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A Generalization of the Sizes of Differential Equations and its Applications to G-Function Theory

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Abstract. The aims of this paper are to introduce a generalization of the notion of the sizes and to present its some applications to G-function theory. We define a new size $\sigma(A, B)$ and estimate it. Furthermore we consider some relations in G-function theory by using our sizes.

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Introduction

In 1929, C. L. Siegel [16] introduced the notion of G-functions: the original definition is as follows.

A function $f(x) = \sum_{i=0}^{\infty} a_i x^i$ is called a G-function if there exists an algebraic number field K of finite degree and a positive constant C with the following properties (i)-(iii) for all i:

(i) $a_i \in K$,
(ii) the absolute values of a_i and its conjugates do not exceed C^i,
(iii) there is a positive integral common denominator of $a_0, a_1, \ldots, a_i$ which does not exceed C^i.

Twenty years ago A. I. Galočhkin [9, Definition 2] mentioned a relation between a differential equation and its G-function solutions. He proposed so-called Galočhkin's condition, which is an assumption about the coefficients of the differential equation. Under this condition, he obtained bounds of the irrational measures of the special values of G-functions.

After some years, E. Bombieri [3] defined the notion of arithmetic type, and suggested to the equivalence between the conditions of Galočhkin and those of himself. He also obtained some irrationality statements about special values of G-functions which are solutions of differential equations of arithmetic type.

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In 1985, D. V. Chudnovsky and G. V. Chudnovsky [6] showed that:

THEOREM A. *If a solution of an irreducible differential equation is a* G*-function, then the differential equation satisfies the Galočhkin condition.*

In other words, they removed the Galočhkin condition from the statements about irrationality, completing Siegel's program.

A G-function is an interesting topic in itself. Y. André [1] proved the following two theorems:

THEOREM B. *The Galočhkin condition and the arithmetic type condition for a differential equation are equivalent.*

THEOREM C. *If a Fuchsian differential equation with only rational exponents at the origin satisfies Galočhkin's condition, then the entries of the normalized uniform part of its solution matrix are* G*-functions.*

In 1994, B. Dwork, G. Gerotto and F. J. Sullivan [8] obtained the converse result to Theorem C by using Chudnovskies' and André's results:

THEOREM D. *If the entries of the normalized uniform part of the solution matrix of a Fuchsian differential equation with only rational exponents at the origin are* G*-functions, then the differential equation satisfies Galočhkin's condition.*

Let K be a number field of finite degree.

We consider the differential equation:

$$\frac{d}{dx}y = Ay \tag{0.1}$$

with $A \in M_n(K(x))$.

The *sizes* and the *global radii* of the function y and the coefficient matrix A of differential equation (0.1) are denoted as $\sigma(y)$, $\sigma(A)$, $\rho(y)$ and $\rho(A)$ respectively (definitions as below). When one uses these notations,

"y is a G-function" is equivalent to $\sigma(y) < \infty$,

"differential equation (0.1) satisfies Galočhkin's condition" is equivalent to $\sigma(A) < \infty$,

"differential equation (0.1) is of arithmetic type" is equivalent to $\rho(A) < \infty$.

The above theorems A-D are the statements on the sizes and the global radii. So, our interest concerns the finiteness and the values of the sizes and the global radii.

In this paper we introduce a generalization of the notion of the sizes and present some applications of it to relations between sizes. This paper is arranged in two chapters as follows:

(I) We consider the differential equation:

$$\frac{d}{dx}X = AX - XB \tag{0.2}$$

with $A,\ B \in M_n(K(x))$.

We attach to (0.2) a size $\sigma(A, B)$ which generalizes the usual size (corresponding to the case $B = 0$).

The following proposition for $\sigma(A, B)$ holds.

THEOREM I. *Let* $\hat{\sigma}(A, B) := \frac{1}{2}(\sigma(A, B) + \sigma(B, A))$ *for* $A, B \in M_n(K(x))$. *Then the map*

$$\hat{\sigma} : M_n(K(x))^2 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is a pseudo distance function. Moreover

$$\sigma(A) = \hat{\sigma}(0, A) = \hat{\sigma}(A, 0)\,.$$

(II) We estimate $\sigma(A, B)$ of differential equation (0.2).

THEOREM II-1. *Let* $u := \sum_{i=0}^{m} u_i x^i \in \mathcal{O}_K[x]$ *be a common denominator of* A *and* B *such that* $uA,\ uB \in M_n(\mathcal{O}_K[x])$*, where* $\mathcal{O}_K$ *denotes the integer ring of* K. *Let* $s := \max(\deg u, \deg(uA), \deg(uB))$. *Then for a solution* $X \in GL_n(K[[x]])$ *of differential equation* (0.2) *with* $X_{|x=0} = I$ *(the identity matrix), we have*

$$\sigma(A, B) \leq 9n^3(s+1)\sigma(X) + 2\sigma(0, B) + (s+1)\hbar_\infty(u) + \log(s+1)\,,$$

where $\hbar_\infty(u) := \frac{1}{m+1}\sum_{v|\infty} \max_{i\leq m} \log\max(1, |u_i|_v)$*,* v *being places of* K.

To state an application of this theorem, we put

$$\frac{1}{x}K[x]_{(x)} := \left\{\frac{f(x)}{x}\,\middle|\, f(x) \in \text{ the local ring of } K[x] \text{ at } (x)\right\}.$$

$$\text{Res}(A) := \text{ the residue of } A \text{ at } x = 0\,,$$

and

$$T[A] := TAT^{-1} + \left(\frac{d}{dx}T\right)T^{-1} \quad \text{(this means “a change of basis”)}.$$

Here $A \in M_n(\frac{1}{x}K[x]_{(x)})$ and $T \in GL_n(K(x))$.

According to [1], there exists $T \in GL_n(K(x))$ such that $T[A] \in M_n(\frac{1}{x}K[x]_{(x)})$ and that none of the differences between the eigenvalues of $\text{Res}(T[A])$ is a non-zero integer. By fixing such $T \in GL_n(K(x))$, there exists the unique solution $Y \in GL_n(K[[x]])$ of the differential equation:

$$\frac{d}{dx}Y = T[A]Y - Y\frac{1}{x}\text{Res}(T[A]) \tag{0.3}$$

with $Y_{|x=0} = I$. We call this unique solution of differential equation (0.3) the *normalized uniform part* of solution of differential equation (0.1). Indeed, $T^{-1}Yx^{\text{Res}(T[A])}$ is a matrix solution of differential equation (0.1).

We now state an application as the following corollary, which is a quantitative version of Theorem D without using Shidlovskii's lemma [1, Chapt. VI, Sect. 2]:

COROLLARY II-2. *Let $A \in M_n(\frac{1}{x}K[x]_{(x)})$ and let Y be the normalized uniform part of the solution of differential equation* (0.1) *with above $T \in GL_n(K(x))$. Let $u \in \mathcal{O}_K[x]$ be a common denominator of $T[A]$, and let $s := \max(\deg u, \deg(uT[A]))$. Suppose that*

$$\mathcal{E} := \{\text{eigenvalues of } \operatorname{Res}(A)\} \subset \mathbb{Q}.$$

Then

$$\sigma(0, A) \leq 9n^3(s+1)\sigma(Y) + 3\log N_{\mathcal{E}} + 3 \sum_{\substack{p \mid N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + (s+1)\hbar_\infty(u)$$

$$+ \log(s+1) + 3(n-1),$$

where $N_{\mathcal{E}} \in \mathbb{N}$ is a common denominator of $\mathcal{E}$.

Notation and terminology

We fix K as a number field of finite degree. For a place v of K we put

$$\begin{cases} |p|_v := p^{\frac{-d_v}{d}} & \text{if} \quad v \mid p \quad (p : \text{prime}), \\ |\xi|_v := |\xi|^{\frac{d_v}{d}} & \text{if} \quad v \mid \infty \quad (\xi \in K), \end{cases}$$

where $d = [K : \mathbb{Q}]$ and $d_v = [K_v : \mathbb{Q}_v]$.

We define a pseudo valuation on $M_{n_1,n_2}(K)$, the set of $n_1 \times n_2$ -matrices of K, as usual: for $M = (m_{i,j})_{j=1,\dots,n_2}^{i=1,\dots,n_1} \in M_{n_1,n_2}(K)$,

$$|M|_v := \max_{\substack{i=1,\dots,n_1 \\ j=1,\dots,n_2}} |m_{i,j}|_v .$$

For $Y_i \in M_{n_1,n_2}(K)$, we consider the Laurent series $Y = \sum_{i=-N}^{\infty} Y_i x^i \in M_{n_1,n_2}(K((x)))$ with $N \in \mathbb{N} \cup \{0\}$.

We write $\log^+ a := \log\max(1, a)$ ($a \in \mathbb{R}_{\geq 0}$). André's symbol in [1], $h_{\cdot,\cdot}(\cdot)$, is defined by

$$h_{v,0}(Y) = \max_{i \leq 0} \log^+ |Y_i|_v ,$$

$$h_{v,m}(Y) = \frac{1}{m} \max_{i \leq m} \log^+ |Y_i|_v \quad (m \neq 0).$$

DEFINITION (Cf. [1, Chapt. I]). We define the *size* of $Y \in M_{n_1,n_2}(K((x)))$ as

$$\sigma(Y) := \varlimsup_{m \to \infty} \sum_v h_{v,m}(Y)$$

and the *global radius* of Y as

$$\rho(Y) := \sum_v \overline{\lim_{m\to\infty}} h_{v,m}(Y),$$

where $\sum_v$ means that v runs over all places of K.

The following definition is equivalent to the conditions (i), (ii), (iii) in introduction in the case of $Y \in K[[x]]$. (Cf. [1, Chapt. I])

DEFINITION. We say that $Y \in M_{n_1,n_2}(K((x)))$ is a (matrix of) *G-function(s)* if $\sigma(Y) < \infty$.

For $f = f(x) = \sum_{i=0}^{N} f_i x^i \in K[x]$ and for every place v of K, the *Gauss absolute value* is defined by $|f|_v := \max_{i=0,\dots,N} |f_i|_v$.

From here we discuss only non-Archimedean valuations, that is, $v \nmid \infty$. For every place v with $v \nmid \infty$ and for $f, g \in K[x]$ with $g \not\equiv 0$, the Gauss absolute value is extended to $K(x)$ by

$$\left|\frac{f}{g}\right|_v := \frac{|f|_v}{|g|_v}.$$

We also define a pseudo valuation on $M_n(K(x))$ as before: for $M = (m_{i,j})_{i,j=1,\dots,n} \in M_n(K(x))$,

$$|M|_v := \max_{i,j=1,\dots,n} |m_{i,j}|_v.$$

Henceforth $\mathcal{D}$ denotes the differential operator $\frac{d}{dx}$. From [1, Chapt. IV, Subsect. 1.5] we have

$$\left|\frac{\mathcal{D}^m}{m!} M\right|_v \le |M|_v$$

for $m = 0, 1, \dots$. Here v is an arbitrary non-Archimedean valuation of K and $M \in M_n(K(x))$.

For a sequence $\{F_i\}_{i=0,1,\dots} \subset M_n(K(x))$ and for every place $v \nmid \infty$, we put

$$h_{v,0}(\{F_i\}) = \log^+ |F_0|_v,$$
$$h_{v,m}(\{F_i\}) = \frac{1}{m} \max_{i\le m} \log^+ |F_i|_v \quad (m = 1, 2, \dots).$$

DEFINITION. We define the *size* of $\{F_i\}_{i=0,1,\dots} \subset M_n(K(x))$ as

$$\sigma(\{F_i\}) := \overline{\lim_{m\to\infty}} \sum_{v\nmid\infty} h_{v,m}(\{F_i\})$$

and the *global radius* of $\{F_i\}$ as

$$\rho(\{F_i\}) := \sum_{v\nmid\infty} \overline{\lim_{m\to\infty}} h_{v,m}(\{F_i\}),$$

where $\sum_{v\nmid\infty}$ means that v runs over all finite places of K.

Chapter 1: the sizes of differential equations

1.1. – Notation

We introduce our original symbols and definitions.

Let $\mathbb{K}$ be a differential extension of K. Suppose that $\mathfrak{J}, \mathfrak{A}, \mathfrak{B}$ are elements in $M_n(\mathbb{K})$. A sequence $\{(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle i\rangle}\}_{i=0,1,\dots} \subset M_n(\mathbb{K})$ is defined by

$$(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle 0\rangle} = \mathfrak{J}$$

and recursively for $i = 1, 2, \dots,$

$$(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle i+1\rangle} = \frac{1}{i+1}(\mathcal{D}(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle i\rangle} - \mathfrak{A}(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle i\rangle} + (\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle i\rangle}\mathfrak{B}).$$

We write $\sigma(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})$ (resp. $\rho(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})$) as an abbreviation for $\sigma(\{(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle i\rangle}\})$ (resp. $\rho(\{(\mathfrak{J}, \mathfrak{A}, \mathfrak{B})^{\langle i\rangle}\})$).

DEFINITIONS 1.1.1. For the differential equation

$$(\mathcal{D} - A)y = 0 \tag{1.1.1.1}$$

with $A \in M_n(K(x))$, we call $\sigma(I, 0, A)$ $(= \sigma(I, A, 0)$, See Subsect. 1.2 below) the *size* of A (or of differential equation (1.1.1.1)) and $\rho(I, 0, A)$ $(= \rho(I, A, 0))$ the *global radius* of A (or of differential equation (1.1.1.1)), where I is the identity matrix and 0 is the zero matrix in $M_n(K(x))$.

The following definition coincides with that of André [1, Chapt. IV, Subsect. 5.2].

DEFINITION 1.1.2. We say that $(\mathcal{D} - A)$ is a *G-operator* if and only if $\sigma(I, 0, A) < \infty$.

REMARK 1.1.3 (Cf. [1], [3]). Differential equation (0.1) with $\rho(I, 0, A) < \infty$ is called *of arithmetic type* by Bombieri.

1.2. – Properties of the sizes

We define *a change of basis* as

$$\mathfrak{J}[A] := \mathfrak{J}A\mathfrak{J}^{-1} + (\mathcal{D}\mathfrak{J})\mathfrak{J}^{-1},$$

where $\mathfrak{J} \in GL_n(K(x))$ and $A \in M_n(K(x))$. One immediately obtains:

PROPOSITION 1.2.1. *For* $\mathfrak{J}_1, \mathfrak{J}_2 \in GL_n(K(x))$ *and for* $A \in M_n(K(x))$, *we have*

$$\mathfrak{J}_1[\mathfrak{J}_2[A]] = (\mathfrak{J}_1\mathfrak{J}_2)[A]. \tag{1.2.1.1}$$

The main purpose of this section is to show the following proposition:

PROPOSITION 1.2.2. *For* $\mathfrak{I}, \mathfrak{I}_1, \mathfrak{I}_2 \in GL_n(K(x))$ *and for* $A, B, C \in M_n(K(x))$, *the following statements hold.*

$$\sigma(I, A, A) = 0, \tag{1.2.2.1}$$

$$\sigma(\mathfrak{I}, A, B) = \sigma(I, A, \mathfrak{I}[B]) = \sigma(I, \mathfrak{I}^{-1}[A], B), \tag{1.2.2.2}$$

$$\sigma(\mathfrak{I}_1, A, 0) = \sigma(\mathfrak{I}_2, 0, A), \tag{1.2.2.3}$$

$$\sigma(\mathfrak{I}_1\mathfrak{I}_2, A, B) \le \sigma(\mathfrak{I}_1, A, C) + \sigma(\mathfrak{I}_2, C, B). \tag{1.2.2.4}$$

We shall often write simply $\sigma(A, B)$ instead of $\sigma(I, A, B)$. We define a function on $M_n(K(x))^2$ by

$$\hat{\sigma}(A, B) := \frac{1}{2}(\sigma(A, B) + \sigma(B, A)).$$

The following statements show its characteristic properties, and give Theorem I in Introduction.

THEOREM 1.2.3. *The map*

$$\hat{\sigma} : M_n(K(x))^2 \longrightarrow \mathbb{R}_{\ge 0} \cup \{\infty\}$$

is a pseudo distance function, that is, it satisfies the following three conditions:

$$\hat{\sigma}(A, A) = 0, \tag{1.2.3.1}$$

$$\hat{\sigma}(A, B) = \hat{\sigma}(B, A) \tag{1.2.3.2}$$

and

$$\hat{\sigma}(A, B) \le \hat{\sigma}(A, C) + \hat{\sigma}(C, B). \tag{1.2.3.3}$$

Moreover we have

$$\sigma(0, A) = \hat{\sigma}(0, A) = \hat{\sigma}(A, 0) \tag{1.2.3.4}$$

and

$$\hat{\sigma}(A, B) = \hat{\sigma}(\mathfrak{I}[A], \mathfrak{I}[B]) \tag{1.2.3.5}$$

for $\mathfrak{I} \in GL_n(K(x))$.

PROOF. The assertions (1.2.3.1), (1.2.3.2) and (1.2.3.3) are trivial by Proposition 1.2.2. As for the identities (1.2.3.4), we use the identity (1.2.2.3) in Proposition 1.2.2. Then we have

$$\sigma(0, A) = \frac{1}{2}(\sigma(I, 0, A) + \sigma(I, A, 0)) = \hat{\sigma}(0, A) = \hat{\sigma}(A, 0) .$$

Finally for the identity (1.2.3.5), we apply Proposition 1.2.1 and the identities (1.2.2.2) in Proposition 1.2.2. Then we have

$$\begin{aligned}\sigma(I, A, B) &= \sigma(I, A, \mathfrak{J}^{-1}[\mathfrak{J}[B]]) = \sigma(\mathfrak{J}^{-1}, A, \mathfrak{J}[B]) \\ &= \sigma(I, \mathfrak{J}[A], \mathfrak{J}[B]) .\end{aligned}$$

Therefore $\hat{\sigma}(A, B) = \hat{\sigma}(\mathfrak{J}[A], \mathfrak{J}[B])$. □

REMARK 1.2.4. One defines an equivalence relation $A \sim B$ in $M_n(K(x))$ as $\hat{\sigma}(A, B) = 0$, then a metric is induced as usual. However it is not known what this equivalence relation $A \sim B$ means. For instance, even in the case of $A \sim 0$, we do not know whether there *always* exists $\mathfrak{J} \in GL_n(K(x))$ such that $A = \mathfrak{J}[0]$. The equality $A = \mathfrak{J}[0]$ implies that all solutions of differential equation (0.1) belong to $M_n(K(x))$. (Cf. [1, Chapt. IV, Subsect. 4.2]).

Our proof of Proposition 1.2.2 requires some preparatory propositions, which are results concerning the properties of the sequence $\{(\mathfrak{J}, A, B)^{\langle i\rangle}\}$. We shall use them later again.

LEMMA 1.2.5. *For* $\mathfrak{J} \in GL_n(K(x))$, $A, B \in M_n(K(x))$ *and for* $m = 0, 1, \ldots,$ *we have*

$$(I, A, B)^{\langle m\rangle} \mathfrak{J} = \left(\mathfrak{J}, A, \mathfrak{J}^{-1}[B]\right)^{\langle m\rangle} , \tag{1.2.5.1}$$

$$\mathfrak{J}(I, A, B)^{\langle m\rangle} = (\mathfrak{J}, \mathfrak{J}[A], B)^{\langle m\rangle} . \tag{1.2.5.2}$$

PROOF. We show the identity (1.2.5.1) by induction on m. For $m = 0$, both sides of the identity (1.2.5.1) are $\mathfrak{J}$ thus it is true in this case. Assume that the identity (1.2.5.1) is true for a given $m \geq 0$. We now prove it for $m + 1$. We differentiate each side of the identity (1.2.5.1). Because $(\mathcal{D}(\mathfrak{J}^{-1}))\mathfrak{J} = -\mathfrak{J}^{-1}\mathcal{D}\mathfrak{J}$, we have

$$\begin{aligned}\mathcal{D}((I, A, B)^{\langle m\rangle} \mathfrak{J}) &= (\mathcal{D}(I, A, B)^{\langle m\rangle})\mathfrak{J} + (I, A, B)^{\langle m\rangle} \mathcal{D}\mathfrak{J} \\ &= ((m+1)(I, A, B)^{\langle m+1\rangle} + A(I, A, B)^{\langle m\rangle} \\ &\quad - (I, A, B)^{\langle m\rangle} B)\mathfrak{J} + (I, A, B)^{\langle m\rangle} \mathcal{D}\mathfrak{J} \\ &= (m+1)(I, A, B)^{\langle m+1\rangle} \mathfrak{J} + A(I, A, B)^{\langle m\rangle} \mathfrak{J} \\ &\quad - (I, A, B)^{\langle m\rangle} \mathfrak{J}(\mathfrak{J}^{-1}B\mathfrak{J} - \mathfrak{J}^{-1}\mathcal{D}\mathfrak{J}) \\ &= (m+1)(I, A, B)^{\langle m+1\rangle} \mathfrak{J} + A\left(\mathfrak{J}, A, \mathfrak{J}^{-1}[B]\right)^{\langle m\rangle} \\ &\quad - \left(\mathfrak{J}, A, \mathfrak{J}^{-1}[B]\right)^{\langle m\rangle} \mathfrak{J}^{-1}[B] .\end{aligned} \tag{1.2.5.3}$$

On the other hand,

$$\mathcal{D}\left(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]\right)^{\langle m\rangle} = (m+1)\left(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]\right)^{\langle m+1\rangle} + A\left(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]\right)^{\langle m\rangle} - \left(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]\right)^{\langle m\rangle}\mathfrak{I}^{-1}[B]. \tag{1.2.5.4}$$

Therefore we find that

$$(I, A, B)^{\langle m+1\rangle}\,\mathfrak{I} = \left(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]\right)^{\langle m+1\rangle}.$$

A similar argument yields the identity (1.2.5.2). □

LEMMA 1.2.6. *Let $\mathbb{L}$ be an algebra over $\mathbb{Z}$. For sequences $\{a_i\}_{i=0,1,\ldots}$, $\{b_i\}_{i=0,1,\ldots}$ $\subset \mathbb{L}$ and for $m = 0, 1, \ldots,$*

$$\sum_{\substack{i+j=m\\ i,j\geq 0}} ((i+1)a_{i+1}b_j + (j+1)a_i b_{j+1}) = (m+1)\sum_{\substack{i+j=m+1\\ i,j\geq 0}} a_i b_j. \tag{1.2.6.1}$$

PROOF. Obvious. □

LEMMA 1.2.7. *For $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, A, B \in M_n(\mathbb{K})$, $c \in \mathbb{K}$ and for $m = 0, 1, \ldots,$ we have*

$$\sum_{i+j=m} (\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle} = \sum_{i+j=m} (-1)^i \frac{\mathcal{D}^j}{j!}(\mathfrak{I}_1\,(\mathfrak{I}_2, A, B)^{\langle i\rangle}\,\mathfrak{I}_3). \tag{1.2.7.1}$$

PROOF. We show the identity (1.2.7.1) by induction on m. For $m = 0$, both sides of the identity (1.2.7.1) are $\mathfrak{I}_1\mathfrak{I}_2\mathfrak{I}_3$ thus it is true in this case. Assume that the identity (1.2.7.1) is true for a given $m \geq 0$. We now show it for $m+1$.

We differentiate the left side of the identity (1.2.7.1). We have

$$
\begin{aligned}
&\mathcal{D}\left(\sum_{i+j=m}(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\right)\\
&= \sum_{i+j=m} (\mathcal{D}\,(\mathfrak{I}_1, cI, A)^{\langle i\rangle})\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad + (\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,(\mathcal{D}\mathfrak{I}_2)\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad + (\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2(\mathcal{D}\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle})\\
&= \sum_{i+j=m} (i+1)\,(\mathfrak{I}_1, cI, A)^{\langle i+1\rangle}\,\mathfrak{I}_2(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad + (j+1)\,(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2(\mathfrak{I}_3, B, cI)^{\langle j+1\rangle}\\
&\quad + c\,(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle} - (\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,A\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad + (\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2 B\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle} - (\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\,c\\
&\quad + (\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,(\mathcal{D}\mathfrak{I}_2)\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
(1.2.7.2)\quad &= (m+1)\sum_{i+j=m+1}(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad + \sum_{i+j=m}(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,(\mathfrak{I}_2, A, B)^{\langle 1\rangle}\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad \text{(by Lemma 1.2.6)}\\
&= (m+1)\sum_{i+j=m+1}(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad + \sum_{i+j=m}(-1)^i\frac{\mathcal{D}^j}{j!}(\mathfrak{I}_1\left((\mathfrak{I}_2, A, B)^{\langle 1\rangle}, A, B\right)^{\langle i\rangle}\mathfrak{I}_3)\\
&= (m+1)\sum_{i+j=m+1}(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\
&\quad + \sum_{i+j=m}(-1)^i\frac{\mathcal{D}^j}{j!}(\mathfrak{I}_1(i+1)\,(\mathfrak{I}_2, A, B)^{\langle i+1\rangle}\,\mathfrak{I}_3)\,.
\end{aligned}
$$

Where $((\mathfrak{I}_2, A, B)^{\langle 1\rangle}, A, B)^{\langle i\rangle} = (i+1)(\mathfrak{I}_2, A, B)^{\langle i+1\rangle}$ can be shown by induction.

On the other hand, the differentiated right side of the identity (1.2.7.1) is

$$
\begin{aligned}
(1.2.7.3)\quad &\mathcal{D}\left(\sum_{i+j=m}(-1)^i\frac{\mathcal{D}^j}{j!}(\mathfrak{I}_1\,(\mathfrak{I}_2, A, B)^{\langle i\rangle}\,\mathfrak{I}_3)\right)\\
&= \sum_{i+j=m}(-1)^i\frac{\mathcal{D}^{j+1}}{j!}(\mathfrak{I}_1\,(\mathfrak{I}_2, A, B)^{\langle i\rangle}\,\mathfrak{I}_3)\,.
\end{aligned}
$$

Therefore we obtain that

$$(1.2.7.4)\quad \begin{aligned}&(m+1)\sum_{i+j=m+1}(\mathfrak{I}_1, cI, A)^{\langle i\rangle}\,\mathfrak{I}_2\,(\mathfrak{I}_3, B, cI)^{\langle j\rangle}\\ &=\sum_{i+j=m}(-1)^{i+1}\frac{\mathcal{D}^j}{j!}(\mathfrak{I}_1(i+1)\,(\mathfrak{I}_2, A, B)^{\langle i+1\rangle}\,\mathfrak{I}_3)\\ &\quad+(-1)^i(j+1)\frac{\mathcal{D}^{j+1}}{(j+1)!}(\mathfrak{I}_1\,(\mathfrak{I}_2, A, B)^{\langle i\rangle}\,\mathfrak{I}_3)\\ &=(m+1)\sum_{i+j=m+1}(-1)^i\frac{\mathcal{D}^j}{j!}(\mathfrak{I}_1\,(\mathfrak{I}_2, A, B)^{\langle i\rangle}\,\mathfrak{I}_3)\end{aligned}$$

by Lemma 1.2.6. □

LEMMA 1.2.8. *For* $\mathfrak{I}_1, \mathfrak{I}_2, A, B, C \in M_n(\mathbb{K})$ *and for* $m = 0, 1, \ldots,$

$$(1.2.8.1)\quad (\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m\rangle} = \sum_{i+j=m}(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\,.$$

PROOF. We show the identity (1.2.8.1) by induction on m. For $m = 0$, both sides of the identity (1.2.8.1) are $\mathfrak{I}_1\mathfrak{I}_2$ hence it is true in this case. Assume that the identity (1.2.8.1) is true for a given $m \geq 0$. We now prove it for $m+1$. We differentiate the right side of the identity (1.2.8.1). Then we have

$$(1.2.8.2)\quad \begin{aligned}&\mathcal{D}\left(\sum_{i+j=m}(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\right)\\ &=\sum_{i+j=m}(\mathcal{D}\,(\mathfrak{I}_1, A, C)^{\langle i\rangle})\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\\ &\quad+(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,(\mathcal{D}\,(\mathfrak{I}_2, C, B)^{\langle j\rangle})\\ &=\sum_{i+j=m}((i+1)\,(\mathfrak{I}_1, A, C)^{\langle i+1\rangle}+A\,(\mathfrak{I}_1, A, C)^{\langle i\rangle}\\ &\quad-(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,C)\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\\ &\quad+(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,((j+1)\,(\mathfrak{I}_2, C, B)^{\langle j+1\rangle}\\ &\quad+C\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}-(\mathfrak{I}_2, C, B)^{\langle j\rangle}\,B)\\ &=\sum_{i+j=m}((i+1)\,(\mathfrak{I}_1, A, C)^{\langle i+1\rangle}\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\\ &\quad+(j+1)\,(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,(\mathfrak{I}_2, C, B)^{\langle j+1\rangle}\\ &\quad+A\,(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\\ &\quad-(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\,B)\\ &=(m+1)\sum_{i+j=m+1}(\mathfrak{I}_1, A, C)^{\langle i\rangle}\,(\mathfrak{I}_2, C, B)^{\langle j\rangle}\\ &\quad+A\,(\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m\rangle}-(\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m\rangle}\,B\\ &\qquad\text{(by Lemma 1.2.6.)}\end{aligned}$$

The last expression is equal to the differentiated left side of the identity (1.2.8.1),

$$\mathcal{D}(\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m\rangle} = (m+1)(\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m+1\rangle} + A(\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m\rangle} - (\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m\rangle} B.$$

Therefore

$$(\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m+1\rangle} = \sum_{i+j=m+1} (\mathfrak{I}_1, A, C)^{\langle i\rangle}(\mathfrak{I}_2, C, B)^{\langle j\rangle}. \qquad \square$$

We now prove Proposition 1.2.2.

PROOF OF PROPOSITION 1.2.2. First we show the identity (1.2.2.1). It is clear that

$$(I, A, A)^{\langle 0\rangle} = I, \quad (I, A, A)^{\langle i\rangle} = 0 \quad (i = 1, 2, \dots).$$

Therefore $\sigma(I, A, A) = 0$.

For the proof of (1.2.2.2), assume that the following identities hold:

$$\sigma(I, A, B) = \sigma(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]) \tag{1.2.9.1}$$

and

$$\sigma(I, A, B) = \sigma(\mathfrak{I}, \mathfrak{I}[A], B). \tag{1.2.9.2}$$

We substitute $\mathfrak{I}[B]$ for B in the identity (1.2.9.1) and $\mathfrak{I}^{-1}[A]$ for A in the identity (1.2.9.2). Then using Proposition 1.2.1, we obtain that

$$\sigma(I, A, \mathfrak{I}[B]) = \sigma(\mathfrak{I}, A, \mathfrak{I}^{-1}[\mathfrak{I}[B]]) = \sigma(\mathfrak{I}, A, B) \tag{1.2.9.3}$$

and

$$\sigma(I, \mathfrak{I}^{-1}[A], B) = \sigma(\mathfrak{I}, \mathfrak{I}[\mathfrak{I}^{-1}[A]], B) = \sigma(\mathfrak{I}, A, B), \tag{1.2.9.4}$$

hence we conclude the identities (1.2.2.2). It suffices to show the identities (1.2.9.1) and (1.2.9.2). We prove the identity (1.2.9.1). Using the identity (1.2.5.1) in Lemma 1.2.5, we find

$$\begin{aligned}\max_{i\le m}\log^+|(\mathfrak{I}, A, \mathfrak{I}^{-1}[B])^{\langle i\rangle}|_v &= \max_{i\le m}\log^+|(I, A, B)^{\langle i\rangle}\mathfrak{I}|_v \\ &\le \max_{i\le m}\log^+|(I, A, B)^{\langle i\rangle}|_v + \log^+|\mathfrak{I}|_v\end{aligned} \tag{1.2.9.5}$$

for every place v with $v \nmid \infty$. Hence

$$\begin{aligned}&\overline{\lim_{m\to\infty}} \sum_{v\nmid\infty} h_{v,m}(\{(\mathfrak{I}, A, \mathfrak{I}^{-1}[B])^{\langle i\rangle}\}) \\ &\le \overline{\lim_{m\to\infty}} \sum_{v\nmid\infty} h_{v,m}(\{(I, A, B)^{\langle i\rangle}\}) + \overline{\lim_{m\to\infty}} \frac{1}{m} \sum_{v\nmid\infty} \log^+|\mathfrak{I}|_v.\end{aligned}$$

The number of places v with $v \nmid \infty$ such that $|\mathfrak{I}|_v \neq 1$ is finite and it follows that

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \sum_{v\nmid\infty} \log^+ |\mathfrak{I}|_v = 0 .$$

It yields

$$\sigma(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]) \leq \sigma(I, A, B) . \tag{1.2.9.6}$$

On the other hand, by the identity (1.2.5.1) in Lemma 1.2.5, we also find

$$\begin{aligned} \max_{i\leq m} \log^+ |(I, A, B)^{\langle i\rangle}|_v &= \max_{i\leq m} \log^+ |(\mathfrak{I}, A, \mathfrak{I}^{-1}[B])^{\langle i\rangle}\mathfrak{I}^{-1}|_v \\ &\leq \max_{i\leq m} \log^+ |(\mathfrak{I}, A, \mathfrak{I}^{-1}[B])^{\langle i\rangle}|_v + \log^+ |\mathfrak{I}^{-1}|_v \end{aligned} \tag{1.2.9.7}$$

for every place v with $v \nmid \infty$. Hence we have as above

$$\sigma(I, A, B) \leq \sigma(\mathfrak{I}, A, \mathfrak{I}^{-1}[B]) . \tag{1.2.9.8}$$

Consequently we obtain the identity (1.2.9.1) from the inequalities (1.2.9.6) and (1.2.9.8). As for the identity (1.2.9.2), one finds it in a similar way.

Next we show the identity (1.2.2.3). Because of the identity (1.2.7.1) for $c = 0$, $\mathfrak{I}_3 = I$ and $B = 0$ in Lemma 1.2.7, we have

$$|(\mathfrak{I}_1, 0, A)^{\langle m\rangle}|_v = \left| \sum_{i=0}^{m} (-1)^{m-i} \left(\frac{\mathcal{D}^i}{i!} (\mathfrak{I}_1 (\mathfrak{I}_2, A, 0)^{\langle m-i\rangle}) \right) \mathfrak{I}_2^{-1} \right|_v \tag{1.2.9.9}$$

for every place v with $v \nmid \infty$. Thus

$$|(\mathfrak{I}_1, 0, A)^{\langle m\rangle}|_v \leq \max_{i\leq m} |(\mathfrak{I}_2, A, 0)^{\langle i\rangle}|_v |\mathfrak{I}_1|_v |\mathfrak{I}_2^{-1}|_v .$$

It follows

$$\sigma(\mathfrak{I}_1, 0, A) \leq \sigma(\mathfrak{I}_2, A, 0) + \overline{\lim_{m\to\infty}} \frac{1}{m} \sum_{v\nmid\infty} \log^+ |\mathfrak{I}_1|_v |\mathfrak{I}_2^{-1}|_v . \tag{1.2.9.10}$$

The number of places v with $v \nmid \infty$ such that $|\mathfrak{I}_1|_v |\mathfrak{I}_2^{-1}|_v \neq 1$ is finite and it follows that

$$\overline{\lim_{m\to\infty}} \frac{1}{m} \sum_{v\nmid\infty} \log^+ |\mathfrak{I}_1|_v |\mathfrak{I}_2^{-1}|_v = 0 .$$

Therefore we obtain

$$\sigma(\mathfrak{I}_1, 0, A) \leq \sigma(\mathfrak{I}_2, A, 0) . \tag{1.2.9.11}$$

We also have

$$\sigma(\mathfrak{I}_2, A, 0) \leq \sigma(\mathfrak{I}_1, 0, A). \tag{1.2.9.12}$$

Consequently we deduce the identity (1.2.2.3) from (1.2.9.11) and (1.2.9.12).

We conclude this proof by showing the inequality (1.2.2.4). By Lemma 1.2.8, one has

$$\begin{aligned} |(\mathfrak{I}_1\mathfrak{I}_2, A, B)^{\langle m\rangle}|_v &= \left| \sum_{i+j=m} (\mathfrak{I}_1, A, C)^{\langle i\rangle} (\mathfrak{I}_2, C, B)^{\langle j\rangle} \right|_v \\ &\leq \max_{i\leq m} |(\mathfrak{I}_1, A, C)^{\langle i\rangle}|_v \cdot \max_{i\leq m} |(\mathfrak{I}_2, C, B)^{\langle i\rangle}|_v . \end{aligned}$$

Then we obtain

$$\sigma(\mathfrak{I}_1\mathfrak{I}_2, A, B) \leq \sigma(\mathfrak{I}_1, A, C) + \sigma(\mathfrak{I}_2, C, B). \qquad \square$$

The statements for the global radii instead of the sizes in Proposition 1.2.2 and Theorem 1.2.3 hold similarly.

PROPOSITION 1.2.10. *For $\mathfrak{I}, \mathfrak{I}_1, \mathfrak{I}_2 \in GL_n(K(x))$ and for $A, B, C \in M_n(K(x))$, the following statements hold.*

$$\rho(I, A, A) = 0, \tag{1.2.10.1}$$

$$\rho(\mathfrak{I}, A, B) = \rho(I, A, \mathfrak{I}[B]) = \rho(I, \mathfrak{I}^{-1}[A], B), \tag{1.2.10.2}$$

$$\rho(\mathfrak{I}_1, A, 0) = \rho(\mathfrak{I}_2, 0, A), \tag{1.2.10.3}$$

$$\rho(\mathfrak{I}_1\mathfrak{I}_2, A, B) \leq \rho(\mathfrak{I}_1, A, C) + \rho(\mathfrak{I}_2, C, B). \tag{1.2.10.4}$$

THEOREM 1.2.11. *Let*

$$\hat{\rho}(A, B) := \frac{1}{2}(\rho(I, A, B) + \rho(I, B, A)).$$

Then the map

$$\hat{\rho} : M_n(K(x))^2 \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is a pseudo distance function. Moreover we have

$$\rho(0, A) = \hat{\rho}(0, A) = \hat{\rho}(A, 0) \tag{1.2.11.1}$$

and

$$\hat{\rho}(A, B) = \hat{\rho}(\mathfrak{I}[A], \mathfrak{I}[B]) \tag{1.2.11.2}$$

for $\mathfrak{I} \in GL_n(K(x))$.

Chapter 2: estimations of the sizes

In this chapter we consider a relation of differential equation (0.2) and its solution.

2.1. – Diophantine approximations

We assume that $X \in GL_n(K[[x]])$ with $X_{|x=0} = I$ is a solution of differential equation (0.2). Hence for $A,\ B \in M_n(K(x))$

$$0 = \mathcal{D}X - AX + XB\,. \tag{2.1.0.1}$$

Moreover we write for $q \in K[x]$ and $P \in M_n(K[x])$, $R := qX - P \in M_n(K[[x]])$.

In this section, we show some properties of simultaneous approximations of the solution X.

The following lemma is fundamental.

LEMMA 2.1.1. *For* $m = 0, 1, \ldots,$

$$(P, A, B)^{\langle m\rangle} = \left(\frac{\mathcal{D}^m}{m!}q\right)X - (R, A, B)^{\langle m\rangle}\,. \tag{2.1.1.1}$$

PROOF. We show the identity (2.1.1.1) by induction on m. For $m = 0$, both sides of the identity (2.1.1.1) are P and it is true in this case. Assume that the identity (2.1.1.1) is true for a given $m \geq 0$. We now prove it for $m + 1$. One has

$$\begin{aligned}
&(P, A, B)^{\langle m+1\rangle}\\
&= \frac{1}{m+1}\left(\mathcal{D}(P, A, B)^{\langle m\rangle} - A\,(P, A, B)^{\langle m\rangle} + (P, A, B)^{\langle m\rangle}\,B\right)\\
&= \frac{1}{m+1}\left(\mathcal{D}\left(\left(\frac{\mathcal{D}^m}{m!}q\right)X - (R, A, B)^{\langle m\rangle}\right) - A\left(\left(\frac{\mathcal{D}^m}{m!}q\right)X - (R, A, B)^{\langle m\rangle}\right)\right.\\
&\quad \left. + \left(\left(\frac{\mathcal{D}^m}{m!}q\right)X - (R, A, B)^{\langle m\rangle}\right)B\right)\\
&= \frac{1}{m+1}\left(\left(\frac{\mathcal{D}^{m+1}}{m!}q\right)X + \left(\frac{\mathcal{D}^m}{m!}q\right)\mathcal{D}X - \mathcal{D}(R, A, B)^{\langle m\rangle} - \left(\frac{\mathcal{D}^m}{m!}q\right)AX\right.\\
&\quad \left. + A\,(R, A, B)^{\langle m\rangle} + \left(\frac{\mathcal{D}^m}{m!}q\right)XB - (R, A, B)^{\langle m\rangle}\,B\right)\\
&= \left(\frac{\mathcal{D}^{m+1}}{(m+1)!}q\right)X - (R, A, B)^{\langle m+1\rangle} + \left(\frac{\mathcal{D}^m}{(m+1)!}q\right)(X, A, B)^{\langle 1\rangle}\,.
\end{aligned}$$

Since $(X, A, B)^{\langle 1\rangle} = \mathcal{D}X - AX + XB = 0$, we obtain

$$(P, A, B)^{\langle m+1\rangle} = \left(\frac{\mathcal{D}^{m+1}}{(m+1)!}q\right) X - (R, A, B)^{\langle m+1\rangle} . \qquad \square$$

NOTATIONS 2.1.2. Throughout this section and the next, we denote by N, L arbitrary integers such that

$$\max(\deg q, \deg P) < N \tag{2.1.2.1}$$

and

$$\operatorname{ord}_0 R \geq L . \tag{2.1.2.2}$$

Moreover for $A, B \in M_n(K(x))$, we write $u \in \mathcal{O}_K[x]$ as a common denominator of A and B and

$$s := \max(\deg u, \deg(uA), \deg(uB)) ,$$

where $\mathcal{O}_K$ is the integer ring of K.

LEMMA 2.1.3. *For $m = 0, 1, \ldots, L$,*

$$u^m (P, A, B)^{\langle m\rangle} \in M_n(K[x]) , \tag{2.1.3.1}$$

$$\deg u^m (P, A, B)^{\langle m\rangle} < N + ms , \tag{2.1.3.2}$$

$$u^m (R, A, B)^{\langle m\rangle} \in M_n(K[[x]]) \tag{2.1.3.3}$$

and

$$\operatorname{ord}_0 u^m (R, A, B)^{\langle m\rangle} \geq L - m . \tag{2.1.3.4}$$

PROOF. We show the four assertions by induction on m. For $m = 0$, we have $(P, A, B)^{\langle 0\rangle} = P$, $\deg P < N$, $(R, A, B)^{\langle 0\rangle} = R$, $\operatorname{ord}_0 R \geq L$ and then (2.1.3.1), (2.1.3.2), (2.1.3.3) and (2.1.3.4) are true in this case. Assume that these four assertions are true for a given $m \geq 0$. We prove them for $m + 1$. Now one has

$$\mathcal{D}(u^m (P, A, B)^{\langle m\rangle}) = m(\mathcal{D}u)u^{m-1} (P, A, B)^{\langle m\rangle} + u^m \mathcal{D} (P, A, B)^{\langle m\rangle} ,$$

or

$$\mathcal{D} (P, A, B)^{\langle m\rangle} = \frac{1}{u^m}\mathcal{D}(u^m (P, A, B)^{\langle m\rangle}) - \frac{m(\mathcal{D}u)}{u} (P, A, B)^{\langle m\rangle} . \tag{2.1.3.5}$$

Substituting the last equation into

$$u^{m+1}(P,A,B)^{\langle m+1\rangle} = \frac{u^{m+1}}{m+1}(\mathcal{D}(P,A,B)^{\langle m\rangle} - A(P,A,B)^{\langle m\rangle} + (P,A,B)^{\langle m\rangle}B), \tag{2.1.3.6}$$

we have the right side of the identity (2.1.3.6)

$$= \frac{u^{m+1}}{m+1}\left(\frac{1}{u^m}\mathcal{D}(u^m(P,A,B)^{\langle m\rangle}\right) - \frac{m(\mathcal{D}u)}{u}(P,A,B)^{\langle m\rangle} - A(P,A,B)^{\langle m\rangle} + (P,A,B)^{\langle m\rangle}B),$$

hence

$$\begin{aligned} u^{m+1}(P,A,B)^{\langle m+1\rangle} = {} & \frac{1}{m+1}u\mathcal{D}(u^m(P,A,B)^{\langle m\rangle}) \\ & - \frac{m}{m+1}(\mathcal{D}u)u^m(P,A,B)^{\langle m\rangle} \\ & - \frac{1}{m+1}(uAu^m(P,A,B)^{\langle m\rangle} - u^m(P,A,B)^{\langle m\rangle}uB). \end{aligned} \tag{2.1.3.7}$$

By the induction assumption, we find that every term of the right side of the last equation belongs to $M_n(K[x])$. Hence

$$u^{m+1}(P,A,B)^{\langle m+1\rangle} \in M_n(K[x]). \tag{2.1.3.8}$$

Moreover by the identity (2.1.3.7) we find

$$\begin{aligned} \deg u^{m+1}(P,A,B)^{\langle m+1\rangle} & \le s + \deg u^m(P,A,B)^{\langle m\rangle} \\ & \le N + (m+1)s. \end{aligned} \tag{2.1.3.9}$$

Similarly we have

$$\begin{aligned} u^{m+1}(R,A,B)^{\langle m+1\rangle} = {} & \frac{1}{m+1}u\mathcal{D}(u^m(R,A,B)^{\langle m\rangle}) \\ & - \frac{m}{m+1}(\mathcal{D}u)u^m(R,A,B)^{\langle m\rangle} \\ & - \frac{1}{m+1}(uAu^m(R,A,B)^{\langle m\rangle} - u^m(R,A,B)^{\langle m\rangle}uB). \end{aligned} \tag{2.1.3.10}$$

By the induction assumption, we find that every term of the right side of the last equation belongs to $M_n(K[[x]])$. Hence

$$u^{m+1}(R,A,B)^{\langle m+1\rangle} \in M_n(K[[x]]). \tag{2.1.3.11}$$

By the identity (2.1.3.10) we find

$$\begin{aligned}\operatorname{ord}_0 u^{m+1}(R,A,B)^{\langle m+1\rangle} &\geq \operatorname{ord}_0 u^m (R,A,B)^{\langle m\rangle} - 1\\ &\geq L-(m+1).\end{aligned}$$

□

DEFINITIONS 2.1.4. For $Z = \sum_{i=0}^{\infty} Z_i x^i \in M_{n_1,n_2}(K[[x]])$, we write

$$Z_{<m} := \sum_{i=0}^{m-1} Z_i x^i \in M_{n_1,n_2}(K[x])$$

with $m = 1, 2, \ldots$. Moreover we write

$$\hbar_{f,m}(Z) := \frac{1}{m}\sum_{v\nmid\infty} \max_{i<m} \log^+ |Z_i|_v ,$$

$$\hbar_{\infty,m}(Z) := \frac{1}{m}\sum_{v\mid\infty} \max_{i<m} \log^+ |Z_i|_v$$

and

$$\hbar_m(Z) := \hbar_{f,m}(Z) + \hbar_{\infty,m}(Z).$$

For $Z \in M_{n_1,n_2}(K[x])$ we write

$$\hbar_f(Z) := \hbar_{f,1+\deg Z}(Z),$$
$$\hbar_\infty(Z) := \hbar_{\infty,1+\deg Z}(Z)$$

and

$$\hbar(Z) := \hbar_f(Z) + \hbar_\infty(Z).$$

For $Z \in M_{n_1,n_2}(K[[x]])$, one obtains immediately that

(2.1.4.1)
$$\begin{aligned}\overline{\lim_{m\to\infty}}\, \hbar_m(Z) &= \overline{\lim_{m\to\infty}}\, \frac{m}{m+1}\sum_v h_{v,m}(Z)\\ &= \sigma(Z).\end{aligned}$$

LEMMA 2.1.5 (Siegel's lemma [3]). *Let D_K be the discriminant of K. Let $d = [K : \mathbb{Q}]$ and $\gamma = 4d^{2d}|D_K|^{\frac{1}{2}}$. Suppose that $M < N$ with $M, N \in \mathbb{N}$. Then for the system of linear equations with coefficients in K*

(2.1.5.1)
$$\sum_{j=1}^{N} a_{i,j} x_j = 0 \quad (i = 1, 2, \ldots, M),$$

there exists a non-trivial solution $\{x_i\} \subset K$ such that

(2.1.5.2)
$$\begin{aligned}&\sum_v \log\max(1, |x_1|_v, |x_2|_v, \ldots, |x_N|_v)\\ &\leq \frac{1}{N-M}\left(\sum_{i=1}^{M}\sum_v \log\max_{j\leq N} |a_{i,j}|_v\right)\\ &\quad + \frac{M}{N-M}\log(2N\gamma) + \log\gamma,\end{aligned}$$

where $\sum_v$ means that v runs over all places of K.

PROOF. See [3, Siegel's lemma]. □

LEMMA 2.1.6. *Suppose that* $N, L \in \mathbb{N}$ *satisfy*

$$N < L < \left(1 + \frac{1}{n^2}\right) N. \tag{2.1.6.1}$$

Then for $X = \sum_{i=0}^{\infty} X_i x^i \in GL_n(K[[x]])$ *with* $X_{|x=0} = I$, *there exist non-zero* q *and* P *with* $q \in K[x]$ *and* $P \in M_n(K[x])$ *which satisfy the following properties:*

$$\max(\deg q, \deg P) < N, \tag{2.1.6.2}$$

$$\mathrm{ord}_0(qX - P) \geq L, \tag{2.1.6.3}$$

$$\begin{aligned} \hbar_N(q) \leq & \frac{n^2(L-N)L}{(N - n^2(L-N))N} \hbar_L(X) \\ & + \frac{n^2(L-N)}{(N - n^2(L-N))N} (\log N + \log 2\gamma) + \frac{1}{N} \log \gamma, \end{aligned} \tag{2.1.6.4}$$

$$\hbar_{f,N}(P) \leq \hbar_{f,N}(q) + \hbar_{f,N}(X), \tag{2.1.6.5}$$

$$\hbar_{\infty,N}(P) \leq \hbar_{\infty,N}(q) + \hbar_{\infty,N}(X) + \frac{\log N}{N}, \tag{2.1.6.6}$$

and

$$\det P \neq 0. \tag{2.1.6.7}$$

Here γ *is the constant as in Lemma* 2.1.5, *depending only on* K.

PROOF. We show the existence of q and P satisfying these conditions. Put

$$P := (qX)_{<N}. \tag{2.1.6.8}$$

If q $(0 \neq q \in K[x])$ satisfies $\deg q < N$, then one has

$$0 \neq P \in M_n(K[x]), \tag{2.1.6.9}$$

since $X_{|x=0} = X_0 = I$. Now we write $q = \sum_{i=0}^{N-1} q_i x^i \in K[x]$. The conditions (2.1.6.3) and (2.1.6.8) require that

$$\sum_{\substack{i+j=l \\ i<N}} q_i X_j = 0 \quad (l = N, N+1, \ldots, L-1). \tag{2.1.6.10}$$

The system of linear equations (2.1.6.10) has N unknowns, $q_0, q_1, \dots, q_{N-1}$, and $n^2(L-N)$ equations. The condition (2.1.6.1) gives $N > n^2(L-N)$ and then from Lemma 2.1.5 there exists a non-trivial solution $\{q_0, q_1, \dots, q_{N-1}\}$ of (2.1.6.10) with

$$\begin{aligned}
&\sum_v \log\max(1, |q_0|_v, |q_1|_v, \dots, |q_{N-1}|_v) \\
&\quad \leq \frac{n^2(L-N)}{N-n^2(L-N)} \sum_v \log\max_{j<L} |X_j|_v \\
&\qquad + \frac{n^2(L-N)}{N-n^2(L-N)} \log(2N\gamma) + \log\gamma\,.
\end{aligned} \tag{2.1.6.11}$$

Therefore we conclude the existence of $0 \neq q \in K[x]$ with $\deg q < N$ satisfying the condition (2.1.6.4). We also have

$$\begin{aligned}
\hbar_{f,N}(P) &= \hbar_{f,N}(qX) \\
&= \frac{1}{N} \sum_{v\nmid\infty} \log^+ \max_{l<N} \left| \sum_{i+j=l} q_i X_j \right|_v \\
&\leq \frac{1}{N} \sum_{v\nmid\infty} \log^+ \max_{l<N} \max_{i+j=l} |q_i X_j|_v \\
&\leq \frac{1}{N} \sum_{v\nmid\infty} \log^+ \max_{i<N} |q_i|_v + \frac{1}{N} \sum_{v\nmid\infty} \log^+ \max_{j<N} |X_j|_v \\
&\leq \hbar_{f,N}(q) + \hbar_{f,N}(X)
\end{aligned}$$

and

$$\begin{aligned}
\hbar_{\infty,N}(P) &= \hbar_{\infty,N}(qX) \\
&= \frac{1}{N} \sum_{v|\infty} \log^+ \max_{l<N} \left| \sum_{i+j=l} q_i X_j \right|_v \\
&\leq \frac{1}{N} \sum_{v|\infty} \log^+ \max_{l<N} (l+1)^{\frac{d_v}{d}} \max_{i+j=l} |q_i X_j|_v \\
&\leq \frac{1}{N} \sum_{v|\infty} \log^+ \max_{i<N} |q_i|_v + \frac{1}{N} \sum_{v|\infty} \log^+ \max_{j<N} |X_j|_v + \frac{\log N}{N} \\
&\leq \hbar_{\infty,N}(q) + \hbar_{\infty,N}(X) + \frac{\log N}{N}\,.
\end{aligned}$$

Therefore we conclude the existence of $0 \neq P \in M_n(K[x])$ with $\deg P < N$ satisfying the conditions (2.1.6.5) and (2.1.6.6).

Finally we prove the assertion (2.1.6.7). Put

$$R := qX - P \in M_n(K[[x]])\,. \tag{2.1.6.12}$$

From the conditions (2.1.6.1), (2.1.6.2) and (2.1.6.3), one has

$$\max(\mathrm{ord}_0\, q, \mathrm{ord}_0\, P) < \mathrm{ord}_0\, R\,. \tag{2.1.6.13}$$

This implies $\mathrm{ord}_0\, q = \mathrm{ord}_0\, P$ by (2.1.6.12) and $X_{|x=0} = I$. Then

$$x^{-\mathrm{ord}_0\, q} R = x^{-\mathrm{ord}_0\, q} qX - x^{-\mathrm{ord}_0\, P} P \tag{2.1.6.16}$$

and $(x^{-\mathrm{ord}_0\, q} R)_{|x=0} = 0$. Hence $(x^{-\mathrm{ord}_0\, P} P)_{|x=0} = (x^{-\mathrm{ord}_0\, q} qX)_{|x=0}$ $= ((x^{-\mathrm{ord}_0\, q} q)_{|x=0} I$ by $X_{|x=0} = I$. Therefore $\det P \neq 0$. □

2.2. – Proof of Theorem II-1

Here we prove Theorem II-1 in Introduction.
The following inequality is required for our proof.

LEMMA 2.2.1. *For $f \in K[x] \setminus \{0\}$, we have*

$$\sum_{v\nmid\infty} \log \left|\frac{1}{f}\right|_v \leq \sum_{v|\infty} \log |f|_v + \log(1 + \deg f)\,. \tag{2.2.1.1}$$

PROOF. We take a root of unity ξ such that $f_{|x=\xi} \neq 0$. Put $F := K(\xi)$. Let v be a place of K and w be a place of F. For $f \in K[x]$ and for $w \nmid \infty$, one has $|f_{|x=\xi}|_w \leq |f|_w$, or

$$\left|\frac{1}{f}\right|_w \leq \frac{1}{|f_{|x=\xi}|_w}\,. \tag{2.2.1.2}$$

The product formula in F gives

$$\sum_{w\nmid\infty} \log \frac{1}{|f_{|x=\xi}|_w} = \sum_{w|\infty} \log |f_{|x=\xi}|_w\,. \tag{2.2.1.3}$$

Furthermore for $w \mid \infty$,

$$|f_{|x=\xi}|_w \leq (1 + \deg f)^{\frac{d_{Fw}}{d_F}} |f|_w\,, \tag{2.2.1.4}$$

where $d_F := [F : Q]$, $d_{F_w} := [F_w : Q_w]$. Therefore we obtain

$$\begin{aligned}\sum_{v\nmid\infty} \log\left|\frac{1}{f}\right|_v &= \sum_{w\nmid\infty} \log\left|\frac{1}{f}\right|_w \\ &\le \sum_{w\mid\infty} \log|f|_w + \log(1+\deg f) \\ &\le \sum_{v\mid\infty} \log|f|_v + \log(1+\deg f).\end{aligned}$$ □

Proof of Theorem II-1. Let $N \in \mathbb{N}$ satisfy

$$N > 12n^2(s+1) \tag{2.2.2.1}$$

and put

$$\begin{aligned} L :&= \left[\left(1+\frac{1}{2n^2}\right)N\right] \\ &= \left(1+\frac{1}{2n^2}\right)N - \epsilon, \end{aligned} \tag{2.2.2.2}$$

where $\epsilon \in \mathbb{R}$ with $0 \le \epsilon < 1$. As in Lemma 2.1.6, we choose $q \in K[x]$ and $P \in M_n(K[x]) \cap GL_n(K(x))$ with

$$\begin{cases} \max(\deg q, \deg P) < N, \\ \mathrm{ord}_0(qX - P) \ge L. \end{cases}$$

We write as in Subsect. 2.1 $R := qX - P$. By Lemma 2.1.1 and Lemma 2.1.3 we have

$$u^m\,(P, A, B)^{\langle m\rangle} = u^m\left(\frac{\mathcal{D}^m}{m!}q\right)X - u^m\,(R, A, B)^{\langle m\rangle} \quad (m \ge 0), \tag{2.2.2.3}$$

$$\begin{cases} u^m\,(P, A, B)^{\langle m\rangle} \in M_n(K[x]), \\ \deg u^m\,(P, A, B)^{\langle m\rangle} < N + ms \quad (0 \le m \le L) \end{cases} \tag{2.2.2.4}$$

and

$$\begin{cases} u^m\,(R, A, B)^{\langle m\rangle} \in M_n(K[[x]]), \\ \mathrm{ord}_0\, u^m\,(R, A, B)^{\langle m\rangle} \ge L - m \quad (0 \le m \le L). \end{cases} \tag{2.2.2.5}$$

Hence for any $m \ge 0$ with

$$N + ms \le L - m, \tag{2.2.2.6}$$

we obtain

$$u^m (P, A, B)^{\langle m\rangle} = \left(u^m \left(\frac{\mathcal{D}^m}{m!} q\right) X\right)_{<N+ms} . \tag{2.2.2.7}$$

On the other hand, by Lemma 1.2.8 we have

$$(I, A, B)^{\langle m\rangle} = \sum_{i+j=m} (P, A, B)^{\langle i\rangle}(P^{-1}, B, B)^{\langle j\rangle} \quad (m \geq 0) \tag{2.2.2.8}$$

and

$$(P^{-1}, B, B)^{\langle j\rangle} = \sum_{k+l=j} (P^{-1}, B, 0)^{\langle k\rangle}(I, 0, B)^{\langle l\rangle} \quad (j \geq 0) . \tag{2.2.2.9}$$

Moreover by the identity (1.2.7.1) for $A = 0$, $c = 0$, $\mathfrak{I}_1 = \mathfrak{I}_2 = I$ and $\mathfrak{I}_3 = P^{-1}$ in Lemma 1.2.7 we have

$$(P^{-1}, B, 0)^{\langle k\rangle} = \sum_{r=0}^{k}(-1)^{k-r} \frac{\mathcal{D}^r}{r!}((I, 0, B)^{\langle k-r\rangle} P^{-1}) \quad (k \geq 0) . \tag{2.2.2.10}$$

Therefore under the hypothesis (2.2.2.6), we obtain by the equalities (2.2.2.7)-(2.2.2.10)

$$\begin{aligned} &(I, A, B)^{\langle m\rangle} \\ &= \sum_{\substack{i+j=m \\ k+l=j \\ 0\leq r\leq k}} u^{-i} \left(u^i \left(\frac{\mathcal{D}^i}{i!} q\right) X\right)_{<N+is} (-1)^{k-r} \\ &\quad \times \left(\frac{\mathcal{D}^r}{r!}((I, 0, B)^{\langle k-r\rangle} P^{-1})\right) (I, 0, B)^{\langle l\rangle} . \end{aligned} \tag{2.2.2.11}$$

Let $\tilde{P}$ denote the adjoint matrix of P. Then for $v \nmid \infty$,

$$\begin{aligned} \log^+ \max_{i\leq m} | (I, A, B)^{\langle i\rangle} |_v &= \log \max_{i\leq m} | (I, A, B)^{\langle i\rangle} |_v \\ &\leq \log \max_{i\leq m} |u^{-i}|_v + \log^+ \max_{i\leq m} \left| \left(u^i \left(\frac{\mathcal{D}^i}{i!} q\right) X\right)_{<N+is} \right|_v \\ &\quad + 2\log^+ \max_{i\leq m} | (I, 0, B)^{\langle i\rangle} |_v + \log |(\det P)^{-1}|_v + \log |\tilde{P}|_v. \end{aligned} \tag{2.2.2.12}$$

By Lemma 2.2.1, for $u \in \mathcal{O}_K[x]$ we have

$$\begin{aligned} \sum_{v\nmid\infty} \max_{i\leq m} \log |u^{-i}|_v &= m \sum_{v\nmid\infty} \log |u^{-1}|_v \\ &\leq m \left(\sum_{v|\infty} \log |u|_v + \log(s + 1)\right) \end{aligned} \tag{2.2.2.13}$$

and similarly

$$
\begin{aligned}
\sum_{v\nmid\infty} \log |(\det P)^{-1}|_v &\le \sum_{v|\infty} \log|\det P|_v + \log(n^2N+1) \\
&\le \sum_{v|\infty} \log(n!^{\frac{d_v}{d}}|P|_v^n) + \log(n^2N+1) \\
&\le n\sum_{v|\infty} \log^+|P|_v + \log n!(n^2N+1)\,.
\end{aligned} \tag{2.2.2.14}
$$

Moreover

$$
\sum_{v\nmid\infty} \log^+|\tilde{P}|_v \le (n-1)\sum_{v\nmid\infty}\log^+|P|_v\,. \tag{2.2.2.15}
$$

Since $u \in \mathcal{O}_K[x]$,

$$
\begin{aligned}
&\sum_{v\nmid\infty} \log^+ \max_{i\le m} \left|\left(u^i\left(\frac{\mathcal{D}^i}{i!}q\right)X\right)_{<N+is}\right|_v \\
&\quad\le \sum_{v\nmid\infty}\log^+|q|_v + \sum_{v\nmid\infty}\log^+|X_{<N+ms}|_v\,.
\end{aligned} \tag{2.2.2.16}
$$

Substituting the inequalities (2.2.2.13)-(2.2.2.16) into the inequality (2.2.2.12), we obtain

$$
\begin{aligned}
\frac{1}{m}\sum_{v\nmid\infty}\max_{i\le m}\log^+|(I,A,B)^{\langle i\rangle}|_v &\le \frac{N}{m}\hbar_{f,N}(q) \\
&+\frac{N+ms}{m}\hbar_{f,N+ms}(X) \\
&+\frac{2}{m}\sum_{v\nmid\infty}\max_{i\le m}\log^+|(I,0,B)^{\langle i\rangle}|_v \\
&+(n-1)\frac{N}{m}\hbar_{f,N}(P)+(s+1)\hbar_\infty(u) \\
&+\log(s+1)+n\frac{N}{m}\hbar_{\infty,N}(P) \\
&+\frac{1}{m}\log n!(n^2N+1)\,.
\end{aligned} \tag{2.2.2.17}
$$

By the inequalities (2.1.6.5) and (2.1.6.6) in Lemma 2.1.6, we find

$$
\begin{aligned}
&\hbar_{fmN}(q)+(n-1)\hbar_{f,N}(P)+n\hbar_{\infty,N}(P) \\
&\quad\le \hbar_{f,N}(q)+(n-1)(\hbar_{f,N}(q)+\hbar_{f,N}(X)) \\
&\qquad+n\left(\hbar_{\infty,N}(q)+\hbar_{\infty,N}(X)+\frac{\log N}{N}\right) \\
&\quad\le n\hbar_N(q)+n\hbar_N(X)+n\frac{\log N}{N}\,.
\end{aligned} \tag{2.2.2.18}
$$

Applying the last inequalities (2.2.2.18) to the inequality (2.2.2.17), we deduce

$$\begin{aligned}\frac{1}{m}\sum_{v\nmid\infty}\max_{i\le m}\log^+|(I,A,B)^{\langle i\rangle}|_v &\le n\frac{N}{m}\hbar_N(q)+n\frac{N}{m}\hbar_N(X)\\ &+\frac{N+ms}{N}\hbar_{N+ms}(X)+\frac{2}{m}\sum_{v\nmid\infty}\max_{i\le m}\log^+|(I,0,B)^{\langle i\rangle}|_v\\ &+\frac{2}{m}\log n!(n^2N+1)+\log(s+1)+(s+1)\hbar_\infty(u)\,.\end{aligned}\tag{2.2.2.19}$$

Now by the identities (2.2.2.2) it is easy to see that

$$\frac{n^2(L-N)}{N-n^2(L-N)}=\frac{n^2\left(\frac{1}{2n^2}N-\epsilon\right)}{N-n^2\left(\frac{1}{2n^2}N-\epsilon\right)}=\frac{\frac{1}{2}N-n^2\epsilon}{\frac{1}{2}N+n^2\epsilon}\le 1$$

and

$$\frac{L}{N}=\frac{\left(1+\frac{1}{2n^2}\right)N-\epsilon}{N}\le 1+\frac{1}{2n^2}\,.$$

Then from Lemma 2.1.6 one has

$$\hbar_N(q)\le\left(1+\frac{1}{2n^2}\right)\hbar_L(X)+\frac{1}{N}(\log N+\log 2\gamma)+\frac{1}{N}\log\gamma\,.\tag{2.2.2.20}$$

The last inequality and the inequality (2.2.2.19) yield

$$\begin{aligned}&\frac{1}{m}\sum_{v\nmid\infty}\max_{i\le m}\log^+|(I,A,B)^{\langle i\rangle}|_v\\ &\le n\frac{N}{m}\left(1+\frac{1}{2n^2}\right)\hbar_L(X)\\ &+n\frac{N}{m}\left(\frac{1}{N}(\log N+\log 2\gamma)+\frac{1}{N}\log\gamma\right)\\ &+n\frac{N}{m}\hbar_N(X)+\frac{N+ms}{N}\hbar_{N+ms}(X)\\ &+\frac{2}{m}\sum_{v\nmid\infty}\max_{i\le m}\log^+|(I,0,B)^{\langle i\rangle}|_v\\ &+\frac{2}{m}\log n!(n^2N+1)+\log(s+1)+(s+1)\hbar_\infty(u)\,.\end{aligned}\tag{2.2.2.21}$$

The last inequality holds for m satisfying the inequality (2.2.2.6). For the validity of the inequality (2.2.2.6), we need

$$N+ms\le L-m=\left(1+\frac{1}{2n^2}\right)N-\epsilon-m$$

by (2.2.2.2), hence

$$m \leq \frac{N}{2n^2(s+1)} - \frac{\epsilon}{s+1}. \tag{2.2.2.22}$$

We choose m as the maximal integer satisfying the inequality (2.2.2.22). Then

$$\frac{N}{2n^2(s+1)} - \frac{\epsilon}{s+1} - 1 < m \leq \frac{N}{2n^2(s+1)} - \frac{\epsilon}{s+1},$$

or more weakly

$$\frac{1}{2n^2(s+1)} - \frac{2}{N} < \frac{m}{N} \leq \frac{1}{2n^2(s+1)}.$$

By the inequality (2.2.2.1), one has

$$2n^2(s+1) \leq \frac{N}{m} < 3n^2(s+1). \tag{2.2.2.23}$$

We recall the identities (2.1.4.1) after Definitions 2.1.4. By the identities (2.2.2.2) and the inequalities (2.2.2.23), each limit $\overline{\lim}_{N\to\infty} \hbar_L(X)$, $\overline{\lim}_{N\to\infty} \hbar_N(X)$ and $\overline{\lim}_{N\to\infty} \hbar_{N+ms}(X)$ is equal to $\sigma(X)$ and each limit

$$\overline{\lim_{N\to\infty}}\, n\frac{N}{m}\left(\frac{1}{N}(\log N + \log 2\gamma) + \frac{\log\gamma}{N}\right)$$

and

$$\overline{\lim_{N\to\infty}}\, \frac{2}{m}\log(n^2N+1)$$

vanishes. Using the inequalities (2.2.2.23), we apply $\overline{\lim}_{N\to\infty}$ (or equivalently $\overline{\lim}_{m\to\infty}$) to the inequality (2.2.2.21). Then we conclude

$$\begin{aligned}\sigma(I, A, B) \leq \Bigg(& n\cdot 3n^2(s+1)\cdot\frac{2n^2+1}{2n^2} \\ &+ n\cdot 3n^2(s+1) + 1 + \frac{s}{2n^2(s+1)}\Bigg)\sigma(X) \\ &+ 2\sigma(0, B) + (s+1)\hbar_\infty(u) + \log(s+1) \\ \leq\ & 9n^3(s+1)\sigma(X) + 2\sigma(0, B) + (s+1)\hbar_\infty(u) + \log(s+1). \qquad \square\end{aligned}$$

2.3. – Proof of Corollary II-2

Before stating a proof of Corollary II-2, we need the definition of the normalized uniform part of solution of differential equation (0.1).

We use the notations as in Introduction: $\frac{1}{x}K[x]_{(x)}$ and $\mathrm{Res}(A)$.

Until the end of this section, assume that

$$\text{every eigenvalue of } \mathrm{Res}(A) \text{ is contained in } \mathbb{Q}. \tag{2.3.0.1}$$

LEMMA 2.3.1 (Shearing transformation [5]). *Suppose that $A \in M_n(\frac{1}{x}K[x]_{(x)})$ satisfies the condition* (2.3.0.1). *Let α be an eigenvalue of* $\mathrm{Res}(A)$. *Then there exists $T_1 \in GL_n(K(x))$ (resp. $T_2 \in GL_n(K(x))$) such that*

$$T_1[A] \in M_n\left(\frac{1}{x}K[x]_{(x)}\right), \tag{2.3.1.1}$$

(2.3.1.2) *T_1 and T_1^{-1} (resp. T_2 and T_2^{-1}) have no poles except at the origin*

and that

(2.3.1.3) *the eigenvalues of* $\mathrm{Res}(T_1[A])$ *coincide with those of* $\mathrm{Res}(A)$ *except the eigenvalue α is replaced $\alpha + 1$ (resp. $\alpha - 1$) with multiplicity.*

PROOF. Since all eigenvalues are rational numbers, there exists $P_1 \in GL_n(K)$ such that $P_1\,\mathrm{Res}(A)P_1^{-1} = \mathrm{Res}(P_1[A])$ is the upper Jordan normal form. Then one can immediately obtain Lemma 2.3.1 by the procedure of [5, Proposition 2.3]. □

Repeated application of Lemma 2.3.1 yields:

PROPOSITION 2.3.2. *Suppose that $A \in M_n(\frac{1}{x}K[x]_{(x)})$ satisfies the condition* (2.3.0.1). *Then there exists $T \in GL_n(K(x))$ such that*

$$T[A] \in M_n\left(\frac{1}{x}K[x]_{(x)}\right), \tag{2.3.2.1}$$

(2.3.2.2) *every eigenvalue of* $\mathrm{Res}(T[A])$ *is less than 1 and is not less than 0*

and that

(2.3.2.3) *T and T^{-1} have no poles except at the origin.*

The following proposition is the uniqueness of the solution of differential equation (0.3).

PROPOSITION 2.3.3. *Let $A \in M_n(\frac{1}{x}K[x]_{(x)})$ satisfy the condition* (2.3.0.1). *Let $T \in GL_n(K(x))$. Suppose that*

(2.3.3.1)
$$T[A] \in M_n\left(\frac{1}{x}K[x]_{(x)}\right)$$

and that

(2.3.3.2) *none of the differences between the eigenvalues of* $\mathrm{Res}(T[A])$ *is a non-zero integer.*

Then there exists the unique solution $Y \in GL_n(K[[x]])$ of the differential equation:

(2.3.3.3)
$$\frac{d}{dx}Y = T[A]Y - Y\frac{1}{x}\mathrm{Res}(T[A])$$

with $Y_{|x=0} = I$.

PROOF. See [1, Chapt. III, Subsect. 1.4].

DEFINITION 2.3.4. Let $A \in M_n(\frac{1}{x}K[x]_{(x)})$. For a fixed $T \in GL_n(K(x))$ satisfying the conditions (2.3.3.1) and (2.3.3.2), we call the unique solution $Y \in GL_n(K[[x]])$ of differential equation (2.3.3.3) with $Y_{|x=0} = I$ the *normalized uniform part* of solution of differential equation (0.1).

The following lemma is required for our proof.

LEMMA 2.3.5. *Let $A \in M_n(\frac{1}{x}K[x]_{(x)})$ and put $B := \frac{1}{x}\mathrm{Res}(A)$. Suppose that*

(2.3.5.1)
$$\mathcal{E} := \{\textit{eigenvalues of } \mathrm{Res}(A)\} \subset \mathbb{Q}.$$

Then one has

(2.3.5.2)
$$\sigma(0, B) \leq \sum_{\substack{p|N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + \log N_{\mathcal{E}} + n - 1,$$

where $N_{\mathcal{E}} \in \mathbb{N}$ is a common denominator of $\mathcal{E}$.

PROOF. Let $C \in M_n(K)$ be the normal Jordan form of $\mathrm{Res}(A)$. Then there exists $H \in GL_n(K)$ with $HBH^{-1} = \frac{1}{x}C$. Since $H[B] = \frac{1}{x}C$, one has

$$\sigma\left(0, \frac{1}{x}C\right) = \sigma(I, 0, H[B]) = \sigma(H, 0, B) = \sigma(I, B, 0) = \sigma(I, 0, B)$$

by Proposition 1.2.2. Consequently it is enough to show that

(2.3.5.3)
$$\sigma\left(0, \frac{1}{x}C\right) \leq \sum_{\substack{p|N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + \log N_{\mathcal{E}} + n - 1.$$

Now we write

$$\binom{C}{0} = I,$$
(2.3.5.4)
$$\binom{C}{m} = \frac{1}{m}\binom{C}{m-1}(C-(m-1)I) \quad (m = 1, 2, \ldots).$$

First we show by induction

$$\left(I, 0, \frac{1}{x}C\right)^{\langle m\rangle} = \frac{1}{x^m}\binom{C}{m} \tag{2.3.5.5}$$

for $m = 0, 1, \ldots$. For $m = 0$, both sides of the identity (2.3.5.5) are I and it is true in this case. Assume that the identity (2.3.5.5) is true for a given $m \geq 0$. Then one has

$$\begin{aligned} x^{m+1}\left(I, 0, \frac{1}{x}C\right)^{\langle m+1\rangle} &= \frac{x^{m+1}}{m+1}\left(\mathcal{D}\left(I, 0, \frac{1}{x}C\right)^{\langle m\rangle} + \left(I, 0, \frac{1}{x}C\right)^{\langle m\rangle}\frac{1}{x}C\right) \\ &= \frac{x^{m+1}}{m+1}\left(\mathcal{D}\left(\frac{1}{x^m}\binom{C}{m}\right) + \frac{1}{x^m}\binom{C}{m}\frac{1}{x}C\right) \\ &= \frac{1}{m+1}\binom{C}{m}(C - mI) \\ &= \binom{C}{m+1}. \end{aligned}$$

Therefore the identity (2.3.5.5) holds. Next we handle two cases.

CASE 1. $v \nmid \infty$ with $|C|_v > 1$. There exists only a finite number of such cases. Identity (2.3.5.5) gives

$$\left|\left(I, 0, \frac{1}{x}C\right)^{\langle m\rangle}\right|_v = \left|\frac{1}{x^m}\binom{C}{m}\right|_v \leq \left|\frac{1}{m!}\right|_v \max(1, |C|_v^m).$$

Since

$$\left|\frac{1}{m!}\right|_v \leq |p|_v^{-\frac{m}{p-1}} \quad (v \mid p),$$

we obtain

$$\frac{1}{m}\max_{i\le m}\log^+\left|\left(I,0,\frac{1}{x}C\right)^{\langle i\rangle}\right|_v \le \frac{-\log|p|_v}{p-1}+\log^+|C|_v. \tag{2.3.5.6}$$

CASE 2. $v\nmid\infty$ with $|C|_v\le 1$. We have

$$\left|\left(I,0,\frac{1}{x}C\right)^{\langle m\rangle}\right|_v = \left|\frac{1}{x^m}\binom{C}{m}\right|_v = \left|\binom{C}{m}\right|_v. \tag{2.3.5.7}$$

Let C_D denote the diagonal part of C and C_N the nilpotent part of C, hence $C=C_D+C_N$. By the commutativity of multiplication, we find that $\binom{C}{m}$ is equal to

$$\sum_{i=0}^{n-1}\frac{1}{m!}C_N^i\sum_{0\le j_1<\cdots<j_i<m}\prod_{\substack{j=0\\ j\ne j_1,\dots,j_i}}^{m-1}(C_D-jI).$$

Hence by $|C_N|_v=1$, one has

$$\left|\binom{C}{m}\right|_v \le \max_{\substack{0\le i\le n-1\\ 0\le j_1<\cdots<j_i<m\\ \alpha\in\mathcal{E}}}\left|\frac{1}{m!}\prod_{\substack{j=0\\ j\ne j_1,\dots,j_i}}^{m-1}(\alpha-j)\right|_v. \tag{2.3.5.8}$$

We put $\alpha=a_\alpha/b_\alpha$ with $a_\alpha,\ b_\alpha\in\mathbb{Z}$, $(a_\alpha,b_\alpha)=1$ and $b_\alpha>0$. By a standard argument (cf. [1, Chapt. I, Appendix]) one finds that the following inequality holds for sufficiently large m and for every prime p with $v\mid p$ in this case:

$$\sum_{v|p}\max_{\substack{0\le i\le n-1\\ 0\le j_1<\cdots<j_i<m\\ \alpha\in\mathcal{E}}}\left|\frac{1}{m!}\prod_{\substack{j=0\\ j\ne j_1,\dots,j_i}}^{m-1}(\alpha-j)\right|_v \le \max_{\alpha\in\mathcal{E}}|b_\alpha-a_\alpha(m-1)|^{n-1}, \tag{2.3.5.9}$$

where $\sum_{v|p}$ means that v runs over all finite places of K with $v\mid p$.

Consequently we obtain

$$\begin{aligned}
\sigma\left(0, \frac{1}{x}C\right) &= \overline{\lim_{m\to\infty}} \frac{1}{m} \sum_{v\nmid\infty} \max_{i\le m} \log^+ \left| \left(I, 0, \frac{1}{x}C\right)^{\langle m\rangle} \right|_v \\
&\le \overline{\lim_{m\to\infty}} \left(\sum_{p:\text{prime}} \sum_{\substack{v\mid p \\ |C|_v>1}} \left(\frac{-\log|p|_v}{p-1} + \log^+ |C|_v \right) \right. \\
&\quad \left. + \sum_{\substack{p\le m \\ p:\text{prime}}} \frac{n-1}{m} \log \max_{\alpha\in\mathcal{E}} |b_\alpha - a_\alpha(m-1)| \right) \\
&\le \overline{\lim_{m\to\infty}} \left(\sum_{\substack{p\mid N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + \sum_{\substack{v\nmid\infty \\ |C|_v>1}} \log^+ |C|_v \right. \\
&\quad \left. + \pi(m) \frac{n-1}{m} \log \max_{\alpha\in\mathcal{E}} |b_\alpha - a_\alpha(m-1)| \right) \\
&\le \sum_{\substack{p\mid N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + \log N_{\mathcal{E}} + n - 1 .
\end{aligned}$$

Here $\pi(m)$ is the number of primes which are not greater than m and we used the prime number theorem. □

Finally we obtain Corollary II-2 as follows:

COROLLARY 2.3.6. *Let $A \in M_n(\frac{1}{x}K[x]_{(x)})$ and Y be the normalized uniform part of the solution of differential equation* (2.3.3.3) *with $T \in GL_n(K(x))$ as in Proposition* 2.3.2. *Let u, s be denoted as in Theorem* II-1 *with $T[A]$ instead of A and $B = \frac{1}{x}\,\mathrm{Res}(T[A])$ and $\mathcal{E} \subset \mathbb{Q}$, $N_{\mathcal{E}}$ as in Lemma* 2.3.5. *Then*

(2.3.6.1)
$$\sigma(0, A) \le 9n^3(s+1)\sigma(Y) + 3\log N_{\mathcal{E}} + 3 \sum_{\substack{p\mid N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + (s+1)\hbar_\infty(u) + \log(s+1) + 3(n-1) .$$

PROOF. We choose $T \in GL_n(K(x))$ as in Proposition 2.3.2. Since each eigenvalue of $\mathrm{Res}(T[A])$ corresponds to one of the eigenvalues of $\mathrm{Res}(A)$ mod 1, we find by Lemma 2.3.5

(2.3.6.2)
$$\sigma\left(\frac{1}{x}\mathrm{Res}(T[A])\right) \le \sum_{\substack{p\mid N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} + \log N_{\mathcal{E}} + n - 1 .$$

Furthermore by Proposition 1.2.2 we have

$$\begin{aligned}\sigma(0, A) &= \sigma(I, A, 0) = \sigma(T, 0, A) = \sigma(I, 0, T[A]) = \sigma(I, T[A], 0)\\ &\leq \sigma\left(I, T[A], \frac{1}{x}\mathrm{Res}(T[A])\right) + \sigma\left(I, \frac{1}{x}\mathrm{Res}(T[A]), 0\right)\\ &\leq \sigma\left(I, T[A], \frac{1}{x}\mathrm{Res}(T[A])\right) + \sigma\left(\frac{1}{x}\mathrm{Res}(T[A])\right).\end{aligned} \tag{2.3.6.3}$$

Consequently, by the inequalities (2.3.6.2), (2.3.6.3) and Theorem II-1, we obtain

$$\begin{aligned}\sigma(0, A) &\leq 9n^3(s+1)\sigma(Y) + 3\sigma\left(\frac{1}{x}\mathrm{Res}(T[A])\right) + (s+1)\hbar_\infty(u) + \log(s+1)\\ &\leq 9n^3(s+1)\sigma(Y) + 3\log N_{\mathcal{E}} + 3\sum_{\substack{p \mid N_{\mathcal{E}}\\ p:\text{prime}}} \frac{\log p}{p-1} + (s+1)\hbar_\infty(u)\\ &\quad + \log(s+1) + 3(n-1).\end{aligned}$$

□

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