ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 30, nº 1 (2001), p. 225-249
<http://www.numdam.org/item?id=ASNSP_2001_4_30_1_225_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXX (2001), pp. 225-249

# Approximation Measures for Logarithms of Algebraic Numbers

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**Abstract.** Given a number field $\mathbb{K}$ and a number $\xi \notin \mathbb{K}$, we say that $\mu > 0$ is a $\mathbb{K}$-irrationality measure of $\xi$ if, for any $\varepsilon > 0$, $\log|\xi - \beta| > -(1+\varepsilon)\,\mu\,\mathrm{h}(\beta)$ for all $\beta \in \mathbb{K}$ with sufficiently large Weil logarithmic height $\mathrm{h}(\beta)$. We find $\mathbb{Q}(\alpha)$-irrationality measures of $\log\alpha$ for suitable algebraic numbers $\alpha$, where $\log\alpha$ denotes the principal value of the logarithm. We combine the saddle point method in complex analysis with an arithmetic method based on the $p$-adic valuation of the gamma-factors occurring in the Euler integral representation of Gauss's hypergeometric function.

**Mathematics Subject Classification (2000):** 11J82 (primary), 11J17, 11J72 (secondary).

## 1. – Introduction

Let $\alpha \in \mathbb{C}$ be an algebraic number satisfying $\alpha \neq 1$ and $\alpha \notin (-\infty, 0]$, and let $\log\alpha$ denote the principal value of its logarithm, i.e.,

$$\log\alpha = \log|\alpha| + i\arg\alpha \quad \text{with} \quad -\pi < \arg\alpha < \pi.$$

In this paper we obtain $\mathbb{Q}(\alpha)$-irrationality measures of $\log\alpha$ if the degree of $\alpha$ is not too large. For instance, taking $\alpha = \sqrt{2}$, we get

$$\left|\log 2 - \frac{a+b\sqrt{2}}{c+d\sqrt{2}}\right| > \text{constant} \cdot \max\{|a|,|b|,|c|,|d|\}^{-12.4288}$$

for all $a, b, c, d \in \mathbb{Z}$ with $(c,d) \neq (0,0)$. A further interesting example is given by the choice $\alpha = e^{i\pi/6}$. We prove

$$\left|\pi - \frac{a+b\sqrt{3}}{c+d\sqrt{3}}\right| > \text{constant} \cdot \max\{|a|,|b|,|c|,|d|\}^{-46.9075}$$

Pervenuto alla Redazione il 23 giugno 2000.

for all $a, b, c, d \in \mathbb{Z}$ with $(c, d) \neq (0, 0)$. In particular, taking $d = 0$, we get the following linear independence measures of 1, $\sqrt{2}$, $\log 2$ and of 1, $\sqrt{3}$, $\pi$ over $\mathbb{Q}$:

$$\left| a + b\sqrt{2} + c\log 2 \right| > \text{constant} \cdot |c| \cdot \max\{|a|, |b|, |c|\}^{-12.4288}$$

and

$$\left| a + b\sqrt{3} + c\pi \right| > \text{constant} \cdot |c| \cdot \max\{|a|, |b|, |c|\}^{-46.9075},$$

for all $a, b, c \in \mathbb{Z}$. We remark that all our results are effective.

In order to obtain $\mathbb{K}$-irrationality measures of $\log\alpha$ for $\alpha \in \mathbb{K}$, where $\mathbb{K}$ is a suitable number field, it is natural to consider the Padé approximants to the function $\log(1 + z)$ of the complex variable $z$, i.e., pairs $\big(p_n(z), q_n(z)\big)$ of non-zero polynomials of degree $\leq n$ with rational coefficients such that

$$p_n(z) + q_n(z)\log(1 + z) \tag{1.1}$$

vanishes at $z = 0$ to the order $2n + 1$. Changing for convenience $1 + z$ into $z$, we write (1.1) in the form

$$a_n(z) + b_n(z)\log z, \tag{1.2}$$

with $a_n, b_n \in \mathbb{Q}[z]$ of degree $\leq n$. It is well known that (1.2) has the representation

$$\begin{aligned} a_n(z) + b_n(z)\log z &= \\ I_n(z) &= (z-1)^{2n+1}\int_0^1 \left(\frac{x(1-x)}{1+x(z-1)}\right)^n \frac{dx}{1+x(z-1)} \end{aligned} \tag{1.3}$$

(see e.g. [AR], formula (17)), where we assume again $z \notin (-\infty, 0]$ and $\log z = \log|z| + i\arg z$ with $-\pi < \arg z < \pi$. The integral representation (1.3) for $z = \alpha$ allows one to compute $\mathbb{K}$-irrationality measures of $\log\alpha$ for $\alpha \in \mathbb{K}$, when $\mathbb{K}$ is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$ (see [AR], Theorem 1).

The main contribution of the present paper is to extend such methods of computation to more general number fields $\mathbb{K}$. As above, let $\alpha \in \mathbb{C}$ be an algebraic number satisfying $\alpha \neq 1$ and $\alpha \notin (-\infty, 0]$. In order to get $\mathbb{Q}(\alpha)$-irrationality measures of $\log\alpha$, we employ asymptotic estimates, as $n \to \infty$, of

$$I_n(\alpha) = a_n(\alpha) + b_n(\alpha)\log\alpha,$$

of $b_n(\alpha)$, and of the Weil height $\mathrm{h}\big(a_n(\alpha)/b_n(\alpha)\big)$. Note that, with the change of variable $1 + x(z - 1) = t$, (1.3) can be written in the form

$$I_n(z) = \int_1^z \left(\frac{(t-1)(z-t)}{t}\right)^n \frac{dt}{t}, \tag{1.4}$$

where the integration path is the segment in $\mathbb{C}$ joining 1 with $z$. By Cauchy's theorem, in the integral representation (1.4) for $z = \alpha$ we can move the path of fixed endpoints 1 and $\alpha$ through the closer of the two stationary points $t = \pm\sqrt{\alpha}$ of the function $(t-1)(\alpha - t)/t$, and then a straightforward application of the saddle point method in complex analysis yields the required asymptotic estimate of $I_n(\alpha)$. As for $b_n(\alpha)$, we have the Cauchy integral representation

$$(1.5) \qquad b_n(z) = \frac{1}{2\pi i}\oint \left(\frac{(t-1)(z-t)}{t}\right)^n \frac{dt}{t}$$

of the coefficient $b_n(z)$ in (1.3), where now the integration path is a contour enclosing the pole $t = 0$, whence the asymptotic estimate of $b_n(\alpha)$ is obtained by taking $z = \alpha$ in (1.5) and applying again the saddle point method. Finally, the required estimate of $\mathrm{h}\big(a_n(\alpha)/b_n(\alpha)\big)$ is reduced to an upper asymptotic estimate of $|a_n(\alpha)|_v$ and $|b_n(\alpha)|_v$, for every normalized absolute value $v$ of $\mathbb{Q}(\alpha)$. If $v$ is ultrametric, we do this by a crude estimate of the denominators. If $v$ is Archimedean, $v$ is associated with an embedding $\sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$, so that $|a_n(\alpha)|_v = |\sigma(a_n(\alpha))|$ and $|b_n(\alpha)|_v = |\sigma(b_n(\alpha))|$. Since $a_n(z), b_n(z) \in \mathbb{Q}[z]$, we have $\sigma(a_n(\alpha)) = a_n(\sigma(\alpha))$ and $\sigma(b_n(\alpha)) = b_n(\sigma(\alpha))$. Thus, for the estimate of $|b_n(\alpha)|_v = |b_n(\sigma(\alpha))|$ we take $z = \sigma(\alpha)$ in (1.5) and apply the saddle point method. It is worth noting that, even though $\alpha \notin (-\infty, 0]$, for some $\sigma$ the conjugate $\sigma(\alpha)$ of $\alpha$ may be a negative real number, so that the saddle point method must be also applied to (1.5), with some additional complications, in the case $z < 0$. Since $a_n(z) = I_n(z) - b_n(z)\log z$, the estimate of $|a_n(\alpha)|_v = |a_n(\sigma(\alpha))|$ is obtained from the estimates of $I_n(\sigma(\alpha))$ and $b_n(\sigma(\alpha))$, and, again because $\sigma(\alpha)$ may be negative, the integration in (1.4) when $z < 0$ is made over a simple path from 1 to $z$ contained in the upper half-plane.

Along these lines, we prove a general result (Theorem 2.2) which yields $\mathbb{Q}(\alpha)$-irrationality measures of $\log\alpha$, for suitable algebraic numbers $\alpha$. In order to improve our numerical results and to enlarge the set of the algebraic numbers to which Theorem 2.2 applies, we extend the method given in [V] for $\alpha \in \mathbb{Q}$, to the case of algebraic numbers $\alpha$ of higher degree. Such an extension (Theorem 2.3) is obtained by replacing the function

$$\frac{(t-1)(z-t)}{t},$$

appearing in (1.4) and (1.5), with

$$\frac{(t-1)^j(z-t)^j}{t^{j-l}},$$

where $j$ and $l$ are integer parameters such that $j > l \geq 0$, and by combining the saddle point method with an arithmetic method introduced in [RV], based on the $p$-adic valuation of the gamma-factors occurring in the Euler integral representation of Gauss's hypergeometric function. In fact, the introduction of the above parameters $j$ and $l$ in (1.3) yields a modified integral

$$\begin{aligned} J_n(z) &= (z-1)^{2jn+1} \int_0^1 \left( \frac{x^j(1-x)^j}{\big(1+x(z-1)\big)^{j-l}} \right)^n \frac{dx}{1+x(z-1)} \\ &= A_n(z) + B_n(z) \log z, \end{aligned}$$

where now $\big(A_n(z), B_n(z)\big)$ are Padé-type approximants to $\log z$, and where

$$(1.6) \qquad \int_0^1 \frac{x^{jn}(1-x)^{jn}}{\big(1+x(z-1)\big)^{(j-l)n}} \frac{dx}{1+x(z-1)}$$

is known to be related to the Gauss hypergeometric function. Using the invariance of this function under the interchange of the two parameters appearing in the numerator of the hypergeometric series, one transforms (1.6) into an integral of a similar type multiplied by a quotient of factorials, and the $p$-adic valuation of such factorials yields further arithmetic information on the Padé-type approximants mentioned above.

## 2. – Notation, preliminaries and statement of the main results

Let $\mathbb{K} \subset \mathbb{C}$ be a number field, and let $\mathcal{M}_{\mathbb{K}}$ be the set of places of $\mathbb{K}$. For any $v \in \mathcal{M}_{\mathbb{K}}$ we denote by $|\cdot|_v$ the normalized absolute value associated with $v$, and we let $\eta_v = [\mathbb{K}_v : \mathbb{Q}_v]$. Thus, if $\sigma$ is an embedding of $\mathbb{K}$ in $\mathbb{C}$ and if $v = v_\sigma$ is the associated place, we have $|\beta|_v = |\sigma(\beta)|$ for $\beta \in \mathbb{K}$, and $\eta_v = 1$ if $\sigma$ is real and $\eta_v = 2$ otherwise. If, instead, $v \mid p$ where $p$ is a prime number, the absolute value $|\cdot|_v$ is normalized by $|p|_v = p^{-1}$.

We recall the definition of the Weil absolute logarithmic height:

$$(2.1) \qquad \mathrm{h}(\beta) = \frac{1}{[\mathbb{K}:\mathbb{Q}]} \sum_{v \in \mathcal{M}_{\mathbb{K}}} \eta_v \log^+ |\beta|_v \qquad \text{for } \beta \in \mathbb{K},$$

where $\log^+ x = \log\max\{x, 1\}$ for $x \geq 0$. It is easy to see that $\mathrm{h}(\beta)$ depends only on the algebraic number $\beta$, and is independent of the number field $\mathbb{K}$ containing $\beta$. For any $\beta_1, \beta_2 \in \mathbb{K}$ we plainly have

$$(2.2) \qquad \mathrm{h}(\beta_1 + \beta_2) \leq \mathrm{h}(\beta_1) + \mathrm{h}(\beta_2) + \log 2,$$

since $\max\{x+y, 1\} \leq 2\max\{x, 1\}\max\{y, 1\}$ for all $x \geq 0$, $y \geq 0$.

We also recall that, for any $\beta \in \mathbb{K}^\times$, we have the product formula

$$\sum_{v \in \mathcal{M}_{\mathbb{K}}} \eta_v \log |\beta|_v = 0, \tag{2.3}$$

and the Liouville inequality

$$\log |\beta| \geq -\delta \, \mathrm{h}(\beta), \tag{2.4}$$

where

$$\delta = \frac{[\mathbb{K}:\mathbb{Q}]}{[\mathbb{K}_\infty:\mathbb{R}]} = \begin{cases} [\mathbb{K}:\mathbb{Q}], & \text{if } \mathbb{K} \subset \mathbb{R} \\ \frac{1}{2}[\mathbb{K}:\mathbb{Q}], & \text{otherwise.} \end{cases} \tag{2.5}$$

As usual, $\mathbb{K}_\infty$ denotes the completion of the field $\mathbb{K}$ with respect to the euclidean absolute value, i.e., $\mathbb{K}_\infty = \mathbb{R}$ if $\mathbb{K} \subset \mathbb{R}$ and $\mathbb{K}_\infty = \mathbb{C}$ otherwise.

Definition 2.1. Let $\mathbb{K} \subset \mathbb{C}$ be a number field, let $\xi \in \mathbb{K}_\infty \setminus \mathbb{K}$, and let $\mu$ be a positive real number. We say that $\mu$ is a $\mathbb{K}$-irrationality measure of $\xi$ if for any $\varepsilon > 0$ there exists a constant $h_0 = h_0(\varepsilon) > 0$ such that

$$\log |\xi - \beta| > -(1+\varepsilon)\,\mu\,\mathrm{h}(\beta)$$

for all $\beta \in \mathbb{K}$ with $\mathrm{h}(\beta) > h_0$. The least $\mathbb{K}$-irrationality measure of $\xi$ is denoted by $\mu_{\mathbb{K}}(\xi)$.

We recall that, by the Dirichlet box principle,

$$\mu_{\mathbb{K}}(\xi) \geq 2\frac{[\mathbb{K}:\mathbb{Q}]}{[\mathbb{K}_\infty:\mathbb{R}]} = 2\delta \tag{2.6}$$

(see [S], pp. 253 and 255).

Our first result is the following.

Theorem 2.2. *Let $\alpha \in \mathbb{C}$ be an algebraic number with $\alpha \neq 1$ and $\alpha \notin (-\infty, 0]$, let*

$$\delta = \frac{[\mathbb{Q}(\alpha):\mathbb{Q}]}{[\mathbb{Q}(\alpha)_\infty:\mathbb{R}]},$$

*and let*

$$\varrho(\alpha) = \log \frac{\max |1 \pm \sqrt{\alpha}|^2}{\min |1 \pm \sqrt{\alpha}|^2} \tag{2.7}$$

*(note that $\varrho(\alpha) > 0$ since $\alpha \notin (-\infty, 0]$). Moreover, let*

$$\mathrm{h}^*(\alpha) = \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \sum_{v \in \mathcal{M}_{\mathbb{Q}(\alpha)}} \eta_v \log^+ |\alpha|_v^*, \tag{2.8}$$

*where*

$$|\alpha|_v^* = \begin{cases} \max\left\{\left|1+\sqrt{\sigma(\alpha)}\right|^2, \left|1-\sqrt{\sigma(\alpha)}\right|^2\right\}, & \text{if } v \mid \infty, \ v = v_\sigma \\ |\alpha|_v, \quad \text{if } v \nmid \infty, \end{cases}$$

*and let*

$$\lambda(\alpha) = \frac{1}{\delta} - \frac{1+\mathrm{h}^*(\alpha)}{\varrho(\alpha)}.$$

*If* $\lambda(\alpha) > 0$, *we have* $\log\alpha \notin \mathbb{Q}(\alpha)$ *and*

$$\mu_{\mathbb{Q}(\alpha)}(\log\alpha) \le \lambda(\alpha)^{-1}.$$

Using the arithmetic method of [RV] alluded to in the introduction (compare with [V]), we can improve the above result. Let $j$ and $l$ be two fixed integers with $j > l \ge 0$, and let $z \ne 0, 1$ be a complex parameter. Consider the function of the complex variable $t$:

$$K_{j,l}(z;t) = \frac{(t-1)^j(z-t)^j}{t^{j-l}}. \tag{2.9}$$

The stationary points of $K_{j,l}(z;t)$, with respect to $t$, satisfying

$$\frac{\partial}{\partial t}K_{j,l}(z;t) = 0 \quad \text{and} \quad K_{j,l}(z;t) \ne 0$$

are the roots $t = t_+(z)$ and $t = t_-(z)$ of the quadratic equation

$$(j+l)t^2 - l(z+1)t - (j-l)z = 0,$$

i.e.,

$$t_\pm(z) = \frac{l(z+1) \pm \sqrt{l^2(z+1)^2 + 4(j^2-l^2)z}}{2(j+l)}. \tag{2.10}$$

If $l^2(z+1)^2 + 4(j^2-l^2)z \ne 0$, we choose the square root in (2.10), i.e., the notation $t_+(z)$ or $t_-(z)$ for each of the two stationary points, according to the following construction. Let $\mathcal{S}$ be the surface in $\mathbb{R}^3 = \{(\mathrm{Re}(t), \mathrm{Im}(t), u)\}$ given by the equation $u = |K_{j,l}(z;t)|$. The stationary points $t_\pm(z)$ are the projections on $\mathbb{R}^2 = \{(\mathrm{Re}(t), \mathrm{Im}(t))\}$ of two saddle points of $\mathcal{S}$, which, by abuse of notation, we also denote $t_\pm(z)$. If $z \notin (-\infty, 0]$, let $\sigma_z$ denote the segment in $\mathbb{C}$ of endpoints 1 and $z$. If $z \in \mathbb{R}$, $z < 0$, let $\sigma_z$ denote a simple path from 1 to $z$ entirely contained (except for the endpoints) in the upper half-plane $\mathrm{Im}(t) > 0$. Since

$$K_{j,l}(z;1) = K_{j,l}(z;z) = 0$$

and

$$\lim_{t\to 0} K_{j,l}(z;t) = \lim_{t\to\infty} K_{j,l}(z;t) = \infty,$$

keeping the endpoints 1 and $z$ fixed we can deform continuously $\sigma_z$ in $\mathbb{C}\setminus\{0\}$ (i.e., without crossing the origin) into a path $\gamma_z$ of endpoints 1 and $z$ such that the maximum of $|K_{j,l}(z;t)|$ on $\gamma_z$ is attained at a saddle point $t_{\max}$ of $\mathcal{S}$, and the tangent to the curve $\gamma_z$ at $t_{\max}$ has the direction of the steepest descent over the surface $\mathcal{S}$. We denote

$$t_+(z) = t_{\max}. \tag{2.11}$$

For brevity, let

$$K_+(z) = K_{j,l}(z;\ t_+(z)) \quad\text{and}\quad K_-(z) = K_{j,l}(z;\ t_-(z)). \tag{2.12}$$

Let $\alpha$ and $\delta$ be as in the statement of Theorem 2.2, and assume

$$|K_+(\alpha)| < |K_-(\alpha)|. \tag{2.13}$$

Moreover, let

$$\varrho_{j,l}(\alpha) = \log\frac{|K_-(\alpha)|}{|K_+(\alpha)|},$$

and

$$\mathrm{h}^*_{j,l}(\alpha) = \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]}\sum_{v\in\mathcal{M}_{\mathbb{Q}(\alpha)}} \eta_v \log^+ |\alpha|^*_{j,l;v},$$

where

$$|\alpha|^*_{j,l;v} = \begin{cases} \left(\max\{|K_+(\sigma(\alpha))|,\ |K_-(\sigma(\alpha))|\}\right)^{1/(j+l)}, & \text{if } v\mid\infty,\ v=v_\sigma \\ |\alpha|_v, \quad \text{if } v\nmid\infty. \end{cases}$$

Finally, let $\Omega_{j,l}$ be the set of the real numbers $\omega\in[0,1)$ satisfying

$$[(j-l)\omega]+[(j+l)\omega] < 2[j\omega],$$

where $[x]$ denotes the integral part of $x$. We prove:

THEOREM 2.3. *With the above notation and assumptions, if*

$$\lambda_{j,l}(\alpha) = \frac{1}{\delta} - \frac{(j+l)\left(1+\mathrm{h}^*_{j,l}(\alpha)\right) - \int_{\Omega_{j,l}} d\psi(x)}{\varrho_{j,l}(\alpha)} > 0,$$

*where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma-function, then $\log\alpha\notin\mathbb{Q}(\alpha)$ and*

$$\mu_{\mathbb{Q}(\alpha)}(\log\alpha) \le \lambda_{j,l}(\alpha)^{-1}.$$

Since $j$ and $l$ are integers, the set $\Omega_{j,l}$ is the union of finitely many intervals $[r,s)$ with rational endpoints $r$ and $s$, whence $\int_{\Omega_{j,l}} d\psi(x)$ is a linear combination with coefficients $\pm1$ of the values of $\psi(x)$ at finitely many rational points.

Note that $\varrho_{1,0}(\alpha)=\varrho(\alpha)$, $\mathrm{h}^*_{1,0}(\alpha)=\mathrm{h}^*(\alpha)$ and $\Omega_{1,0}=\varnothing$, whence $\lambda_{1,0}(\alpha)=\lambda(\alpha)$. Thus Theorem 2.2 is a special case of Theorem 2.3.

For the computation of $\mathbb{K}$-irrationality measures, we shall employ the following lemma.

LEMMA 2.4. *Let $\mathbb{K} \subset \mathbb{C}$ be a number field, and let $\delta$ be defined by* (2.5). *Let $\xi \in \mathbb{C}$, and let $(\vartheta_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathbb{K}$ satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \log|\xi - \vartheta_n| = -\varrho \quad \textit{and} \quad \limsup_{n\to\infty} \frac{1}{n} \,\mathrm{h}(\vartheta_n) \le c \tag{2.14}$$

*for positive real numbers $\varrho$ and $c$. If*

$$\lambda = \frac{1}{\delta} - \frac{c}{\varrho} > 0, \tag{2.15}$$

*then $\xi \notin \mathbb{K}$ and $\mu_{\mathbb{K}}(\xi) \le \lambda^{-1}$.*

PROOF. First we prove that $\xi \notin \mathbb{K}$. If we had $\xi \in \mathbb{K}$, then $\xi - \vartheta_n \in \mathbb{K}$ and, by (2.14), $\xi - \vartheta_n \neq 0$. Hence, by the Liouville inequality (2.4) and by (2.2),

$$-\log|\xi - \vartheta_n| \le \delta \,\mathrm{h}(\xi - \vartheta_n) \le \delta(\mathrm{h}(\xi) + \mathrm{h}(\vartheta_n) + \log 2).$$

Dividing by $n$ and letting $n \to \infty$ yield, by (2.14), $\varrho \le \delta c$. This contradicts (2.15). Therefore $\xi \notin \mathbb{K}$.

Since $0 < \delta\lambda < 1$, we can choose $\varepsilon$ such that

$$0 < \varepsilon < \frac{1}{8} \quad \text{and} \quad 1 + 5\varepsilon < (\delta\lambda)^{-1}.$$

Let $\beta \in \mathbb{K}$. All the inequalities written below hold for $\mathrm{h}(\beta) > h_0 = h_0(c,\varrho,\delta,\varepsilon) > 0$, where $h_0$ can be effectively computed. By (2.14), there exists an integer $n_0 > 0$ such that

$$\varrho(1 - \delta\lambda\varepsilon)n < -\log|\xi - \vartheta_n| < \varrho(1 + \delta\lambda\varepsilon)n$$

and

$$\mathrm{h}(\vartheta_n) < (c + \varrho\lambda\varepsilon)n$$

for all $n > n_0$. Let

$$m = \left[\frac{(1+4\varepsilon)\,\mathrm{h}(\beta)}{\varrho\lambda}\right],$$

whence

$$n_0 < (1+4\varepsilon)\left(\frac{\mathrm{h}(\beta)}{\varrho\lambda} - 1\right) < m \le \frac{(1+4\varepsilon)\,\mathrm{h}(\beta)}{\varrho\lambda}.$$

We distinguish two cases.

*First case*: $\vartheta_m \neq \beta$. By (2.2) and (2.4) we have

$$\begin{aligned} -\log|\vartheta_m - \beta| &\le \delta \,\mathrm{h}(\vartheta_m - \beta) \\ &\le \delta(\mathrm{h}(\vartheta_m) + \mathrm{h}(\beta) + \log 2) \\ &< \delta(c + \varrho\lambda\varepsilon)m + \delta(\mathrm{h}(\beta) + \log 2). \end{aligned} \tag{2.16}$$

Therefore, since $\varrho - \delta c = \varrho\lambda\delta$ and $1+\varepsilon < (1+4\varepsilon)(1-2\varepsilon) < 2$, we obtain

$$\begin{aligned}\log|\xi - \vartheta_m| - \log|\vartheta_m - \beta| &< -m(\varrho - \delta c - 2\varrho\lambda\delta\varepsilon) + \delta(\mathrm{h}(\beta) + \log 2)\\ &< -(1+4\varepsilon)(1-2\varepsilon)\left(\frac{\mathrm{h}(\beta)}{\varrho\lambda} - 1\right)\varrho\lambda\delta + \delta(\mathrm{h}(\beta) + \log 2)\\ &= (1-(1+4\varepsilon)(1-2\varepsilon))\delta\,\mathrm{h}(\beta) + \delta((1+4\varepsilon)(1-2\varepsilon)\varrho\lambda + \log 2)\\ &< -\varepsilon\delta\,\mathrm{h}(\beta) + \delta(2\varrho\lambda + \log 2)\\ &< -\log 2,\end{aligned}$$

whence $|\xi - \vartheta_m| < \frac{1}{2}|\vartheta_m - \beta|$. By the triangle inequality we get $\frac{1}{2}|\vartheta_m - \beta| < |\xi - \beta|$ and, again by (2.16),

$$\begin{aligned}-\log|\xi - \beta| &< \delta((c + \varrho\lambda\varepsilon)m + \mathrm{h}(\beta) + \log 2) + \log 2\\ &< (1+4\varepsilon)\delta\left(\frac{c + \varrho\lambda\varepsilon}{\varrho\lambda} + 1\right)\mathrm{h}(\beta) + (\delta+1)\log 2\\ &= (1+4\varepsilon)(\lambda^{-1} + \delta\varepsilon)\,\mathrm{h}(\beta) + (\delta+1)\log 2\\ &< (1+5\varepsilon)(\lambda^{-1} + \delta\varepsilon)\,\mathrm{h}(\beta)\\ &< (1+6\varepsilon)\lambda^{-1}\,\mathrm{h}(\beta).\end{aligned}$$

*Second case*: $\vartheta_m = \beta$. Then $-\log|\xi - \beta| = -\log|\xi - \vartheta_m| < \varrho(1+\delta\lambda\varepsilon)m$, whence

$$\begin{aligned}-\log|\xi - \beta| &< (1+4\varepsilon)(1+\delta\lambda\varepsilon)\lambda^{-1}\,\mathrm{h}(\beta)\\ &= (1+4\varepsilon)(\lambda^{-1} + \delta\varepsilon)\,\mathrm{h}(\beta)\\ &< (1+6\varepsilon)\lambda^{-1}\,\mathrm{h}(\beta).\end{aligned}$$

□

## 3. – Proof of Theorem 2.2 and first examples

Let $z \in \mathbb{C}$, $z \neq 1$, $z \notin (-\infty, 0]$, and let $n$ be a positive integer. Define

$$I_n(z) = (z-1)^{2n+1}\int_0^1 \left(\frac{x(1-x)}{1+x(z-1)}\right)^n \frac{dx}{1+x(z-1)}. \tag{3.1}$$

With the change of variable $1 + x(z-1) = t$ we obtain

$$I_n(z) = \int_1^z \left(\frac{(t-1)(z-t)}{t}\right)^n \frac{dt}{t}, \tag{3.2}$$

where the integration path is the segment in $\mathbb{C}$ joining 1 with $z$. As in [V] pp. 354–355, from (3.2) we easily get, by the binomial theorem,

$$I_n(z) = a_n(z) + b_n(z) \log z, \tag{3.3}$$

where $\log z = \int_1^z dt/t$ with the same integration path as in (3.2), whence $\log z = \log |z| + i \arg z$ with $-\pi < \arg z < \pi$, and where

$$a_n(z) = 2 \sum_{0 \le h < k \le n} (-1)^{h+k} \binom{n}{h} \binom{n}{k} \frac{z^k - z^h}{k - h} \tag{3.4}$$

and

$$b_n(z) = \sum_{k=0}^{n} \binom{n}{k}^2 z^k. \tag{3.5}$$

Thus $a_n(z)$ and $b_n(z)$ are polynomials of degree $n$ satisfying $b_n(z) \in \mathbb{Z}[z]$ and $d_n a_n(z) \in \mathbb{Z}[z]$, where

$$d_n = \operatorname{lcm}\{1, \ldots, n\}. \tag{3.6}$$

Again using the binomial theorem, we get

$$\frac{1}{2\pi i} \oint \left( \frac{(t-1)(z-t)}{t} \right)^n \frac{dt}{t} = \sum_{k=0}^{n} \binom{n}{k}^2 z^k,$$

where the integration is made over a contour enclosing $t = 0$. Hence, by (3.5),

$$b_n(z) = \frac{1}{2\pi i} \oint \left( \frac{(t-1)(z-t)}{t} \right)^n \frac{dt}{t}. \tag{3.7}$$

LEMMA 3.1. *Let* $z \in \mathbb{C}$, $z \neq 1$, $z \notin (-\infty, 0]$, *and let* $I_n(z)$, $a_n(z)$ *and* $b_n(z)$ *be as in* (3.1)- (3.5). *Then*

$$\lim_{n \to \infty} |I_n(z)|^{1/n} = \min \left| 1 \pm \sqrt{z} \right|^2$$

*and*

$$\lim_{n \to \infty} |a_n(z)|^{1/n} = \lim_{n \to \infty} |b_n(z)|^{1/n} = \max \left| 1 \pm \sqrt{z} \right|^2.$$

*Moreover, for any* $z \in \mathbb{R}$, $z < 0$, *the polynomials* $a_n(z)$ *and* $b_n(z)$ *satisfy*

$$\max \left\{ \limsup_{n \to \infty} |a_n(z)|^{1/n}, \limsup_{n \to \infty} |b_n(z)|^{1/n} \right\} \le \left| 1 + \sqrt{z} \right|^2 = \left| 1 - \sqrt{z} \right|^2.$$

PROOF. We apply the saddle point method (see, e.g., [D] pp. 279–285) to evaluate asymptotically $I_n(z)$ and $b_n(z)$ as $n \to \infty$. Using the notation (2.9), we abbreviate

$$K(z;t) = K_{1,0}(z;t) = \frac{(t-1)(z-t)}{t}.$$

The saddle points of the surface $u = |K(z;t)|$ are now given by (2.10) for $j=1$, $l=0$, i.e., $t_\pm(z) = \pm\sqrt{z}$. We distinguish two cases, depending on whether the complex parameter $z \neq 0, 1$ satisfies $z \notin (-\infty, 0]$ or $z \in (-\infty, 0)$.

If $z \notin (-\infty, 0]$, it is easily seen that, according to (2.11), $t_+(z) = \sqrt{z}$ is the square root satisfying $\mathrm{Re}(\sqrt{z}) > 0$, so that $\mathrm{Im}(z)$ and $\mathrm{Im}(\sqrt{z})$ have the same sign. With the notation (2.12) we have

$$|K_+(z)| = \left|K(z;\sqrt{z})\right| = \left|1-\sqrt{z}\right|^2$$

and

$$|K_-(z)| = \left|K(z;-\sqrt{z})\right| = \left|1+\sqrt{z}\right|^2,$$

whence $|K_+(z)| < |K_-(z)|$. In the integral (3.2) we deform the segment of fixed endpoints 1 and $z$ into the integration path $\gamma_z$ described in Section 2, so that the maximum of $|K(z;t)|$ on $\gamma_z$ is attained at $t = \sqrt{z}$. By the saddle point method we immediately obtain

$$\lim_{n\to\infty} |I_n(z)|^{1/n} = |K_+(z)| = \min\left|1 \pm \sqrt{z}\right|^2. \tag{3.8}$$

Similarly, we can deform the integration contour in (3.7) so that the maximum of $|K(z;t)|$ on the contour is attained at $t = -\sqrt{z}$. Again by the saddle point method we get

$$\lim_{n\to\infty} |b_n(z)|^{1/n} = |K_-(z)| = \max\left|1 \pm \sqrt{z}\right|^2. \tag{3.9}$$

By (3.3), (3.8) and (3.9) we have

$$\begin{aligned}\lim_{n\to\infty} |a_n(z)|^{1/n} &= \lim_{n\to\infty} |b_n(z)|^{1/n} \lim_{n\to\infty} \left|\frac{I_n(z)}{b_n(z)} - \log z\right|^{1/n} \\ &= \max\left|1 \pm \sqrt{z}\right|^2.\end{aligned} \tag{3.10}$$

If $z \in \mathbb{R}$, $z < 0$, by (2.11) $t_+(z) = \sqrt{z}$ is the square root satisfying $\mathrm{Im}(\sqrt{z}) > 0$. We now define

$$I_n^+(z) = \int_{\gamma_z} \left(K(z;t)\right)^n \frac{dt}{t}, \tag{3.11}$$

where $\gamma_z$ is the path from 1 to $z$, described in Section 2, such that the maximum of $|K(z;t)|$ on $\gamma_z$ is attained at $t = \sqrt{z}$. Again by the binomial theorem, we obviously have

$$I_n^+(z) = a_n(z) + b_n(z)\log z, \tag{3.12}$$

where now $\log z = \log|z| + i\pi$, and where $a_n(z)$ and $b_n(z)$ are the polynomials (3.4) and (3.5). In particular, (3.7) still holds. Since, in the present case, $\mathrm{Re}(\sqrt{z}) = 0$, we have $|1-\sqrt{z}|^2 = |1+\sqrt{z}|^2$, i.e.,

$$|K_+(z)| = |K_-(z)|. \tag{3.13}$$

From (3.11) we get, by the saddle point method,

$$\lim_{n\to\infty} |I_n^+(z)|^{1/n} = \left|1-\sqrt{z}\right|^2 = \left|1+\sqrt{z}\right|^2. \tag{3.14}$$

In the integral representation (3.7) for $b_n(z)$ we change the integration contour into $\gamma_z \cup \overline{\gamma_z}$, where $\gamma_z$ is the path oriented from 1 to $z$ considered above in (3.11), and where the conjugate $\overline{\gamma_z}$ is oriented from $z$ to 1. The contour $\gamma_z \cup \overline{\gamma_z}$ passes through the saddle points $\pm\sqrt{z}$, and, by (3.13), we cannot apply the saddle point method for the asymptotic estimate of $|b_n(z)|^{1/n}$ as we did in the previous case. Since now $z \in \mathbb{R}$, we have $\overline{K(z;t)} = K(z;\bar{t})$. Thus, by (3.11),

$$\overline{I_n^+(z)} = -\int_{\overline{\gamma_z}} (K(z;t))^n \,\frac{dt}{t},$$

whence

$$\begin{aligned} b_n(z) &= \frac{1}{2\pi i}\left(\int_{\gamma_z} (K(z;t))^n \,\frac{dt}{t} + \int_{\overline{\gamma_z}} (K(z;t))^n \,\frac{dt}{t}\right) \\ &= \frac{1}{2\pi i}\left(I_n^+(z) - \overline{I_n^+(z)}\right) = \frac{1}{\pi}\,\mathrm{Im}(I_n^+(z)). \end{aligned}$$

Therefore

$$|b_n(z)| \le \frac{1}{\pi}\,|I_n^+(z)|$$

and, by (3.12),

$$|a_n(z)| \le |I_n^+(z)| + |b_n(z)|\,|\log z| \le \left(1 + \frac{1}{\pi}\,|\log z|\right)|I_n^+(z)|.$$

Hence, by (3.14),

$$\limsup_{n\to\infty} |b_n(z)|^{1/n} \le \left|1-\sqrt{z}\right|^2 = \left|1+\sqrt{z}\right|^2$$

and

$$\limsup_{n\to\infty} |a_n(z)|^{1/n} \le \left|1-\sqrt{z}\right|^2 = \left|1+\sqrt{z}\right|^2. \qquad \square$$

PROOF OF THEOREM 2.2. Since $\alpha \notin (-\infty, 0]$, Lemma 3.1 implies in particular that $a_n(\alpha) \neq 0$ and $b_n(\alpha) \neq 0$ for every sufficiently large $n$. Thus we may apply Lemma 2.4 with $\mathbb{K} = \mathbb{Q}(\alpha)$, $\xi = \log\alpha$ and $\vartheta_n = -a_n(\alpha)/b_n(\alpha)$, whence, by (3.3), $\xi - \vartheta_n = I_n(\alpha)/b_n(\alpha)$. By Lemma 3.1 we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\log|\xi - \vartheta_n| &= \lim_{n\to\infty} \frac{1}{n}\log|I_n(\alpha)| - \lim_{n\to\infty}\frac{1}{n}\log|b_n(\alpha)| \\ &= -\log\frac{\max|1\pm\sqrt{\alpha}|^2}{\min|1\pm\sqrt{\alpha}|^2} = -\varrho(\alpha).\end{aligned}$$

For the required upper asymptotic estimate of the Weil height $\mathrm{h}(\vartheta_n)$, take any $v \in \mathcal{M}_{\mathbb{Q}(\alpha)}$. If $v \mid \infty$, let $v = v_\sigma$ be associated with an embedding $\sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$. As we remarked in the introduction, we have $|a_n(\alpha)|_v = |a_n(\sigma(\alpha))|$ and $|b_n(\alpha)|_v = |b_n(\sigma(\alpha))|$. Let $d_n$ be defined by (3.6), whence, by the prime number theorem, $\lim\limits_{n\to\infty} \frac{1}{n}\log d_n = 1$. Then, again by Lemma 3.1,

$$\begin{aligned}(3.15)\qquad &\limsup_{n\to\infty} \frac{1}{n}\log\max\{|d_n a_n(\alpha)|_v,\ |d_n b_n(\alpha)|_v\} \\ &\quad= 1 + \limsup_{n\to\infty}\frac{1}{n}\log\max\{|a_n(\sigma(\alpha))|,\ |b_n(\sigma(\alpha))|\} \\ &\quad\le 1 + \log\max\left|1\pm\sqrt{\sigma(\alpha)}\right|^2 = 1 + \log^+|\alpha|_v^*.\end{aligned}$$

If $v \mid p$ where $p$ is a prime, we use the ultrametric inequality $|x_1 + \cdots + x_r|_v \le \max\{|x_1|_v, \ldots, |x_r|_v\}$, and in particular $|m|_v \le 1$ for $m \in \mathbb{Z}$. Since $d_n a_n$ and $d_n b_n$ are polynomials of degrees $\le n$ with integer coefficients, we get

$$|d_n a_n(\alpha)|_v \le \left(\max\{1, |\alpha|_v\}\right)^n$$

and

$$|d_n b_n(\alpha)|_v \le \left(\max\{1, |\alpha|_v\}\right)^n,$$

whence

$$(3.16)\qquad \limsup_{n\to\infty}\frac{1}{n}\log\max\{|d_n a_n(\alpha)|_v,\ |d_n b_n(\alpha)|_v\} \le \log^+|\alpha|_v^*.$$

By the product formula (2.3) with $\beta = d_n b_n(\alpha)$ we obtain

$$\begin{aligned}\frac{1}{n}\,\mathrm{h}(\vartheta_n) &= \frac{1}{n[\mathbb{Q}(\alpha):\mathbb{Q}]}\left(\sum_{v\in\mathcal{M}_{\mathbb{Q}(\alpha)}} \eta_v \log^+\left|\frac{d_n a_n(\alpha)}{d_n b_n(\alpha)}\right|_v + \sum_{v\in\mathcal{M}_{\mathbb{Q}(\alpha)}}\eta_v\log|d_n b_n(\alpha)|_v\right) \\ &= \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]}\sum_{v\in\mathcal{M}_{\mathbb{Q}(\alpha)}}\eta_v\,\frac{1}{n}\log\max\{|d_n a_n(\alpha)|_v,\ |d_n b_n(\alpha)|_v\},\end{aligned}$$

whence, by (3.15) and (3.16),

$$\limsup_{n\to\infty} \frac{1}{n}\,\mathrm{h}(\vartheta_n) \le \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]}\left(\sum_{v\mid\infty}\eta_v(1+\log^+|\alpha|_v^*)+\sum_{v\nmid\infty}\eta_v\log^+|\alpha|_v^*\right)$$
$$= \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]}\sum_{v\mid\infty}\eta_v + \mathrm{h}^*(\alpha) \;=\; 1+\mathrm{h}^*(\alpha).$$

Lemma 2.4 with $\varrho = \varrho(\alpha)$ and $c = 1+\mathrm{h}^*(\alpha)$ yields the desired conclusion. □

In Theorem 2.2, the cases where either $\alpha \in \mathbb{Q}$, or $\mathbb{Q}(\alpha)$ is an imaginary quadratic extension of $\mathbb{Q}$, are covered by Theorem 1 of [AR]. Thus the first interesting case is that of real quadratic irrationals, for which $\delta = 2$. Let $\alpha = a + b\sqrt{m} > 0$, with a square-free $m \in \mathbb{N}$ and with $a, b \in \mathbb{Z}$, and let $\alpha' = a - b\sqrt{m}$. Then

$$\mathrm{h}^*(\alpha) = \log(1+\sqrt{\alpha}) + \log\left|1+\sqrt{\alpha'}\,\right|,$$
$$\varrho(\alpha) = 2\log(1+\sqrt{\alpha}) - 2\log|1-\sqrt{\alpha}\,|,$$

and

$$\lambda(\alpha) = -\,\frac{1+\log|1-\sqrt{\alpha}\,|+\log\left|1+\sqrt{\alpha'}\,\right|}{2\log(1+\sqrt{\alpha}) - 2\log|1-\sqrt{\alpha}\,|}\,.$$

Therefore, if $\lambda(\alpha) > 0$ then $\lambda(\alpha)^{-1}$ is a $\mathbb{Q}(\alpha)$-irrationality measure of $\log\alpha$.

Take for instance $\alpha = \sqrt{2}$. We get:

$$\mu_{\mathbb{Q}(\sqrt{2})}(\log\sqrt{2}) = \mu_{\mathbb{Q}(\sqrt{2})}(\log 2) < 21.8391. \tag{3.17}$$

We recall that, by a result of Reyssat [R],

$$\log|\log 2 - \beta\,| \ge -210\,\mathrm{h}(\beta),$$

for every quadratic irrational $\beta$ of sufficiently large logarithmic height (note that the heights employed by Reyssat are not normalized, so that, using our notation, the non-quadraticity measure 105 of $\log 2$ found by Reyssat must be multiplied by the factor $2 = [\,\mathbb{Q}(\beta):\mathbb{Q}\,]$). Recently, Hata [H] has improved this result by replacing 210 with 50.0926 (again, in our notation Hata's exponent 25.0463 must be doubled). Our result (3.17) is quantitatively better, but holds only for quadratic irrationals $\beta \in \mathbb{Q}(\sqrt{2})$.

Other interesting examples are obtained by taking $\alpha = 1 + p - q\sqrt{2}$, where $p/q$ is a convergent in the continued fraction expansion of $\sqrt{2}$. For instance, we get

| $\alpha =$ | $\mu_{\mathbb{Q}(\sqrt{2})}(\log\alpha) <$ |
|---|---|
| $8 - 5\sqrt{2}$ | 10.8912 |
| $18 - 12\sqrt{2}$ | 7.6234 |
| $3364 - 2378\sqrt{2}$ | 4.9905. |

Other examples are given by the square roots of rationals close to 1:

| $r =$ | $\mu_{\mathbb{Q}(\sqrt{r})}(\log r) <$ |
|---|---|
| 3/2 | 12.1383 |
| 4/3 | 10.0343 |
| 5/4 | 9.0645 |
| 100/99 | 5.7096. |

We now consider the case of cubic irrationals. We only give two examples: the first with a cubic irrational $\alpha \notin \mathbb{R}$, and the second with $\alpha \in \mathbb{R}$. Let $\alpha$ be one of the two complex conjugate roots of the polynomial $x^3 - 5x + 5$. Then

$$\mu_{\mathbb{Q}(\alpha)}(\log \alpha) < 9.5039.$$

Let now $\alpha = \sqrt[3]{7/6}$. Then

$$\mu_{\mathbb{Q}(\sqrt[3]{7/6})}(\log(7/6)) < 101.8212.$$

In order to estimate how far the above numerical results are from the best that one may expect, we prove the following proposition (compare with (2.6)).

PROPOSITION 3.2. *Let $\mathbb{K} \subset \mathbb{C}$ be a number field, and let $\delta$ be defined by* (2.5). *There exists a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{K}$ such that $\alpha_n \neq 1$, $\alpha_n \notin (-\infty, 0]$, and*

$$\lim_{n \to \infty} \lambda(\alpha_n)^{-1} = 2\delta,$$

*where*

$$\lambda(\alpha) = \frac{1}{\delta} - \frac{1 + \mathrm{h}^*(\alpha)}{\varrho(\alpha)}. \tag{3.18}$$

The proof of Proposition 3.2 is an immediate consequence of the following Lemmas 3.3 and 3.4.

LEMMA 3.3. *Let $(\beta_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathbb{K}$ such that $\beta_n \neq 0$, $\lim_{n \to \infty} \beta_n = 0$, and, for a real number $\tau > 0$,*

$$\lim_{n \to \infty} \frac{\log |\beta_n|}{\mathrm{h}(\beta_n)} = -\tau. \tag{3.19}$$

*Then*

$$\lim_{n \to \infty} \lambda(1 + \beta_n) = \frac{1}{\delta} - \frac{1}{2\tau}.$$

PROOF. For any $z \in \mathbb{C}$ we plainly have

$$\max\{|z|, 1\} \leq \max \left|1 \pm \sqrt{z}\right|^2 \leq 4 \max\{|z|, 1\}. \tag{3.20}$$

Also, for any $z \in \mathbb{C}$, $z \neq 1$, let, as in (2.7),

$$\varrho(z) = \log \frac{\max|1 \pm \sqrt{z}\,|^2}{\min|1 \pm \sqrt{z}\,|^2} = -2\log|1-z| + 2\log\max\left|1 \pm \sqrt{z}\,\right|^2.$$

Then, by (3.20),

$$0 \leq \varrho(z) + 2\log|1-z| \leq 2\log\bigl(4\max\{|z|, 1\}\bigr) = 4\log 2 + 2\log^+|z|.$$

Therefore

$$-2\log|1-z| \leq \varrho(z) \leq -2\log|1-z| + 4\log 2 + 2\log^+|z|. \tag{3.21}$$

Let now $\alpha \in \mathbb{K}$, $\alpha \neq 0, 1$. Taking $\mathbb{K} = \mathbb{Q}(\alpha)$ in the definition (2.1) for $\mathrm{h}(\alpha)$, we get, by (2.8),

$$\mathrm{h}^*(\alpha) - \mathrm{h}(\alpha) = \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \sum_{\substack{v|\infty \\ v=v_\sigma}} \eta_v \left( \log\max\left|1 \pm \sqrt{\sigma(\alpha)}\right|^2 - \log\max\{|\sigma(\alpha)|, 1\} \right).$$

Hence, by (3.20),

$$0 \leq \mathrm{h}^*(\alpha) - \mathrm{h}(\alpha) \leq \frac{2\log 2}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \sum_{v|\infty} \eta_v = 2\log 2.$$

For $\alpha = 1 + \beta_n$ we get

$$\mathrm{h}(1+\beta_n) \leq \mathrm{h}^*(1+\beta_n) \leq \mathrm{h}(1+\beta_n) + 2\log 2,$$

whence, by (2.2),

$$\mathrm{h}(\beta_n) - \log 2 \leq \mathrm{h}^*(1+\beta_n) \leq \mathrm{h}(\beta_n) + 3\log 2. \tag{3.22}$$

Taking $z = 1 + \beta_n$ in (3.21), we get, for any sufficiently large $n$,

$$-2\log|\beta_n| \leq \varrho(1+\beta_n) \leq -2\log|\beta_n| + 5\log 2. \tag{3.23}$$

Therefore, by (3.19), (3.22) and (3.23),

$$\lim_{n\to\infty} \frac{1 + \mathrm{h}^*(1+\beta_n)}{\varrho(1+\beta_n)} = \frac{1}{2\tau},$$

whence, by (3.18),

$$\lim_{n\to\infty} \lambda(1+\beta_n) = \frac{1}{\delta} - \frac{1}{2\tau}. \qquad \square$$

LEMMA 3.4. *There exists a sequence $(\beta_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{K}$ satisfying the assumptions of Lemma 3.3 with $\tau = \delta$.*

PROOF. If $\delta = 1$, we take $\beta_n = 1/n$. If $\delta > 1$, let $\nu = [\mathbb{K} : \mathbb{Q}]$ and let $\mathcal{B} = \{u_1, \dots, u_\nu\}$ be a basis in the ring of integers of $\mathbb{K}$. We apply Lemma 1.3.2 of [W], p. 11, with $X = X_n = n^2$, $l = l_n = n^{2\delta}$ and $U \geq |u_1| + \cdots + |u_\nu|$. If $\mathbb{K} \subset \mathbb{R}$, we take $\mu = 1$ and $u_{i,1} = u_i$ $(i = 1, \dots, \nu)$. If $\mathbb{K} \not\subset \mathbb{R}$, we take $\mu = 2$ and

$$\begin{cases} u_{i,1} = \mathrm{Re}(u_i) \\ u_{i,2} = \mathrm{Im}(u_i) \end{cases} \qquad (i = 1, \dots, \nu).$$

Then there exist $\xi_1^{(n)}, \dots, \xi_\nu^{(n)} \in \mathbb{Z}$, not all zero, with $\left|\xi_i^{(n)}\right| \leq X_n$ $(i = 1, \dots, \nu)$, such that

$$\beta_n = u_1\xi_1^{(n)} + \cdots + u_\nu\xi_\nu^{(n)}$$

satisfies

$$|\beta_n| \leq \sqrt{2}\, \frac{UX_n}{l_n} = c_1 n^{-2(\delta-1)},$$

where $c_1 = c_1(\mathcal{B}) > 0$. Note that $\beta_n \neq 0$ and

$$\mathrm{h}(\beta_n) \leq \frac{\delta - 1}{\delta} \log X_n + c_2 = \frac{2(\delta - 1)}{\delta} \log n + c_2,$$

with $c_2 = c_2(\mathcal{B}) > 0$. Therefore

$$\limsup_{n\to\infty} \frac{\log|\beta_n|}{\mathrm{h}(\beta_n)} \leq -\delta.$$

By Liouville's inequality (2.4) we obtain

$$\lim_{n\to\infty} \frac{\log|\beta_n|}{\mathrm{h}(\beta_n)} = -\delta. \qquad \square$$

## 4. – Proof of Theorem 2.3 and further examples

For $z \in \mathbb{C}$, $z \neq 1$, $z \notin (-\infty, 0]$, let

$$J_{h,j,l}(z) = (z-1)^{h+j+1} \int_0^1 \frac{x^h(1-x)^j}{(1+x(z-1))^{j-l}}\, \frac{dx}{1+x(z-1)}, \tag{4.1}$$

where $h$, $j$, $l$ are integers such that

$$h > 0 \quad \text{and} \quad j > l \geq 0. \tag{4.2}$$

As in Section 3, the change of variable $1+x(z-1)=t$ and the binomial theorem yield

$$(4.3)\qquad J_{h,j,l}(z)=\int_1^z \frac{(t-1)^h(z-t)^j}{t^{j-l}}\,\frac{dt}{t}=A_{h,j,l}(z)+B_{h,j,l}(z)\log z,$$

where $A_{h,j,l}(z)\in\mathbb{Q}[z]$ and $B_{h,j,l}(z)\in\mathbb{Z}[z]$ are polynomials of degrees not exceeding $\max\{j,\,h+l\}$. Moreover, if we let

$$(4.4)\qquad M=\max\{j-l,\,h+l\},$$

we easily obtain

$$(4.5)\qquad d_M A_{h,j,l}(z)\in\mathbb{Z}[z],$$

where $d_M=\operatorname{lcm}\{1,\ldots,M\}$. Also, as in (3.7),

$$(4.6)\qquad B_{h,j,l}(z)=\frac{1}{2\pi i}\oint \frac{(t-1)^h(z-t)^j}{t^{j-l}}\,\frac{dt}{t}\,.$$

We now apply the arithmetic method introduced in [V] for the search of irrationality measures of logarithms of rational numbers. Let ${}_2F_1(a,b;c;y)$ denote the Gauss hypergeometric function:

$${}_2F_1(a,b;c;y)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\,\frac{y^n}{n!}\,,$$

where $y$ is a complex variable satisfying $|y|<1$, and $a,\,b,\,c$ are any complex parameters with $c\neq 0,-1,-2,\ldots$. Here

$$(a)_0=1,\qquad (a)_n=a(a+1)\cdots(a+n-1)\quad (n=1,2,\ldots),$$

and similarly for $(b)_n$ and $(c)_n$. Obviously

$$(4.7)\qquad {}_2F_1(a,b;c;y)={}_2F_1(b,a;c;y).$$

It is well known that the integral representation

$$(4.8)\qquad {}_2F_1(a,b;c;y)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-xy)^a}\,dx,$$

where $\Gamma$ denotes the Euler gamma-function, holds for any $a,\,b,\,c$ such that $\operatorname{Re}(c)>\operatorname{Re}(b)>0$. Moreover, the integral in (4.8) is clearly a holomorphic function of $y$ in $\mathbb{C}\setminus[1,+\infty)$, so that (4.8) gives the analytic continuation of ${}_2F_1(a,b;c;y)$ for $y \notin [1,+\infty)$. Choosing $a = j-l+1$, $b = h+1$, $c = h+j+2$, $y = 1-z$, we get, by (4.1), (4.7) and (4.8),

$$J_{h,j,l}(z) = \frac{h!\,j!}{(j-l)!\,(h+l)!}\,J_{j-l,h+l,l}(z). \tag{4.9}$$

From (4.3) and (4.9) we have

$$(j-l)!\,(h+l)!\,A_{h,j,l}(z) = h!\,j!\,A_{j-l,h+l,l}(z). \tag{4.10}$$

We choose three fixed integers $h$, $j$, $l$ satisfying (4.2) and

$$j \le h+l,$$

so that, by (4.4),

$$\max\{h,j\} \le h+l = M.$$

Thus, by (4.5) applied to the polynomial $A_{j-l,h+l,l}(z)$, we obtain

$$d_M A_{j-l,h+l,l}(z) \in \mathbb{Z}[z]. \tag{4.11}$$

For any $n = 1,2,\ldots$ we have, by (4.10),

$$\big((j-l)n\big)!\,\big((h+l)n\big)!\,d_{Mn}A_{hn,jn,ln}(z) = (hn)!\,(jn)!\,d_{Mn}A_{(j-l)n,(h+l)n,ln}(z), \tag{4.12}$$

where

$$d_{Mn}A_{hn,jn,ln}(z),\ d_{Mn}A_{(j-l)n,(h+l)n,ln}(z) \in \mathbb{Z}[z]$$

by (4.5) and (4.11). For a prime $p$, let

$$\omega = \{n/p\} = n/p - [n/p]$$

denote the fractional part of $n/p$. Applying to (4.12) the arguments of [RV], pp. 44–45, it is easy to see that any prime $p > \sqrt{Mn}$ for which

$$[(j-l)\omega] + [(h+l)\omega] < [h\omega] + [j\omega] \tag{4.13}$$

divides all the coefficients of the polynomial

$$d_{Mn}A_{hn,jn,ln}(z).$$

Let $\Omega$ be the set of the real numbers $\omega \in [0,1)$ satisfying (4.13), and let

$$\Delta_n = \prod_{\substack{p>\sqrt{Mn}\\ \{n/p\}\in\Omega}} p \quad \text{and} \quad D_n = \frac{d_{Mn}}{\Delta_n} \quad (n = 1,2,\ldots).$$

Clearly

$$D_n A_{hn,jn,ln}(z) \in \mathbb{Z}[z].$$

By [RV], pp. 50-51, or by [V], p. 357, we easily obtain $D_n \in \mathbb{Z}$ and $\lim_{n\to\infty} \frac{1}{n}\log D_n = M - \int_\Omega d\psi(x) = h+l-\int_\Omega d\psi(x)$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma-function.

We have proved

LEMMA 4.1. *Let $h$, $j$, $l$ be integers such that $j > l \geq 0$ and $h \geq j - l$, let $A_{h,j,l}(z)$ be the polynomial defined by* (4.3), *and let $\Omega$ be the set of the real numbers $\omega \in [0, 1)$ satisfying* (4.13). *For any $n = 1, 2, \ldots$ there exists a positive integer $D_n$ such that*

$$D_n A_{hn,jn,ln}(z) \in \mathbb{Z}[z]$$

*and*

$$\lim_{n\to\infty} \frac{1}{n} \log D_n = h + l - \int_{\Omega} d\psi(x),$$

*where $\psi(x) = \Gamma'(x)/\Gamma(x)$.*

In the sequel, we apply Lemma 4.1 with $h = j$. We abbreviate $J_n(z)$, $A_n(z)$ and $B_n(z)$ for $J_{jn,jn,ln}(z)$, $A_{jn,jn,ln}(z)$ and $B_{jn,jn,ln}(z)$ respectively. Thus we write (4.3) and (4.6) as

$$J_n(z) = \int_1^z \left(K_{j,l}(z;t)\right)^n \frac{dt}{t} = A_n(z) + B_n(z) \log z \tag{4.14}$$

and

$$B_n(z) = \frac{1}{2\pi i} \oint \left(K_{j,l}(z;t)\right)^n \frac{dt}{t}, \tag{4.15}$$

with $K_{j,l}(z;t)$ defined by (2.9).

LEMMA 4.2. *Let $z \in \mathbb{C}$, $z \neq 1$, $z \notin (-\infty, 0]$, and let $J_n(z)$, $A_n(z)$ and $B_n(z)$ be as in* (4.14) *and* (4.15). *If $z$ satisfies*

$$|K_+(z)| < |K_-(z)|, \tag{4.16}$$

*with $K_+(z)$ and $K_-(z)$ given by* (2.12) *with the notation* (2.11), *then*

$$\lim_{n\to\infty} |J_n(z)|^{1/n} = |K_+(z)| \tag{4.17}$$

*and*

$$\lim_{n\to\infty} |A_n(z)|^{1/n} = \lim_{n\to\infty} |B_n(z)|^{1/n} = |K_-(z)|. \tag{4.18}$$

*Moreover, for any $z \in \mathbb{C}$, $z \neq 0, 1$, and $z$ not necessarily satisfying* (4.16), *the polynomials $A_n(z)$ and $B_n(z)$ satisfy*

$$\max\left\{\limsup_{n\to\infty} |A_n(z)|^{1/n}, \limsup_{n\to\infty} |B_n(z)|^{1/n}\right\} \leq \max\{|K_+(z)|, |K_-(z)|\}. \tag{4.19}$$

PROOF. The proof of (4.17) and (4.18), by virtue of the assumption (4.16), is exactly similar to the proof of (3.8), (3.9) and (3.10) in Lemma 3.1. As for

(4.19), we can deform the integration contour in (4.15) so that the maximum of $|K_{j,l}(z;t)|$ on the contour is attained either at just one of the two saddle points $t_{\pm}(z)$, or at both. In the former case, by the saddle point method the limit of $|B_n(z)|^{1/n}$ exists, and is either $|K_+(z)|$ or $|K_-(z)|$. In particular

$$\limsup_{n\to\infty} |B_n(z)|^{1/n} \le \max\{|K_+(z)|,\ |K_-(z)|\}.$$

In the latter case we have

$$|K_+(z)| = |K_-(z)| = k,$$

say, and the integration contour can be written as $\gamma_+ \cup \gamma_-$, where $\gamma_+$ is the path $\gamma_z$ described in Section 2, oriented from 1 to $z$ and passing through $t_+(z)$, and $\gamma_-$ is a path oriented from $z$ to 1 and passing through $t_-(z)$. Then

$$\begin{aligned} B_n(z) &= \pm\frac{1}{2\pi i}\left(\int_{\gamma_+} \left(K_{j,l}(z;t)\right)^n \frac{dt}{t} + \int_{\gamma_-} \left(K_{j,l}(z;t)\right)^n \frac{dt}{t}\right) \\ &= \pm\frac{1}{2\pi i}\left(J_n^+(z) + J_n^-(z)\right), \end{aligned}$$

say, where, by the saddle point method,

$$\lim_{n\to\infty} |J_n^+(z)|^{1/n} = \lim_{n\to\infty} |J_n^-(z)|^{1/n} = k.$$

Therefore

$$\limsup_{n\to\infty} |B_n(z)|^{1/n} \le k = |K_+(z)| = |K_-(z)|.$$

Finally, the required upper asymptotic estimate for $|A_n(z)|^{1/n}$ is obtained from

$$J_n^+(z) = \int_{\gamma_z} \left(K_{j,l}(z;t)\right)^n \frac{dt}{t} = A_n(z) + B_n(z)\log z,$$

and from the upper estimates for $|J_n^+(z)|$ and $|B_n(z)|$. This completes the proof of (4.19). □

Proof of Theorem 2.3 The arguments are similar to the proof of Theorem 2.2. Since $\alpha \notin (-\infty, 0]$, and by the assumption (2.13), Lemma 4.2 implies that $A_n(\alpha) \ne 0$ and $B_n(\alpha) \ne 0$ for every sufficiently large $n$. We apply Lemma 2.4 with $\mathbb{K} = \mathbb{Q}(\alpha)$, $\xi = \log\alpha$ and $\vartheta_n = -A_n(\alpha)/B_n(\alpha)$. By (4.14), $\xi - \vartheta_n = J_n(\alpha)/B_n(\alpha)$. By (4.17) and (4.18) we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\log|\xi - \vartheta_n| &= \lim_{n\to\infty} \frac{1}{n}\log|J_n(\alpha)| - \lim_{n\to\infty} \frac{1}{n}\log|B_n(\alpha)| \\ &= \log|K_+(\alpha)| - \log|K_-(\alpha)| = -\varrho_{j,l}(\alpha). \end{aligned}$$

For any $v \in \mathcal{M}_{\mathbb{Q}(\alpha)}$, $v \mid \infty$, $v = v_\sigma$, we have $|A_n(\alpha)|_v = |A_n(\sigma(\alpha))|$ and $|B_n(\alpha)|_v = |B_n(\sigma(\alpha))|$. Let $D_n$ be the integer in Lemma 4.1, where we choose $h = j$. Then, by Lemma 4.1 and by (4.19),

$$
\begin{aligned}
(4.20)\qquad & \limsup_{n\to\infty} \frac{1}{n} \log\max\{|D_n A_n(\alpha)|_v,\ |D_n B_n(\alpha)|_v\} \\
&= j + l - \int_{\Omega_{j,l}} d\psi(x) + \limsup_{n\to\infty} \frac{1}{n} \log\max\{|A_n(\sigma(\alpha))|,\ |B_n(\sigma(\alpha))|\} \\
&\le j + l - \int_{\Omega_{j,l}} d\psi(x) + (j+l)\log^+ |\alpha|^*_{j,l;v}\,.
\end{aligned}
$$

We recall that $D_n A_n(\alpha)$ and $D_n B_n(\alpha)$ are polynomials of degrees $\le (j+l)n$ with integer coefficients. Hence, for any $v \in \mathcal{M}_{\mathbb{Q}(\alpha)}$ with $v \nmid \infty$, we have

$$
(4.21)\qquad \limsup_{n\to\infty} \frac{1}{n} \log\max\{|D_n A_n(\alpha)|_v,\ |D_n B_n(\alpha)|_v\} \le (j+l)\log^+ |\alpha|^*_{j,l;v}\,.
$$

As in the proof of Theorem 2.2 we get, by the product formula (2.3) with $\beta = D_n B_n(\alpha)$,

$$
\frac{1}{n}\,\mathrm{h}(\vartheta_n) = \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \sum_{v\in\mathcal{M}_{\mathbb{Q}(\alpha)}} \eta_v \frac{1}{n} \log\max\{|D_n A_n(\alpha)|_v,\ |D_n B_n(\alpha)|_v\},
$$

whence, by (4.20) and (4.21),

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\,\mathrm{h}(\vartheta_n) &\le \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \Bigg( \sum_{v\mid\infty} \eta_v \Big(j + l - \int_{\Omega_{j,l}} d\psi(x) + (j+l)\log^+ |\alpha|^*_{j,l;v}\Big) \\
&\qquad\qquad + \sum_{v\nmid\infty} \eta_v (j+l)\log^+ |\alpha|^*_{j,l;v} \Bigg) \\
&= \frac{1}{[\mathbb{Q}(\alpha):\mathbb{Q}]} \sum_{v\mid\infty} \eta_v \Big(j + l - \int_{\Omega_{j,l}} d\psi(x)\Big) + (j+l)\,\mathrm{h}^*_{j,l}(\alpha) \\
&= (j+l)\big(1 + \mathrm{h}^*_{j,l}(\alpha)\big) - \int_{\Omega_{j,l}} d\psi(x).
\end{aligned}
$$

Theorem 2.3 follows, by Lemma 2.4 with $\varrho = \varrho_{j,l}(\alpha)$ and

$$
c = (j+l)\big(1 + \mathrm{h}^*_{j,l}(\alpha)\big) - \int_{\Omega_{j,l}} d\psi(x). \qquad \square
$$

Theorem 2.3 allows us to improve the numerical results given in Section 3, and to find further approximation measures.

Take $\alpha = \sqrt{2}$, and let $j = 5$ and $l = 1$. With the notation (2.11) we have

$$t_+(\sqrt{2}) = 1.19279135\ldots, \qquad t_-(\sqrt{2}) = -0.79042242\ldots,$$

and

$$\varrho_{5,1}(\sqrt{2}) = 24.28015944\ldots, \qquad \mathrm{h}^*_{5,1}(\sqrt{2}) = 1.02377484\ldots.$$

The set $\Omega_{5,1}$ is the union of the intervals

$$\left[\frac{1}{5}, \frac{1}{4}\right), \quad \left[\frac{2}{5}, \frac{1}{2}\right), \quad \left[\frac{3}{5}, \frac{2}{3}\right), \quad \left[\frac{4}{5}, \frac{5}{6}\right),$$

whence

$$\int_{\Omega_{5,1}} d\psi(x) = 1.95612483\ldots.$$

Theorem 2.3 yields

$$\mu_{\mathbb{Q}(\sqrt{2})}(\log 2) < 12.4288.$$

Let now $\alpha$ be one of the complex conjugate roots of the polynomial $x^3 - 5x + 5$, and let $j = 7$ and $l = 1$. If we take, e.g., $\alpha$ to be the root with $\mathrm{Im}(\alpha) > 0$, we have

$$\begin{aligned} \alpha &= 1.31368254\ldots + 0.42105280\ldots i, \\ t_+(\alpha) &= 1.15991035\ldots + 0.18557845\ldots i, \\ t_-(\alpha) &= -0.87070003\ldots - 0.13294685\ldots i. \end{aligned}$$

Also

$$\varrho_{7,1}(\alpha) = 30.54305490\ldots, \qquad \mathrm{h}^*_{7,1}(\alpha) = 1.29719805\ldots.$$

The set $\Omega_{7,1}$ is the union of the intervals

$$\left[\frac{1}{7}, \frac{1}{6}\right), \quad \left[\frac{2}{7}, \frac{1}{3}\right), \quad \left[\frac{3}{7}, \frac{1}{2}\right), \quad \left[\frac{4}{7}, \frac{5}{8}\right), \quad \left[\frac{5}{7}, \frac{3}{4}\right), \quad \left[\frac{6}{7}, \frac{7}{8}\right),$$

whence

$$\int_{\Omega_{7,1}} d\psi(x) = 2.31440700\ldots$$

(see [V]). By Theorem 2.3 we get

$$\mu_{\mathbb{Q}(\alpha)}(\log \alpha) < 7.105.$$

For $\alpha = \sqrt[3]{7/6}$, $j = 9$, $l = 1$, we have

$$t_+(\sqrt[3]{7/6}) = 1.02606228\ldots, \qquad t_-(\sqrt[3]{7/6}) = -0.82078962\ldots,$$

and

$$\varrho_{9,1}(\sqrt[3]{7/6}) = 78.27477707\ldots, \qquad \mathrm{h}^*_{9,1}(\sqrt[3]{7/6}) = 1.68890105\ldots.$$

The set $\Omega_{9,1}$ is the union of

$$\left[\tfrac{1}{9},\tfrac{1}{8}\right),\ \left[\tfrac{2}{9},\tfrac{1}{4}\right),\ \left[\tfrac{1}{3},\tfrac{3}{8}\right),\ \left[\tfrac{4}{9},\tfrac{1}{2}\right),\ \left[\tfrac{5}{9},\tfrac{3}{5}\right),\ \left[\tfrac{2}{3},\tfrac{7}{10}\right),\ \left[\tfrac{7}{9},\tfrac{4}{5}\right),\ \left[\tfrac{8}{9},\tfrac{9}{10}\right),$$

and we have

$$\int_{\Omega_{9,1}} d\psi(x) = 2.57871295\ldots.$$

Hence

$$\mu_{\mathbb{Q}(\sqrt[3]{7/6})}(\log(7/6)) < 43.9427.$$

Finally, let $\alpha = e^{i\pi/6} = \frac{1}{2}(\sqrt{3}+i),\ j=5,\ l=1$. Here

$$t_+(e^{i\pi/6}) = 0.95936116\ldots + 0.25706005\ldots i,$$
$$t_-(e^{i\pi/6}) = -0.64835693\ldots - 0.17372671\ldots i.$$

We have

$$\varrho_{5,1}(e^{i\pi/6}) = 20.08135621\ldots, \qquad \mathrm{h}^*_{5,1}(e^{i\pi/6}) = 0.92811618\ldots.$$

Theorem 2.3 gives the following $\mathbb{Q}(\sqrt{3})$-irrationality measure of $\pi$, stated in the introduction:

$$\mu_{\mathbb{Q}(\sqrt{3})}(\pi) = \mu_{\mathbb{Q}(i,\sqrt{3})}(i\pi/6) < 46.9075.$$

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