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Differential Equations on Contact Riemannian Manifolds

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Abstract. Building on work by S. Tanno, [28], we study certain differential equations on a contact Riemannian manifold whose almost CR structure is not integrable, in general. We prove a 'universality' property of *Tanno's equation* $\nabla_\xi \mathcal{L}_\xi g - 2(\mathcal{L}_\xi g) \cdot \phi = 0$. We show that the *sublaplacian* Δ_H (introduced by S. Tanno, cf. op. cit.) is subelliptic of order 1/2 (hence Δ_H is hypoelliptic and has a discrete spectrum tending to $+\infty$). We consider the *tangential Cauchy-Riemann equations* $\overline{\partial}_H \omega = 0$, $\omega \in \Omega^{0,q}(M)$, $q \geq 0$, and associate a *twisted cohomology* (cf. I. Vaisman, [31]) with the corresponding tangential Cauchy-Riemann pseudocomplex. We build a Lorentzian metric G_η on the total space of a certain principal S^1-bundle $\pi : F(M) \to M$ over a contact manifold (M, η). When the almost CR structure of M is integrable, G_η is the Fefferman metric (cf. J.M. Lee, [21]) of $(M, -\eta)$. We show that a C^∞ map $f : M \to N$ of a contact Riemannian manifold M into a Riemannian manifold (N, g') satisfies $\Delta_H f^i + 4g^{\alpha\bar{\beta}} \xi_\alpha(f^j)\xi_{\bar{\beta}}(f^k) \left(\Gamma'^i_{jk} \circ f\right) = 0$ if and only if the vertical lift $f \circ \pi$ of f is a harmonic map with respect to the (generalized) Fefferman metric G_η.

Mathematics Subject Classification (2000): 53B30 (primary), 53D10, 32V20, 32V05 (secondary).

Contents

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1. – Introduction

Let M be a real $(2n+1)$-dimensional C^∞ manifold. A rank n complex subbundle $T_{1,0}(M)$ of the complexified tangent bundle $T(M)\otimes \mathbf{C}$ is an *almost CR structure* (of *CR dimension n*) on M if

$$T_{1,0}(M)\cap T_{0,1}(M)=(0)\,.$$

Here $T_{0,1}(M)=\overline{T_{1,0}(M)}$ and an overbar denotes complex conjugation. A pair $(M, T_{1,0}(M))$ is an *almost CR manifold* (of CR dimension n). For instance, any real hypersurface M of an almost complex manifold (V, J_V) is an almost CR manifold with the almost CR structure

$$T_{1,0}(M)=[T(M)\otimes \mathbf{C}]\cap T'(V)$$

where $T'(V)$ is the holomorphic tangent bundle over V (i.e. the eigenbundle of $J_V: T(V)\otimes\mathbf{C}\to T(V)\otimes\mathbf{C}$ corresponding to the eigenvalue $i=\sqrt{-1}$). An almost CR manifold of this sort (i.e. whose almost CR structure is induced by the almost complex structure of the ambient space) is called *embedded.*

Let $(M, T_{1,0}(M))$ be an almost CR manifold. The *Levi distribution* is the rank $2n$ real subbundle $H(M)\subset T(M)$ given by

$$H(M)=\mathrm{Re}\{T_{1,0}(M)\oplus T_{0,1}(M)\}\,.$$

It carries the complex structure

$$J: H(M)\to H(M),\ J(Z+\overline{Z})=i(Z-\overline{Z}),\ Z\in T_{1,0}(M)\,.$$

In particular, $H(M)$ is oriented. Let $E=H(M)^\perp\subset T^*(M)$ be the conormal bundle of $H(M)$, i.e.

$$E_x=\{\omega\in T^*_x(M):\mathrm{Ker}(\omega)\supseteq H(M)_x\}\,,\ x\in M\,.$$

If M is orientable, an assumption we adopt from now on, then

$$E\approx T(M)/H(M)$$

is an orientable real line bundle over M, hence trivial. Therefore, E admits globally defined nowhere vanishing C^∞ sections $\theta\in\Gamma^\infty(E)$, each of which is referred to as a *pseudohermitian structure* on M. Given a pseudohermitian structure θ on M, the *Levi form* L_θ is given by

$$L_\theta(Z,W)=-i(d\theta)(Z,\overline{W})\,,\ Z,W\in T_{1,0}(M)\,.$$

Any other pseudohermitian structure $\hat\theta$ on M is given by $\hat\theta=\lambda\theta$, for some C^∞ function $\lambda: M\to\mathbf{R}\setminus\{0\}$. Also $L_{\hat\theta}=\lambda L_\theta$. An almost CR manifold M is *nondegenerate* if L_θ is nondegenerate for some θ, and thus for all. Also, M is *strictly pseudoconvex* if L_θ is positive definite for some θ.

We say $T_{1,0}(M)$ is a *CR structure* (and $(M, T_{1,0}(M))$ a *CR manifold*) if $T_{1,0}(M)$ is *(formally) integrable*, i.e.

$$[\Gamma^\infty(T_{1,0}(M)), \Gamma^\infty(T_{1,0}(M))] \subseteq \Gamma^\infty(T_{1,0}(M))$$

(cf. [13]). For instance, if (V, J_V) is a complex manifold, then the almost CR structure of any embedded almost CR manifold $M \subset V$ is actually integrable (as a consequence of the integrability of J_V). Nondegenerate CR manifolds are the objects studied in *pseudohermitian geometry*, cf. e.g. N. Tanaka, [27], and S. Webster, [32]. A pseudohermitian structure θ on a nondegenerate CR manifold is actually a *contact form*, i.e. $\theta \wedge (d\theta)^n$ is a volume form on M. Once a contact form θ has been fixed, there is a unique globally defined tangent vector field $T \in \mathcal{X}(M)$ so that

$$\theta(T) = 1\,, \quad T \rfloor d\theta = 0$$

(the *characteristic direction* of (M, θ)) and $T(M) = H(M) \oplus \mathbf{R}T$. Thus one may build the *Webster metric*

$$\begin{aligned} g_\theta(X, Y) &= (d\theta)(X, JY) \\ g_\theta(X, T) &= 0 \\ g_\theta(T, T) &= 1 \end{aligned}$$

for any $X, Y \in H(M)$. It is a semi-Riemannian metric on M (of signature $(2r+1, 2s)$, where (r, s) is the signature of L_θ). Moreover, by a result in [27] (cf. also [32]) there is a unique linear connection ∇^* on M (the *Tanaka-Webster connection* of (M, θ)) so that 1) $H(M)$ is parallel with respect to ∇^*, 2) $\nabla^* J = 0$ and $\nabla^* g_\theta = 0$, and 3) the torsion T^* of ∇^* is *pure*, i.e.

$$\begin{aligned} T^*(Z, W) &= 0 \\ T^*(Z, \overline{W}) &= 2i L_\theta(Z, W) T \\ \tau \circ J + J \circ \tau &= 0 \end{aligned}$$

for any $Z, W \in T_{1,0}(M)$, where

$$\tau : T(M) \to T(M), \; \tau(X) = T^*(T, X), \; X \in T(M)$$

is the *pseudohermitian torsion* of ∇^*. The Tanaka-Webster connection is in many respects similar to the Levi-Civita connection of a Riemannian manifold and to the Chern connection of a Hermitian manifold, a similarity which allows the generalization (to the realm of pseudohermitian geometry) of an array of concepts and results originating in Riemannian or Hermitian geometry, cf. e.g. E. Barletta & al, [1]-[4], [7]-[9], D. Jerison & J.M. Lee, [16]-[17], J.M. Lee, [22], H. Urakawa, [29]-[30], and S. Webster, [33]-[34]. The construction of the

Tanaka-Webster connection relies however on the (formal) integrability property of the CR structure, and the tools of pseudohermitian geometry may not be applied to an almost CR manifold. Nevertheless, in view of the approach of S. Tanno, [28], a sort of pseudohermitian geometry is available on a class of almost CR manifolds, arising in contact Riemannian geometry, cf. e.g. D.E. Blair, [6]. Precisely, let (M, η) be a contact manifold and $g \in \mathcal{M}(\eta)$ an associated metric (cf. the next section for a reminder of contact geometry). Let ∇ be the Levi-Civita connection of (M, g). S. Tanno considered (cf. [28]) the $(1,2)$-tensor field

$$Q^i_{jk} = \nabla_k \phi^i_j + \xi^i \phi^r_j \nabla_k \eta_r + \phi^i_r \eta_j \nabla_k \xi^r$$

(the *Tanno tensor* field) and the linear connection ∇^* (the *(generalized) Tanaka-Webster connection* of (M, η)) given by

$$\Gamma^{*i}_{jk} = \Gamma^i_{jk} + \eta_j \phi^i_k - \eta_k \nabla_j \xi^i + \xi^i \nabla_j \eta_k$$

where Γ^i_{jk} are the (local) coefficients of ∇. Then (cf. [28], p. 353-354) the almost CR structure of M is integrable if and only if $Q = 0$ and, if this is the case, then ∇^* is the ordinary Tanaka-Webster connection of $(M, -\eta)$ (hence the adopted terminology).

In the present paper, we start from the results of [28]. Namely, S. Tanno has considered the Dirichlet energy

$$E(g) = \int_M \|\mathcal{L}_\xi g\|^2 \, dv_g \, , \quad g \in \mathcal{M}(\eta)$$

and showed that an associated metric $g \in \mathcal{M}(\eta)$ is a critical point of E if and only if $T(g) = 0$, where T is the *Tanno operator*

$$T(g) = \nabla_\xi \mathcal{L}_\xi g - 2(\mathcal{L}_\xi g) \cdot \phi$$

(cf. Theorem 5.1 in [28], p. 357). Moreover, S. Tanno has introduced a second order differential operator Δ_H (coinciding with the sublaplacian Δ_b of [21], p. 414, in the integrable case) and showed that

$$-\frac{4(n+1)}{n} \Delta_H u + S^* u = \tilde{S}^* u^{(n+2)/n} . \tag{1}$$

Here S^* and $\tilde{S}^*$ are respectively the (generalized) Tanaka-Webster scalar curvature functions of (M, η) and $(M, u^{2/n} \eta)$, and $u : M \to (0, +\infty)$ is a C^∞ function. Hence one should solve for u in (1) to determine a contact form $\tilde{\eta} = u^{2/n} \eta$ with $\tilde{S}^* = \lambda = \text{const.}$ (the contact analogue of the Yamabe problem in Riemannian geometry).

Our results and expectations through the present paper may be briefly described, as follows. We prove a 'universality' property of Tanno's equation $T(g) = 0$ on a contact manifold (M, η), i.e. for $f \in C^\omega(\mathbf{R})$ with $Z(f') = \emptyset$

and $Z(F) \neq \mathbf{R}$, $T(g) = 0$ is shown to be the Euler-Lagrange system of the variational principle

$$\delta \int_M f(\|\mathcal{L}_\xi g\|^2)\eta \wedge (d\eta)^n = 0.$$

Moreover, we show that Δ_H is a subelliptic differential operator of order 1/2, i.e. for any $x \in M$ there is a neighborhood U and a constant $C > 0$ so that

$$\|f\|_{1/2}^2 \leq C\{|(\Delta_H f, f)| + \|f\|^2\}$$

for any $f \in C_0^\infty(U)$.

We consider the *tangential Cauchy-Riemann pseudocomplex*

$$\cdots \to \Omega^{0,q-1}(M) \xrightarrow{\overline{\partial}_H} \Omega^{0,q}(M) \xrightarrow{\overline{\partial}_H} \Omega^{0,q+1}(M) \to \cdots \tag{2}$$

This is a complex (i.e. $\overline{\partial}_H^2 = 0$) if and only if $T_{1,0}(M)$ is integrable. By a result of I. Vaisman, [31], one may associate with $(\Omega^{0,*}(M), \overline{\partial}_H)$ a complex $(\mathcal{D}^*(M), \overline{\delta}_H)$. The corresponding cohomology $H^*(\mathcal{D}^*(M), \overline{\delta}_H)$ (the *twisted cohomology* of (2)) is the usual Kohn-Rossi cohomology when $T_{1,0}(M)$ is integrable.

Finally, we build a Lorentzian metric G_η on the total space $F(M)$ of a certain principal circle bundle over M. When $T_{1,0}(M)$ is integrable, our G_η coincides with the Fefferman metric (cf. [21]) of $(M, -\eta)$. As an application, we show that the vertical lift $f \circ \pi : (F(M), G_\eta) \to (N, g')$ (cf. Section 6 for definitions), of any pseudoharmonic (in the sense of [10], p. 108) map $f : M \to N$ of a contact Riemannian manifold M into a Riemannian manifold (N, g'), is a harmonic map.

Our program, for a forthcoming paper, will be to investigate whether 1) the restricted conformal class of G_η is a gauge invariant, and whether 2) the Yamabe problem for G_η is equivalent to (1). This is of course the case when $T_{1,0}(M)$ is integrable. The problem of solving (1) (with $\tilde{S}^* =$ const.) is left open (cf. [16]-[17] for the integrable case). The Authors are grateful to the Referee, whose suggestions improved the original form of the manuscript.

2. – Contact geometry

Let M be a $(2n+1)$-dimensional manifold. An *almost contact structure* on M (cf. [6], p. 19) is a synthetic object (ϕ, ξ, η) consisting of a $(1,1)$-tensor field ϕ, a vector field ξ, and a 1-form η so that (in classical tensor notation)

$$\phi_k^i \phi_j^k = -\delta_j^i + \eta_j \xi^i \tag{3}$$

$$\eta_i \phi^i_j = 0\,, \qquad \phi^i_j \xi^j = 0\,, \qquad \eta_i \xi^i = 1\,. \tag{4}$$

A Riemannian metric g on M is *associated* (or *compatible*) to the almost contact structure (ϕ, ξ, η) (and (ϕ, ξ, η, g) is an *almost contact metric structure*) if

$$g_{ij} \phi^i_k \phi^j_\ell = g_{k\ell} - \eta_k \eta_\ell \tag{5}$$

$$g_{ij} \xi^j = \eta_i\,. \tag{6}$$

Associated metrics always exist (cf. [6], p. 21). With any almost contact metric structure one associates a 2-form Ω given by $\Omega_{ij} = g_{ik} \phi^k_j$. A *contact metric structure* is an almost contact metric structure (ϕ, ξ, η, g) for which $\Omega = d\eta$ (cf. [6], p. 25). Contact metric structures possess the following elementary properties (cf. e.g. Lemma 1.1 in [28], p. 351)

$$\nabla_\xi \eta = 0\,, \qquad \nabla_\xi \xi = 0\,, \qquad \xi^j \nabla_i \eta_j = 0 \tag{7}$$

$$\nabla_i \xi^i = 0\,, \qquad \nabla_i \phi^i_j = -2n\eta_j \tag{8}$$

$$\phi^r_i \phi^s_j \, \nabla_r \eta_s = -\, \nabla_j \eta_i \tag{9}$$

$$\phi^k_j \nabla_k \eta_i = \phi^k_i \nabla_k \eta_j\,, \qquad \phi^k_j \nabla_i \eta_k = \phi^k_i \nabla_j \eta_k \tag{10}$$

$$\nabla_\xi \phi = 0 \tag{11}$$

where ∇ is the Levi-Civita connection of (M, g). (Almost) contact metric structures occur for instance on real hypersurfaces of Kählerian manifolds (cf. e.g. [24]) and have been studied by several authors (cf. [6] and references therein; cf. also [26] for results – old and new – not reported on in [6]).

The following (equivalent) approach to contact Riemannian geometry is also useful. Let (M, η) be a *contact manifold*, i.e. a real $(2n+1)$-dimensional C^∞ manifold M endowed with a 1-form η so that $\Psi = \eta \wedge (d\eta)^n$ is a volume form on M. There is a unique tangent vector field $\xi \in \mathcal{X}(M)$ so that $\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0$ (the *characteristic direction* of (M, η)). By a well known result (cf. e.g. [6], p. 25-26) there exist a Riemannian metric g and a $(1,1)$-tensor field ϕ on M so that $g(X, \xi) = \eta(X)$, $\phi^2 = -I + \eta \otimes \xi$, and $g(X, \phi Y) = (d\eta)(X, Y)$, for any $X, Y \in \mathcal{X}(M)$. Such g is referred to as *associated* to η and $\mathcal{M}(\eta)$ will indicate the set of all associated Riemannian metrics; clearly (ϕ, ξ, η, g) is a contact metric structure on M. Note that each $g \in \mathcal{M}(\eta)$ has the same volume form Ψ.

3. – On Tanno's equation in contact Riemannian geometry

Let (M, η) be a contact manifold. We shall need the *Tanno operator*

$$T(g) = \nabla_\xi \mathcal{L}_\xi g - 2(\mathcal{L}_\xi g)\cdot\phi \ , \quad g \in \mathcal{M}(\eta)$$

where ∇ is the Levi-Civita connection of (M, g) and the dot product is given by $S \cdot R = (S_{ik} R^k_j)$ for any tensor fields $S = (S_{ij})$ and $R = (R^i_j)$ on M. Also $\mathcal{L}$ denotes the Lie derivative. When M is compact, S. Tanno considers (cf. [28], p. 356) the Lagrangian

$$L(g) = \|\mathcal{L}_\xi g\|^2 \ , \quad g \in \mathcal{M}(\eta)$$

and shows that an associated metric g is a critical point of $E(g) = \int_M L(g)\Psi$ if and only if $T(g) = 0$ (*Tanno's equation*), cf. [28], p. 357. Our purpose is to show that Tanno's equation is the Euler-Lagrange system of (the variational principles associated to) a large class of analytic Lagrangians depending only on $\|\mathcal{L}_\xi g\|^2$ (in a nonlinear way). Precisely, we state

THEOREM 1. *Let (M, η) be a $(2n+1)$-dimensional compact contact manifold and $f : \mathbf{R} \to \mathbf{R}$ a real analytic function. Set*

$$E_f(g) = \int_M L(g)\, \eta \wedge (d\eta)^n \ , \quad L(g) = f(\|\mathcal{L}_\xi g\|^2) \ , \quad g \in \mathcal{M}(\eta) .$$

Let $F(\rho) = f''(\rho)\rho + \frac{1}{2} f'(\rho)$ and set $Z(F) = \{x \in \mathbf{R} : F(x) = 0\}$. Then either 1) $Z(F) = \mathbf{R}$, and then the Lagrangian $L(g)$ is proportional to $\|\mathcal{L}_\xi g\|$, or 2) $Z(F) \subset \mathbf{R}$ (strict inclusion), and then an associated metric $g \in \mathcal{M}(\eta)$ is a critical point of E_f if and only if $\xi(\|\mathcal{L}_\xi g\|^2) = 0$ everywhere on M and $T(g) = 0$ at all points $x \in M$ where $f'(\|\mathcal{L}_\xi g\|^2_x) \neq 0$.

The proof of Theorem 1 will rely on Theorem 2 below. Note that, if $f \in C^\omega(\mathbf{R})$ satisfies $Z(f') = \emptyset$ and $Z(F) \neq \mathbf{R}$, then $g \in \mathcal{M}(\eta)$ is critical for E_f if and only if $T(g) = 0$. This is the sought after 'universality' property of Tanno's equation. The authors are grateful to M. Francaviglia for sharing with them the ideas in [12].

3.1. – The first variation formula

Let (M, η) be a compact contact manifold and $g \in \mathcal{M}(\eta)$ and set

$$\rho = \|\mathcal{L}_\xi g\|^2 .$$

Given a smooth curve $g(t) \in \mathcal{M}(\eta)$ so that $g(0) = g$, let $(\phi(t), \xi, \eta, g(t))$ be the corresponding contact metric structure. Then

$$g_{ij}(t) = g_{ij} + t h_{ij} + O(t^2) \tag{12}$$

yields

$$g^{ij}(t) = g^{ij} - th^{ij} + O(t^2)$$

$$\mathcal{L}_\xi g_{ij}(t) = \mathcal{L}_\xi g_{ij} + t\mathcal{L}_\xi h_{ij} + O(t^2)$$

where $h^{ij} = g^{ik}g^{j\ell}h_{k\ell}$. Consequently

$$\begin{aligned}\rho(t) &= g^{ik}(t)g^{j\ell}(t)\left(\mathcal{L}_\xi g_{ij}(t)\right)\left(\mathcal{L}_\xi g_{k\ell}(t)\right)\\ &= \rho + 2t\langle \mathcal{L}_\xi g, \mathcal{L}_\xi h\rangle - 2tg^{ik}h^{j\ell}\left(\mathcal{L}_\xi g_{ij}\right)\left(\mathcal{L}_\xi g_{k\ell}\right) + O(t^2)\end{aligned}$$

or, by observing

$$g^{ik}h^{j\ell}\left(\mathcal{L}_\xi g_{ij}\right)\left(\mathcal{L}_\xi g_{k\ell}\right) = 0 \tag{13}$$

we may conclude that

$$\rho(t) = \rho + 2t\langle \mathcal{L}_\xi g, \mathcal{L}_\xi h\rangle + O(t^2)\,. \tag{14}$$

Cf. [28], p. 357, for a proof of (13) (as a consequence of (7)-(11)). Next (by (14))

$$\begin{aligned}\frac{d}{dt}\{E_f(g(t))\}_{t=0} &= 2\int_M f'(\rho)\langle \mathcal{L}_\xi g, \mathcal{L}_\xi h\rangle\ \Psi\\ &= 2\int_M f'(\rho)g^{ik}g^{j\ell}(\mathcal{L}_\xi g_{ij})(\mathcal{L}_\xi h_{k\ell})\ \Psi\\ &= 2\int_M f'(\rho)(\nabla^k\xi^\ell + \nabla^\ell\xi^k)(\mathcal{L}_\xi h_{k\ell})\ \Psi\end{aligned}$$

where

$$\mathcal{L}_\xi h_{k\ell} = \xi^i\nabla_i h_{k\ell} + h_{i\ell}\nabla_k\xi^i + h_{ki}\nabla_\ell\xi^i\,.$$

Set

$$\sigma = h_{ij}(\nabla^i\xi^j + \nabla^j\xi^i)\,.$$

As (by (8)) $\mathrm{div}(\xi) = 0$ we have

$$\mathrm{div}(\sigma f'(\rho)\xi) = \sigma f''(\rho)\xi(\rho) + f'(\rho)\xi^k\nabla_k\sigma$$

hence, by Green's lemma

$$\int_M f'(\rho)\xi^k(\nabla_k\sigma)\ \Psi = -\int_M \sigma f''(\rho)\xi(\rho)\ \Psi\,. \tag{15}$$

Using the identity

$$\nabla_k\sigma = (\nabla_k h_{ij})(\nabla^i\xi^j + \nabla^j\xi^i) + h_{ij}\nabla_k(\nabla^i\xi^j + \nabla^j\xi^i)$$

we may rewrite (15) as

$$\int_M f'(\rho)\xi^k(\nabla_k h_{ij})(\nabla^i\xi^j + \nabla^j\xi^i)\ \Psi$$
$$= -\int_M \left[\sigma f''(\rho)\xi(\rho) + f'(\rho)h_{ij}\xi^k\nabla_k(\nabla^i\xi^j + \nabla^j\xi^i)\right]\ \Psi .$$

Consequently

$$\frac{d}{dt}\{E_f(g(t))\}_{t=0} = 2\int_M h_{ij}\left\{-f''(\rho)\xi(\rho)(\nabla^i\xi^j + \nabla^j\xi^i)\right. \tag{16}$$
$$\left. - f'(\rho)\left[\xi^k\nabla_k(\nabla^i\xi^j + \nabla^j\xi^i) - 2(\nabla_k\xi^i)(\nabla^k\xi^j + \nabla^j\xi^k)\right]\right\}\ \Psi .$$

Finally, using the identities

$$h_{ij}(\nabla^i\xi^j + \nabla^j\xi^i) = \langle h, \mathcal{L}_\xi g\rangle$$
$$h_{ij}\xi^k\nabla_k(\nabla^i\xi^j + \nabla^j\xi^i) = \langle h, \nabla_\xi\mathcal{L}_\xi g\rangle$$
$$h_{ij}(\nabla_k\xi^i)(\nabla^k\xi^j + \nabla^j\xi^k) = \langle h, (\mathcal{L}_\xi g)\cdot\phi\rangle$$

we may write (16) as

$$\frac{d}{dt}\{E_f(g(t))\}_{t=0} = 2\int_M \langle h, -f''(\rho)\xi(\rho)\mathcal{L}_\xi g - f'(\rho)T(g)\rangle\ \Psi \tag{17}$$

where T is the Tanno operator. This suggests the following

THEOREM 2. *Let (M, η) be a compact contact manifold. An associated metric $g \in \mathcal{M}(\eta)$ is a critical point of $E_f(g) = \int_M f(\rho)\ \Psi$ if and only if*

$$f''(\rho)\xi(\rho)\mathcal{L}_\xi g + f'(\rho)T(g) = 0 . \tag{18}$$

We shall need

LEMMA 1 (cf. [28], p. 356). *Let $g(t)$ be a smooth curve in $\mathcal{M}(\eta)$ so that $g(0) = g$. Then $h = (h_{ij})$ given by* (12) *satisfies*

$$h^+\xi = 0\ ,\quad h^+\phi = -\phi\, h^+ \tag{19}$$

where $h^+ = \left(h^i_j\right)$. Viceversa, let h be a symmetric $(0,2)$*-tensor field satisfying* (19). *Then*

$$g(t) = g\cdot\exp(th^+)\ ,\quad |t| < \epsilon$$

is a smooth curve in $\mathcal{M}(\eta)$ with $g(0) = g$.

At this point, we may prove Theorem 2. To this end, we set

$$h = -f''(\rho)\xi(\rho)\mathcal{L}_\xi g - f'(\rho)T(g)$$

and show that h obeys to (19) in Lemma 1. Already $T(g)$ is symmetric and $T(g)^+\xi = 0$, cf. [28], p. 358. Thus, to check that $h^+\xi = 0$ we need only to see that (by (7))

$$g^{ik}\xi^j\mathcal{L}_\xi g_{jk} = g^{ik}\xi^j(\nabla_j\eta_k + \nabla_k\eta_j) = g^{ik}\xi^j\nabla_j\eta_k = \xi^j\nabla_j\xi^i = (\nabla_\xi\xi)^i = 0\,.$$

Moreover, as already $T(g)^+\phi = -\phi\, T(g)^+$ (cf. [28], p. 358), to check that $h^+\phi = -\phi\, h^+$ we need only to see that (by (10))

$$\begin{aligned} g^{i\ell}\phi^k_j\mathcal{L}_\xi g_{k\ell} + g^{k\ell}\phi^i_k\mathcal{L}_\xi g_{\ell j} &= g^{i\ell}\phi^k_j(\nabla_k\eta_\ell + \nabla_\ell\eta_k) + g^{k\ell}\phi^i_k(\nabla_\ell\eta_j + \nabla_j\eta_\ell) \\ &= g^{i\ell}\phi^k_\ell(\nabla_k\eta_j + \nabla_j\eta_k) + g^{k\ell}\phi^i_k(\nabla_\ell\eta_j + \nabla_j\eta_\ell) = 0 \end{aligned}$$

because of

$$g^{i\ell}\phi^k_\ell = -g^{\ell k}\phi^i_\ell\,.$$

By Lemma 1 one may conduct the above computation (leading to the first variation formula (17)) with the variation $g(t) = g\cdot\exp(th^+)$ and show that g is critical if and only if $\int_M \|h\|^2\,\Psi = 0$. □

3.2. – Proof of Theorem 1

If $F(\rho) = 0$ for all $\rho > 0$, then $f(\rho) = a\sqrt{\rho} + b$, $a, b \in \mathbf{R}$. To prove the second statement in Theorem 1, let $g \in \mathcal{M}(\eta)$. Sufficiency follows from Theorem 2. To establish necessity, assume g to be critical. If $\xi(\rho) = 0$ then (18) yields $T(g) = 0$ on $\{x \in M : f'(\rho_x) \neq 0\}$. Let us show that the alternative $\xi(\rho) \neq 0$ actually does not occur. The proof is by contradiction. Assume that $\xi(\rho) \neq 0$ and let $U = \{x \in M : \xi(\rho)_x \neq 0\}$ (an open set). We have

(20) $$\xi(\rho) = 2\langle\nabla_\xi\mathcal{L}_\xi g, \mathcal{L}_\xi g\rangle\,.$$

Indeed

$$\begin{aligned} \langle\nabla_\xi\mathcal{L}_\xi g, \mathcal{L}_\xi g\rangle &= (\nabla_\xi\mathcal{L}_\xi g_{ij})(\mathcal{L}_\xi g_{k\ell})g^{ik}g^{j\ell} \\ &= \xi^s\nabla_s(\mathcal{L}_\xi g_{ij}\cdot\mathcal{L}_\xi g_{k\ell}\cdot g^{ik}g^{j\ell}) - \xi^s\mathcal{L}_\xi g_{ij}\cdot(\nabla_s\mathcal{L}_\xi g_{k\ell})g^{ik}g^{j\ell} \\ &= \xi^s\nabla_s\rho - \langle\nabla_\xi\mathcal{L}_\xi g, \mathcal{L}_\xi\rangle \end{aligned}$$

and (20) is proved. Next, note that $(\mathcal{L}_\xi g)_x \neq 0$ for any $x \in U$. Indeed $Z(\mathcal{L}_\xi g) \subseteq Z(\xi(\rho))$ (by (20)) hence $M \setminus Z(\mathcal{L}_\xi g) \supseteq U$.

As g is critical, the identity (19) holds on M, and therefore on U. Take the inner product with $\mathcal{L}_\xi g$ and use (20) as well as

$$\langle T\cdot\phi, T\rangle = 0$$

for any $(0,2)$-tensor field $T = (T_{ij})$ on M. We obtain

$$f''(\rho)\rho + \frac{1}{2}f'(\rho) = 0$$

everywhere on U, i.e. $\rho(U) \subseteq Z(F)$ (in particular $Z(F) \neq \emptyset$). As $F(\rho) = f''(\rho)\rho + \frac{1}{2}f'(\rho)$ is real analytic and $Z(F) \neq \mathbf{R}$ it follows that the set $Z(F)$ is at most countable. Consequently ρ is constant on each connected component A of U. In particular $\xi(\rho) = 0$ on A, a contradiction. □

4. – The sublaplacian of a contact Riemannian manifold

Let (M, η) be a contact manifold and $g \in \mathcal{M}(\eta)$ an associated metric. S. Tanno has considered a second order differential operator Δ_H given by (cf. [28], p. 363)

$$\Delta_H f = \Delta f - \xi(\xi f) = (g^{ij} - \xi^i \xi^j)\nabla_i \nabla_j f \,, \quad f \in C^\infty(M)$$

where Δ is the Laplace-Beltrami operator of (M, g). Then Δ_H is referred to as the *sublaplacian* of M. The purpose of the present section is to show that Δ_H is subelliptic of order $1/2$. We recall that a formally selfadjoint second order differential operator $\mathcal{L} : C^\infty(M) \to C^\infty(M)$ on (M, g) is *subelliptic of order* ϵ $(0 < \epsilon < 1)$ at a point $x \in M$ if there exist a neighborhood U of x and a constant $C > 0$ so that

$$\|f\|_\epsilon^2 \leq C\{|(\mathcal{L}f, f)| + \|f\|^2\}$$

for any $f \in C_0^\infty(U)$. Here $\|\cdot\|$ is the L^2 norm and $\|\cdot\|_\epsilon$ is the usual Sobolev norm of order ϵ (cf. e.g. [18], p. 46).

Let $\{X_j : 1 \leq j \leq 2n\} = \{X_\alpha, \phi X_\alpha : 1 \leq \alpha \leq n\}$ be a local orthonormal frame of $H(M) = \mathrm{Ker}(\eta)$ (where $X_{\alpha+n} = \phi X_\alpha$). Then $\{X_A : 0 \leq A \leq 2n\} = \{X_j, \xi\}$ is a (local) orthonormal frame of $T(M)$ (here $X_0 = \xi$) and

$$\Delta f = \sum_{A=0}^{2n} \{X_A(X_A f) - (\nabla_{X_A} X_A) f\} = \xi^2(f) + \sum_{j=1}^{2n} \{X_j^2(f) - (\nabla_{X_j} X_j) f\}$$

by (7). Hence

$$\Delta_H f = \text{trace}\ \{\pi_H \nabla^2 f\}\,. \tag{21}$$

As to the notation in (21), if B is a bilinear form on $T(M)$ then $\pi_H B$ denotes the restriction of B to $H(M) \otimes H(M)$ and

$$\text{trace}\ \{\pi_H B\} = \sum_{j=1}^{2n} B(X_j, X_j)\,.$$

Also $\nabla^2 f$ is the Hessian of f with respect to ∇, i.e.

$$(\nabla^2 f)(X, Y) = (\nabla_X df) Y = X(Y(f)) - (\nabla_X Y) f\,.$$

S. Tanno has considered (cf. (2.1) in [28], p. 353) a $(1, 2)$-tensor field Q given by

$$Q^i_{jk} = \nabla_k \phi^i_j + \xi^i \phi^r_j \nabla_k \eta_r + \phi^i_r \eta_j \nabla_k \xi^r\,.$$

Extend ϕ by $\mathbf{C}$-linearity to $H(M) \otimes \mathbf{C}$ and set $T_{1,0}(M) = Eigen(i)$ (the eigenbundle of ϕ corresponding to the eigenvalue $i = \sqrt{-1}$). Then $T_{1,0}(M)$ is an almost CR structure and (by Prop. 2.1 in [28], p. 353) $(M, T_{1,0}(M))$ is a CR manifold if and only if $Q = 0$. On the other hand, set

$$N_\phi = [\phi, \phi] + 2(d\eta) \otimes \xi$$
$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi\{[\phi X, Y] + [X, \phi Y]\}$$

and recall that an almost contact structure (ϕ, ξ, η) is *normal* (cf. [6], p. 49) if $N_\phi = 0$. If (ϕ, ξ, η) is normal then (by a result in [15]) $(M, T_{1,0}(M))$ is a CR manifold. Therefore Q and N_ϕ are related. Indeed Q may be written as

$$Q(X, Y) = (\nabla_Y \phi)X + [(\nabla_Y \eta)\phi X]\,\xi + \eta(X)\phi(\nabla_Y \xi)$$

or (by $\eta \circ \phi = 0$, $\phi\xi = 0$) in terms of the covariant derivative of ϕ

$$Q(X, Y) = \pi_{H(M)}(\nabla_Y \phi)X - \eta(X)(\nabla_Y \phi)\xi \tag{22}$$

where $\pi_{H(M)} : T(M) \to H(M)$ is the natural projection (associated with the direct sum decomposition $T(M) = H(M) \oplus \mathbf{R}\xi$). Moreover (by (7) in [6], p. 54)

$$2g((\nabla_X \phi)Y, Z) = g(N_\phi(Y, Z), \phi X) + 2\Omega(\phi Y, X)\eta(Z) - 2\Omega(\phi Z, X)\eta(Y)$$

hence (22) may be written as

$$2g(Q(X, Y), Z) = g(N_\phi(X, Z) - \eta(X)N_\phi(\xi, Z) - \eta(Z)N_\phi(X, \xi), \phi Y) \tag{23}$$

for any $X, Y, Z \in \mathcal{X}(M)$. Clearly normality yields $Q = 0$ hence the Ianus theorem (cf. [15], or [6], p. 62) is a corollary of Prop. 2.1 in [28], p. 353. However, the converse is not true as normality is known (cf. [6], p. 51) to be equivalent to $\mathcal{L}_\xi g = 0$ while there are several examples of strictly pseudoconvex CR manifolds of nonzero pseudohermitian torsion (cf. e.g. [7], p. 41).

As recalled in the Introduction, for any nondegenerate CR manifold M on which a pseudohermitian structure η has been fixed there is a unique linear connection ∇^* (the *Tanaka-Webster connection* of (M, η)) compatible with both the maximal complex structure and the Levi form of M, cf. [27] and [32] (the torsion tensor of ∇^* is always $\neq 0$ and obeys a *purity* condition, cf. [8], p. 173). Also ∇^* is related (cf. (4) in [8], p. 174) to the Levi-Civita connection of the *Webster metric* $g_{-\eta}$ (given by (2.18) in [32], p. 34). This relation may be taken as a definition of ∇^* on a contact Riemannian manifold (on which, in general, $T_{1,0}(M)$ is only an almost CR structure). Indeed, together with S. Tanno we set (cf. (3.1) in [28], p. 354)

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\phi Y - \eta(Y)\nabla_X \xi + [(\nabla_X \eta)Y]\,\xi \tag{24}$$

for any $X, Y \in \mathcal{X}(M)$. Then (by Prop. 3.1 in [28], p. 354) ∇^* is the unique linear connection obeying to i) $\nabla^*\eta = 0$, $\nabla^*\xi = 0$, ii) $\nabla^* g = 0$, iii) $T^*(X, Y) =$

$\Omega(X, Y)\xi$ for any $X, Y \in H(M)$, and $T^*(\xi, \phi Z) = -\phi T^*(\xi, Z)$ for any $Z \in T(M)$, and iv) $(\nabla^*_X \phi)Y = Q(Y, X)$ for any $X, Y \in T(M)$. We refer to ∇^* as the *(generalized) Tanaka-Webster connection* of the contact Riemannian manifold M. By (24) the Hessians of f with respect to ∇ and ∇^* are related by

$$(\nabla^{*2} f)(X, Y) = (\nabla^2 f)(X, Y) - [(\nabla_X \eta)Y]\xi(f).$$

Yet (by (7)-(8))

$$\text{trace } \{\pi_{H(M)} \nabla \eta\} = \text{div}(\xi) = 0$$

hence (21) may be also written as

$$\Delta_H f = \text{trace } \{\pi_H \nabla^{*2} f\}. \tag{25}$$

Set $\xi_\alpha = \frac{1}{2}\{X_\alpha - i\phi X_\alpha)$ and $\xi_{\bar\alpha} = \overline{\xi_\alpha}$. Let Γ^A_{BC} be the coefficients of ∇^* with respect to $\{\xi_A\} = \{\xi_\alpha, \xi_{\bar\alpha}, \xi\}$, i.e.

$$\nabla^*_{\xi_B} \xi_C = \Gamma^A_{BC} \xi_A.$$

Then (25) may be written as

$$\begin{aligned} \Delta_H f &= 2\sum_{\alpha=1}^{n} \{\xi_\alpha \xi_{\bar\alpha} f + \xi_{\bar\alpha} \xi_\alpha f\} \\ &\quad - 2\sum_{\alpha,\beta} \{(\Gamma^\beta_{\alpha\bar\alpha} + \Gamma^\beta_{\bar\alpha\alpha})\xi_\beta f + (\Gamma^{\bar\beta}_{\alpha\bar\alpha} + \Gamma^{\bar\beta}_{\bar\alpha\alpha})\xi_{\bar\beta} f + (\Gamma^0_{\alpha\bar\alpha} + \Gamma^0_{\bar\alpha\alpha})\xi f\}. \end{aligned} \tag{26}$$

As a consequence of $(\nabla^*_X \phi)Y = Q(Y, X)$ one has

$$\phi \nabla^*_{\xi_\alpha} \xi_\beta = i \nabla_{\xi_\alpha} \xi_\beta - Q_{\beta\alpha} \tag{27}$$

$$\phi \nabla^*_{\xi_\alpha} \xi_{\bar\beta} = -i \nabla^*_{\xi_\alpha} \xi_{\bar\beta} - Q_{\bar\beta\alpha} \tag{28}$$

$$\phi \nabla^*_{\xi_\alpha} \xi = -Q_{0\alpha} \tag{29}$$

$$\phi \nabla^*_\xi \xi_\beta = i \nabla^*_\xi \xi_\beta - Q_{\beta 0} \tag{30}$$

where $Q_{BC} = Q(\xi_B, \xi_C) = Q^A_{BC}\xi_A$. Note that (by (23))

$$Q(\xi, Y) = Q(X, \xi) = 0$$

hence $Q_{0B} = 0$, $Q_{A0} = 0$ (with the corresponding simpler form of (29)-(30)). Let $\{\eta^\alpha\}$ be the (local) complex 1-forms determined by

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad \eta^\alpha(\xi_{\bar\beta}) = \eta^\alpha(\xi) = 0.$$

Apply η^γ (respectively $\eta^{\bar\gamma}$, η) to the identities (27)-(30). As

$$\eta^\alpha \circ \phi = i\eta^\alpha , \qquad \eta^{\bar\alpha} \circ \phi = -i\eta^{\bar\alpha} , \qquad \eta \circ \phi = 0$$

we obtain

$$Q^\gamma_{\beta\alpha} = 0 , \ \Gamma^{\bar\gamma}_{\alpha\beta} = -\frac{i}{2} Q^{\bar\gamma}_{\beta\alpha} , \ \Gamma^0_{\alpha\beta} = 0 \tag{31}$$

$$\Gamma^\gamma_{\alpha\bar\beta} = \frac{i}{2} Q^\gamma_{\bar\beta\alpha} , \ Q^{\bar\gamma}_{\bar\beta\alpha} = 0 , \ \Gamma^0_{\alpha\bar\beta} = 0 \tag{32}$$

$$\Gamma^\gamma_{\alpha 0} = 0 , \ \Gamma^{\bar\gamma}_{\alpha 0} = 0 \tag{33}$$

$$\Gamma^{\bar\gamma}_{0\beta} = 0 , \ \Gamma^0_{0\beta} = 0 . \tag{34}$$

Then (26) becomes

$$\begin{aligned} \Delta_H f &= 2 \sum_\alpha \{ \xi_\alpha \xi_{\bar\alpha} f + \xi_{\bar\alpha} \xi_\alpha f \} \\ &\quad - 2 \sum_{\alpha,\gamma} \left\{ \left(\Gamma^\gamma_{\bar\alpha\alpha} + \frac{i}{2} Q^\gamma_{\bar\alpha\alpha} \right) \xi_\gamma f + \left(\Gamma^{\bar\gamma}_{\alpha\bar\alpha} - \frac{i}{2} Q^{\bar\gamma}_{\alpha\bar\alpha} \right) \xi_{\bar\gamma} f \right\} . \end{aligned} \tag{35}$$

We shall establish the following

THEOREM 3. *Let M be a real $(2n+1)$-dimensional C^∞ manifold with a contact metric structure (ϕ, ξ, η, g). Then the sublaplacian Δ_H of M is a subelliptic operator of order $\frac{1}{2}$, at any $x \in M$. Consequently Δ_H is hypoelliptic and satisfies the a priori estimates*

$$\| f \|^2_{s+1} \le C_s \left(\| \Delta_H f \|^2_s + \| f \|^2 \right) , \ f \in C^\infty_0(U) \tag{36}$$

for each $s \ge 0$. In particular Δ_H has a discrete spectrum $0 < \lambda_1 < \lambda_2 < \cdots \to +\infty$.

The proof of Theorem 3 relies on a lemma by E.V. Radkevic, [25] (cf. Lemma 2 below) and is presented in the remainder of this section. Note firstly that $\xi^* = -\xi$ (indeed, as $\mathrm{div}(\xi) = 0$, one has

$$(\xi f, g) = \int_M [\mathrm{div}(f \bar g \xi) - f \xi(\bar g)] \Psi = -(f, \xi g)$$

(by Green's lemma) for any $f, g \in C^\infty(M) \otimes \mathbf{C}$ (at least one of compact support)). Thus Δ_H is selfadjoint. Next, note that

$$\xi^*_\alpha = -\xi_{\bar\alpha} - \left(\Gamma^{\bar\beta}_{\bar\beta\bar\alpha} + \frac{i}{2} Q^\beta_{\bar\alpha\beta} \right) \tag{37}$$

$$\xi^*_{\bar\alpha} = -\xi_\alpha - \left(\Gamma^\beta_{\beta\alpha} - \frac{i}{2} Q^{\bar\beta}_{\alpha\bar\beta} \right) \tag{38}$$

Indeed, as $\nabla^*\Psi = 0$

$$\mathrm{div}(f\bar{g}\xi_\alpha) = \text{trace } \{Z \mapsto \nabla^*_Z(f\bar{g}\xi_\alpha)\} = \xi_\alpha(f\bar{g}) + (\Gamma^\beta_{\beta\alpha} + \Gamma^{\bar{\beta}}_{\bar{\beta}\alpha} + \Gamma^0_{0\alpha}\}f\bar{g}$$

hence (by Green's lemma)

$$(\xi_\alpha f, g) = -\int_M f\left[\xi_\alpha(\bar{g}) + (\Gamma^\beta_{\beta\alpha} + \Gamma^{\bar{\beta}}_{\bar{\beta}\alpha} + \Gamma^0_{0\alpha})\bar{g}\right]\Psi$$

and (by (31)-(34)) the identity (37) is proved. Using (37)-(38) one gets

$$\begin{aligned}(\Delta_H f, f) = &-2\sum_\alpha\{\|\xi_\alpha f\|^2 + \|\xi_{\bar{\alpha}} f\|^2\}\\ &-2\sum_{\alpha,\beta}\left\{\left(\xi_{\bar{\alpha}} f, \left(\Gamma^{\bar{\beta}}_{\bar{\beta}\bar{\alpha}} + \frac{i}{2}Q^\beta_{\bar{\alpha}\beta}\right)f\right) + \left(\xi_\alpha f, \left(\Gamma^\beta_{\beta\alpha} - \frac{i}{2}Q^{\bar{\beta}}_{\alpha\bar{\beta}}\right)f\right)\right.\\ &\left.+\left(\xi_\beta f, \left(\Gamma^{\bar{\beta}}_{\alpha\bar{\alpha}} - \frac{i}{2}Q^{\bar{\beta}}_{\alpha\bar{\alpha}}\right)f\right) + \left(\xi_{\bar{\beta}} f, \left(\Gamma^\beta_{\bar{\alpha}\alpha} + \frac{i}{2}Q^\beta_{\bar{\alpha}\alpha}\right)f\right)\right\}.\end{aligned} \tag{39}$$

At this point, one may observe the cancellation of Christoffel symbols (as $\nabla^* g = 0$ yields $\xi_A(g_{\alpha\bar{\beta}}) = \Gamma^\gamma_{A\alpha}g_{\gamma\bar{\beta}} + \Gamma^{\bar{\gamma}}_{A\bar{\beta}}g_{\alpha\bar{\gamma}}$, i.e. $\Gamma^\beta_{A\alpha} = -\Gamma^{\bar{\alpha}}_{A\bar{\beta}}$). Also, if $N_\phi(\xi_B, \xi_C) = N^A_{BC}\xi_A$ then $Q^{\bar{\gamma}}_{\alpha\bar{\beta}} = -\frac{i}{2}N^\beta_{\alpha\gamma}$ hence

$$Q^{\bar{\beta}}_{\alpha\bar{\beta}} + Q^{\bar{\alpha}}_{\beta\bar{\beta}} = -\frac{i}{2}(N^\beta_{\alpha\beta} + N^\beta_{\beta\alpha}) = 0\,.$$

Finally (39) becomes

$$(\Delta_H f, f) = -2\sum_\alpha\{\|\xi_\alpha f\|^2 + \|\xi_{\bar{\alpha}} f\|^2\}\,. \tag{40}$$

We need to recall the following

LEMMA 2 (E.V. Radkevic, [25]). *Let (M, g) be a Riemannian manifold and $K \subset M$ a compact set. Let $Z_1, \cdots, Z_N$ be complex vector fields on M whose linear span is closed under complex conjugation and so that*

$$\{Z_1, \cdots, Z_N\} \cup \{[Z_j, Z_k] : 1 \le j, k \le N\}$$

spans the tangent space at each $x \in K$. Then there is a constant $C > 0$ so that

$$\|f\|^2_{1/2} \le C\left(\sum_{j=1}^N \|Z_j f\|^2 + \|f\|^2\right)$$

for any $f \in C^\infty_0(K)$.

As $T^*(X, Y) = \Omega(X, Y)\xi$ for any $X, Y \in H(M)$, we have

$$(\Gamma^A_{\alpha\bar\beta} - \Gamma^A_{\bar\beta\alpha})\xi_A + \frac{i}{2}\delta_{\alpha\beta}\xi = [\xi_\alpha, \xi_{\bar\beta}]$$

hence (by (32))

$$[\xi_\alpha, \xi_{\bar\beta}] \equiv \frac{i}{2}\delta_{\alpha\beta}\xi \ , \ \bmod \xi_\gamma, \xi_{\bar\gamma}$$

Therefore, the hypothesis of Lemma 2 are satisfied if we take the Z_j's to be $\{\xi_1, \cdots, \xi_n, \xi_{\bar 1}, \cdots, \xi_{\bar n}\}$. Finally, Lemma 2 and (40) lead to

$$\|f\|^2_{1/2} \leq C\{|(\Delta_H f, f)| + \|f\|^2\}$$

i.e. Δ_H is subelliptic of order $\frac{1}{2}$. Then (by a result in [19]) Δ_H is hypoelliptic and satisfies the *a priori* estimates (36). The statement on the spectrum of Δ_H follows from a result in [23].

The problem of finding lower bounds on the first nonzero eigenvalue λ_1 of the sublaplacian Δ_H of a contact Riemannian manifold (with a generally nonintegrable almost CR structure $T_{1,0}(M)$) is open. Cf. [14] and [4] for such lower bounds on a strictly pseudoconvex (integrable) CR manifold. As we shall see in Section 6, Δ_H is related to the (generalized) Fefferman metric G_η (Δ_H is the push forward of the Laplace-Beltrami operator of G_η).

5. – Tangential Cauchy-Riemann equations

5.1. – The Cauchy-Riemann pseudocomplex

Let (M, η) be a contact manifold and $\xi \in \mathcal{X}(M)$ its characteristic direction. A complex valued p-form ω on M is a *$(p, 0)$-form* if $T_{0,1}(M) \rfloor \omega = 0$. Let $\Lambda^{p,0}(M)$ denote the bundle of all $(p, 0)$-forms on M. Similarly, a complex valued q-form ω on M is a *$(0, q)$-form* if $T_{1,0}(M) \rfloor \omega = 0$ and $\xi \rfloor \omega = 0$. We denote by $\Lambda^{0,q}(M)$ the bundle of all $(0, q)$-forms on M and set

$$\Lambda^{p,q}(M) = \Lambda^{p,0}(M) \wedge \Lambda^{0,q}(M) .$$

Next, set $\Omega^{p,q}(M) = \Gamma^\infty(\Lambda^{p,q}(M))$. The complex de Rham algebra of M admits the decomposition

$$\Omega(M) = \oplus_{r\geq 0} \oplus_{p+q=r} \Omega^{p,q}(M) .$$

Let $\pi^{p,q} : \Omega^r(M) \to \Omega^{p,q}(M)$ the natural projections, $p + q = r$, and set

$$\partial_H\omega = \pi^{p+1,q} d\,\omega , \qquad \overline\partial_H\omega = \pi^{p,q+1} d\,\omega$$

$$\mathcal{N}\omega = \pi^{p+2,q-1} d\,\omega , \qquad \overline{\mathcal{N}}\omega = \pi^{p-1,q+2} d\,\omega$$

for any $\omega \in \Omega^{p,q}(M)$. Then

$$d\omega = \partial_H \omega + \overline{\partial}_H \omega + \mathcal{N}\omega + \overline{\mathcal{N}}\omega \, , \quad \omega \in \Omega^{p,q}(M)$$

and $d^2 = 0$ gives

$$\begin{gathered} \partial_H^2 + \mathcal{N}\overline{\partial}_H + \overline{\partial}_H \mathcal{N} = 0 \\ \overline{\partial}_H^2 + \partial_H \overline{\mathcal{N}} + \overline{\mathcal{N}}\partial_H = 0 \\ \mathcal{N}^2 = \overline{\mathcal{N}}^2 = 0 \\ \partial_H \overline{\partial}_H + \overline{\partial}_H \partial_H + \mathcal{N}\overline{\mathcal{N}} + \overline{\mathcal{N}}\mathcal{N} = 0 \\ \partial_H \mathcal{N} + \mathcal{N}\partial_H = \overline{\partial}_H \overline{\mathcal{N}} + \overline{\mathcal{N}}\,\overline{\partial}_H = 0 \, . \end{gathered}$$

In particular, if $\omega \in \Omega^{0,q}(M)$ then $\overline{\partial}_H \omega$ is the unique $(0, q+1)$-form coinciding with $d\omega$ on $T_{0,1}(M) \otimes \cdots \otimes T_{0,1}(M)$ ($q+1$ terms). On functions $f \in \Omega^{0,0}(M) = C^\infty(M) \otimes \mathbf{C}$ one has $(\overline{\partial}_H f)\overline{Z} = \overline{Z}(f)$, $Z \in S$. We refer to

$$\overline{\partial}_H \omega = 0$$

as the *tangential Cauchy-Riemann equations.* A C^∞ function f satisfying the tangential Cauchy-Riemann equations is a *CR function.* Let $CR(M)$ be the set of all CR functions on M.

PROPOSITION 1. *Let M be a contact Riemannian manifold and $f : M \to \mathbf{C}$ a C^∞ function. Then $f \in CR(M)$ if and only if $(X + i\phi X)f = 0$ for any $X \in H(M)$ and, if this is the case, $N_\phi(X, Y)f = 0$ for any $X, Y \in H(M)$. Moreover, if $f \in CR(M)$ and 0 is not a critical value of f then $f^{-1}(0)$ is a ϕ-invariant submanifold of M if and only if $\xi(f) = 0$.*

The proof is elementary. It shows however that, as well as in almost complex geometry, unless N_ϕ is sufficiently degenerated, (M, η) may not have even local nonconstant CR functions.

We refer to

$$C^\infty(M) \otimes \mathbf{C} \xrightarrow{\overline{\partial}_H} \Omega^{0,1}(M) \xrightarrow{\overline{\partial}_H} \cdots \xrightarrow{\overline{\partial}_H} \Omega^{0,n}(M) \to 0 \tag{41}$$

as the *tangential Cauchy-Riemann pseudocomplex.*

PROPOSITION 2. *$\overline{\partial}_H^2 = 0$ if and only if $T_{1,0}(M)$ is integrable, hence in general (41) is only a pseudocomplex.*

Indeed, let $\omega \in \Omega^{0,q}(M)$. Then

$$\begin{aligned} &(\overline{\partial}_H^2 \omega)(\xi_{\bar\alpha_0}, \cdots, \xi_{\bar\alpha_{q+1}}) \\ &= \frac{i}{2} \sum_{j<k} (-1)^{j+k+1} (Q^\gamma_{\bar\alpha_k \bar\alpha_j} - Q^\gamma_{\bar\alpha_j \bar\alpha_k})(d\omega)(\xi_\gamma, \xi_{\bar\alpha_0}, \cdots, \hat{\xi}_{\bar\alpha_j}, \cdots, \hat{\xi}_{\bar\alpha_k}, \cdots, \xi_{\bar\alpha_{q+1}}). \end{aligned}$$

Together with

$$Q(\overline{Z}, \overline{W}) - Q(\overline{W}, \overline{Z}) = 2i[\overline{Z}, \overline{W}]_{1,0} \, , \quad Z, W \in \Gamma^\infty(T_{1,0}(M))$$

this shows that $\overline{\partial}_H^2 = 0$ iff $Q = 0$. □

5.2. – The Kohn-Rossi cohomology

Let M be a contact Riemannian manifold and set

$$H^{0,q}_{\overline{\partial}_H}(M) = \frac{\operatorname{Ker}\{\overline{\partial}_H : \Omega^{0,q}(M) \to \cdot\}}{[\overline{\partial}_H \Omega^{0,q-1}(M)] \cap \operatorname{Ker}\{\overline{\partial}_H : \Omega^{0,q}(M) \to \cdot\}} .$$

It is the ordinary Kohn-Rossi cohomology (cf. [20]) of $(M, T_{1,0}(M))$, when $T_{1,0}(M)$ is integrable. Let us consider the complex

$$\cdots \to \Omega^{0,q}(M) \times \Omega^{0,q+1}(M) \xrightarrow{\overline{\delta}_H} \Omega^{0,q+1}(M) \times \Omega^{0,q+2}(M) \to \cdots \tag{42}$$

with the coboundary operator

$$\overline{\delta}_H(\lambda, \mu) = (\overline{\partial}_H \lambda - \mu, \overline{\partial}_H^2 \lambda - \overline{\partial}_H \mu) .$$

By the general theory of subcomplexes (cf. e.g. [31], p. 356) the complex (42) is acyclic and the cohomology of any canonically defined subcomplex of (42) is usually referred to as a *twisted cohomology* of (41). Set

$$A^{p,q}(M) = \overline{\mathcal{N}}(\Omega^{p+1,q-2}(M)) \subseteq \Omega^{p,q}(M)$$
$$\mathcal{D}^q(M) = \Omega^{0,q}(M) \times A^{0,q+1}(M) .$$

Then

$$\overline{\delta}_H(\lambda, \overline{\mathcal{N}}\mu) = (\overline{\partial}_H \lambda - \overline{\mathcal{N}}\mu, \overline{\mathcal{N}}(\overline{\partial}_H \mu - \partial_H \lambda))$$

hence $(\mathcal{D}^*(M), \overline{\delta}_H)$ is a subcomplex of $(\Omega^{0,*}(M) \times \Omega^{0,*+1}(M), \overline{\delta}_H)$. We may state

THEOREM 4. *Let M be a contact Riemannian manifold. There is an isomorphism of* $\Omega^{0,0}(M)$*-modules*

$$H^{0,q}_{\overline{\partial}_H}(M) \approx H^q(\mathcal{D}^*(M), \overline{\delta}_H) , \quad q \geq 0 .$$

6. – The Fefferman metric of a contact Riemannian manifold

6.1. – The canonical bundle

Let (M, η) be a contact manifold and $\xi \in \mathcal{X}(M)$ its characteristic direction. Set $K(M) = \Lambda^{n+1,0}(M)$ (the *canonical bundle* of (M, η)). Let $\pi_0 : K(M) \to M$ be the projection. Set $K^0(M) = K(M) \setminus \{zero\ section\}$. There is a natural action of $\mathbf{R}_+ = (0, +\infty)$ on $K^0(M)$. Let $F(M) = K^0(M)/\mathbf{R}_+$ be the quotient space. Then $F(M)$ is a principal S^1-bundle over M. Indeed, for any $x \in M$

let $\{\xi^\alpha\}$ be a local frame of the almost CR structure $T_{1,0}(M)$ defined on an open neighborhood U of x and $\{\eta^\alpha\}$ the corresponding admissible frame. Let $\pi : F(M) \to M$ be the projection. Then

$$\pi^{-1}(U) \to U \times S^1 \, , \; [\omega] \mapsto \left(x, \frac{\lambda}{|\lambda|}\right) ,$$
$$\omega = \lambda(\eta \wedge \eta^1 \wedge \cdots \wedge \eta^n)_x \in \pi^{-1}(U), \; x \in U, \; \lambda \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$$

is a local trivialization chart of $F(M)$. We refer to $S^1 \to F(M) \to M$ as the *Fefferman bundle* of M. We shall need the *tautologous form*

$$\Xi \in \Gamma^\infty(\Lambda^{n+1} T^*(K(M)) \otimes \mathbf{C})$$

given by

$$\Xi_\omega(Z_1, \cdots, Z_{n+1}) = \omega((d_\omega \pi_0) Z_1, \cdots, (d_\omega \pi_0) Z_{n+1})$$
$$Z_1, \cdots, Z_{n+1} \in T_\omega(K(M)), \; \omega \in K(M) .$$

We establish

LEMMA 3. *For any* $\omega \in \Gamma^\infty(K^0(M))$ *there is a unique* C^∞ *function* $\lambda : M \to (0, +\infty)$ *so that*

$$2^n i^{n(n+2)} n! \eta \wedge (\xi \rfloor \omega) \wedge (\xi \rfloor \bar{\omega}) = \lambda \Psi . \tag{43}$$

Before proving Lemma 3, we need some local calculations (with respect to an admissible coframe). Let M be a real $(2n+1)$-dimensional C^∞ manifold and (ϕ, ξ, η, g) a contact metric structure on M. Let $\{X_\alpha, \phi X_\alpha\}$ be a local frame of $H(M) = \mathrm{Ker}(\eta)$ and set $\xi_\alpha = \frac{1}{2}(X_\alpha - i\phi X_\alpha)$ and $\xi_{\bar{\alpha}} = \overline{\xi_\alpha}$. The *admissible coframe* associated with $\{\xi_\alpha\}$ is the local frame $\{\eta^\alpha\}$ of $T_{1,0}(M)^*$ determined by

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta \, , \; \eta^\alpha(\xi_{\bar{\beta}}) = \eta^\alpha(\xi) = 0 .$$

Note that, with respect to an admissible coframe

$$d\eta = -2i g_{\alpha\bar{\beta}} \eta^\alpha \wedge \eta^{\bar{\beta}} \tag{44}$$

as a consequence of the contact condition $d\eta = \Omega$. Let ∇^* be the (generalized) Tanaka-Webster connection of $(M, (\phi, \xi, \eta, g))$ and T^* its torsion tensor field. Next, set $\tau X = -T^*(\xi, X)$, $X \in T(M)$ (the *pseudohermitian torsion*). As $\phi\tau + \tau\phi = 0$ (cf. (iii-2) in Prop. 3.1, [28], p. 354) it follows that $\tau(T_{1,0}(M)) \subseteq T_{0,1}(M)$. For local calculations, set $\tau\xi_\alpha = A^{\bar{\beta}}_\alpha \xi_{\bar{\beta}}$ and $\tau^\alpha = A^\alpha_{\bar{\beta}} \eta^{\bar{\beta}}$, where $A^\alpha_{\bar{\beta}} = \overline{A^{\bar{\alpha}}_\beta}$. Define local 1-forms ω^A_B by setting

$$\nabla^* \xi_C = \omega^A_C \otimes \xi_A$$
$$\omega^A_C = \Gamma^A_{BC} \eta^B .$$

Then

$$d\eta^\alpha = \eta^\beta \wedge \omega_\beta^\alpha + \eta \wedge \tau^\alpha + \frac{i}{2}\eta^{\bar\beta} \wedge \eta^\alpha(Q(\xi_{\bar\beta}, \cdot)) . \tag{45}$$

Indeed, we may look for $d\eta^\alpha$ in the form

$$d\eta^\alpha = B^\alpha_{\beta\gamma}\eta^\beta \wedge \eta^\gamma + B^\alpha_{\beta\bar\gamma}\eta^\beta \wedge \eta^{\bar\gamma} + B^\alpha_{\bar\beta\bar\gamma}\eta^{\bar\beta} \wedge \eta^{\bar\gamma} + (B^\alpha_{\beta 0}\eta^\beta + B^\alpha_{\bar\beta 0}\eta^{\bar\beta}) \wedge \eta .$$

Applying this identity to the pairs (ξ_β, ξ_γ), $(\xi_\beta, \xi_{\bar\gamma})$, $(\xi_{\bar\beta}, \xi_{\bar\gamma})$, (ξ_β, ξ) and $(\xi_{\bar\beta}, \xi)$, respectively, leads to (by (iii-1) in Prop. 3.1, [28], p. 354, and by our (31)-(34))

$$\begin{aligned}
B^\alpha_{\beta\gamma} &= B^\alpha_{\gamma\beta} - \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\beta} \\
B^\alpha_{\beta\bar\gamma} &= \Gamma^\alpha_{\bar\gamma\beta} - \frac{i}{2}Q^\alpha_{\bar\gamma\beta} \\
B^\alpha_{\bar\beta\bar\gamma} &= B^\alpha_{\bar\gamma\bar\beta} + \frac{i}{2}(Q^\alpha_{\bar\beta\bar\gamma} - Q^\alpha_{\bar\gamma\bar\beta}) \\
B^\alpha_{\beta 0} &= \Gamma^\alpha_{0\beta} \\
B^\alpha_{\bar\beta 0} &= -A^\alpha_{\bar\beta} .
\end{aligned}$$

Finally

$$d\eta^\alpha = \eta^\beta \wedge (\Gamma^\alpha_{\gamma\beta}\eta^\gamma + \Gamma^\alpha_{\bar\gamma\beta}\eta^{\bar\gamma} + \Gamma^\alpha_{0\beta}\eta) + \eta \wedge (A^\alpha_{\bar\beta}\eta^{\bar\beta}) - \frac{i}{2}(Q^\alpha_{\bar\gamma\beta}\eta^\beta + Q^\alpha_{\bar\gamma\bar\beta}\eta^{\bar\beta}) \wedge \eta^{\bar\gamma}$$

and (45) is proved.

Let us go back to the proof of Lemma 3. By (44) the volume form $\Psi = \eta \wedge (d\eta)^n$ may be expressed as

$$\Psi = 2^n i^{n(n+2)} n! \det(g_{\alpha\bar\beta})\eta \wedge \eta^1 \wedge \cdots \wedge \eta^n \wedge \eta^{\bar 1} \wedge \cdots \wedge \eta^{\bar n} . \tag{46}$$

Any $\omega \in \Gamma^\infty(K^0(M))$ may be expressed (locally) as $\omega = f\,\eta \wedge \eta^1 \wedge \cdots \wedge \eta^n$ for some C^∞ function $f : U \to \mathbf{C}^*$. Then

$$\xi \rfloor \omega = \frac{f}{n+1}\eta^1 \wedge \cdots \wedge \eta^n .$$

Set

$$\lambda_U = \frac{1}{(n+1)^2}\frac{|f|^2}{\det(g_{\alpha\bar\beta})} > 0 .$$

Then, on one hand (by (46))

$$2^n i^{n(n+2)} n!\, \eta \wedge (\xi \rfloor \omega) \wedge (\xi \rfloor \bar\omega) = \lambda_U\, \eta \wedge (d\eta)^n .$$

On the other hand, given another (local) frame $\{\xi'^\alpha\}$ of $T_{1,0}(M)$, defined on some open set U' so that $U \cap U' \neq \emptyset$, one has $\lambda_U|_{U\cap U'} = \lambda_{U'}|_{U\cap U'}$ hence the (local) functions λ_U glue up to a (globally defined) C^∞ function $\lambda : M \to (0, +\infty)$ satisfying (43). Lemma 3 is completely proved.

There is a natural embedding $i_\eta : F(M) \to K(M)$. Indeed, let $[\omega] \in F(M)$ with $\pi_0(\omega) = x$. By Lemma 3 there is a unique $\lambda \in (0, +\infty)$ so that

$$2^n i^{n(n+2)} n!\, \eta_x \wedge (\xi_x \rfloor \omega) \wedge (\xi_x \rfloor \bar{\omega}) = \lambda\, \Psi_x .$$

Then we set

$$i_\eta([\omega]) = \frac{1}{\sqrt{\lambda}}\omega .$$

If ω' is another representative of $[\omega]$ then $\omega' = a\omega$ for some $a \in (0, +\infty)$, hence $\lambda' = a^2\lambda$ (so that $i_\eta([\omega])$ is well defined).

Using the embedding i_η we may define the form

$$\mathcal{Z} \in \Gamma^\infty(\Lambda^{n+1} T^*(F(M)) \otimes \mathbf{C})$$

as the pullback of the tautologous $(n+1)$-form Ξ on $K(M)$, i.e. we set

$$\mathcal{Z} = \frac{1}{n+1} i_\eta^*\, \Xi .$$

Let $\{\eta^\alpha\}$ be a frame of $T_{1,0}(M)^*$ on U. Define the (local) form $\Xi_0 \in \Gamma^\infty(U, K(M))$ by

$$\Xi_0 = \det(g_{\alpha\bar{\beta}})^{1/2} \eta \wedge \eta^1 \wedge \cdots \wedge \eta^n .$$

Also, consider

$$\mathcal{Z}_0 \in \Gamma^\infty(U, \Lambda^{n+1} T^*(F(M)) \otimes \mathbf{C})$$

given by

$$\mathcal{Z}_0 = \pi^*\, \Xi_0 .$$

If $c \in F(M)$ is fixed, define the C^∞ curve $a_c : \mathbf{R} \to F(M)$ by $a_c(\theta) = e^{i\theta} c$, $\theta \in \mathbf{R}$. Next, let $\Gamma \in \mathcal{X}(F(M))$ be given by

$$\Gamma_c = \frac{da_c}{d\theta}(0) \, , \; c \in F(M) .$$

Then $\Gamma \in \operatorname{Ker}(d\pi)$. Let $\{\eta^\alpha\}$ be a frame for $T_{1,0}(M)^*$ on U, as above. Define $\gamma : \pi^{-1}(U) \to \mathbf{R}$ by setting

$$\gamma([\omega]) = \arg\left(\frac{f}{|f|}\right)$$

$$\omega = f(\eta \wedge \eta^1 \wedge \cdots \wedge \eta^n \wedge \eta^{\bar{1}} \wedge \cdots \wedge \eta^{\bar{n}})_x \, , \; \pi_0(\omega) = x \, , \; f \in \mathbf{C}^* .$$

Here $\arg : S^1 \to [0, 2\pi)$. We shall need the following

LEMMA 4.
1) $\mathcal{Z} = e^{i\gamma}\mathcal{Z}_0$
2) $(d\gamma)\Gamma = 1$.

PROOF. Let $\omega = f(\eta \wedge \eta^1 \wedge \cdots \wedge \eta^n \wedge \eta^{\bar{1}} \wedge \cdots \wedge \eta^{\bar{n}})_x \in K^0(M)_x$. As $\pi_0 \circ i_\eta = \pi$ one may perform the calculation

$$\begin{aligned}\mathcal{Z}_{[\omega]} &= \frac{1}{n+1}(i_\eta^* \Xi)_{[\omega]} = \frac{1}{n+1}\Xi_{i_\eta([\omega])}(d_{[\omega]}i_\eta)\\ &= \frac{1}{n+1} i_\eta([\omega]) \circ (d_{i_\eta([\omega])}\pi_0) \circ (d_{[\omega]}i_\eta) = \frac{f}{|f|}\Xi_{0,x}(d_{[\omega]}\pi)\\ &= \exp(i \arg(f/|f|))\ (\pi^*\Xi_0)_{[\omega]} = \exp(i\gamma([\omega]))\mathcal{Z}_{0,[\omega]}\end{aligned}$$

and 1) is proved. To prove 2) one notes that $\exp(i\gamma(a_{[\omega]}(\theta))) = \exp(i(\theta + \gamma([\omega])))$ hence differentiation with respect to θ yields $\frac{d}{d\theta}(\gamma \circ a_{[\omega]}) = 1$. □

6.2. – The Fefferman metric

We shall need the following

LEMMA 5. *There is a unique complex n-form ρ on $F(M)$ so that*

$$V \rfloor \rho = 0\,, \quad \mathcal{Z} = (\pi^*\eta) \wedge \rho$$

for any lift V of ξ to $F(M)$, i.e. $\pi_* V = \xi$.

PROOF. Let V be a lift of ξ to $F(M)$ and set $\rho = (n+1)V \rfloor \mathcal{Z}$. The definition of ρ doesn't depend upon the choice of lift. Indeed, if V' is another vector field on $F(M)$ with $\pi_* V' = \xi$ then $V' - V \in \mathrm{Ker}(\pi_*)$, i.e. $V' = V + f\Gamma$ for some $f \in C^\infty(F(M))$. On the other hand (by Lemma 4)

$$\begin{aligned}\Gamma \rfloor \mathcal{Z} &= \Gamma \rfloor (e^{i\gamma}\mathcal{Z}_0)\\ &= e^{i\gamma}\Gamma \rfloor (\pi^*\mathcal{Z}_0) = e^{i\gamma}\pi^*((\pi_*\Gamma) \rfloor \mathcal{Z}_0) = 0\,.\end{aligned}$$

Clearly $V \rfloor \rho = 0$ (as $\mathcal{Z}$ is skew). To establish the second requirement, we conduct the calculation

$$\begin{aligned}(\pi^*\eta) \wedge \rho &= (n+1)(\pi^*\eta) \wedge (V \rfloor \mathcal{Z}) = (n+1)e^{i\gamma}(\pi^*\eta) \wedge (V \rfloor \mathcal{Z}_0)\\ &= (n+1)e^{i\gamma}(\pi^*\eta) \wedge (V \rfloor \pi^*\Xi_0) = (n+1)e^{i\gamma}\pi^*(\eta \wedge (\xi \rfloor \Xi_0))\,.\end{aligned}$$

On the other hand

$$\xi \rfloor \Xi_0 = \frac{1}{n+1}\det(g_{\alpha\bar{\beta}})^{1/2}\eta^1 \wedge \cdots \wedge \eta^n$$

and the proof is complete.

For any differential 2-form ω on M there is a natural concept of trace, defined as follows. Let $\tilde{\omega} : T_{1,0}(M) \to T_{1,0}(M)$ be the bundle endomorphism naturally induced by the $(1,1)$-component of ω, i.e.

$$g(\tilde{\omega}Z, \overline{W}) = i\,\omega(Z, \overline{W})$$

for any $Z, W \in T_{1,0}(M)$. Locally, with respect to some (local) frame $\{\xi_\alpha\}$ of $T_{1,0}(M)$, we may write

$$\omega \equiv i\,\omega_{\alpha\bar{\beta}}\eta^\alpha \wedge \eta^{\bar{\beta}}\ , \ \bmod \eta^\alpha \wedge \eta^\beta\,,\ \eta^\alpha \wedge \eta\,,\ \eta^{\bar{\alpha}} \wedge \eta\,.$$

Thus $\tilde{\omega}\xi_\alpha = \omega_\alpha^\beta \xi_\beta$, where $\omega_\alpha^\beta = -\frac{1}{2}\omega_{\alpha\bar{\gamma}}g^{\bar{\gamma}\beta}$, and trace (ω) is defined to be trace $(\tilde{\omega}) = \omega_\alpha^\alpha$.

PROPOSITION 3. *There is a unique real* 1*-form* $\sigma \in \Gamma^\infty(T^*(F(M)))$ *so that*

$$dZ = i(n+2)\sigma \wedge Z + e^{i\gamma}\pi^*\left[\det(g_{\alpha\bar{\beta}})^{1/2}\mathcal{W}\right] \tag{47}$$

$$\sigma \wedge d\rho \wedge \bar{\rho} = \text{trace }(d\sigma)\ i\,\sigma \wedge (\pi^*\eta) \wedge \rho \wedge \bar{\rho} \tag{48}$$

where $\mathcal{W}$ *is the complex* $(n+2)$*-form on* M *given by*

$$\mathcal{W} = \frac{i}{2}\eta \wedge \sum_{\alpha=1}^{n}(-1)^\alpha \eta^1 \wedge \cdots \wedge \left(Q^\alpha_{\bar{\beta}\bar{\gamma}}\eta^{\bar{\beta}} \wedge \eta^{\bar{\gamma}}\right) \wedge \cdots \wedge \eta^n$$

and ρ *is the complex* n*-form on* $F(M)$ *given by Lemma* 5.

PROOF. As the proof of Proposition 3 is rather involved, we organize it in several steps, as follows.

STEP 1. *We have*

$$d\eta^{01\cdots n} = \left(-\omega_\alpha^\alpha + \frac{i}{2}Q^\alpha_{\bar{\beta}\alpha}\eta^{\bar{\beta}}\right) \wedge \eta^{01\cdots n} + \mathcal{W}\,. \tag{49}$$

Firstly, we make use of the identities (44)-(45) to compute the exterior differential of $\eta^{01\cdots n} = \eta \wedge \eta^{1\cdots n}$, where $\eta^{1\cdots n} = \eta^1 \wedge \cdots \wedge \eta^n$. We have

$$\begin{aligned}
d\eta^{01\cdots n} &= d\eta \wedge \eta^{1\cdots n} - \eta \wedge \sum_\alpha (-1)^{\alpha-1}\eta^1 \wedge \cdots \wedge (d\eta^\alpha) \wedge \cdots \wedge \eta^n \\
&= -2ig_{\alpha\bar{\beta}}\,\eta^\alpha \wedge \eta^{\bar{\beta}} \wedge \eta^{1\cdots n} + \eta \wedge \sum_\alpha (-1)^\alpha \eta^1 \wedge \cdots \wedge \Big(\eta^\beta \wedge \omega_\beta^\alpha + \eta \wedge \tau^\alpha \\
&\quad + \frac{i}{2}Q^\alpha_{\bar{\beta}\gamma}\eta^{\bar{\beta}} \wedge \eta^\gamma + \frac{i}{2}Q^\alpha_{\bar{\beta}\bar{\gamma}}\eta^{\bar{\beta}} \wedge \eta^{\bar{\gamma}}\Big) \wedge \cdots \wedge \eta^n \\
&= \eta \wedge \sum_\alpha (-1)^\alpha \eta^1 \wedge \cdots \wedge \left(\eta^\alpha \wedge \omega_\alpha^\alpha + \frac{i}{2}Q^\alpha_{\bar{\beta}\alpha}\eta^{\bar{\beta}} \wedge \eta^\alpha\right) \wedge \cdots \wedge \eta^n + \mathcal{W}
\end{aligned}$$

hence

$$d\eta^{01\cdots n} = \left(-\omega_\alpha^\alpha + \frac{i}{2} Q_{\bar\beta\alpha}^\alpha \eta^{\bar\beta}\right) \wedge \eta^{01\cdots n} + \mathcal{W}$$

and Step 1 is proved.

STEP 2. *We have*

$$d\Xi_0 = \left(d \log G - \omega_\alpha^\alpha + \frac{i}{2} Q_{\bar\beta\alpha}^\alpha \eta^{\bar\beta}\right) \wedge \Xi_0 + G\mathcal{W}. \tag{50}$$

Set $G = \det(g_{\alpha\bar\beta})^{1/2}$ for simplicity. Then (by (49))

$$\begin{aligned} d\Xi_0 &= (dG) \wedge \eta^{01\cdots n} + G\, d\eta^{01\cdots n} \\ &= \left[dG + G(-\omega_\alpha^\alpha + \frac{i}{2} Q_{\bar\beta\alpha}^\alpha \eta^{\bar\beta})\right] \wedge \eta^{01\cdots n} + G\mathcal{W} \end{aligned}$$

hence

$$d\Xi_0 = \left(d \log G - \omega_\alpha^\alpha + \frac{i}{2} Q_{\bar\beta\alpha}^\alpha \eta^{\bar\beta}\right) \wedge \Xi_0 + G\mathcal{W}$$

and Step 2 is proved. At this point we may prove (47). Set $h = \log G \in C^\infty(U)$. Define the local form $\omega \in \Gamma^\infty(U, \Lambda^{1,0}(M))$ by setting

$$\omega = \left(h_\beta - \Gamma_{\beta\bar\alpha}^{\bar\alpha} - \frac{i}{2} Q_{\beta\bar\alpha}^{\bar\alpha}\right) \eta^\beta \ , \ h_\beta = \xi_\beta(h).$$

Then (by (50))

$$d\Xi_0 = \bar\omega \wedge \Xi_0 + G\mathcal{W} = (\bar\omega - \omega) \wedge \Xi_0 + G\mathcal{W}$$

(as $\omega \wedge \Xi_0 = 0$). As $\bar\omega - \omega$ is purely imaginary we may define a real 1-form σ_0 on M by setting

$$\bar\omega - \omega = i(n+2)\sigma_0 .$$

Hence

$$d\Xi_0 = i(n+2)\sigma_0 \wedge \Xi_0 + G\,\mathcal{W}. \tag{51}$$

At this point, we differentiate $\mathcal{Z} = e^{i\gamma}\mathcal{Z}_0$ and use (51) so that to obtain

$$d\mathcal{Z} = i(n+2)\sigma \wedge \mathcal{Z} + e^{i\gamma}\pi^*(G\,\mathcal{W})$$

where

$$\sigma = \frac{1}{n+1} d\gamma + \pi^*\sigma_0 \tag{52}$$

hence (47) is proved.

Let Ω be a 2-form on M. Then, by definition, trace $(\pi^*\Omega) =$ trace $(\Omega)\circ\pi$. As (by (52)) $d\sigma = \pi^* d\sigma_0$ and $d\sigma_0$ is a 2-form on M, this definition may be used to make sense of trace $(d\sigma)$.

STEP 3. *Let $f \in C^\infty(M)$ be a real valued function and define a 1-form σ_f on $F(M)$ by setting*

$$\sigma_f = \sigma + \pi^*(f\eta).$$

Then we have

$$\sigma_f \wedge d\rho \wedge \bar{\rho} = \sigma \wedge d\rho \wedge \bar{\rho} - i(n+2)(f\circ\pi)\sigma \wedge (\pi^*\eta) \wedge \rho \wedge \bar{\rho} \tag{53}$$

where ρ is the complex n-form on $F(M)$ furnished by Lemma 5.

Differentiate $\mathcal{Z} = (\pi^*\eta) \wedge \rho$ so that to get

$$d\mathcal{Z} = (\pi^* d\eta) \wedge \rho - (\pi^*\eta) \wedge d\rho$$

hence (by (47))

$$(\pi^*\eta) \wedge d\rho = (\pi^* d\eta) \wedge \rho - i(n+2)\sigma \wedge \mathcal{Z} - e^{i\gamma}\pi^*(G\,\mathcal{W}).$$

Consequently

$$\sigma_f \wedge d\rho \wedge \bar{\rho} = \sigma \wedge d\rho \wedge \bar{\rho} + (f\circ\pi)\left[(\pi^* d\eta) \wedge \rho - i(n+2)\sigma \wedge \mathcal{Z} - e^{i\gamma}\pi^*(G\,\mathcal{W})\right] \wedge \bar{\rho}$$

hence

$$\sigma_f \wedge d\rho \wedge \bar{\rho} = \sigma \wedge d\rho \wedge \bar{\rho} - i(n+2)(f\circ\pi)\sigma \wedge (\pi^*\eta) \wedge \rho \wedge \bar{\rho}$$

because of

$$(\pi^* d\eta) \wedge \rho \wedge \bar{\rho} = 0$$
$$(\pi^*\mathcal{W}) \wedge \bar{\rho} = 0.$$

Indeed, to check the last two identities, it suffices to look at the explicit expression of ρ, i.e. (cf. the proof of Lemma 5)

$$\rho = e^{i\gamma}\pi^*(G\,\eta^{1\cdots n}).$$

STEP 4. *Computing* trace $(d\sigma_f)$.

Let us differentiate in $\sigma_f = \sigma + \pi^*(f\eta)$ so that to get

$$d\sigma_f = d\sigma + \pi^*(df \wedge \eta + f d\eta).$$

As

$$df \wedge \eta + f d\eta = (f_\alpha \eta^\alpha + f_{\bar{\alpha}} \eta^{\bar{\alpha}}) \wedge \eta - 2if g_{\alpha\bar{\beta}} \eta^\alpha \wedge \eta^{\bar{\beta}}$$

it follows that

$$\text{trace } (df \wedge \eta + f d\eta) = nf$$

hence

$$\text{trace } (d\sigma_f) = \text{trace } (d\sigma) + n\, f \circ \pi\,. \tag{54}$$

During the next calculations, for the sake of simplicity, we do not distinguish notationally between f and $f \circ \pi$, respectively η and $\pi^*\eta$.

STEP 5. *There is a C^∞ function $u : F(M) \to \mathbf{C}$ so that*

$$\sigma \wedge d\rho \wedge \bar{\rho} = u\, i\, \sigma \wedge \eta \wedge \rho \wedge \bar{\rho}\,. \tag{55}$$

Using (54) we find

$$\text{trace } (d\sigma_f) i\, \sigma_f \wedge \eta \wedge \rho \wedge \bar{\rho} = \text{trace } (d\sigma) i\, \sigma \wedge \eta \wedge \rho \wedge \bar{\rho} + i\, n\, f\, \sigma \wedge \eta \wedge \rho \wedge \bar{\rho}. \tag{56}$$

We wish to determine $f \in C^\infty(M)$ so that

$$\sigma_f \wedge d\rho \wedge \bar{\rho} = \text{trace } (d\sigma_f) i\, \sigma_f \wedge \eta \wedge \rho \wedge \bar{\rho}$$

(then σ_f would be the real 1-form on $F(M)$ we are looking for, i.e. whose existence and uniqueness is claimed in Proposition 3, because of $(f\eta) \wedge \mathcal{Z} = 0$). To solve (57) for f we substitute from (53) and (56). We obtain

$$\sigma \wedge d\rho \wedge \bar{\rho} - \text{trace } (d\sigma) i\, \sigma \wedge \eta \wedge \rho \wedge \bar{\rho} = 2i(n+1) f\, \sigma \wedge \eta \wedge \rho \wedge \bar{\rho} \tag{58}$$

which uniquely determines f because $\sigma \wedge \eta \wedge \rho \wedge \bar{\rho}$ is a volume form on $F(M)$. However, we need to check that f (determined by (58)) is real valued. As $\sigma \wedge d\rho \wedge \bar{\rho}$ is a top degree form on $F(M)$, there is a C^∞ function $u : F(M) \to \mathbf{C}$ so that

$$\sigma \wedge d\rho \wedge \bar{\rho} = u\, i\, \sigma \wedge \eta \wedge \rho \wedge \bar{\rho}$$

and Step 5 is proved.

STEP 6. *u is real valued.*

Indeed, note that

$$\rho \wedge \bar{\rho} = G^2 \eta^{1\cdots n} \wedge \eta^{\bar{1}\cdots\bar{n}}$$

or

$$(d\eta)^n = 2^n i^{n(n+2)} n!\, \rho \wedge \bar{\rho}$$

hence (by differentiation)

$$0 = d\rho \wedge \bar{\rho} + (-1)^n \rho \wedge d\bar{\rho}$$

which may be also written as

$$d\rho \wedge \bar{\rho} = (-1)^{n^2+1} d\bar{\rho} \wedge \rho\,. \tag{59}$$

Finally, it is easy to see that (55)-(59) lead to $u = \bar{u}$.

To complete the proof of Proposition 3, let us substitute from (55) into (58). Since $\sigma \wedge \eta \wedge \rho \wedge \bar{\rho}$ is a volume form, we obtain

$$u - \text{trace } (d\sigma) = 2(n+1) f$$

hence f is real valued. □

Let us remark that, in the proof of Proposition 3, we made use several times of the fact that $\sigma \wedge (\pi^*\eta) \wedge \rho \wedge \bar{\rho}$ is a volume form. This follows by observing that

$$2^n i^{n(n+2)} n!(n+2)\sigma \wedge (\pi^*\eta) \wedge \rho \wedge \bar{\rho} = d\gamma \wedge (\pi^*\Psi).$$

Consider the (degenerate) bilinear form L_η on $T(M)$ given by i) $L_\eta(X, Y) = -g(X, Y)$ for any $X, Y \in H(M)$, and ii) $L_\eta(\xi, X) = 0$ for any $X \in T(M)$ (in particular $L_\eta(\xi, \xi) = 0$). At this point, we may define the semi-Riemannian metric G_η on $F(M)$ by setting

$$G_\eta = \pi^* L_\eta + 2(\pi^*\eta) \odot \sigma \tag{60}$$

where σ is the real 1-form furnished by Proposition 3. Also $\odot$ is the symmetric tensor product, i.e. $\lambda \odot \mu = \frac{1}{2}\{\lambda \otimes \mu + \mu \otimes \lambda\}$. We claim that G_η is a Lorentz metric on $F(M)$. Since the 1-forms $\{\pi^*\eta^A, \sigma\}$ are pointwise linearly independent, we may consider the dual frame $\{V_A, \Sigma\}$ (a local frame of $T(F(M)) \otimes \mathbf{C}$ on $\pi^{-1}(U)$). Then $\pi_* V_A = \xi_A$ and $\pi_*\Sigma = 0$. Set $H_\eta(Z, W) = G_\eta(Z, \bar{W})$ for any $Z, W \in T(F(M)) \otimes \mathbf{C}$. Then H_η is represented (with respect to the chosen frame $\{V_A, \Sigma\}$) as

$$H_\eta : \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & g_{\alpha\bar{\beta}} & 0 & 0 \\ 0 & 0 & g_{\bar{\alpha}\beta} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation is

$$(1 - t^2)\left|\det(g_{\alpha\bar{\beta}} - t\delta_{\alpha\beta})\right|^2 = 0$$

hence G_η has index 1.

The Lorentz metric G_η (given by (60)) is called the *Fefferman metric* of the contact Riemannian manifold $(M, (\phi, \xi, \eta, g))$. When $Q = 0$ (i.e. $T_{1,0}(M)$ is integrable) G_η coincides with the ordinary Fefferman metric (cf. [11] and [21]).

6.3. – Pseudoharmonic maps

We begin by reviewing the concept of *pseudoharmonic map*. Let (M, ∇) and (N, ∇') be two manifolds with linear connection, and $P \subset T(M)$ a subbundle. Let g be a semi-Riemannian (bundle) metric in P. Let $f : M \to N$ be a C^∞ map and $f^{-1}TN \to M$ the pullback of $T(N)$ by f. Let $f^{-1}\nabla'$ be the connection in $f^{-1}TN$ induced by ∇'. This is most easily described in local coordinates, as follows. The *natural lift* $\tilde{Y} : f^{-1}(V) \to f^{-1}TN$ of a tangent vector field $Y : V \to T(N)$ (with $V \subseteq N$ open) is given by $\tilde{Y}(x) =$

$Y(f(x))$, $x \in f^{-1}(V)$. Let (V, y^i) be a local coordinate system on N and let Y_i be the natural lift of $\partial/\partial y^i$. Then

$$(f^{-1}\nabla')_X Y_i = X(f^i)({\Gamma'}^k_{ij} \circ f)Y_k \, , \; X \in \mathcal{X}(M)$$

where ${\Gamma'}^i_{jk}$ are the local coefficients of ∇' with respect to (V, y^i) and $f^i = y^i \circ f$. Moreover, let ∇^f be the connection in $T^*(M) \otimes f^{-1}TN$ determined by

$$\nabla^f_X(\omega \otimes s) = (\nabla_X \omega) \otimes s + \omega \otimes (f^{-1}\nabla')_X s \, .$$

For any bilinear form B on $T(M)$ let $\pi_P B$ denote the restriction of B to $P \otimes P$. Finally, set

$$\tau(f; P, g, \nabla, \nabla') = \text{trace } \{\pi_P \nabla^f df\} \in \Gamma^\infty(f^{-1}TN)$$

where the trace is computed with respect to g. We say that f is *pseudoharmonic*, with respect to the data (P, g, ∇, ∇'), if

$$\tau(f; P, g, \nabla, \nabla') = 0 \, .$$

This slightly generalizes the notion in [10], p. 108-109, where $P = T(M)$.

Let (N, g') be a Riemannian manifold and $F : (F(M), G_\eta) \to (N, g')$ a C^∞ map. The *energy* of F over a compact domain $D \subseteq F(M)$ is

$$\mathbf{E}(F; D) = \frac{1}{2} \int_D \text{trace } (F^* g') dv_{G_\eta}$$

where dv_{G_η} is the volume element of $(F(M), G_\eta)$ (and the trace is computed with respect to G_η). Then F is *harmonic* if, for any compact domain $D \subseteq F(M)$, it is an extremal of the energy $\mathbf{E}(\cdot; D)$ with respect to all variations of F supported in D. Therefore, F is harmonic if and only if it satisfies the Euler-Lagrange equations

$$\Box F^i + \sum_{p,q=1}^{2n+2} G^{pq} \left({\Gamma'}^i_{jk} \circ F\right) \frac{\partial F^j}{\partial u^p} \frac{\partial F^k}{\partial u^q} = 0 \tag{61}$$

for some local coordinate system (U, x^a) on M, respectively (V, y^i) on N, and $F^i = y^i \circ F$. Here $\Box$ is the wave operator (the Laplace-Beltrami operator associated with the Lorentzian metric G_η). One endows $F(M)$ with the induced local coordinate system $(\pi^{-1}(U), u^p)$, with $u^a = x^a \circ \pi$ and $u^{2n+2} = \gamma$. We may state the following

THEOREM 5. *Let (M, η) be a contact manifold and $g \in \mathcal{M}(\eta)$ an associated metric. Let $f : M \to N$ be a C^∞ map of M into a Riemannian manifold (N, g'). Let ∇' be the Levi-Civita connection of (N, g'). Then the following statements are equivalent*

i) *f is pseudoharmonic with respect to the data $(H(M), g, \nabla^*, \nabla')$.*

ii) *f satisfies the PDEs*

$$\Delta_H f^i + 4g^{\alpha\bar\beta}(\Gamma'^i_{jk} \circ f)\xi_\alpha(f^j)\xi_{\bar\beta}(f^k) = 0 \tag{62}$$

for some local frame $\{\xi_\alpha\}$ of $T_{1,0}(M)$ and some local coordinate system (V, y^i) on N, where $f^i = y^i \circ f$.

iii) *The vertical lift $F = f \circ \pi : (F(M), G_\eta) \to (N, g')$ is a harmonic map.*

This generalizes a result in [5] (from strictly pseudoconvex CR manifolds to contact Riemannian manifolds (whose almost CR structure is not necessarily integrable)). To prove Theorem 5, we firstly compute $\tau(f; H(M), g, \nabla^*, \nabla')$. Let $\{X_\alpha, \phi X_\alpha\}$ be a local orthonormal frame of $H(M)$, with respect to g, and set $Z_\alpha = \frac{1}{2}\{X_\alpha - i\phi X_\alpha\}$. For any bilinear form B on $T(M)$

$$\text{trace }\{\pi_{H(M)}B\} = 2\sum_{\alpha=1}\{B(Z_\alpha, Z_{\bar\alpha}) + B(Z_{\bar\alpha}, Z_\alpha)\}$$

Let $\{\theta^\alpha\}$ be an admissible coframe. Taking into account

$$\nabla^*\eta = 0$$

$$\nabla^*\theta^\alpha = -\theta^B \otimes \left(\Gamma^\alpha_{B\gamma}\theta^\gamma + \frac{i}{2}Q^\alpha_{\bar\gamma B}\theta^{\bar\gamma}\right)$$

and (35) we obtain

$$\tau(f; H(M), g, \nabla^*, \nabla') = \left(\Delta_H f^i + 4\sum_{\alpha=1}^{n} Z_\alpha(f^j)Z_{\bar\alpha}(f^k)(\Gamma'^i_{jk} \circ f)\right) Y_i\,.$$

Finally, if $\{\xi_\alpha\}$ is an arbitrary local frame of $T_{1,0}(M)$, then $Z_\alpha = U^\beta_\alpha \xi_\beta$ for some C^∞ functions $U^\beta_\alpha : U \to \mathbf{C}$ and $\sum_\beta U^\lambda_\beta U^{\bar\mu}_{\bar\beta} = g^{\lambda\bar\mu}$, and i) $\Longleftrightarrow$ii) is proved. To prove ii) $\Longleftrightarrow$ iii) we need

LEMMA 6. $S^1 \subset Isom(F(M), G_\eta)$.

PROOF. Let $a \in S^1$. Then $i_\eta \circ R_a = a i_\eta$ hence

$$R^*_a \mathcal{Z} = \frac{1}{n+1} R^*_a i^*_\eta \Xi = a\mathcal{Z}\,.$$

Moreover

$$\gamma \circ R_a = \gamma + \arg(a) + 2k\pi\ , \ k \in \mathbf{Z}$$

hence, by taking into account Proposition 3

$$(\sigma - R^*_a\sigma) \wedge \mathcal{Z} = 0\,. \tag{63}$$

Let $\{V_A, \Sigma\}$ be dual to $\{\pi^*\eta^A, \sigma\}$. Then $(dR_a)V_A \in Span\{V_A\}$ and then, by applying $i_{V_0} i_{V_1} \cdots i_{V_n}$ (where $i_X = X \rfloor$) to (63) we obtain $R^*_a\sigma = \sigma$. Finally (by (60)) $R^*_a G_\eta = G_\eta$. □

Going back to the proof of Theorem 5, let $F : F(M) \to N$ be a S^1-invariant C^∞ map. Then F descends to a map $f : M \to N$ (so that $F = f \circ \pi$). The wave operator $\Box$ is S^1-invariant (as $\Box$ is invariant under any isometry of $(F(M), G_\eta)$ and, by Lemma 6, each R_a, $a \in S^1$, is an isometry) hence, for any $u \in C^\infty(M)$, the function $\Box(u \circ \pi)$ descends to a function on M, denoted by the same symbol. Thus $\Box$ pushes forward to a differential operator $\pi_*\Box : C^\infty(M) \to C^\infty(M)$ given by $(\pi_*\Box)u = \Box(u \circ \pi)$. A verbatim transcription of the proof of Prop. 6.1 in [22], p. 425, gives $\pi_*\Box = \frac{1}{2}\Delta_H$ (the $\frac{1}{2}$ factor comes from our convention as to the definition of the sublaplacian). Hence, the S^1-invariant map F is harmonic if and only if (by (61))

$$(64) \quad (\Delta_H f^i) \circ \pi + 2 \sum_{a,b=1}^{2n+1} G^{ab} \left(\Gamma'^i_{jk} \circ f \circ \pi\right) \left(\frac{\partial f^j}{\partial x^a} \circ \pi\right) \left(\frac{\partial f^k}{\partial x^b} \circ \pi\right) = 0 .$$

Let $\{\xi_\alpha\}$ be a local frame of $T_{1,0}(M)$ and $\{\eta^\alpha\}$ the corresponding admissible coframe. Let $\{V_A, \Sigma\}$ be dual to $\{\pi^*\eta^A, \sigma\}$. Relabel the variables x^a, $1 \leq a \leq 2n+1$, as x^A, $A \in \{0, 1, \cdots, n, \bar{1}, \cdots, \bar{n}\}$, where $x^{\bar{\alpha}} = x^{\alpha+n}$, $x^0 = x^{2n+1}$. Then $\xi_A = \lambda_A^B \partial/\partial x^A$, for some C^∞ function $\lambda = [\lambda_A^B] : U \to GL(2n+1, \mathbf{C})$. It follows that

$$V_A = (\lambda_A^B \circ \pi) \frac{\partial}{\partial u^B} , \quad \Sigma = (n+2) \frac{\partial}{\partial \gamma} .$$

Set $\mu = \lambda^{-1}$. As $L_\eta = 2g_{\alpha\bar{\beta}}\eta^\alpha \odot \eta^{\bar{\beta}}$ we get

$$G_\eta : \begin{bmatrix} g_{\alpha\bar{\beta}}(\mu_A^\alpha \mu_B^{\bar{\beta}} + \mu_A^{\bar{\beta}} \mu_B^\alpha) & \frac{1}{n+2}\mu_A^0 \\ \frac{1}{n+2}\mu_B^0 & 0 \end{bmatrix}$$

with respect to the frame $\{\partial/\partial u^A, \partial/\partial \gamma\}$. The inverse is denoted by

$$\begin{bmatrix} G^{AB} & G^{A,2n+2} \\ G^{2n+2,B} & G^{2n+2,2n+2} \end{bmatrix}$$

and a calculation shows that

$$G^{AB}\mu_A^\alpha \mu_B^{\bar{\beta}} = g^{\alpha\bar{\beta}}$$

$$G^{AB}\mu_A^0 = 0 , \qquad G^{AB}\mu_A^\alpha \mu_B^\beta = 0 .$$

Consequently

$$G^{AB}\frac{\partial f^j}{\partial x^A}\frac{\partial f^k}{\partial x^B} = g^{\alpha\bar{\beta}}\{\xi_\alpha(f^j)\xi_{\bar{\beta}}(f^k) + \xi_{\bar{\beta}}(f^j)\xi_\alpha(f^k)\}$$

hence (64) yields (62). $\Box$

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