0 \left(\frac{M_i^{\frac{n}{n-2}}}{g(M_i)} \right)^{\frac{n-2}{2}} \end{aligned}$$

By Lemma 2.3, the left hand side tends to zero as $i \rightarrow +\infty$. Thus, (2.19) follows immediately.

In the next section, we will apply the MMP to prove that v is a constant function. Thus it yields a contradiction to Step 1. In the proof, we will see that (2.19) is important in the process of the MMP. We remark that under the hypothesis of Lemma 2.4, (2.19) holds also without assuming that Γ is a compact set of B_2 .

3. – Proof of Theorem 1.1, Part II

In this section, we are going to apply the MMP to conclude that v is radially symmetric with respect to *any* point of \mathbb{R}^n . Thus, v is a constant function in \mathbb{R}^n and it yields a contradiction to Step 1 of Section 2.

STEP 3. For any \hat{y} , we let \bar{v} and $\bar{v}_i(y)$ respectively be the Kelvin transformation of v and v_i with respect to the unit ball with center \hat{y} , i.e.,

$$(3.1) \quad \bar{v}(y) = |y|^{2-n} v \left(\frac{y}{|y|^2} + \hat{y} \right), \text{ and } \bar{v}_i(y) = |y|^{2-n} v_i \left(\frac{y}{|y|^2} + \hat{y} \right).$$

Obviously \bar{v}_i satisfies

$$(3.2) \quad \Delta \bar{v}_i + \bar{h}_i(y, \bar{v}_i) = 0,$$

for $|y| \geq \tilde{M}_i^{-1}$ except at the singularity \bar{y}_i with $|\bar{y}_i| = (\tilde{M}_i |z_i|)^{-1} (1 + o(1))$, where

$$(3.3) \quad \bar{h}_i(y, t) = |y|^{-(n+2)} h_i(|y|^{n-2} t).$$

Clearly, $\bar{v}_i(y)$ converges in $C_{loc}^{1,\alpha}(\mathbb{R}^n \setminus \{0\})$ to \bar{v} . To prove radial symmetry of \bar{v} , it suffices for us to prove that \bar{v} is symmetric with respect to y_1 -direction. Following conventional notations, we let for any $\lambda < 0$, $\Sigma_\lambda = \{y \mid y_1 > \lambda\}$, $T_\lambda = \{y \mid y_1 = \lambda\}$ and y^λ denote the reflection point of y with respect to T_λ . Let $\bar{v}_i^\lambda(y) = \bar{v}_i(y^\lambda)$ for $y \in \Sigma_\lambda$, and

$$(3.4) \quad w_\lambda(y) = \bar{v}_i(y) - \bar{v}_i^\lambda(y).$$

For any fixed $\lambda_0 < 0$, we want to prove

$$(3.5) \quad w_\lambda(y) > 0 \text{ for } y \in \Sigma'_\lambda \text{ and } \lambda \leq \lambda_0,$$

where $\Sigma'_\lambda = \{y \in \Sigma_\lambda \mid |y| \geq (\tilde{M}_i)^{-1} \text{ and } y \neq \bar{y}_i\}$.

To make the MMP work, we should first establish the following two facts.

$$(3.6) \quad w_\lambda(y) \rightarrow +\infty \text{ for } |y| = \tilde{M}_i^{-1} \text{ as } i \rightarrow +\infty, \text{ and}$$

$$(3.7) \quad \bar{h}_i(y^\lambda, \bar{v}_i(y)) \leq \bar{h}_i(y, \bar{v}_i(y)) \text{ for } |y| \geq 1.$$

Note that since $M_i |y|^{n-2} \bar{v}_i(y) \geq t_0$ for $|y| \geq 1$ if i is large, by the expression (3.3) of $\bar{h}_i(y, t)$, we have

$$\begin{aligned} & \bar{h}_i(y^\lambda, \bar{v}_i(y)) - \bar{h}_i(y, \bar{v}_i(y)) \\ &= g(M_i)^{-1} \left\{ |y^\lambda|^{-(2+n)} g(M_i |y^\lambda|^{n-2} \bar{v}_i(y)) - |y|^{-(2+n)} g(M_i |y|^{n-2} \bar{v}_i(y)) \right\} \geq 0. \end{aligned}$$

This proves (3.7).

To prove (3.6), by (3.1) and Lemma 2.1 we have for $|y| = (\tilde{M}_i)^{-1}$,

$$(3.8) \quad \bar{v}_i(y) \geq (\tilde{M}_i)^{n-2} M_i^{-1} \inf_{\partial B_2} u \geq m (g(M_i)/M_i^{\frac{n}{n-2}})^{\frac{n-2}{2}} \rightarrow +\infty,$$

by step 2, where $m = \inf_{\partial B_2} u > 0$. Clearly, there exists a positive constant $c = c(\lambda_0)$ such that

$$\bar{v}_i(y^\lambda) \leq c(\lambda_0) \quad \text{for } y \in \Sigma_\lambda \quad \text{and } \lambda \leq \lambda_0.$$

Hence, (3.6) follows readily.

By equation (3.1), $w_\lambda(y)$ satisfies

$$(3.9) \quad \Delta w_\lambda(y) + \bar{h}_i(y^\lambda, \bar{v}_i(y)) - \bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y)) = \bar{h}_i(y^\lambda, \bar{v}_i(y)) - \bar{h}_i(y, \bar{v}_i(y)) \leq 0$$

for $y \in \mathbb{R}^n \setminus B_1 = \{y \mid |y| \geq 1\}$. Assume $\bar{v}_i(y) \leq \bar{v}_i^\lambda(y)$ for some $y \in \mathbb{R}^n \setminus B_1$. Then by the assumption on g ,

$$\begin{aligned} \bar{h}_i(y^\lambda, \bar{v}_i(y)) - \bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y)) &\geq \bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y)) \left[\left(\frac{\bar{v}_i}{\bar{v}_i^\lambda} \right)^{\frac{n+2}{n-2}} - 1 \right] \\ &\geq \frac{\bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y))}{\bar{v}_i^\lambda} \left(\frac{\bar{v}_i^{\frac{n+2}{n-2}} - (\bar{v}_i^\lambda)^{\frac{n+2}{n-2}}}{(\bar{v}_i^\lambda)^{\frac{4}{n-2}}} \right) \geq \frac{n+2}{n-2} \frac{\bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y))}{\bar{v}_i^\lambda} (\bar{v}_i - \bar{v}_i^\lambda). \end{aligned}$$

By (3.3) and (2.11) or (2.17), one has

$$\frac{\bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y))}{\bar{v}_i^\lambda} = |y^\lambda|^{-4} \left(\frac{h_i(v_i)}{v_i} \right) \leq c_1 |y|^{-4},$$

where $h_i(v_i)/v_i$ is evaluated at $(y^\lambda/|y^\lambda|^2) + \hat{y}$. Thus,

$$\frac{\bar{h}_i(y^\lambda, \bar{v}_i(y)) - \bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y))}{v_i(y) - \bar{v}_i(y^\lambda)} \leq \frac{n+2}{n-2} \frac{\bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y))}{\bar{v}_i^\lambda(y)} \leq c_2 |y|^{-4}.$$

Therefore, we have by (3.9)

$$(3.10) \quad \begin{aligned} \Delta w_\lambda(y) + c_2 |y|^{-4} w_\lambda(y) &\leq \Delta w_\lambda(y) + \bar{h}_i(y^\lambda, \bar{v}_i(y)) - \bar{h}_i(y^\lambda, \bar{v}_i^\lambda(y)) \\ &= \bar{h}_i(y^\lambda, \bar{v}_i(y)) - \bar{h}_i(y, \bar{v}_i(y)) \leq 0 \end{aligned}$$

whenever i is large, $w_\lambda(y) < 0$ and $|y| \geq 1$.

Now we are in the position to apply the MMP. By the maximum principle and (3.10), we have the following.

$$(3.11) \quad \text{There exists a constant } R_0 > 1 \text{ such that if } y_0 \text{ achieves the minimum of } w_\lambda \text{ and } w_\lambda(y_0) = \inf_{\Sigma_\lambda} w_\lambda(y) < 0 \text{ then } |y_0| \leq R_0.$$

The proof of (3.11) is well-known. See [4] for the proof.

Since $\bar{v}_i(y)$ is superharmonic, by Lemma 2.1 and (3.6),

$$\bar{v}_i(z) \geq \inf_{|y|=R_0} \bar{v}_i(y) > 0, \quad \text{for } |z| < R_0.$$

Together with (3.11), there is $\lambda_1 \ll -1$ such that

$$w_\lambda(y) > 0 \quad \text{for } y \in \Sigma'_\lambda \text{ and } \lambda \leq \lambda_1.$$

Now let $\bar{\lambda}_0 = \sup\{\lambda \leq \lambda_0 \mid w_\mu(y) > 0 \text{ for } y \in \Sigma'_\mu \text{ and } \mu \leq \lambda\}$. Clearly, $\bar{\lambda}_0$ depends on i . We claim

STEP 4. $\bar{\lambda}_0 = \lambda_0$ for large i .

By continuity, $w_{\bar{\lambda}_0}(y) \geq 0$ in $\Sigma'_{\bar{\lambda}_0}$. Since $w_{\bar{\lambda}_0}(y) > 0$ for $|y| = (\bar{M}_i)^{-1}$, $w_{\bar{\lambda}_0} \not\equiv 0$ in $\Sigma'_{\bar{\lambda}_0}$. To prove Step 4, we first claim that there exists i_0 such that for $i \geq i_0$,

$$(3.12) \quad w_{\bar{\lambda}_0}(y) > 0 \text{ in } \Sigma'_{\bar{\lambda}_0} \text{ and } \frac{\partial w_{\bar{\lambda}_0}}{\partial y_1}(y) < 0 \text{ for } y \in T_{\bar{\lambda}_0}.$$

When the nonlinear term $g(t)$ is Lipschitz, the claim (3.12) follows readily from the strong maximum principle, because $w_{\bar{\lambda}_0}$ can be written as a nonnegative supersolution of a linear elliptic equation. Due to the lack of regularity of $g(t)$, the proof of (3.12) is unusually long.

Our method is based on a simple observation. Suppose there exists y_0 such that (3.12) does not hold. If $y_0 \in \Sigma'_{\bar{\lambda}_0}$ such that $w_{\bar{\lambda}_0}(y_0) = 0$, then by noting $\bar{\lambda}_0 \leq \lambda_0 < 0$, there is a neighborhood U_0 of y_0 (of course, the size of U_0 depends on i) such that

$$(3.13) \quad |y^{\bar{\lambda}_0}|^{n-2} \bar{v}_i(y^{\bar{\lambda}_0}) \geq |y|^{n-2} \bar{v}_i(y) \quad \text{for } y \in U_0 \cap \bar{\Sigma}_{\bar{\lambda}_0}.$$

If $\frac{\partial w_{\bar{\lambda}_0}}{\partial y_1}(y_0) = 0$ for some $y_0 \in T_{\bar{\lambda}_0}$, then

$$\frac{\partial \bar{v}_i}{\partial y_1}(y_0) = 0 \quad \text{and} \quad \frac{\partial}{\partial y_1} (|y|^{n-2} \bar{v}_i(y)) = \left(\frac{\partial}{\partial y_1} |y|^{n-2} \right) \bar{v}_i(y) < 0$$

at $y = y_0$. Hence, (3.13) also holds for some neighborhood U_0 of y_0 . Then in the next paragraph, we can see that by (3.13), $w_{\bar{\lambda}_0}(y)$ is superharmonic in $U_0 \cap \Sigma_{\bar{\lambda}_0}$ provided $M_i |y_0|^{n-2} \bar{v}_i(y_0) \geq 2t_0$.

By the monotone assumption of g and the expression (3.1) of \bar{h}_i , we have by (3.13),

$$\begin{aligned}
 \bar{h}_i(y, \bar{v}_i(y)) &= |y|^{-(n+2)} h_i(|y|^{n-2} \bar{v}_i(y)) \\
 &\geq |y^{\bar{\lambda}_0}|^{-(n+2)} h_i(|y^{\bar{\lambda}_0}|^{n-2} \bar{v}_i(y^{\bar{\lambda}_0})) \left(\frac{\bar{v}_i(y)}{\bar{v}_i(y^{\bar{\lambda}_0})} \right)^{\frac{n+2}{n-2}} \\
 (3.14) \quad &= \bar{h}_i(y^{\bar{\lambda}_0}, \bar{v}_i(y^{\bar{\lambda}_0})) \left(\frac{\bar{v}_i(y)}{\bar{v}_i(y^{\bar{\lambda}_0})} \right)^{\frac{n+2}{n-2}} \\
 &\geq \bar{h}_i(y^{\bar{\lambda}_0}, \bar{v}_i(y^{\bar{\lambda}_0})) \quad \text{for } y \in U_0 \cap \bar{\Sigma}_{\bar{\lambda}_0},
 \end{aligned}$$

where the last inequality is due to the fact $w_{\bar{\lambda}_0}(y) \geq 0$, provided that

$$(3.15) \quad M_i |y|^{n-2} \bar{v}_i(y) \geq t_0 \quad \text{holds for } y \in U_0 \cap \bar{\Sigma}_{\bar{\lambda}_0}.$$

Thus, if $M_i |y_0|^{n-2} \bar{v}_i(y_0) \geq 2t_0$, U_0 can be chosen so that (3.15) holds for $y \in U_0$. Clearly, $w_{\bar{\lambda}_0}(y)$ is a superharmonic function in $U_0 \cap \bar{\Sigma}_{\bar{\lambda}_0}$. By the strong maximum principle, $w_{\bar{\lambda}_0} \equiv 0$ in $U_0 \cap \bar{\Sigma}_{\bar{\lambda}_0}$. By repeating the same argument, we have $w_{\bar{\lambda}_0}(y) \equiv 0$ in $\Sigma'_{\bar{\lambda}_0}$, which yields a contradiction to (3.6). Hence (3.12) is established if $M_i |y_0|^{n-2} \bar{v}_i(y_0) \geq 2t_0$.

Suppose that $w_{\bar{\lambda}_0}(y_0) = 0$ for some $y_0 \in \Sigma'_{\bar{\lambda}_0}$ and

$$M_i |y_0|^{n-2} \bar{v}_i(y_0) \leq 2t_0.$$

Then, there is a neighborhood U_0 of y_0 such that

$$(3.16) \quad M_i |y|^{n-2} \bar{v}_i(y) \leq 3t_0.$$

Since $\bar{v}_i(y)$ is the Kelvin transformation of v_i , $\bar{v}_i(y^\lambda) \geq c(\lambda_0) > 0$ for some constant independent of i . Thus, $v_i(y) \geq \bar{v}_i(y^\lambda) \geq c(\lambda_0) > 0$ and the set of y satisfying (3.16) is contained in $B_i := \{y \mid \tilde{M}_i^{-1} \leq |y| \leq c_1 M_i^{-\frac{1}{n-2}}\}$ for some $c_1 > 0$ independent of i . Therefore, to establish (3.12), it remains to consider the points in B_i which satisfy (3.16). Two cases are discussed separately.

CASE 1. $\lim_{t \rightarrow +\infty} \frac{g(t)}{t^{\frac{n+2}{n-2}}} = d > 0$. In this case, we will introduce an auxiliary function $\varphi_i(y)$. For any $\varepsilon > 0$, set

$$(3.17) \quad \varphi_i(y) = \frac{\varepsilon}{M_i} G(y, 0) - c_1 \int_{B_i} G(y, x) dx \quad \text{for } y \in B\left(0, \frac{\lambda_0}{2}\right)$$

where $G(y, x)$ is the Green function with singularity at x for the ball $B(0, \frac{\lambda_0}{2})$ and c_1 is a positive constant independent of i , which is chosen later.

Recall the definition of \bar{h}_i in (3.3). Then

$$(3.18) \quad \begin{aligned} \bar{h}_i \left(y^{\bar{\lambda}_0}, \bar{v}_i(y^{\bar{\lambda}_0}) \right) &= g(M_i)^{-1} |y^{\bar{\lambda}_0}|^{-(n+2)} g(M_i |y^{\bar{\lambda}_0}|^{n-2} \bar{v}_i(y^{\bar{\lambda}_0})) \\ &= d \left(\bar{v}_i(y^{\bar{\lambda}_0}) \right)^{\frac{n+2}{n-2}} (1 + o(1)) \quad \text{for } y \in B \left(0, \frac{\lambda_0}{2} \right), \end{aligned}$$

where $o(1)$ tends uniformly to zero as $i \rightarrow +\infty$.

Thus, if $|y|^{n-2} \bar{v}_i(y) \geq |y^{\bar{\lambda}_0}|^{n-2} \bar{v}_i^{\bar{\lambda}_0}(y)$ for some $y \in B(0, \frac{\lambda_0}{2})$, then by (3.18),

$$(3.19) \quad \begin{aligned} \bar{h}_i(y, \bar{v}_i(y)) &= d \left(\bar{v}_i(y) \right)^{\frac{n+2}{n-2}} (1 + o(1)) \\ &\geq d \left(\bar{v}_i(y^{\bar{\lambda}_0}) \right)^{\frac{n+2}{n-2}} (1 + c_0(\lambda_0)) > \bar{h}_i(y^{\bar{\lambda}_0}, \bar{v}_i^{\bar{\lambda}_0}(y)) \end{aligned}$$

for some constant $c_0(\lambda_0)$ independent of i . If $|y|^{n-2} \bar{v}_i(y) \leq |y^{\bar{\lambda}_0}|^{n-2} \bar{v}_i^{\bar{\lambda}_0}(y)$ and $y \notin B_i$, then $\bar{v}_i(y) > \bar{v}_i(y^{\bar{\lambda}_0})$ and by (3.3),

$$(3.20) \quad \bar{h}_i(y, \bar{v}_i(y)) > \bar{h}_i(y^{\bar{\lambda}_0}, \bar{v}_i^{\bar{\lambda}_0}(y))$$

holds also. If $|y|^{n-2} \bar{v}_i(y) \leq |y^{\bar{\lambda}_0}|^{n-2} \bar{v}_i^{\bar{\lambda}_0}(y)$ and $y \in B_i$, we have

$$(3.21) \quad \begin{aligned} \Delta w_{\bar{\lambda}_0}(y) &= \bar{h}_i(y^{\bar{\lambda}_0}, \bar{v}_i^{\bar{\lambda}_0}(y)) - \bar{h}_i(y, \bar{v}_i(y)) < \bar{h}_i(y^{\bar{\lambda}_0}, \bar{v}_i^{\bar{\lambda}_0}(y)) \\ &= d \left(\bar{v}_i(y^{\bar{\lambda}_0}) \right)^{\frac{n+2}{n-2}} (1 + o(1)) \leq c_1 \end{aligned}$$

for some c_1 . From (3.19)-(3.21), we have

$$(3.22) \quad \Delta w_{\bar{\lambda}_0}(y) \leq \begin{cases} c_1 & \text{if } y \in B_i, \\ 0 & \text{if } y \in B \left(0, \frac{\lambda_0}{2} \right) \setminus B_i. \end{cases}$$

Now the constant c_1 of (3.17) is chosen as the same in (3.22). Then

$$\Delta(w_{\bar{\lambda}_0} - \varphi_i)(y) \leq 0$$

for $y \in B(0, \frac{\lambda_0}{2})$ except for a set of capacity 0.

By a straightforward computation, the second term of φ_i can be estimated by

$$\left| \int_{B_i} G(y, z) dz \right| \leq c_2 M_i^{-\frac{n}{n-2}} G(y, 0)$$

for $\tilde{M}_i^{-1} \leq |y| \leq \frac{\lambda_0}{2}$. Thus, for any $\varepsilon > 0$, there exists $i_0 = i_0(\varepsilon)$, such that for $i \geq i_0$,

$$\frac{\varepsilon}{2M_i} G(y, 0) \leq \varphi_i(y) \leq \frac{2\varepsilon}{M_i} G(y, 0) \quad \text{for } \tilde{M}_i^{-1} \leq |y| \leq \frac{\lambda_0}{2}.$$

Choose $\varepsilon \ll m = \inf_{\partial B_2} u$. By (3.8), we have

$$w_{\tilde{\lambda}_0}(y) \geq \varphi_i(y) \quad \text{for } |y| = \tilde{M}_i^{-1}.$$

Since $w_{\tilde{\lambda}_0} - \varphi_i(y)$ is superharmonic in $\tilde{M}_i^{-1} < |y| < \frac{\lambda_0}{2}$ and $w_{\tilde{\lambda}_0}(y) - \varphi_i(y)$ has a lower bound for each i , by Lemma 2.1, the maximum principle gives

$$\bar{w}_0(y) > \varphi_i(y) \geq 0 \quad \text{for } y \in B_{\tilde{\lambda}_0} \setminus \{|y| \leq \tilde{M}_i^{-1}\}.$$

Therefore, (3.12) is proved in case 1.

CASE 2. $\lim_{t \rightarrow +\infty} \frac{g(t)}{t^{\frac{n+2}{n-2}}} = 0$. The case is simpler. By (3.18), we have

$$(3.23) \quad \bar{h}_i(y^{\tilde{\lambda}_0}, \bar{v}_i^{\tilde{\lambda}_0}(y)) = g(M_i)^{-1} M_i^{\frac{n+2}{n-2}} (\bar{v}_i^{\tilde{\lambda}_0}(y))^{\frac{n+2}{n-2}} o(1)$$

for $y \in U_0$, where (3.16) holds for $y \in U_0$. On the other hand,

$$(3.24) \quad \begin{aligned} \bar{h}_i(y, \bar{v}_i(y)) &= g(M_i)^{-1} |y|^{-(n+2)} g(M_i |y|^{n-2} \bar{v}_i(y)) \\ &= g(M_i)^{-1} M_i^{\frac{n+2}{n-2}} \bar{v}_i^{\frac{n+2}{n-2}}(y) \left\{ (|y|^{n-2} M_i \bar{v}_i(y))^{-\frac{n+2}{n-2}} g(M_i |y|^{n-2} \bar{v}_i(y)) \right\} \\ &\geq g(M_i)^{-1} M_i^{\frac{n+2}{n-2}} \bar{v}_i^{\frac{n+2}{n-2}}(y) \inf_{m \leq s \leq 3t_0} \frac{g(s)}{s^{\frac{n+2}{n-2}}} > \bar{h}_i(y^{\tilde{\lambda}_0}, \bar{v}_i^{\tilde{\lambda}_0}(y)) \end{aligned}$$

for $y \in U_0$. Thus, $w_{\tilde{\lambda}_0}(y)$ is a nonnegative superharmonic function in U_0 with $w_{\tilde{\lambda}_0}(y_0) = 0$. By the strong maximum principle, $w_{\tilde{\lambda}_0}(y) \equiv 0$ in U_0 , which yields a contradiction. Therefore, the proof of (3.12) is completely finished.

For each $i \geq i_0$, let $N_i \subseteq \Sigma'_{\tilde{\lambda}_0}$ be a neighborhood of the singularity set of \bar{v}_i , and set

$$\epsilon_i = \inf_{\partial N_i} \bar{v}_i(y) > 0.$$

Let $\epsilon'_i \leq \epsilon_i$ be a small positive number such that if y is a regular point of \bar{v}_i and $w_{\tilde{\lambda}_0}(y) \leq \epsilon'_i$, then (3.13) holds for a neighborhood U_0 of y . By the same argument as above, we can show that $\Delta w_{\tilde{\lambda}_0}(y) < 0$ for $y \in U_0$. Thus, by Lemma 2.1,

$$(3.25) \quad w_{\tilde{\lambda}_0}(y) \geq \epsilon'_i \quad \text{for } y \in B_{\lambda_0} \cap \Sigma'_{\tilde{\lambda}_0}.$$

Once (3.12) is established for $i \geq i_0$, we want to prove $\bar{\lambda}_0 = \lambda_0$ for all $i \geq i_0$. Now suppose $\bar{\lambda}_0 < \lambda_0$ for some $i \geq i_0$. Let i be fixed. By the definition of $\bar{\lambda}_0$ there exists a sequence $\lambda_j \downarrow \bar{\lambda}_0$ such that

$$\inf_{\Sigma'_{\lambda_j}} w_j(y) < 0,$$

where $w_j := w_{\lambda_j}$. By (3.25), $\inf w_j$ can be achieved at some interior point $y_j^* \in \Sigma'_{\lambda_j}$ such that y_j^* is away from the singular set of \bar{v}_i .

By (3.11), $|y_j^*| \leq R_0$. Let y_0 be a limit point of y_j^* . If $y_0 \in \Sigma'_{\bar{\lambda}_0}$, then $w_{\bar{\lambda}_0}(y_0) = 0$, which contradicts to (3.12). If $y_0 \in T_{\bar{\lambda}_0}$, then $\frac{\partial w_{\bar{\lambda}_0}}{\partial y_1}(y_0) = 0$, which yields a contradiction to (3.12) again. Hence $\lambda_0 = \bar{\lambda}_0$ is proved. By passing $i \rightarrow +\infty$ first and then $\lambda_0 \rightarrow 0$, we obtain $\bar{v}(y^\lambda) \leq \bar{v}(y)$ for $y_1 > 0$ and $\lambda < 0$. Since we can apply the method of moving planes to \bar{v} in any direction, \bar{v} is radially symmetric with respect to the origin. Going back to the original v , v is radially symmetric with respect to \hat{y} . It follows that v is identically equal to 1, which yields a contradiction to Step 1. Therefore the proof of Theorem 1.1 is finished. \square

As the remark at the end of Section 2, Theorem 1.1 still holds when Γ is not contained in B_2 , provided the assumption of Lemma 2.4 is satisfied. For the reference, we state it as follows.

THEOREM 3.1. *Suppose that g satisfies the hypothesis of Lemma 2.4 and Γ be a closed set of \bar{B}_2 with $\text{Cap}(\Gamma) = 0$. Let u be a positive singular solution of (1.1) satisfying*

$$u(x) \geq m > 0 \text{ for } x \in B_2 \setminus \Gamma$$

for some positive constant m . Then there exists a constant $c > 0$ such that

$$\frac{g(u(x))}{u(x)} \leq c d^{-2}(x) \text{ for } |x| \leq 1,$$

where $d(x)$ denotes the distance of x to Γ .

4. – Radial symmetry

In this section we will prove two results about the radial symmetry of positive singular solutions: Theorem 1.3 and Lemma 4.1. Both of them are useful when we apply the blow-up argument in the proof of asymptotic symmetry.

PROOF OF THEOREM 1.3. Suppose that u is a nonconstant positive solution of

$$(4.1) \quad \Delta u + g(u) = 0 \text{ in } \mathbb{R}^n.$$

We claim that $u(x) = O(|x|^{2-n})$ and $|\nabla u(x)| = O(|x|^{1-n})$ for $|x|$ large. This claim will be proved via the method of moving planes.

Since u is not a constant in \mathbb{R}^n , we may assume that u is not radially symmetric about the origin. Let u^* be the Kelvin transformation with respect to the unit ball, namely, $u^*(y) = |y|^{2-n} u(\frac{y}{|y|^2})$. Then u^* satisfies

$$(4.2) \quad \begin{cases} \Delta u^* + g^*(y, u^*(y)) = 0 & \text{in } \mathbb{R}^n \setminus \{0\} \\ u^*(y) = O(|y|^{2-n}) & \text{at } \infty, \end{cases}$$

where $g^*(y, t) = |y|^{-(n+2)}g(|y|^{n-2}t)$. Since u^* is not radially symmetric about the origin, we assume that u^* is *not* symmetric with respect to the hyperplane $T_0 = \{y \mid y_1 = 0\}$. In what follows, we want to prove that $u^*(y)$ is symmetric with respect to some hyperplane $T_{\lambda_0} = \{y \mid y_1 = \lambda_0\}$ for $\lambda_0 \neq 0$. Thus, $u^*(y)$ is continuous near the origin and the claim follows readily.

Using the notations in Section 2, we let

$$(4.3) \quad w_\lambda(y) = u^*(y) - u^*(y^\lambda)$$

for $y \in \Sigma_\lambda$ and $\lambda \leq 0$. Since u^* is superharmonic in $\mathbb{R}^n \setminus \{0\}$, u^* satisfies

$$(4.4) \quad u^*(y) \geq \inf_{\partial N} u^* > 0 \quad \text{for } y \in N,$$

where N is a small neighborhood of 0. Thus, it is not difficult to see that there exists a $\lambda_1 \ll 0$ such that $w_\lambda(y) > 0$ for $y \in \Sigma_\lambda$ and $\lambda \leq \lambda_1$. Let $c_1 > 0$ such that

$$\left(\frac{g(u(x))}{u(x)} \right) \leq c_1 \quad \text{for } |x| \leq 1.$$

Thus, we can apply the same argument as in (3.10) to obtain that there exists a constant $c_2 > 0$ such that for $|y| \geq 1$,

$$(4.5) \quad \Delta w_\lambda(y) + c_2|y|^{-4}w_\lambda(y) \leq 0$$

whenever $w_\lambda(y) \leq 0$. From (4.5), the following holds: *There exists $R_0 > 0$ such that if $\lambda < 0$ and $w_\lambda(y) < 0$ for some point in Σ_λ and $w_\lambda(y_0) = \inf_{\Sigma_\lambda} w_\lambda(y)$ for some point $y_0 \in \Sigma_\lambda \setminus \{0\}$, then*

$$(4.6) \quad |y_0| \leq R_0.$$

Let $\lambda_0 = \sup\{\lambda \leq 0 \mid w_\mu(y) > 0 \text{ in } \Sigma_\mu \text{ for all } \mu \leq \lambda\}$. We claim that if $\lambda_0 < 0$, then $w_{\lambda_0}(y) \equiv 0$ in Σ_{λ_0} , and then u^* is symmetric with respect to T_{λ_0} .

Suppose that $w_{\lambda_0}(y) \not\equiv 0$ in Σ_{λ_0} . By continuity, $w_{\lambda_0}(y) \geq 0$ in Σ_{λ_0} . Furthermore, we claim

$$(4.7) \quad w_{\lambda_0}(y) > 0 \text{ in } \Sigma_{\lambda_0} \text{ and } \frac{\partial w_{\lambda_0}(y)}{\partial y_1} < 0 \text{ on } T_{\lambda_0}.$$

If g is a Lipschitz function, then w_{λ_0} satisfies $\Delta w_{\lambda_0} + C(y)w_{\lambda_0} \leq 0$ for some locally bounded function $C(y)$. Thus, (4.7) follows from the strong maximum principle and the Hopf boundary point lemma. In case without any regularity assumption of g , we can prove (4.7) by a similar argument (but simpler because (1.2) holds for all $t > 0$) as the proof of (3.12). The idea is to show that $w_{\lambda_0}(y)$ is superharmonic in a neighborhood of points where (4.7) fails. We leave the detail to the readers.

By the definition of λ_0 , there exists a sequence of $\lambda_j \downarrow \lambda_0$ such that

$$w_j(y_j) = \inf_{\Sigma_{\lambda_j}} w_j(y) < 0 \text{ for some } y_j \in \Sigma_{\lambda_j} \setminus \{0\}$$

where $w_j := w_{\lambda_j}$ in Σ_{λ_j} . By (4.6), y_j is bounded. Let y_0 be a limit point of y_j . By Lemma 2.1, $y_0 \neq 0$ and either $y_0 \in \Sigma_{\lambda_0}$ and $w_{\lambda_0}(y_0) = 0$ or $y_0 \in T_{\lambda_0}$ and $\frac{\partial w_{\lambda_0}}{\partial y_1}(y_0) = 0$, which both yield a contradiction to (4.7). Thus $w_{\lambda_0}(y) \equiv 0$, i.e., $u^*(y)$ is symmetric with respect to T_{λ_0} .

If $\lambda_0 = 0$, then $u^*(-y_1, y_2, \dots, y_n) \leq u^*(y_1, y_2, \dots, y_n)$ for $y_1 \geq 0$. By moving the hyperplane T_λ from the positive y_1 -direction, we can use the same argument to prove either u^* is symmetric with respect to $T_{\mu_0} = \{y \mid y_1 = \mu_0 > 0\}$ or $u^*(y_1, y_2, \dots, y_n) \leq u^*(-y_1, y_2, \dots, y_n)$ for $y_1 \geq 0$. But, the latter case could not happen because $u^*(y)$ is not symmetric with respect to y_1 . Hence u^* is symmetric with respect to some T_{λ_0} with $\lambda_0 \neq 0$. In particular, $u(x) = O(|x|^{2-n})$ and $|\nabla u(x)| = O(|x|^{1-n})$ for $|x|$ large.

Since $u^*(y)$ is symmetric with respect to T_{λ_0} , by equation (4.1), we have

$$(4.8) \quad |y^{\lambda_0}|^{-(n+2)} g(|y^{\lambda_0}|^{n-2} u^*(y)) \equiv |y|^{-(n+2)} g(|y|^{n-2} u^*(y))$$

for $y_1 \geq \lambda_0$. Without loss of generality, we may assume $\lambda_0 < 0$. First, we see that (4.8) implies that there is $\delta > 0$ such that $g(t)t^{-\frac{n+2}{n-2}} = \text{constant}$ for $t \in [0, \delta]$ because $|y|^{n-2} u^*(y) \rightarrow 0$ as $y \rightarrow 0$.

Applying the Pohozaev identity, we have

$$\begin{aligned} & \int_{B_R} \left(nG(u) - \frac{n-2}{2} g(u)u \right) dx \\ &= \int_{\partial B_R} \left(R \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{R}{2} |\nabla u|^2 + RG(u) + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} \right) ds, \end{aligned}$$

where $G(t) = \int_0^t g(s)ds$. Note that for $|x|$ large, $G(u(x)) = c u(x)^{\frac{n+2}{n-2}} = O(|x|^{-2n})$. Thus, the right-hand side tends to 0 as $R \rightarrow +\infty$. On the other hand, since

$$G(t) \geq g(t)t^{-\frac{(n+2)}{n-2}} \int_0^t s^{\frac{n+2}{n-2}} ds = \frac{n-2}{2n} g(t)t,$$

we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^n} \left\{ nG(u(x)) - \frac{n-2}{2} g(u)u(x) \right\} dx, \\ &= \lim_{R \rightarrow +\infty} \int_{\partial B_R} \left(R \left| \frac{\partial u}{\partial \nu} \right|^2 - \frac{R}{2} |\nabla u|^2 + RG(u(x)) + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} \right) d\sigma = 0 \end{aligned}$$

which implies that $nG(u(x)) \equiv \frac{n-2}{2} g(u(x))u(x)$ in \mathbb{R}^n . This proves Theorem 1.3. □

LEMMA 4.1. *Suppose that $g(t)$ is positive for $t > 0$ and $g(t)t^{-\frac{n+2}{n-2}}$ is nonincreasing for $t > 0$. Let u be a positive singular solution of*

$$(4.9) \quad \Delta u(x) + g(u(x)) = 0 \text{ in } \mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^k),$$

where $0 \leq k \leq n - 2$, and $\{0\} \times \mathbb{R}^k = \{(0, y) \mid y \in \mathbb{R}^k\}$. Assume that u can not be extended as a C^1 function to the whole \mathbb{R}^n . Then $u(x, y)$ is radially symmetric in $|x|$.

The proof of Lemma 4.1 is the same as Theorem 1.3. For the case $k = 0$, we perform the Kelvin transformation:

$$u_\delta^*(y) = |y|^{-n+2} u \left(\frac{y}{|y|^2} - \delta e_n \right),$$

where $\delta > 0$ and $e_n = (0, 0, \dots, 1)$. Obviously, u_δ^* has singularities at 0 and e_n/δ . Since e_n/δ is a nonremovable singularity, by using the same argument of Theorem 1.3, $u_\delta^*(y)$ can be shown to be symmetric with respect to y_1 . By letting $\delta \rightarrow 0$, we conclude that the Kelvin transformation of u with respect to the unit ball centered at 0 is radially symmetric. Thus, u is radially symmetric.

For $k \geq 1$, we have to apply the Kelvin transformation twice i.e for $\delta > 0$,

$$u_\delta^*(y) = |y|^{2-n} \left| \frac{y}{|y|^2} - \delta e_2 \right|^{2-n} u(I_\delta(y)),$$

and

$$I_\delta(y) = \left(\frac{y}{|y|^2} - \delta e_2 \right) \left| \frac{y}{|y|^2} - \delta e_2 \right|^{-2},$$

where $e_2 = (0, 1, 0, \dots, 0)$. Obviously, the singular set of u_δ^* is contained in the hyperplane $\{y_1 = 0\}$. Thus, by applying the same argument, $u_\delta^*(y)$ is shown to be symmetric with respect to y_1 . By passing the limit $\delta \rightarrow 0$, we prove Lemma 4.1. For the details of the proof, we should leave to the readers. \square

5. – Asymptotic symmetry

PROOF OF COROLLARY 1.2. Clearly, (1.6) holds by the spherical Harnack inequality (1.3). Suppose (1.7) fails. Then there exist points x_i, \bar{x}_i and $\epsilon_0 > 0$ such that $|x_i| = |\bar{x}_i| = r_i, \lim_{i \rightarrow \infty} r_i = 0$ and

$$(5.1) \quad \frac{u(\bar{x}_i)}{u(x_i)} \geq 1 + \epsilon_0.$$

Set $M_i = u_i(x_i), v_i(x) = M_i^{-1}u(r_i x)$ and

$$(5.2) \quad \Delta v_i + h_i(v_i) = 0,$$

where $h_i(t) = \frac{r_i^2 g(M_i t)}{M_i}$.

By Theorem 1.1 and the Harnack inequality (1.3), $v_i(y)$ is uniformly bounded in any compact set of \mathbb{R}^n . Thus, by passing to a subsequence, v_i converges to v in $C_{loc}^{1,\alpha}(\mathbb{R}^n \setminus \{0\})$. By (1.2), $h_i(t)$ can be written as

$$(5.3) \quad h_i(t) = f_i(t)t^{\frac{n+2}{n-2}}$$

for some nonincreasing function $f_i(t)$ in $t > M_i^{-1}t_0$ with some $t_0 \geq 0$. Since $h_i(v_i(y))/v_i(y)$ is uniformly bounded in any compact set of $\mathbb{R}^n \setminus \{0\}$ by Theorem 1.1, $f_i(t)$ is uniformly bounded in any compact set of (v_-, ∞) , where $v_- = \inf_{y \in \mathbb{R}^n} v(y)$. Without loss of generality, we may assume that $f_i(t)$ converges to $f(t)$ for almost everywhere of t . Acturally, if $f(t)$ is continuous at s and $s_i \rightarrow s$, then we have $\lim_{i \rightarrow +\infty} f_i(s_i) = f(s)$. Since $f(t)$ is nonincreasing, there are at most countable many discontinuous points $\{t_j\}_{j=1}^\infty$. Let $E_j = \{y \mid v(y) = t_j\}$. If E_j has a positive measure, then $\Delta v(x) = 0$ and $\lim_{i \rightarrow +\infty} \Delta v_i(x) = \Delta v = 0$ on E_j in the distribution sense. By equation (5.2), $h_i(v_i(x))$ converges to 0 for a.e. $x \in E_j$. Obviously, it implies $f(t) = 0$ for $t > t_j$. In particular, f is continuous for $t > t_j$. Let f be defined with $f(t_j) = 0$. Then we conclude that the measure $|E_j| = 0$ for all j except one at most, and v satisfies

$$(5.4) \quad \Delta v + h(v) = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

in the distribution sense where $h(t) = f(t)t^{\frac{n+2}{n-2}}$. By (5.1), v is not a radially symmetric function. Thus, Lemma 4.1 yields that 0 is a removable singularity of v . Since $f(t)$ is nonincreasing in t for all $t > 0$, by Theorem 1.3, v must be radially symmetric with respect to some point y_0 and v has the form

$$(5.5) \quad v(y) = c_0 \left(\frac{\lambda}{\lambda^2 + |y - y_0|^2} \right)^{\frac{n-2}{2}}$$

for positive constants c_0 and λ . By (5.1) again, $y_0 \neq 0$. Without loss of generality, we may assume $y_0 = (-2a_0, 0, \dots, 0)$ for some $a_0 < 0$.

Let $v^*(y)$ be the Kelvin transformation of

$$v^*(y) = \left(\frac{a_0}{|y|} \right)^{n-2} v \left(\frac{a_0^2 y}{|y|^2} + y_0^* \right),$$

where $y_0^* = (-a_0, 0, \dots, 0)$. By a straightforward computation, $v^*(y)$ is radially symmetric with respect to $-\beta_0 e_1$ for some $\beta_0 > 0$.

Before we proceed to use the MMP, we first claim

$$(5.6) \quad \lim_{x \rightarrow 0} u(x)|x|^{n-2} = 0.$$

To see it, by (1.1), we have for $0 < s < r$

$$\bar{u}'(r)r^{n-1} - \bar{u}'(s)s^{n-1} = \int_{B_r \setminus B_s} g(u(x))dx,$$

where $\bar{u}(r) = \int_{|x|=r} u d\sigma$ denotes the average of u over the sphere $|x| = r$. Since $g(u(x)) \in L^1(B_1)$, $\lim_{r \rightarrow 0} \bar{u}'(r)r^{n-1}$ always exists. If $\lim_{r \rightarrow 0} \bar{u}'(r)r^{n-1} \neq 0$, then $\bar{u}(r)r^{n-2}$ is bounded by two positive constants as $r \rightarrow 0$. Applying the Harnack inequality (1.5), we have $c_1 \leq u(x)|x|^{n-2} \leq c_2$ for two positive constants c_1 and c_2 . Then

$$v_i(y) = \frac{u_i(r_i y)}{u_i(x_i)} \geq \left(\frac{c_2}{c_1}\right) |y|^{2-n}$$

which implies that 0 is a nonremovable singularity of v . Clearly, it yields a contradiction to (5.5), where v is smooth at 0. Thus, $\lim_{r \rightarrow 0} \bar{u}'(r)r^{n-1} = 0$ and by the Harnack inequality again, $u(x) = o(1)|x|^{2-n}$ as $|x| \rightarrow 0$.

Set $v_i^*(y) = (\frac{a_0}{|y|})^{n-2} v_i(\frac{a_0^2 y}{|y|^2} + y_0^*)$. Obviously, $v_i^*(y)$ is defined for $|y| \geq a_0^2 r_i$ and has a singularity at $-y_0^*$ which lies in the right-hand side of the hyperplane $\{y \mid y_1 = 0\}$. Since $v_i(y)$ uniformly converges to v in any compact set of $\mathbb{R}^n \setminus \{0\}$, v_i^* has a local maximum at y_i which tends to $-\beta_0 e_1$ as $i \rightarrow +\infty$. For $|y| = a_0^2 r_i$,

$$(5.7) \quad v_i^*(y) \geq a_0^{n-2} (M_i r_i^{n-2})^{-1} m \rightarrow +\infty$$

as $i \rightarrow +\infty$ by (5.6) where $m = \inf_{\partial B_2} u$. Thus, we can apply the MMP as in the proofs of Theorem 1.1 and Theorem 1.3 to show that

$$\frac{\partial v_i^*}{\partial y_1}(y) < 0$$

for $y_1 < -\beta_0/2$ and for i large. And it yields a contradiction to the fact that v_i^* has a local maximum near $-\beta_0 e_1$. Therefore, the proof of Corollary 1.2 is complete. □

Finally, we want to consider that Γ is a smooth k -dimensional submanifold with $1 \leq k \leq n - 2$, we let N be a tube neighborhood of Γ such that any point of N can be uniquely expressed as a sum $x + v$ where $x \in \Gamma$ and $v \in (T_x \Gamma)^\perp$, the orthogonal complement of the tangent space of Γ at x . Denote π to be the orthogonal projection of N onto Γ . For small $r > 0$ and $z \in \Gamma$, let $\pi_r^{-1}(z) = \{y \in N : \pi(y) = z \text{ and } |y - z| = r\}$.

THEOREM 5.1. *Let Γ be a k -dimensional submanifold in \bar{B}_1 and let N and π be described as above. In addition to (1.2), g satisfies $\lim_{t \rightarrow +\infty} t^{-\frac{n}{n-2}} g(t) = +\infty$. Then for any solution u to (1.1) in B_2 , we have $u(x) = u(x')(1 + o(1))$ for $x, x' \in \pi_r^{-1}(z)$ as $r \rightarrow 0^+$.*

PROOF. Suppose that the conclusion of Theorem 5.1 does not hold. Then there exist x_i, \bar{x}_i and $\epsilon_0 > 0$ such that $d(x_i, \Gamma) = d(\bar{x}_i, \Gamma) = r_i \rightarrow 0$ and (5.1) holds. Assume $\lim_{i \rightarrow +\infty} x_i = \lim_{i \rightarrow +\infty} \bar{x}_i = 0 \in \Gamma$ and $T_0\Gamma = \{0\} \times \mathbb{R}^k$. Set $M_i = u_i(x_i)$ and $v_i(x) = M_i^{-1}u(r_i x)$. Note that $M_i \rightarrow +\infty$, otherwise v_i converges to v and v is a positive harmonic function in $\mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^k)$. No matter whether v can be extended smoothly in \mathbb{R}^n or not, $v(x, y) = v(|x|, y)$ which yields a contradiction to (5.1). Thus, without loss of generality, we may assume that $g(t)t^{-\frac{n+2}{n-2}}$ is nonincreasing for $t > 0$. By passing to a subsequence, v_i converges to v in $C_{loc}^{1,\alpha}(\mathbb{R}^n \setminus \{0\} \times \mathbb{R}^k)$. Due to (5.1), by Lemma 4.1, $v(y)$ can be extended as a smooth function in \mathbb{R}^n . Thus, $v(y)$ has the form of (5.5) with $y_0 \notin \{0\} \times \mathbb{R}^k$. Assume $y_0 = -a_0 e_1 + z_0$ for some $a_0 > 0$ and $z_0 \in \{0\} \times \mathbb{R}^k$. Choose δ to be a positive small number such that the Kelvin transformation v^* defined by $v^*(y) = |y|^{2-n}v(\frac{y}{|y|^2} - \delta e_1)$ is radially symmetric about some point y^* and the first coordinate $y_1^* \leq -c_1$ for some $c_1 > 0$ independent of δ . Let $v_i^*(y) = |y|^{2-n}v_i(y/|y|^2 - \delta e_1)$. Obviously, $v_i^*(y)$ is defined in $|y| \geq r_i$, and the singular set Γ_i of v_i is a bounded set for any $\delta > 0$. Since $T_0\Gamma = \{0\} \times \mathbb{R}^k$, we can choose i large so that $\Gamma_i \subset \{y \mid y_1 \geq -\frac{c_1}{4}\}$. From now on, δ will be fixed. Thus, the hyperplane $T_\lambda = \{y \mid y_1 = \lambda\}$ never touches with the singular set Γ_i for $\lambda \leq -\frac{c_1}{2}$. Since $g(t)t^{-\frac{n}{n-2}} \rightarrow 0$ as $t \rightarrow +\infty$, we have

$$M_i^{\frac{2}{n-2}} r_i^2 = o(1) r_i^2 \left(\frac{g(M_i)}{M_i} \right) = o(1) \frac{g(u(x_i))}{u(x_i)} |x_i|^2 = o(1)$$

by Theorem 1.1 as $i \rightarrow +\infty$. Thus, on the boundary $|y| = r_i$, v_i satisfies

$$v_i^*(y) \geq m (r_i^{n-2} M_i)^{-1} \rightarrow +\infty$$

as $i \rightarrow +\infty$, where $m = \inf_{\partial B_2} u$. Then MMP gives $\frac{\partial v_i^*}{\partial y_1}(y) < 0$ for $y_1 \leq -\frac{c_1}{2}$, which yields a contradiction to the fact that $v_i^*(y)$ has a local maximum in the half-plane $\{y \mid y_1 \leq -c_1\}$. Hence, the proof of Theorem 5.1 is complete. \square

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