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The Arithmetic Hyperbolic 3-Manifold of Smallest Volume

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Abstract. We show that the arithmetic hyperbolic 3-manifold of smallest volume
is the Weeks manifold. The next smallest one is the Meyerhoff manifold.

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0. – Introduction

A hyperbolic 3-manifold is a 3-manifold admitting a complete Riemannian
metric all of whose sectional curvatures are -1. The universal cover of such
a manifold can therefore be identified with the hyperbolic 3-space, that is, the
unique connected and simply connected hyperbolic 3-manifold. We denote by

$$\mathbb{H}^3 = \{(z, t) \in \mathbb{C} \oplus \mathbb{R} \mid t > 0\},$$

the upper half space model of hyperbolic 3-space. With this convention, the
full group of orientation-preserving isometries of $\mathbb{H}^3$ is simply PGL$(2, \mathbb{C})$. A
Kleinian group Γ is a discrete subgroup of PGL$(2, \mathbb{C})$. Hence an orientable
hyperbolic 3-manifold is the quotient of $\mathbb{H}^3$ by a torsion-free Kleinian group,
since this acts properly discontinuously and freely on $\mathbb{H}^3$. If we relax the
condition that Γ act freely, allowing Γ to contain torsion elements, we obtain a
complete orientable hyperbolic 3-orbifold (cf. [Th] for further details). We will
only be concerned with the case that $M = \mathbb{H}^3/\Gamma$ has a hyperbolic structure
of finite volume (we say Γ has finite covolume). Therefore in what follows,
by a hyperbolic 3-manifold or 3-orbifold we shall always mean a complete
orientable hyperbolic 3-manifold or 3-orbifold of finite volume. By the Mostow
and Prasad Rigidity Theorems [Mo] [Pr], hyperbolic volume is a topological
invariant of a hyperbolic 3-orbifold and is therefore a natural object to study.

It is known from the work of Jørgensen and Thurston [Th] [Gr] that there is
a hyperbolic 3-manifold of minimal volume V_0 and that there are at most finitely

many non-isometric hyperbolic 3-manifolds attaining this minimum. Much work has been done in trying to identify V_0. Most recently, inspired by work of Gabai, Meyerhoff and Thurston, [GMT], work of [Prz] has given the best current estimate as $V_0 > 0.28\dots$. In addition, the program initiated by Culler and Shalen [CS1], [CS2], together with Hersonsky [CHS], uses topological information to help in estimating the volume. At present this work has culminated in showing that the closed hyperbolic 3-manifold of smallest volume has $b_1 \leq 2$, where b_1 is the rank of the first homology with coefficients in $\mathbb{Q}$.

The hyperbolic 3-manifold conjectured to be of smallest volume is the Weeks manifold, first defined in [We1], which is obtained by $(5, 1)$, $(5, 2)$ Dehn surgery on the complement of the Whitehead link in S^3 (as shown below). In particular, the Weeks manifold has $b_1 = 0$.
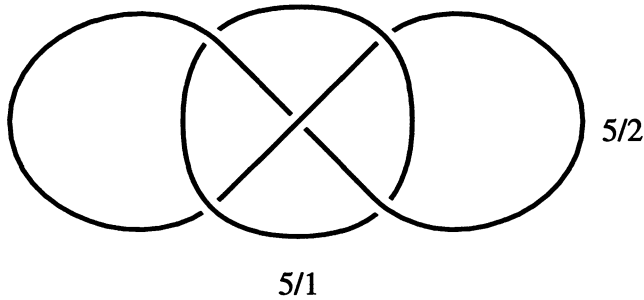

5/2

5/1

Fig. 1.

THEOREM 0.1. *The Weeks manifold has the smallest volume among all arithmetic hyperbolic 3-manifolds. Up to isometry, it is the unique arithmetic hyperbolic 3-manifold of that volume.*

The Weeks manifold has volume $0.9427073627769\cdots$ and is well-known to be arithmetic. We shall also show that the arithmetic hyperbolic 3-manifold having the next smallest volume is the Meyerhoff manifold, namely the manifold obtained by $(5, 1)$-surgery on the figure eight knot complement (see [Ch]), which has volume $0.9813688288922\cdots$, and is again the unique arithmetic hyperbolic 3-manifold of that volume. There are no other arithmetic hyperbolic 3-manifolds having volume less than 1. Below we recall the definition of an arithmetic hyperbolic 3-manifold and give the arithmetic data associated to the Weeks manifold.

We now sketch the proof of Theorem 0.1. Let $M' = \mathbb{H}^3/\Gamma'$ be an arithmetic 3-manifold of volume at most 1 and let $M = \mathbb{H}^3/\Gamma$ be a minimal orbifold covered by M'. Thus Γ' is a torsion-free subgroup of finite index in the maximal arithmetic Kleinian group Γ and

(0.1) $\text{Vol}(\mathbb{H}^3/\Gamma') = [\Gamma : \Gamma']\, \text{Vol}(\mathbb{H}^3/\Gamma) \leq 1.$

The advantage of passing from a torsion-free Γ' to a maximal, but not necessarily torsion-free Γ, is that Borel [Bo] classified such Γ and gave a formula for the volume of $\mathbb{H}^3/\Gamma$. With the help of this formula and results from [CF1], we first show that if $\text{Vol}(\mathbb{H}^3/\Gamma) \leq 1$, then the degree of the number field k used to define Γ must satisfy $[k : \mathbb{Q}] \leq 8$.

When $[k : \mathbb{Q}]$ is small, there are abundantly (but finitely) many arithmetic 3-orbifolds of volume smaller than 1. Hence we also look for lower bounds on the index $[\Gamma : \Gamma']$ appearing in (0.1). The easiest way to do this is by finding finite subgroups $H \subset \Gamma$ and noting that the order of such an H must divide the index $[\Gamma : \Gamma']$. Here we draw on the results of [CF2] and on exhaustive lists of number fields of small discriminant. By these purely number-theoretic arguments we are able to narrow the list of possible Γ's in (0.1) to just the nine groups G_i, listed in Theorem 2.0.1. The second half of the proof is devoted to studying these nine orbifolds, and is done using a package of computer programs developed by the third and fourth authors for studying the geometry of arithmetic hyperbolic 3-orbifolds (see [JR2] for more details).

For eight of these, neither volume nor finite subgroup considerations can rule out a manifold cover of $\mathbb{H}^3/G_i$ having volume less than 1. For the remaining one, finite subgroup considerations do work (see [CF4]), but the number theoretic approach becomes cumbersome. The computer packages allow us to eliminate all but the Weeks manifold and the Meyerhoff manifold as we now discuss. To handle the first eight groups described in Theorem 2.0.1, we use the computer packages to generate presentations for certain of Borel's maximal arithmetic groups by constructing a Dirichlet polyhedron for these arithmetic Kleinian groups. Once these presentations are obtained, it can be checked directly that these "candidate" presentations are indeed the presentations of the required groups, by computing the faithful discrete representations involved and applying the results of [MR1] and [Bo].

Next we use the presentations to try to either compute all torsion-free subgroups of each G_i of the appropriate indices, or to show that such subgroups do not exist. This we do first via the group theoretic language Cayley (or its more recent upgrade Magma), but we then sketch a more direct method. By inspecting the data that Cayley/Magma produced, we finally arrive at two arithmetic 3-manifolds of volume less than 1. As these correspond to the Weeks and Meyerhoff manifolds, we are then done with the proof.

The remaining group described in Theorem 2.0.1 is ruled out by using the computer packages to construct a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ subgroup in the group of units in a certain maximal order of a quaternion algebra defined over a sextic field with one complex place and discriminant -215811 (see Section 2.5). This then allows us to conclude there is no manifold of volume less than 1 arising in this case.

Early attempts to construct the orbifold groups G_i used Weeks' list of volumes of hyperbolic 3-manifolds created by SnapPea [We2]. Although not used in the proof, we wish to acknowledge the role this data played in helping complete the proof.

1. – Definitions and preliminaries

We recall some basic facts about arithmetic Kleinian groups. See [Bo], [Vi] and [CF3] for further details on this section. Recall that a quaternion algebra B over a field k is a 4-dimensional central simple algebra over k. When k has characteristic different from 2, we can describe B as follows. Let a and b be non-zero elements of k. There is a basis for B of the form $\{1, i, j, ij\}$ where $i^2 = a$, $j^2 = b$ and $ij = -ji$. B is then said to have Hilbert symbol $\{a, b\}$.

Now let k be a number field, that is, a finite extension of $\mathbb{Q}$. Let v be a place of k, and k_v the completion of k at v. We let $B_v = B \otimes_k k_v$, which is a quaternion algebra over k_v. The set of places for which B_v is a division algebra ("B ramifies"), denoted herein by $\mathrm{Ram}(B)$, is finite, of even cardinality and contains no complex place of k. Conversely, any such set R of places of k determines a unique quaternion algebra B over k satisfying $R = \mathrm{Ram}(B)$. We denote by $\mathrm{Ram}_f(B)$ the subset of $\mathrm{Ram}(B)$ consisting of all finite places in $\mathrm{Ram}(B)$.

Let B be a quaternion algebra over k, where k is a number field or a completion of such a field at a finite place, and let $\mathcal{O}_k$ the ring of integers of k. An order of B is a finitely generated $\mathcal{O}_k$-submodule of B which contains a k-basis for B and which is a ring with 1. An order of B is maximal if it is not properly contained in any other order of B.

One way to define arithmetic Kleinian groups [Bo] is to begin with a number field k having exactly one complex place and a quaternion algebra B over k ramified at all real places of k. We shall use the notation $\bar{x}$ to denote an element of B^*/k^* represented by $x \in B^*$. Let $\mathfrak{D}$ be a maximal order of B and let

$$\Gamma_{\mathfrak{D}} = \{\bar{x} \in B^*/k^* \mid x\mathfrak{D}x^{-1} = \mathfrak{D}\}.$$

Via the complex place of k we get an embedding $\rho : B \hookrightarrow M(2, \mathbb{C})$ and hence a $\bar{\rho} : B^*/k^* \hookrightarrow \mathrm{PGL}(2, \mathbb{C})$. For simplicity we identify $\Gamma_{\mathfrak{D}}$ with $\bar{\rho}(\Gamma_{\mathfrak{D}})$. Then $\Gamma_{\mathfrak{D}} \subset \mathrm{PGL}(2, \mathbb{C})$ is a Kleinian group giving rise to a hyperbolic 3-orbifold $\mathbb{H}^3/\Gamma_{\mathfrak{D}}$ of finite volume. The class of arithmetic Kleinian groups is that obtained from the commensurability classes in $\mathrm{PGL}(2, \mathbb{C})$ of all such $\Gamma_{\mathfrak{D}}$. For Kleinian groups, this definition of arithmeticity coincides with the usual notion of arithmetic groups [Bo]. We recall that two subgroups Γ and Γ' of $\mathrm{PGL}(2, \mathbb{C})$ are commensurable if some conjugate $\tilde{\Gamma}$ of Γ is such that $\tilde{\Gamma} \cap \Gamma'$ has finite index in both $\tilde{\Gamma}$ and Γ'. A hyperbolic 3-orbifold, or 3-manifold, $\mathbb{H}^3/\Gamma$ is called arithmetic if Γ is an arithmetic Kleinian group.

Here we summarize the arithmetic data associated to the Weeks manifold.

PROPOSITION 1.1. *The Weeks manifold is the unique hyperbolic 3-manifold which covers with degree 12 the orbifold $\mathbb{H}^3/\Gamma_{\mathfrak{D}}$, where $\mathfrak{D}$ is any maximal order in the quaternion algebra B defined over the cubic field k of discriminant -23, ramified at the real place and at the prime of norm 5 of k.*

Proposition 1.1 will be proved in Section 3. Explicitly, the field k in Proposition 1.1 can be given as $k = \mathbb{Q}(\theta)$ where θ satisfies $\theta^3 - \theta + 1 = 0$. Borel's volume formula (see (2.1.1) below) shows that the volume of the Weeks manifold is

$$\frac{3 \cdot 23^{\frac{3}{2}} \zeta_k(2)}{4\pi^4} = 0.9427073627769 \cdots ,$$

where ζ_k denotes the Dedekind zeta function of k. The numerical value of the volume can be obtained by computing $\zeta_k(2)$, which is incorporated in the PARI number theory package [Co], or from the geometric definition of the manifold. Of course, it was this coincidence of volumes that pointed immediately to the above characterization.

A Kleinian group $\Gamma \subset PGL(2, \mathbb{C})$ is maximal if it is maximal, with respect to inclusion, within its commensurability class. Borel [Bo] proved that any maximal arithmetic Kleinian group is isomorphic to some group $\Gamma_{S,\mathfrak{D}}$, which we now define. Let $\mathfrak{D}$ be a maximal order of B and S a finite (possibly empty) set of primes of k disjoint from $\mathrm{Ram}_f(B)$. For each $\mathfrak{p} \in S$ choose a local maximal order $E_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ such that $[\mathfrak{D}_{\mathfrak{p}} : E_{\mathfrak{p}} \cap \mathfrak{D}_{\mathfrak{p}}] = \mathrm{Norm}_{k/\mathbb{Q}}(\mathfrak{p})$, where $\mathfrak{D}_{\mathfrak{p}} = \mathfrak{D} \otimes_{\mathcal{O}_k} \mathcal{O}_{k_{\mathfrak{p}}}$. We shall say that $x \in B_{\mathfrak{p}}^*$ fixes $\mathfrak{D}_{\mathfrak{p}}$ (resp., $\{\mathfrak{D}_{\mathfrak{p}}, E_{\mathfrak{p}}\}$) if $x\mathfrak{D}_{\mathfrak{p}}x^{-1} = \mathfrak{D}_{\mathfrak{p}}$ (resp., either x fixes $\mathfrak{D}_{\mathfrak{p}}$ and $E_{\mathfrak{p}}$, or $x\mathfrak{D}_{\mathfrak{p}}x^{-1} = E_{\mathfrak{p}}$ and $xE_{\mathfrak{p}}x^{-1} = \mathfrak{D}_{\mathfrak{p}}$). Borel's definition is

$$\Gamma_{S,\mathfrak{D}} = \left\{ \bar{x} \in B^*/k^* \,\middle|\, x \text{ fixes } \mathfrak{D}_{\mathfrak{p}} \text{ for all } \mathfrak{p} \notin S, \text{ and for } \mathfrak{p} \in S, x \text{ fixes } \{\mathfrak{D}_{\mathfrak{p}}, E_{\mathfrak{p}}\} \right\}.$$

When S is empty we find $\Gamma_{S,\mathfrak{D}} = \Gamma_{\mathfrak{D}}$. We remark that $\Gamma_{S,\mathfrak{D}}$ is not necessarily maximal if S is non-empty.

Two maximal orders $\mathfrak{D}$ and $\mathfrak{D}'$ of B are said to be of the same type if they are conjugate by an element of B^*. In this case $\Gamma_{S,\mathfrak{D}}$ is conjugate to $\Gamma_{S,\mathfrak{D}'}$. Thus, to study all the $\Gamma_{S,\mathfrak{D}}$ up to conjugacy, it suffices to select one $\mathfrak{D}$ from each type. Types can be parametrized by the group $T(B)$ defined as the group of fractional ideals of k, modulo the subgroup generated by squares of ideals, by ideals in $\mathrm{Ram}_f(B)$ and by principal ideals (α) generated by an $\alpha \in k^*$ which is positive at all real embeddings of k. The set of types is in bijection with the elements of $T(B)$ [Vi] (recall that we are assuming that $\mathrm{Ram}(B)$ includes all real places). The bijection is obtained [CF3] by fixing any maximal order, say $\mathfrak{D}$, and mapping $\mathfrak{D}'$ to the class of $\rho(\mathfrak{D}, \mathfrak{D}')$ in $T(B)$, where $\rho(\mathfrak{D}, \mathfrak{D}') = \prod_i \mathfrak{a}_i$, the $\mathfrak{a}_i$ being ideals of $\mathcal{O}_k$ such that $\mathfrak{D}/(\mathfrak{D} \cap \mathfrak{D}') \cong \oplus_i \mathcal{O}_k/\mathfrak{a}_i$ as $\mathcal{O}_k$-modules.

For the reader's convenience, we relate our notation to Borel's. The group Borel writes $\Gamma_{S,S'}$ [Bo, p. 9] coincides with our $\Gamma_{S,\mathfrak{D}}$ when our $\mathfrak{D}$ is set equal to his $\mathfrak{D}(S')$ [Bo, p. 12]. It will also be convenient (for the discussion in Section 3) to remark upon an alternative description of Borel's maximal groups. An Eichler

Order of B is the intersection of two maximal orders of B. As in the case for a maximal order above, the normalizer of an Eichler Order $\mathcal{E}$ projects to an arithmetic Kleinian group. Borel's maximal groups can be described as certain of these images of normalizers of Eichler orders. The corresponding S and S', in Borel's notation, depend on divisors of the discriminant of the relevant $\mathcal{E}$ (see [MR2] and [Vi, p. 99, Ex. 5.4] for more details).

2. – Small arithmetic orbifolds with little torsion

2.0. – The list

We shall prove

THEOREM 2.0.1. *If M_0 is an arithmetic hyperbolic 3-manifold with $\mathrm{Vol}(M_0) \leq 1$, then M_0 covers one of the nine orbifolds $\mathbb{H}^3/G_i$ described below, where notation is as follows. $\mathrm{Ram}(B)$ always includes all real places of k, $\mathfrak{D}$ stands for any maximal order of B in the cases $1 \leq i \leq 8$ and $\mathfrak{p}_j$ denotes the unique prime of k of norm j. When $i = 9$, $\mathfrak{D}$ is any maximal order of B not containing a primitive cube root of unity.*

Table 1

(1) $k = \mathbb{Q}(x)$, where $x^4 - 3x^3 + 7x^2 - 5x + 1 = 0$, $\mathrm{disc}_k = -283$, $\mathrm{Ram}_f(B) = \emptyset$, $G_1 = \Gamma_{\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_1) = 0.0408903 \cdots$ and 12 divides the covering degree $[M_0 : \mathbb{H}^3/G_1]$.

(2) $k = \mathbb{Q}(x)$, where $x^4 - 5x^3 + 10x^2 - 6x + 1 = 0$, $\mathrm{disc}_k = -331$, $\mathrm{Ram}_f(B) = \emptyset$, $G_2 = \Gamma_{\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_2) = 0.0526545 \cdots$ and 12 divides $[M_0 : \mathbb{H}^3/G_2]$.

(3) $k = \mathbb{Q}(x)$, where $x^3 + x + 1 = 0$, $\mathrm{disc}_k = -31$, $\mathrm{Ram}_f(B) = \mathfrak{p}_3$, $G_3 = \Gamma_{\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_3) = 0.06596527 \cdots$ and 12 divides $[M_0 : \mathbb{H}^3/G_3]$.

(4) $k = \mathbb{Q}(x)$, where $x^3 - x + 1 = 0$, $\mathrm{disc}_k = -23$, $\mathrm{Ram}_f(B) = \mathfrak{p}_5$, $G_4 = \Gamma_{\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_4) = 0.0785589 \cdots$ and 12 divides $[M_0 : \mathbb{H}^3/G_4]$.

(5) $k = \mathbb{Q}(x)$, where $x^3 - x + 1 = 0$, $\mathrm{disc}_k = -23$, $\mathrm{Ram}_f(B) = \mathfrak{p}_7$, $G_5 = \Gamma_{\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_5) = 0.1178384 \cdots$ and 4 divides $[M_0 : \mathbb{H}^3/G_5]$.

(6) $k = \mathbb{Q}(x)$, where $x^4 - 5x^3 + 10x^2 - 6x + 1 = 0$, $\mathrm{disc}_k = -331$, $\mathrm{Ram}_f(B) = \emptyset$, $S = \mathfrak{p}_5$, $G_6 = \Gamma_{S,\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_6) = 0.1579636 \cdots$ and 4 divides $[M_0 : \mathbb{H}^3/G_6]$.

(7) $k = \mathbb{Q}(x)$, where $x^5 + x^4 - 3x^3 - 2x^2 + x - 1 = 0$, $\mathrm{disc}_k = -9759$, $\mathrm{Ram}_f(B) = \mathfrak{p}_3$, $G_7 = \Gamma_{\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_7) = 0.2280430 \cdots$ and 4 divides $[M_0 : \mathbb{H}^3/G_7]$.

(8) $k = \mathbb{Q}(x)$, where $x^4 - 3x^3 + 7x^2 - 5x + 1 = 0$, $\mathrm{disc}_k = -283$, $\mathrm{Ram}_f(B) = \emptyset$, $S = \mathfrak{p}_{11}$, $G_8 = \Gamma_{S,\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_8) = 0.2453422 \cdots$ and 4 divides $[M_0 : \mathbb{H}^3/G_8]$.

(9) $k = \mathbb{Q}(x)$, where $x^6 - x^5 - 2x^4 - 2x^3 + x^2 + 3x + 1 = 0$, $\mathrm{disc}_k = -215811$, $\mathrm{Ram}_f(B) = \emptyset$, $G_9 = \Gamma_{\mathfrak{D}}$, $\mathrm{Vol}(\mathbb{H}^3/G_9) = 0.27833973 \cdots$ and 2 divides $[M_0 : \mathbb{H}^3/G_9]$.

We will show in Section 3 that orbifolds (1) and (8) are covered by the Meyerhoff manifold and (4) is covered by the Weeks manifold. The other orbifolds will turn out not to be covered by any manifold of volume ≤ 1.

We now sketch the proof of Theorem 2.0.1, hoping it will help guide the reader through the next ten pages. We begin by citing Borel's formula for the volume of a minimal arithmetic orbifold $\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}$ covered by M_0. Of course, $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})$ is a lower bound for the volume of M_0 in Theorem 2.0.1. Our first aim is to show that $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > 1$ for number fields of degree larger than 8.

The paper [CF1] is entirely devoted to the problem of obtaining lower bounds for $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})$, but with the aim of finding which orbifolds satisfy $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) < 0.042$, this being a bound for the smallest arithmetic orbifolds. In Section 2.1 we simply quote inequalities verbatim from [CF1] and use them to show that $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > 1$ unless $[k : \mathbb{Q}] \leq 8$ (Proposition 2.1.2).

For $[k : \mathbb{Q}] \leq 8$, the lower bound $\mathrm{Vol}(M_0) \geq \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})$ is too crude, as there are very many orbifolds $\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}$ with volume less than one. Hence we need to get a hold of a lower bound for the covering degree $[M_0 : \mathbb{H}^3/\Gamma_{S,\mathfrak{D}}]$. As explained in the Introduction and proved in Lemma 2.2.1 below, we bound $[M_0 : \mathbb{H}^3/\Gamma_{S,\mathfrak{D}}]$ from below by the least common multiple $\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}})$ of the order of all the torsion subgroups H of $\Gamma_{S,\mathfrak{D}}$. Thus, $\mathrm{Vol}(M_0) \geq \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}})$. To compute $\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}})$, it suffices to compute the orders of the p-Sylow subgroups of all finite subgroups $H \subset \Gamma_{S,\mathfrak{D}}$. As $\Gamma_{S,\mathfrak{D}} \subset \mathrm{PGL}(2,\mathbb{C})$, and the list of finite subgroups of $\mathrm{PGL}(2,\mathbb{C})$ is well-known, one sees that it suffices to compute the finite cyclic subgroups and the dihedral 2-subgroups of $\Gamma_{S,\mathfrak{D}}$. In Section 2.2 we quote verbatim from [CF2] the conditions that determine when $\Gamma_{S,\mathfrak{D}}$ contains such a finite subgroup (Lemmas 2.2.3-2.2.6). Most of the work (three of the four lemmas) concerns groups of order dividing 8. These are by far the most frequent ones among finite p-Sylow subgroups associated to orbifolds of small volume.

In the subsequent sections Section 2.3-2.7 we start at $[k : \mathbb{Q}] = 8$ and go down degree by degree eliminating orbifolds with $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}}) > 1$. The result is the list given in Table 1. More precisely, the first 8 entries in Table 1 is a complete list of all $\Gamma_{S,\mathfrak{D}}$ satisfying $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}}) \leq 1$. The last orbifold in the table is there because the computation of $\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}})$ is quite complicated by purely number theoretical means. It is carried out in Section 3.4 with the help of a matrix representation obtained geometrically.

2.1. – Reduction to small degrees

Let $\mathbb{H}^3/\Gamma$ be a minimal orbifold covered by a manifold M_0 as in Theorem 2.0.1. Then, as described in Section 1, Γ is isomorphic to some $\Gamma_{S,\mathfrak{D}}$ and $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq \mathrm{Vol}(M_0) \leq 1$. In this subsection, which relies heavily on the volume inequalities in [CF1], we prove that if $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1$, then the number field k defining $\Gamma_{S,\mathfrak{D}}$ satisfies $[k : \mathbb{Q}] \leq 8$ and certain class number restrictions.

Borel proved [Bo], [CF1, Prop. 2.1]

$$(2.1.1) \quad \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) = \frac{2\pi^2\zeta_k(2)d_k^{\frac{3}{2}}\left(\prod_{\mathfrak{p}\in\mathrm{Ram}_f(B)}\frac{N\mathfrak{p}-1}{2}\right)\prod_{\mathfrak{p}\in S}(N\mathfrak{p}+1)}{2^m(4\pi^2)^{[k:\mathbb{Q}]}[k_B:k]},$$

for some integer m with $0 \leq m \leq |S|$. Here ζ_k denotes the Dedekind zeta function of k, d_k is the absolute value of the discriminant of k, $\mathrm{Ram}_f(B)$ and S are as in Section 1, N denotes the absolute norm, and k_B is the class field defined as the maximal abelian extension of k which is unramified at all finite places of k, whose Galois group is 2-elementary and in which all $\mathfrak{p} \in \mathrm{Ram}_f(B)$ are completely decomposed. By class field theory, the Frobenius map induces an isomorphism $T(B) \cong \mathrm{Gal}(k_B/k)$, where $T(B)$ is the group defined in Section 1 which parametrizes the types of maximal orders of B. Thus, $[k_B : k]$ equals the type number of B. The maximal order $\mathfrak{D}$ itself does not enter into the volume formula.

REMARK. When S is empty, we clearly have $m = |S| = 0$. When $S = \{\mathfrak{p}\}$ consists of a single prime, then $m = 1$ if and only if $\mathfrak{p} = (\alpha)\mathfrak{b}\mathfrak{a}^2$ for some $\alpha \in k^*$ which is positive at all real places of k, some integral ideal $\mathfrak{b}$ divisible only by primes in $\mathrm{Ram}_f(B)$ and some fractional ideal $\mathfrak{a}$. This follows from Borel's proof [Bo, Section 5.3-5.5] on noting that if $\mathfrak{p}$ is not as above, then $\Gamma_{S,\mathfrak{D}} \subset \Gamma_{\mathfrak{D}}$. We note that the above condition on $\mathfrak{p}$ is equivalent to $\mathfrak{p}$ being completely decomposed in k_B/k.

From (2.1.1) we conclude as in [CF1, p. 512] that

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \geq \frac{8\pi^2\zeta_k(2)d_k^{\frac{3}{2}}[\mathcal{O}_k^*:\mathcal{O}_{k,+}^*]\left(\prod_{\mathfrak{p}\in\mathrm{Ram}_f(B)}\frac{N\mathfrak{p}-1}{2}\right)\prod_{\mathfrak{p}\in S}(N\mathfrak{p}+1)}{2^m(8\pi^2)^{[k:\mathbb{Q}]}h(k,2,B)},$$

where $\mathcal{O}_k^*$ and $\mathcal{O}_{k,+}^*$ denote respectively the units and the totally positive units of k and $h(k,2,B)$ is the order of the (wide) ideal class group of k modulo the square of all classes and modulo the classes corresponding to primes in $\mathrm{Ram}_f(B)$. In particular,

$$(2.1.2) \quad \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \geq \frac{8\pi^2\zeta_k(2)d_k^{\frac{3}{2}}[\mathcal{O}_k^*:\mathcal{O}_{k,+}^*]}{2^r(8\pi^2)^{[k:\mathbb{Q}]}h(k,2,B)},$$

where $r = r(B)$ is the number of primes in $\mathrm{Ram}_f(B)$ of norm 2. Lower bounds for volumes of arithmetic orbifolds can be obtained using Odlyzko's lower bounds for discriminants of number fields, as follows [CF1, Lemma 3.4].

LEMMA 2.1.1. *Let K/k be a finite extension which is unramified at all finite places. Then*

$$\frac{\log d_k}{[k:\mathbb{Q}]} \geq \gamma + \log(4\pi) + \frac{r_1(K)}{[K:\mathbb{Q}]} - \frac{12\pi}{5\sqrt{y}[K:\mathbb{Q}]}$$

$$- \int_0^\infty (1 - \alpha(x\sqrt{y})) \left(\frac{1}{\sinh(x)} + \frac{r_1(K)}{2[K:\mathbb{Q}]\cosh^2(x/2)} \right) dx$$

$$+ \frac{4}{[K:\mathbb{Q}]} \sum_{\mathfrak{P}} \sum_{j=1}^\infty \frac{\log(N\mathfrak{P})}{1 + N\mathfrak{P}^j} \alpha(j\sqrt{y}\log N\mathfrak{P}),$$

where $\gamma = 0.5772156\cdots$ is Euler's constant, $y > 0$ is arbitrary, $r_1(K)$ denotes the number of real places of K, the sum on $\mathfrak{P}$ is over all the prime ideals of K and

$$\alpha(t) = \frac{9(\sin t - t\cos t)^2}{t^6}.$$

In our applications, the field K above will always be contained in the class field k_B appearing in (2.1.1).

PROPOSITION 2.1.2. *Assume* $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1$. *Here* $\Gamma_{S,\mathfrak{D}}$ *is associated to a quaternion algebra B over a number field k having exactly one complex place, as described in Section 1. Then the following hold*

(1) $[k:\mathbb{Q}] \leq 8$.

(2) $[k_B : k] \leq 2$.

(3) *For $6 \leq [k:\mathbb{Q}] \leq 8$, we have $h(k, 2, B) = 1$.*

(4) *For $[k:\mathbb{Q}] = 2$, we have $d_k \leq 56$.*

It follows from Borel's classification of minimal arithmetic orbifolds [Bo] that the manifold M_0 in Theorem 2.0.1 covers an orbifold $\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}$ associated to a field k of degree at most 8 satisfying the above restrictions.

PROOF. We first deal with claim (4). When $[k:\mathbb{Q}] = 2$, *i.e.* when k is imaginary quadratic, genus theory shows that $h(k, 2, B) \leq 2^{g-1}$, where g is the number of prime factors of d_k. Hence $h(k, 2, B) \leq \sqrt{d_k/3}$. From (2.1.2), taking into account the contribution to $\zeta_k(2)$ of a prime above 2, we find then in the imaginary quadratic case that

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > \frac{d_k}{6\pi^2\sqrt{3}}.$$

Hence, if $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1$, then $d_k \leq 6\pi^2\sqrt{3} = 121.03\cdots$. However, for $d_k \leq 121$, genus theory again gives $h(k, 2, B) \leq 2$ except for $d_k = 60$. In this case k has unique primes $\mathfrak{p}_2$ and $\mathfrak{p}_3$ of norm 2 and 3, respectively. Thus, considering two Euler factors of $\zeta_k(2)$,

$$\frac{60^{\frac{3}{2}}\zeta_k(2)\prod_{\mathfrak{p}\in\mathrm{Ram}_f(B)}\frac{N\mathfrak{p}-1}{2}}{8\pi^2 h(k, 2, B)} > \frac{60^{\frac{3}{2}}(1 - \frac{1}{4})^{-1}(1 - \frac{1}{9})^{-1}}{64\pi^2} > 1.1.$$

This means $d_k \neq 60$. Hence, if $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1$, then $h(k, 2, B) \leq 2$. But then (2.1.2) gives $d_k \leq 56$. This proves (4) in Proposition 2.1.2.

We may now assume $[k : \mathbb{Q}] \geq 3$. Then k has at least one real place and therefore $-1 \notin \mathcal{O}_{k,+}^*$ and $[\mathcal{O}_k^* : \mathcal{O}_{k,+}^*] \geq 2$. A consequence of Lemma 2.1.1 and the volume formula (2.1.1) is [CF1, Lemma 4.3]

$$(2.1.3) \qquad \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > 0.69 \exp\left(0.37[k : \mathbb{Q}] - \frac{19.08}{h(k, 2, B)} \right).$$

From this, we easily deduce that

$$(2.1.4) \qquad \text{if } \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1, \text{ then } h(k, 2, B) \leq 16.$$

Indeed, if $h(k, 2, B) > 16$ then $h(k, 2, B) \geq 32$. Then (2.1.3) shows that

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > 1$$

as soon as $[k : \mathbb{Q}] > 2$.

We now use the inequalities [CF1, eq. (4.6)]

$$
\begin{aligned}
&\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \\
&\qquad > \frac{2}{[k_B : k]} \exp\left([k:\mathbb{Q}]\left(1.4143 + \frac{1.005 r_1(K)}{[K:\mathbb{Q}]} - \frac{1.14475 r_1(k)}{[k:\mathbb{Q}]} - \frac{11.31}{[K:\mathbb{Q}]} \right) \right) \\
(2.1.5) \\
&\qquad > \frac{2[\mathcal{O}_k^* : \mathcal{O}_{k,+}^*]}{h(k,2,B)} \exp\left([k:\mathbb{Q}]\left(1.4143 + \frac{1.005 r_1(K)}{[K:\mathbb{Q}]} - \frac{1.8379 r_1(k)}{[k:\mathbb{Q}]} - \frac{11.31}{[K:\mathbb{Q}]} \right) \right),
\end{aligned}
$$

where K/k is any elementary abelian 2-extension unramified at the finite places and in which all $\mathfrak{p} \in \mathrm{Ram}_f(B)$ are completely decomposed (see also the sentence preceeding [CF1, Lemma 4.6]). We first take $K \subset k_B$ to be the maximal unramified 2-elementary extension of k in which all $\mathfrak{p} \in \mathrm{Ram}_f(B)$ are completely decomposed. Then $r_1(K)/[K : \mathbb{Q}] = r_1(k)/[k:\mathbb{Q}] = 1 - \frac{2}{[k:\mathbb{Q}]}$ and, by class field theory, $[K : k] = h(k, 2, B)$. For $[k : \mathbb{Q}] \geq 3$, we have by (2.1.5)

$$
\begin{aligned}
(2.1.6) \quad & \log\left(\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \right) \\
& > \log 4 - \log h(k, 2, B) + 2(0.8329) + 0.5814[k : \mathbb{Q}] - \frac{11.31}{h(k, 2, B)}.
\end{aligned}
$$

From (2.1.4) we know that $h(k, 2, B) \leq 16$ if $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1$. Hence (2.1.6) yields

$$(2.1.7) \qquad \begin{aligned} &\text{If } \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1, \text{ then } [k : \mathbb{Q}] \leq 14 \text{ and } h(k, 2, B) \leq 2. \\ &\text{If also } [k : \mathbb{Q}] \geq 6, \text{ then } h(k, 2, B) = 1. \end{aligned}$$

Thus we have shown (3) in Proposition 2.1.2.

To prove (2), we need to show $[k_B : k] \leq 2$. Let $2^E := [k_B : k]$ and $2^F := 2^{r_1(k)}/[\mathcal{O}_k^* : \mathcal{O}_{k,+}^*]$. Then $0 \leq F \leq r_1(k) - 1 = [k : \mathbb{Q}] - 3 \leq 11$. Class field theory shows that $[k_B : k] = h(k, 2, B)2^F$. As $h(k, 2, B) \leq 2$, we have $0 \leq E \leq [k : \mathbb{Q}] - 2 \leq 12$. Set $K = k_B$ in the first inequality in (2.1.5). Then, dropping the term $r_1(k_B)/[k_B : \mathbb{Q}]$ as k_B/k may ramify at the infinite places,

$$\log\big(\text{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})\big) > -\log 2^{E-1} + (1.4143 - 1.14475)[k:\mathbb{Q}] + 2(1.14475) - \frac{11.31}{2^E}$$

$$\geq -E\log 2 + 0.26955(E+3) + 2.9826 - \frac{11.31}{2^E}.$$

Thus E can only take the values 0, 1, 9, 10, 11 or 12. Suppose $E \geq 9$. Then $[k : \mathbb{Q}] \geq 11$ and k_B/k is an extension of degree at least 2^9 which is unramified at the finite places. Odlyzko's discriminant bounds, as refined by Serre and Poitou, give for any number field K [Po, eq. 16],

$$\frac{1}{[K : \mathbb{Q}]} \log d_K \geq 3.10823 - \frac{12.644}{[K : \mathbb{Q}]^{\frac{2}{3}}}.$$

Hence

$$d_k^{1/[k:\mathbb{Q}]} = d_{k_B}^{1/[k_B:\mathbb{Q}]} > 21.5,$$

as $[k_B : \mathbb{Q}] \geq 11 \cdot 2^9$. Then (2.1.2), with $h(k, 2, B) \leq 2$, gives

$$\text{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > \frac{8\pi^2(1 - \frac{1}{4})^{-r} d_k^{\frac{3}{2}}}{2^r (8\pi^2)^{[k:\mathbb{Q}]}} \geq 8\pi^2 \Big(\frac{(21.5)^{\frac{3}{2}}}{1.5 \cdot 8\pi^2}\Big)^{[k:\mathbb{Q}]} > 7.07,$$

for $[k : \mathbb{Q}] \leq 14$. Hence $E = 0$ or 1. This proves (2) in Proposition 2.1.2.

By (2.1.7), to conclude the proof of Proposition 2.1.2 we only need to rule out fields k with degrees $9 \leq [k : \mathbb{Q}] \leq 14$. As $[k_B : k] \leq 2$, from (2.1.1) we find

(2.1.8) $$\text{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > \frac{\pi^2 d_k^{\frac{3}{2}} (\frac{2}{3})^r}{(4\pi^2)^{[k:\mathbb{Q}]}},$$

with r as in (2.1.2). We now use the following table.

$[k : \mathbb{Q}]$	=	9	10	11	12	13	14
$\dfrac{1}{[k : \mathbb{Q}]} \log d_k$	>	2.316	2.421	2.511	2.591	2.661	2.724
$\dfrac{\pi^2 d_k^{\frac{3}{2}}}{(4\pi^2)^{[k:\mathbb{Q}]}}$	>	1.605	6.34	26.78	123.7	598.3	3080
Norm 2 gives	>	8.026	8.767	9.476	10.09	10.55	11.00
y parameter	=	0.95	0.825	0.725	0.65	0.6	0.555

The second line gives the lower bound for $\frac{1}{[k:\mathbb{Q}]} \log d_k$, coming from Lemma 2.1.1 with $K = k$ and y as given on the last line. This bound does not consider any contribution from the finite places. The third line shows the corresponding lower bound for the main term appearing in the volume formula (2.1.1). The fourth line gives the factor by which the previous line can be multiplied for each prime of norm 2 in k (coming from its contribution to the lower bound for d_k in Lemma 2.1.1). It is clear from the table above that for $9 \le [k : \mathbb{Q}] \le 14$ the contribution to the lower bound for $d_k^{3/2}$ of a prime of norm 2 greatly exceeds the $\frac{2}{3}$ lost in the estimate (2.1.8). Hence the third line of the table and (2.1.8) complete the proof of Proposition 2.1.2.   □

2.2. – Torsion

In this step of the proof of Theorem 2.0.1, we study torsion in the groups $\Gamma_{S,\mathfrak{D}}$. We begin with a well-known result from group theory.

LEMMA 2.2.1. *Suppose Γ is a group having a finite subgroup $H \subset \Gamma$ and a torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index. Then the index $[\Gamma : \Gamma']$ is divisible by the order of H.*

PROOF. It will suffice to show that the left multiplication action of H on the set of left cosets of Γ' in Γ is free. This follows from the fact that the stabilizer in H of a left coset of Γ' is conjugate in Γ to a finite subgroup of Γ', and Γ' has no non-trivial finite subgroups by assumption.   □

The lemma leads us to define

$$\mathrm{lcmtor}(\Gamma) := \text{ least common multiple}\{|H| : H \subset \Gamma, H \text{ a finite subgroup}\}.$$

We shall often use the following consequence of Lemma 2.2.1.

LEMMA 2.2.2. *If a manifold $M = \mathbb{H}^3 / \Gamma$ of finite volume covers an orbifold $\mathbb{H}^3 / \Gamma_{S,\mathfrak{D}}$, then $\mathrm{Vol}(M)$ is an integral multiple of $\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}}) \cdot \mathrm{Vol}(\mathbb{H}^3 / \Gamma_{S,\mathfrak{D}})$.*

We shall need the following results from [CF2].

LEMMA 2.2.3. *Let $\Gamma_{S,\mathfrak{D}}$ be associated, as above, to a number field k having exactly one complex place and to a quaternion algebra B ramified at all real places of k. Let l be an odd prime number and ζ_l a primitive l-th root of unity in some algebraic closure of k. If $\zeta_l \in k$, then $\Gamma_{S,\mathfrak{D}}$ contains an element of order l if and only if $B \cong M(2, k)$. Assume now $\zeta_l \notin k$. Then $\Gamma_{S,\mathfrak{D}}$ contains an element of order l if and only if the following four conditions hold.*

(1) $\zeta_l + \zeta_l^{-1} \in k$.
(2) *If $\mathfrak{p} \in \mathrm{Ram}_f(B)$, then $N\mathfrak{p} \not\equiv 1$ (modulo l). If $\mathfrak{p} \in \mathrm{Ram}_f(B)$ lies above l, then $\mathfrak{p}$ is not split in $k(\zeta_l)/k$.*
(3) *If $\mathfrak{p} \in S$, then $N\mathfrak{p} \not\equiv -1$ (modulo l).*
(4) *$\mathfrak{D}$ contains an element $y \neq 1$ such that $y^l = 1$.*

Furthermore, condition (4) is implied by (1) and (2) (and so may be dropped), except when $\mathrm{Ram}_f(B)$ is empty and all primes of k lying above l split in $k(\zeta_l)/k$.

Note that $\mathfrak{D}$ appears only in (4) above, and that this condition can be dropped if $[k : \mathbb{Q}]$ is odd, or if some prime of k above l does not split in $k(\zeta_l)/k$, or if the narrow class number of k is odd. Here, and throughout the paper, we take "narrow" in its strictest sense, taking all real places of k into consideration.

There is a similar criterion for $\Gamma_{S,\mathfrak{D}}$ to contain an element of order 4 [CF2].

LEMMA 2.2.4. *Let $\Gamma_{S,\mathfrak{D}}$ be as above and let $i = \sqrt{-1}$ in some algebraic closure of k. If $i \in k$ then $\Gamma_{S,\mathfrak{D}}$ contains an element of order 4 if and only $B \cong M(2,k)$. Assume now that $i \notin k$. Then $\Gamma_{S,\mathfrak{D}}$ contains an element of order 4 if and only the following four conditions hold.*

(1) *If $\mathfrak{p} \in \mathrm{Ram}_f(B)$, then $N\mathfrak{p} \not\equiv 1 \pmod{4}$. If $\mathfrak{p} \in \mathrm{Ram}_f(B)$ lies above 2, then $\mathfrak{p}$ is not split in $k(i)/k$.*

(2) *If $\mathfrak{p} \in S$, then $N\mathfrak{p} \not\equiv -1 \pmod{4}$.*

(3) *Any prime of k lying above 2, and not contained in $S \cup \mathrm{Ram}_f(B)$, has an even absolute ramification index.*

(4) *$\mathfrak{D}$ contains an element y such that $y^2 = -1$.*

Furthermore, condition (4) is implied by (1) (and so may be dropped), except when $\mathrm{Ram}_f(B)$ is empty and all primes of k lying above 2 split in $k(i)/k$.

Next we examine the elements of order 2 in $\Gamma_{S,\mathfrak{D}}$. Up to conjugacy these elements are parametrized by a set $C_2 = C_2(S, \mathfrak{D}) \subset k^*/k^{*2}$ which we now define. The trivial coset k^{*2} is defined to be in C_2 if and only if $B \cong M(2,k)$. The non-trivial cosets in C_2 are those represented by some totally negative $w \in k^*$ satisfying conditions (a) through (d) below:

(a) No $\mathfrak{p} \in \mathrm{Ram}_f(B)$ is split in $k(\sqrt{w})/k$.

(b) Write the principal fractional ideal $(w) = \mathfrak{a}^2\mathfrak{b}$, where $\mathfrak{a}$ is a fractional ideal and $\mathfrak{b}$ is a square-free integral ideal. Then any $\mathfrak{p}$ dividing $\mathfrak{b}$ is in $S \cup \mathrm{Ram}_f(B)$.

(c) For every $\mathfrak{p} \in S$ not lying above 2, either $\mathfrak{p}$ divides the ideal $\mathfrak{b}$ in condition (b) above or $\mathfrak{p}$ is split in $k(\sqrt{w})/k$.

(d) There is an embedding over $\mathcal{O}_k$ of the ring of integers $\mathcal{O}_{k(\sqrt{w})}$ into $\mathfrak{D}$.

As in Lemmas 2.2.3 and 2.2.4, condition (d) is implied by (a), and so can be dropped, except when $\mathrm{Ram}_f(B)$ is empty, $k(\sqrt{w})/k$ is unramified at all finite places and all primes lying above 2 split in $k(\sqrt{w})/k$.

We again cite from [CF2]:

LEMMA 2.2.5. *The group $\Gamma_{S,\mathfrak{D}}$ contains an element of order 2 if and only if the set $C_2(S, \mathfrak{D})$ defined above is non-empty.*

We will also need to know when $\Gamma_{S,\mathfrak{D}}$ contains a dihedral group of order 4 or 8. In general, this may depend on $\mathfrak{D}$ in a complicated way. We state here a simple case which will suffice here.

LEMMA 2.2.6. *Assume that $k_B = k$, where k_B is as in the volume formula (2.1.1). Then $\Gamma_{S,\mathfrak{D}}$ contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if there exists c and $d \in k^*$ satisfying (1), (2) and (3) below:*

(1) *The algebra B has Hilbert symbol $\{c, d\}$.*

(2) *$\{\mathfrak{p} \notin \mathrm{Ram}_f(B) \mid \mathrm{ord}_\mathfrak{p}(c)$ or $\mathrm{ord}_\mathfrak{p}(d)$ is an odd integer$\} \subset S$. Here $\mathrm{ord}_\mathfrak{p}(\alpha)$ denotes the exponent of $\mathfrak{p}$ in the prime ideal factorization of the principal ideal (α).*

(3) *$S \subset \{\mathfrak{p} \notin \mathrm{Ram}_f(B) \mid \mathrm{ord}_\mathfrak{p}(c)$ or $\mathrm{ord}_\mathfrak{p}(d)$ is an odd integer, or $\mathfrak{p}$ divides $2\}$.*

$\Gamma_{S,\mathfrak{D}}$ contains a subgroup isomorphic to a dihedral group of order 8 if and only if $\Gamma_{S,\mathfrak{D}}$ contains an element of order 4 and $\{c, d\}$ above can be taken as $\{-1, d\}$.

REMARK. In Lemma 2.2.6 it suffices to assume, instead of $k_B = k$, that $\mathrm{Gal}(k_B/k)$ is generated by the image under the Frobenius map of the primes in S and of the primes lying above 2 [CF2].

Let the manifold M_0 in Theorem 2.0.1 cover some minimal orbifold $\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}$. By Lemma 2.2.2, we have $1 \geq \mathrm{Vol}(M_0) \geq \mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}}) \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})$. We therefore assume, throughout the rest of Section 2, that

$$(2.2) \qquad \mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}}) \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1 \,.$$

We proceed to make a complete list of all orbifold groups $\Gamma_{S,\mathfrak{D}}$ satisfying (2.2). The resulting orbifolds are the first eight given in Theorem 2.0.1. The ninth one is on the list because in this section we only prove that $\mathrm{lcmtor}(\Gamma_{\mathfrak{D}})$ is even and that $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) = 0.2783\ldots$. In a later section, we shall show by geometric means that $\mathrm{lcmtor}(\Gamma_{\mathfrak{D}}) = 4$, so that it too can be excluded as it violates (2.2).

By Proposition 2.1.2, the field k defining $\Gamma_{S,\mathfrak{D}}$ satisfies $2 \leq [k : \mathbb{Q}] \leq 8$ and $[k_B : k] \leq 2$.

2.3. – Degree 8

For $[k : \mathbb{Q}] = 8$ and $r_1(k) = 6$, Lemma 2.1.1 with $y = 1.1$ and $K = k$ gives the lower bound $d_k > 8.974^8$. Hence (cf. (2.1.1))

$$(2.3.1) \qquad \frac{2\pi^2 d_k^{\frac{3}{2}}}{(4\pi^2)^{[k:\mathbb{Q}]}[k_B : k]} > \frac{0.912}{[k_B : k]}.$$

Moreover, any $\mathfrak{p} \in \mathrm{Ram}_f(B)$ allows us to gain a factor of

$$(2.3.2) \qquad \frac{N\mathfrak{p} - 1}{2}(1 - N\mathfrak{p}^{-2})^{-1} \exp\left(1.5 \cdot 4 \sum_{j=1}^{\infty} \frac{\log(N\mathfrak{p})}{1 + N\mathfrak{p}^j} \alpha(j\sqrt{y} \log N\mathfrak{p})\right)$$

on the lower bound for $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})$ (see formula (2.1.1) and Lemma 2.1.1). For any $\mathfrak{p} \in \mathrm{Ram}_f(B)$, this improves (2.3.1) by a factor of at least 4.864, using $y = 1.1$ in (2.3.2). A $\mathfrak{p} \in S$ contributes even more, as the $\frac{N\mathfrak{p}-1}{2}$ is replaced by

$\frac{Np+1}{2}$. Therefore S and $\mathrm{Ram}_f(B)$ must be empty. We now use Lemmas 2.2.2 and 2.2.3. If $[k_B : k] = 1$ and $\mathrm{Ram}_f(B)$ is empty, then the narrow class number is odd. We find then that 3 divides $\mathrm{lcmtor}(\Gamma_{\mathfrak{D}})$, as $k(\sqrt{-3})/k$ must ramify at some prime above 3. This gives $\mathrm{lcmtor}(\Gamma_{\mathfrak{D}}) \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) > 2.736$, considering (2.3.1).

The case $[k_B : k] = 2$ is more difficult. First we prove the existence of 2-torsion in $\Gamma_{\mathfrak{D}}$.

LEMMA 2.3. *Suppose* $[k_B : k] = 2$, $h(k, 2, B) = 1$ *and that* $\mathrm{Ram}_f(B)$ *is empty. Then* $\Gamma_{\mathfrak{D}}$ *has two-torsion. The same holds for* $\Gamma_{S,\mathfrak{D}}$ *if the product of the primes in S has trivial image in the narrow class group of k.*

PROOF. Let h_k and h_k^+ denote, respectively, the wide and narrow class number of k. From the hypotheses, $h_k^+/h_k = 2$. A short calculation then shows $[\mathcal{O}_{k,+}^* : (\mathcal{O}_{k,+}^*)^2] = 4$. It follows that there is a totally positive unit ε such that $k(\sqrt{-\varepsilon})/k$ is a quadratic extension which is not contained in the narrow Hilbert class field of k. Hence $-\varepsilon \in C_2(\emptyset, \mathfrak{D})$ and, by Lemma 2.2.5, $\Gamma_{\mathfrak{D}}$ has 2-torsion for every $\mathfrak{D}$. To prove the last statement, replace ε above by a totally positive generator of the product of the primes in S. $\qquad\square$

Although Lemmas 2.2.2 and 2.3 show that

$$\mathrm{lcmtor}(\Gamma_{\mathfrak{D}}) \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) \geq 2 \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}),$$

the lower bound $d_k > 8.974^8$ only yields $2 \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) \geq 0.912$. Fortunately, M. Atria [At] has recently succeeded in improving Poitou's bound to $d_k > 9.05^8$. This yields $2 \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) \geq 1.009$. Hence we can rule out the case $[k : \mathbb{Q}] = 8$.

2.4. – Degree 7

In this case the lower bound is $d_k > 7.76^7$, corresponding to $y = 1.35$ and $K = k$, which yields

$$(2.4) \qquad \frac{2\pi^2 d_k^{\frac{3}{2}}}{(4\pi^2)^{[k:\mathbb{Q}]}[k_B : k]} > \frac{0.28}{[k_B : k]}.$$

As $\mathrm{Ram}(B)$ includes all five real places and has even cardinality, there must be a $\mathfrak{p} \in \mathrm{Ram}_f(B)$. Any such prime contributes a factor of at least 4.23 to (2.4). Thus, $\mathrm{Ram}_f(B)$ has exactly one prime $\mathfrak{p}$, S is empty and $[k_B : k] = 2$ (as $4.23 \cdot 0.28 > 1$). We now use:

LEMMA 2.4. *Suppose K/k is a quadratic extension unramified at all finite places and that $r_1(k)$ is odd. Then $r_1(K) \geq 2$.*

PROOF. Considering the image under the reciprocity map to $\mathrm{Gal}(K/k)$ of the global idele $-1 \in k^*$, we find

$$1 = (-1)^{\text{number of real places of } k \text{ ramified in } K/k}.$$

Therefore at least one real place of k splits in K/k.

We return to degree 7 fields. Thus $[k_B : \mathbb{Q}] = 14$, $r_1(k_B) \geq 2$ and $\mathfrak{p} \in \text{Ram}_f(B)$ is split to k_B (by definition of k_B). Then Lemma 2.1.1, with $y = 0.8$ and $K = k_B$ gives $d_k > 8.97^7$. As $\mathfrak{p}$ contributes a factor, as in (2.3.2), of at least 5.95, we conclude $[k : \mathbb{Q}] < 7$.

2.5. – Degree 6

If $[k_B : k] = 2$, then $d_k > 7.41^6$. This follows from Lemma 2.1.1 with $y = 1.1$, $K = k_B$ and taking the worst possible value of $r_1(k_B)$ (which is always 0 [Ma], [Po]). Thus,

$$\frac{2\pi^2 d_k^{\frac{3}{2}}}{(4\pi^2)^{[k:\mathbb{Q}]}[k_B : k]} > 0.1755 .$$

Any prime in $\text{Ram}_f(B)$ (which necessarily splits to k_B), contributes a factor of at least 4.86. As $\text{Ram}(B)$ has even cardinality, it follows that $\text{Ram}_f(B)$ is empty. A prime in S split to k_B yields a factor of 7.51. Hence S is empty or contains only primes inert to k_B. A prime in S inert to k_B contributes at least a factor of 2.855, except for a prime of norm 3 which contributes only 2.767. Hence S contains at most one prime, which must be inert to k_B. By the Remark following (2.1.1), if S is nonempty then $\Gamma_{S,\mathfrak{D}}$ is properly contained in $\Gamma_{\mathfrak{D}}$, and so can be dismissed in the proof of Theorem 2.0.1. We conclude that S is empty. Therefore, by Lemma 2.3, $\Gamma_{\mathfrak{D}}$ has 2-torsion. Also, if the narrow Hilbert class field $k_B \neq k(\sqrt{-3})$, then $\Gamma_{\mathfrak{D}}$ has 3-torsion (Lemma 2.2.3), which suffices to rule out such orbifolds. This leaves us to check only the list of sextic fields with exactly one complex place such that $k(\sqrt{-3})/k$ is unramified above 3 and with $d_k < 332572$. A check of the complete lists of sextic fields of this signature (see [BMO], [O1-2] and the tables available by FTP from megrez.math.u-bordeaux.fr) with $d_k < 332572$, d_k divisible by 3, and with the prime(s) above 3 having an even absolute ramification index, yields the following three fields:

a) $k = \mathbb{Q}(x)$, where $x^6 - x^5 - 2x^4 - 2x^3 + x^2 + 3x + 1 = 0$, $\text{disc}_k = -215811 = -3^3 \, 7993$,

b) $k = \mathbb{Q}(x)$, where $x^6 - 2x^5 - x^2 + 2x + 1 = 0$, $\text{disc}_k = -288576 = -3^3 \, 4^3 \, 167$,

c) $k = \mathbb{Q}(x)$, where $x^6 - 2x^5 - 2x^4 + 5x^3 - 2x^2 - 2x + 1 = 0$, $\text{disc}_k = -309123 = -3^3 \, 107^2$.

Using PARI [Co] we find that the place above 3 splits in $k(\sqrt{-3})/k$ only in case a). Hence, in cases b) and c), any $\Gamma_{\mathfrak{D}}$ has 2- and 3-torsion. This suffices to rule out these two fields.

However, in case a) above, we do not have 3-torsion in all $\Gamma_{\mathfrak{D}}$'s. It follows from the discussion at the end of Section 1 that there are two types of maximal orders in B. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be maximal orders of different types. Since

$\mathrm{Ram}_f(B) = \emptyset$ each of the groups $\Gamma_{\mathfrak{D}}$ and $\Gamma_{\mathfrak{D}'}$ has 2-torsion. Exactly one of these, say $\Gamma_{\mathfrak{D}'}$, also has 3-torsion [CF2]. Since $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}'}) = \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) = 0.27833973\cdots$, this suffices to rule out $\Gamma_{\mathfrak{D}'}$. The last group in Theorem 2.0.1 is the group $\Gamma_{\mathfrak{D}}$.

If $[k_B : k] = 1$, we only have the lower bound $d_k > 6.52^6$, corresponding to $y = 1.7$ and $K = k$ in Lemma 2.1.1. This yields

$$\frac{2\pi^2 d_k^{\frac{3}{2}}}{(4\pi^2)^{[k:\mathbb{Q}]}[k_B : k]} > 0.111\,.$$

Also, any prime in $\mathrm{Ram}_f(B)$ (respectively, in S) contributes at least a factor of 3.6 (respectively, 5.7). It follows that $\mathrm{Ram}_f(B)$ is empty and that S is either empty or contains exactly one prime. Since $\mathrm{Ram}_f(B)$ is empty and $[k_B : k] = 1$, k has odd narrow class number. We now need a result which we will also use frequently for smaller degrees.

LEMMA 2.5. *Suppose the narrow class number h_k^+ is odd. Then $\Gamma_{S,\mathfrak{D}}$ has 2-torsion for any S and $\mathfrak{D}$.*

PROOF. Let $S' = S \cup \mathrm{Ram}_f(B)$ and $\mathfrak{a} = \prod_{\mathfrak{p} \in S'} \mathfrak{p}$. Then we can find a totally positive $c \in k^*$ such that $\mathfrak{a}^{h_k^+} = (c)$. As h_k^+ is odd, $-c$ represents an element of $C_2(S, \mathfrak{D})$ for any $\mathfrak{D}$, notation being as in Lemma 2.2.5. Here we must take a little care with condition (d) in the definition of $C_2(S, \mathfrak{D})$. It can be dropped, as remarked there, if $k(\sqrt{-c})/k$ ramifies at some finite place. This must be the case since h_k^+ is odd, except possibly in the case that $-c$ is a square. This can only happen if S' is empty and $[k : \mathbb{Q}] = 2$, as c is totally positive. But then $B = M(2, k)$, in which case (by definition) $C_2(S, \mathfrak{D})$ always contains the trivial coset. We conclude that $C_2(S, \mathfrak{D})$ is non empty, and so $\Gamma_{S,\mathfrak{D}}$ has 2-torsion, by Lemma 2.2.5. □

By the lemma, $\Gamma_{S,\mathfrak{D}}$ has 2-torsion . Hence S is empty. But then we also have 3-torsion. This leads to $d_k < 100720$. However, for the two fields in this discriminant range, the prime 2 is inert from $\mathbb{Q}$, which means that B has Hilbert symbol $\{-1, -1\}$. This shows, by Lemma 2.2.6, that 12 divides $\mathrm{lcmtor}(\Gamma_{\mathfrak{D}})$ for these fields. Hence we conclude that $[k : \mathbb{Q}] < 6$.

2.6. – Degree 5

Suppose first that $[k_B : k] = 2$. Lemma 2.4, aplied to k_B/k, shows that $r_1(k_B) \geq 2$. Lemma 2.1.1, with $y = 1.1$ and $K = k_B$, now implies that $d_k > 7.49^5$. Hence

(2.6) $$\frac{2\pi^2 d_k^{\frac{3}{2}}}{(4\pi^2)^{[k:\mathbb{Q}]}[k_B : k]} > 0.372\,.$$

Any prime in $\mathrm{Ram}_f(B)$ contributes at least a factor of 4.86. As $\mathrm{Ram}_f(B)$ is not empty, we conclude that $[k_B : k] = 1$.

If the narrow class number h_k^+ is even, which may happen even if $[k_B : k] = 1$, let K/k be a quadratic extension unramified at the finite places. We get the same lower bound $d_k > 7.49^5$. However, now the contribution of any $\mathfrak{p} \in \mathrm{Ram}_f(B)$, which may be inert to K, is only ≥ 1.18 if $N\mathfrak{p} = 2$, and ≥ 1.38 if $N\mathfrak{p} > 2$. Using $[k_B : k] = 1$, (2.6) and $1.38 \cdot 2 \cdot 0.372 > 1$, we conclude that $\mathrm{Ram}_f(B)$ consists of a single prime of norm 2 and S is empty. But then Lemma 2.2.3 shows that $\Gamma_{\mathfrak{D}}$ has 3-torsion. Therefore h_k^+ is odd.

Lemma 2.1.1, with $y = 2.3$ and $K = k$, gives $d_k > 5.26^5$. Hence

$$\frac{2\pi^2 d_k^{\frac{3}{2}}}{(4\pi^2)^{[k:\mathbb{Q}]}} > 0.0525 \,.$$

By Lemma 2.5, $\Gamma_{S,\mathfrak{D}}$ has 2-torsion. Any prime in $\mathrm{Ram}_f(B)$ (respectively, in S) contributes a factor of at least 2.9 (respectively, 4.8). Hence $\mathrm{Ram}_f(B)$, which has odd cardinality, contains exactly one prime and S is empty. This allows us to restrict ourselves to $\Gamma_{\mathfrak{D}}$ coming from quintic fields with $d_k < 18070$, unless $\mathrm{Ram}_f(B)$ consists of a prime of norm 2. But in this case, $\Gamma_{\mathfrak{D}}$ has 3-torsion, so we get the better bound $d_k < 11384$. Examining k with $d_k < 18070$ yields only the field of discriminant -9759 (for which $\Gamma_{\mathfrak{D}}$ has no 3-torsion). This gives the seventh field in Theorem 2.0.1.

2.7. – Degrees 2, 3 and 4

When the degree $[k : \mathbb{Q}]$ is this small, one comes across very many fields and orbifolds that need to be examined individually in order to compute $\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}}) \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}})$, or at least insure that it is > 1. We give a sample of some of these calculations.

When working through imaginary quadratic fields we need not consider unramified (matrix) algebras $B = M(2, k)$, as this corresponds to cusped manifolds. In this case, Adams [Ad] showed that the smallest volume of any cusped manifold (arithmetic or not, orientable or not) is the volume of the regular ideal simplex in $\mathbb{H}^3$, which is approximately $1.01\ldots$. Thus, by Proposition 2.1.2, we need only consider imaginary quadratic fields with $d_k \leq 56$ and with $\mathrm{Ram}_f(B)$ consisting of at least two primes. An interesting example corresponds to $k = \mathbb{Q}(\sqrt{-3})$ and B the algebra ramified at the primes of norm 3 and 4. Then $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) = 0.1268677\cdots$ and $\Gamma_{\mathfrak{D}}$ has no 3-torsion. Lemmas 2.2.4 and 2.2.6, with $\{c, d\} = \{-1, \sqrt{-3}\}$, show that $\Gamma_{\mathfrak{D}}$ contains a subgroup isomorphic to a dihedral group of order 8. As $8 \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{\mathfrak{D}}) = 1.01494\cdots > 1$, this field can be dismissed.

When $[k : \mathbb{Q}] = 4$, for example, one begins by finding an upper bound for d_k by checking from (2.1.1) that $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > 1$ for $d_k \geq 11579$, even in the case that $\mathrm{Ram}_f(B)$ contains four primes of norm 2 and $[k_B : k] = 2$. If $[k_B : k] = 2$ we proceed as follows. An examination, using PARI, of the list of such fields of degree 4 with $r_1 = 2$ and even narrow class number h_k^+ shows that there are none with four primes of norm 2. Hence $\mathrm{Ram}_f(B)$ contains at

most 3 primes of norm 2. Again from (2.1.1), and using the fact that $\mathrm{Ram}_f(B)$ is of even cardinality we find that $\mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) > 1$ for $d_k \geq 8170$. However, the first field k with h_k^+ even and three primes of norm 2 has $d_k = 8712$. Hence $\mathrm{Ram}_f(B)$ contains at most two primes of norm 2. The corresponding bound is now $d_k \leq 6743$. One now examines the six fields with h_k^+ even and two primes of norm 2 in this discriminant range. The conclusion in each case is that either the orbifold volume is large enough (from the contributions of other primes in $\mathrm{Ram}_f(B)$ or in S) or that $\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}})$ is at least 2, 3 or 6, depending on the field. Thus, there is at most one prime of norm 2 in $\mathrm{Ram}_f(B)$. We are then down to fields with h_k^+ even, one place of norm 2 in $\mathrm{Ram}(B)$ and $d_k < 4758$. These have to be examined one by one. The case $[k_B : k] = 1$ is handled similarly.

Cubic fields are treated in much the same way. The one new feature in this case is that, since there is only one real place, we always have $h_k^+ = h_k$.

In this way, after a laborious check of the rigorous lists of small-discriminant number fields (available by FTP from megrez.math.u-bordeaux.fr), we arrive at a list of seven orbifold groups $\Gamma_{S,\mathfrak{D}}$ such that $\mathrm{lcmtor}(\Gamma_{S,\mathfrak{D}}) \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma_{S,\mathfrak{D}}) \leq 1$ and $[k : \mathbb{Q}] \leq 4$. Together with the orbifold coming from the quintic of discriminant -9759 and the sextic of discriminant -215811 discussed in Section 2.5, this is the list appearing in Theorem 2.0.1. Volumes appearing there were calculated using PARI. The covering degree restrictions come from the results on lcmtor in subsection 2.2. This concludes the proof of Theorem 2.0.1.

REMARK. The reader will find in [CF2], [CF4] further information on finite subgroups of arithmetic Kleinian groups, including the arithmetic construction of the Weeks manifold and a computation of torsion in the orbifold groups associated to quintic and sextic fields in Theorem 2.0.1.

3. – Proof of Theorem 0.1

Here we complete the proof of Theorem 0.1. The underlying idea is simply to get presentations for the groups of the orbifolds listed in Table 1 and check that subgroups of the appropriate index all have elements of finite order. This is done for the groups $G_1, \ldots G_8$. G_9 is handled separately.

3.1. – The groups G_i, $i = 1, \cdots, 8$.

In this section we prove,

PROPOSITION 3.1. *Presentations* Γ_i *for the groups* G_i $(i = 1, \ldots 8)$ *in Table 1 are:*

(1) $\Gamma_1 = < a, b, c | a^3 = b^2 = c^2 = (a(bc)^2b)^2 = (abc)^2 = (acbc)^3 = 1 >$;

(2) $\Gamma_2 = < a, b, c | a^3 = b^3 = c^2 = (a^{-1}b)^2 = (baca^{-1}bc)^2 = 1,$
$(cbcb^{-1}aca^{-1}b)^2 = 1 >$;

(3) $\Gamma_3 = < a, b, c | a^2 = b^2 = c^2 = (abacab)^3 = (ca)^2 = ((ba)^3 cbac)^2 = 1 >;$

(4) $\Gamma_4 = < a, b, c | a^2 = b^2 = c^2 = (abc)^3 = (babc)^2 = (acb(ca)^2 c)^2 = 1 >;$

(5) $\Gamma_5 = < a, b, c | a^2 = b^2 = c^2 = (babc)^2 = (ca)^2 (bc)^2 (bac)^2 a = 1 >;$

(6) $\Gamma_6 = < a, b, c | a^2 = b^2 = c^2 = (bc)^2 = ((ca)^2 (ba)^2)^2 = 1,$
$(c(ab)^2)^3 ac(ab)^3 = 1 >;$

(7) $\Gamma_7 = < a, b, c | a^2 = b^2 = c^2 = 1, (c(ab)^2 a)^2 = 1,$
$(ab)^2 acbabc(abac)^2 bab = 1, (abcabacbab)^2 = 1 >;$

(8) $\Gamma_8 = < a, b, c | a^2 = b^2 = c^2 = (abcb)^2 = (b(ca)^2 c)^2 = 1,$
$ab(ca)^3 bab(ca)^3 babc = 1 >.$

PROOF. The presentations in this proposition were obtained by a rigorous process involving computer calculation, which we discuss briefly. More details of this process will be given in [JR2].

The idea is this: from the arithmetic data, we know that there exists a co-compact Kleinian group G_i. In all of the cases at hand, G_i is the normalizer of a certain order $\mathfrak{D}_i$. This is obvious for all values of i except 6 and 8 where S is nonempty. To deal with these, recall from the discussion in Section 1 that $\Gamma_{S,\mathfrak{D}}$ has the description as the image of the normalizer of an Eichler order. Hence, using only integer arithmetic, we can generate subgroups Γ of G_i. This is accomplished by enumerating the elements in the order $\mathfrak{D}_i$, and checking to see which order elements are in the group (by checking whether or not a given element normalizes $\mathfrak{D}_i$). Using this enumeration, we then use a particular representation of G_i into PGL(2, $\mathbb{C}$) (essentially induced by splitting the quaternion algebra at the complex place), together with floating-point interval arithmetic, to calculate an approximate Dirichlet fundamental polyhedron. The representation of Γ into PGL(2, $\mathbb{C}$) is induced by a representation of $\mathfrak{D}_i$, which is readily determined by knowing the trace of each basis element of $\mathfrak{D}_i$. We continue adding new elements to Γ until a volume computation tells us that $\Gamma = G_i$.

The face-pairings of this approximate polyhedron are then used to generate a group presentation for the orbifold (one generator for each face-pair, one relation for each edge class). Note that all conjugacy classes of elements of finite order show up as relators which are a proper power of the word represented by the faces around an edge class, with the total dihedral angle around the edge class used to calculate the exponent associated to each relator. Thus, torsion-free groups are easily recognized from such a presentation.

Note that the vertices of this polyhedron are only calculated with finite precision – that is, there is an uncertainty associated to each vertex. However, because of the use of interval arithmetic, upper bounds on this uncertainty are known (precisely). The true Dirichlet polyhedron of the group G_i (if it were known precisely) could be obtained from the approximate one by adjusting vertices within their "region of uncertainty" as well as possibly splitting approximate vertices into multiple true vertices joined by new edges lying entirely within a region of uncertainty.

However, if the regions of uncertainty are small compared to the injectivity radius of the Kleinian group (suitably defined), then none of these adjustments will affect the presentation calculated from the face-pairings of the polyhedron. Specifically, if the largest diameter of any region of uncertainty is smaller than the minimum of half of the length of the shortest closed geodesic and half of the shortest ortholength between two nonintersecting elliptic axes (using in both cases precisely known lower bounds for these lengths), then the presentation computed from the approximate polyhedron will be isomorphic to the group G_i.

This procedure was carried out for each of the groups G_i ($i = 1, \ldots 8$) in Table 1 and the resulting presentations (after simplification) are the ones given in the statement of this Proposition. The approximate fundamental domains are given in Figure 2 in the Appendix. It might be noted that in these pictorial representations, the uncertainties are sufficiently small that the true fundamental domains would be indistinguishable from the approximate ones to the naked eye.

The simplifications used to obtain the presentations in this Proposition from the face-pairing presentations derived directly from the polyhedra are given in Table 3 of the Appendix. The maximum uncertainties and the various relevant lengths for each polyhedron are given in Table 4 of the Appendix. Note that the uncertainties themselves are highly dependent on the numerical implementation chosen (and on the domain generating algorithm chosen, as well as choice of basepoint and representation) whereas the lengths are geometrically determined and thus independent of all these factors. Another implementation of the same procedure might well produce much more uncertainty, to the point of having an uncertainty that is larger than the bound guaranteeing correctness of the resulting presentation. These numbers are the results of our particular implementation and in these cases the uncertainties are all well within the necessary limits. The reader may notice that in all cases in Table 4 the shortest ortholength is half the length of the shortest geodesic. This is often the case for orbifolds generated by elliptic elements as all of the G_i are.   □

3.2. – Trace calculations

As a further check on the presentations, we can compute the character varieties of the Γ_i and verify that these do correspond to the arithmetic data on G_i with which we started. This allows us to rule out the possibility of an error, either in the implementation of the procedure that calculated the face-pairing presentations for Γ_i or in the simplification process that led to the presentations above. We indicate how this was done specifically for Γ_1 and Γ_4. The others follow similar lines and details are available from the authors.

Throughout this calculation, we will be working with SL(2, $\mathbb{C}$) since it is computationally simpler than PSL(2, $\mathbb{C}$), but since we are working with orbifolds, some care is required when lifting representations from PSL(2, $\mathbb{C}$).

To show that Γ_1 is isomorphic to G_1 we first compute a faithful discrete representation of Γ_1. Note that by construction Γ_1 is isomorphic to a Kleinian group since it arises as face-pairings on a polyhedron. We find it convenient to work with traces.

By standard arguments (cf. [CS3] for example), the traces of a three-generator subgroup $\langle x, y, z \rangle$, of SL(2, $\mathbb{C}$) are completely determined by the traces of x, y, z, xy, yz, zx and xyz. Let $\rho : \Gamma_1 \to$ SL(2, $\mathbb{C}$) be any representation and let

$$t = \mathrm{tr}\rho(a),$$
$$u = \mathrm{tr}\rho(b),$$
$$v = \mathrm{tr}\rho(c),$$
$$x = \mathrm{tr}\rho(ab),$$
$$y = \mathrm{tr}\rho(bc),$$
$$z = \mathrm{tr}\rho(ca),$$
$$w = \mathrm{tr}\rho(abc).$$

Then, we compute that $\mathrm{tr}\rho(a(bc)^2b) = xy^2 - tvy + yz - x$, and $\mathrm{tr}\rho(acbc) = yz - tu + x$. These seven variables are not independent: it is always true that $w^2 - w(ty + uz + vx - tuv) + t^2 + u^2 + v^2 + x^2 + y^2 + z^2 + xyz - 4 - tvz - uvy - tux = 0$. Thus, the relations in the group give us the following seven equations in the seven trace variables:

$$1 = t,$$
$$0 = u,$$
$$0 = v,$$
$$0 = xy^2 - tvy + yz - x,$$
$$\pm 1 = yz - tu + x,$$
$$0 = w,$$
$$0 = w^2 - w(ty + uz + vx - tuv) + t^2 + u^2 + v^2 + x^2 + y^2 + z^2$$
$$+ xyz - 4 - tvz - uvy - tux.$$

These readily simplify to yield $y = \sqrt{\alpha}$, $x = \pm 1/(2 - \alpha)$, $z = \pm(1 - \alpha)/\sqrt{\alpha}(2 - \alpha)$ where α satisfies $\alpha^4 - 7\alpha^3 + 16\alpha^2 - 12\alpha + 1 = 0$. Applying the results of [MR1] to the group generated by a^2 and $ba^{-2}b^{-1}$, we find that the invariant trace field is $\mathbb{Q}(\alpha)$ and the invariant quaternion algebra has Hilbert symbol $\{-3, \alpha^3 - \alpha^2 - 1\}$. The invariant trace field is thus a field with one complex place of discriminant -283 and the invariant quaternion algebra is ramified at both of the real places. Furthermore, the algebra is not ramified at any finite place, since the only primes dividing the entries in the Hilbert symbol are 3 (which is inert from $\mathbb{Q}$) and a prime of norm 13 ($\alpha^3 - \alpha^2 - 1$ has norm 13). Neither of these primes ramifies since -3 is a square modulo 13 and $\alpha^3 - \alpha^2 - 1$ is a square in $\frac{\mathbb{Z}}{3\mathbb{Z}}(\alpha)$, the finite field of order 81. Furthermore, 2 is inert, so it does not ramify since the ramification set must have even cardinality.

Thus, we see that Γ_1 is indeed commensurable with G_1. To see that it is isomorphic to that group, the computer packages allow us to calculate the volume of the approximate fundamental domain for Γ_1 and observe that it

coincides (only a very rough approximation is necessary) with the covolume for G_1 given in Table 1. However, as the algebra has only one type of maximal order (see Section 1), the results of [Bo] show that G_1 is the unique group in its commensurability class achieving the smallest covolume and the next covolume is at least twice as big. Thus, $G_1 \cong \Gamma_1$.

We now show $G_4 \cong \Gamma_4$. Repeating the earlier procedure, we obtain the seven equations

$$0 = t = u = v\,,$$
$$0 = xy - tv + z\,,$$
$$0 = yz^3 - tuz^2 + xz^2 - 2yz + tu - x\,,$$
$$\pm 1 = w\,,$$
$$0 = w^2 - w(ty + uz + vx - tuv) + t^2 + u^2 + v^2 + x^2 + y^2 + z^2$$
$$+ xyz - 4 - tvz - uvy - tux$$

which simplify to yield $y = \sqrt{2 - \alpha^2}$, $x = \pm\sqrt{\alpha^2 + 1}$ and $z = -xy$ where α satisfies $\alpha^3 - \alpha + 1 = 0$ (there is also another solution $x = z = 0$, $y = \sqrt{3}$ which is totally real and hence does not correspond to a discrete, faithful representation into $PSL(2, \mathbb{C})$). Again applying [MR1], this time to the group generated by $(ba)^2$ and $(bc)^2$, we find that the invariant trace field is $\mathbb{Q}(\alpha)$ and the invariant quaternion algebra has Hilbert symbol $\{-3, -2 - \alpha\}$. Thus, the invariant trace field is a field with one complex place of discriminant -23 and the invariant quaternion algebra is ramified at the unique real place. The only possible finite places in the ramification set are primes over $2, 3$ and 5 (since $-2 - \alpha$ has norm 5). In this field, 2 and 3 are inert, while 5 splits as a product of a prime ideal of norm 5 and a prime ideal of norm 25. The algebra is ramified at the prime of norm 5, but not at the prime of norm 25 since -3 is not a square in $\mathbb{Z}/5\mathbb{Z}$ but is a square in the field of order 25. The algebra does not ramify at 3 since $-2 - \alpha$ is a square in $\frac{\mathbb{Z}}{3\mathbb{Z}}(\alpha)$, the finite field of order 81. The algebra does not ramify at 2 since the ramification set must have even cardinality.

So, again we see that Γ_4 is commensurable with G_4. Volume considerations then force $G_4 \cong \Gamma_4$ as before.

The other groups, G_2 to G_8, are treated similarly. Details are available from the authors. We now relate G_4 and G_1 to the Weeks and Meyerhoff manifolds.

PROPOSITION 3.2. *The orbifold $\mathbb{H}^3/G_4$ is covered with degree 12 by the Weeks manifold. The orbifold $\mathbb{H}^3/G_1$ is covered with degree 24 by the Meyerhoff manifold.*

PROOF. We first record the arithmetic nature of the Weeks manifold M_W. Chinburg and Jørgenson stated in Section I of [Ch] that M_W is arithmetic, but a proof was never published. A proof is in fact implicit in several places now, e.g. in [RW]. For completeness we give details, following [RW].

To simplify some of the calculations, it is convenient to work with a slightly different surgery description of the Weeks manifold given in the Introduction. Doing $(5, 1)$-surgery on one component of the Whitehead link produces a once punctured torus bundle X (the sister to the figure eight knot) and the manifold we require can be described as $(-3, 1)$-surgery on X with respect to some framing of the peripheral torus (see for example the census of closed 3-manifolds produced by SnapPea [We2]). Let M_W denote this closed manifold, a presentation for the fundamental group is

$$\pi_1(M_W) = < a, b \mid a^2b^2a^2b^{-1}ab^{-1} = 1, a^2b^2a^{-1}ba^{-1}b^2 = 1 > .$$

By [We1], this manifold is hyperbolic, and we now compute a faithful discrete representation (which is unique up to conjugacy). We shall make use of Mathematica in these calculations.

We begin by noting that the hyperbolic structure on M_W arises from a faithful discrete representation of $\pi_1(M_W)$ into $\mathrm{PSL}(2, \mathbb{C})$, and this can be lifted to a representation of $\pi_1(M_W)$ into $\mathrm{SL}(2, \mathbb{C})$ (see [Cu]). We can conjugate a representation ρ of $\pi_1(M_W)$ into $\mathrm{SL}(2, \mathbb{C})$ so that the images of a and b are the matrices:

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}$$

$$\rho(b) = \begin{pmatrix} y & 0 \\ r & y^{-1} \end{pmatrix}$$

respectively. As we are looking for the faithful discrete representation into $\mathrm{SL}(2, \mathbb{C})$, x and y are always non-zero, and not roots of unity.

Write the first relation as $w = 0$ where $w = a^2b^2a^2 - ba^{-1}b$. This gives the following equations.

$$0 = w_{11} = r x^2 + r x^4 + r x y^2 + r x^2 y^2 + r x^4 y^2 - y^3 + x^5 y^3.$$

$$0 = w_{12} = 1 + x^2 + r x y + 2 r x^3 y + r x^5 y + x^3 y^2 + r x y^3 + 2 r x^3 y^3 \\ + r x^5 y^3 + x^4 y^4 + x^6 y^4.$$

$$0 = w_{21} = r \left(x - x^2 + r x y - y^2 + x y^2 \right).$$

$$0 = w_{22} = 1 - x^5 + r x y + r x^3 y + r x^4 y + r x y^3 + r x^3 y^3.$$

Note from the equation for w_{21} we have either $r = 0$ or we can solve for r in terms of x and y (which as noted above are always non-zero). Since the faithful discrete representation will correspond to a non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ we must have $r \neq 0$. Thus we may assume r is non-zero and from above, is given by:

$$r = \frac{(x^2 - x + y^2 - xy^2)}{xy}.$$

Using this and re-working the above equations gives:

$$w_{11} = (-1+x)\,x\,\left(1 + x^2 + y^2 - x\,y^2 + x^2\,y^2 + y^4 + x^2\,y^4\right),$$

$$w_{12} = \left(1 - x + x^2 - x^3 + x^4\right)\left(1 + x^2 + y^2 - x\,y^2 + x^2\,y^2 + y^4 + x^2\,y^4\right),$$

$$w_{21} = 0,$$

$$w_{22} = (-1+x)\,(-1+x)\,\left(-1 - x^2 - y^2 + x\,y^2 - x^2\,y^2 - y^4 - x^2\,y^4\right).$$

Notice that the expressions for w_{11}, w_{12} and w_{22} all have the common factor

$$p(x, y) = \left(1 + x^2 + y^2 - x\,y^2 + x^2\,y^2 + y^4 + x^2\,y^4\right).$$

The only way we can simultaneously satisfy all the above equations is for $p(x, y) = 0$.

Writing the second relation as $u = a^2 b^2 - b^{-2} a b^{-1} a$, and solving $u = 0$ we deduce that

$$u_{12} = (x - y)\,(-1 + x\,y) = 0,$$

and so we must have $x = y$ or $x = 1/y$. With $x = y$ (a similar argument applies to $y = 1/x$, which also yields the same characters), $p(x, y)$ is simply the polynomial in x given by:

$$p(x) = 1 + 2\,x^2 - x^3 + 2\,x^4 + x^6,$$

which must solve to zero to determine a representation. Solving for $z = x + x^{-1}$ yields the polynomial in z, $z^3 - z - 1 = 0$. From the equation for r we see that $r = 2 - z$.

To see that M_W is arithmetic, and determine the arithmetic data associated to M_W we use [MR1]. Briefly, let Γ denote the faithful discrete representation constructed above. The invariant trace-field of Γ is the cubic field $\mathbb{Q}(z)$ with one complex place and which has discriminant -23. All traces of elements in Γ are integers in $\mathbb{Q}(z)$ since this is true for the images of a, b and ab. The Hilbert symbol for the invariant quaternion algebra is given by $\{\mathrm{tr}^2(a) - 4, \mathrm{tr}([a, b]) - 2\}$ which on calculation gives $\{z^2 - 4, x\}$ where $x = 3z^2 - z - 5$ and satisfies $x^3 + 9x^2 + 32x + 25 = 0$ and $u = z^2 - 4$ satisfies, $u^3 + 10u^2 + 33u + 35 = 0$

Note that the real Galois conjugate of u is approximately -2.24512, so that $z^2 - 4 < 0$, and the real Galois conjugate of x is approximately -1.06008, and hence the invariant quaternion algebra is ramified at the real place of $\mathbb{Q}(z)$. It follows from these remarks that M_W is arithmetic.

To determine the finite places at which the algebra is ramified we proceed as follows. From the description of the invariant quaternion algebra given above the only finite places that can ramify are the unique places of norm 5, 7, 8, and 25. As noted in the introduction, the volume of M_W is approximately

0.9427073627769 (this approximation can be determined from SnapPea). Now the group of isometries of M_W is known to be the dihedral group of order 12 (see [HW] or [MV]), hence in the commensurability class of M_W there is an orbifold of volume approximately 0.078558946. Using the volume formula of Borel discussed in Section 1 the only finite place that can ramify in the invariant quaternion algebra is the one of norm 5. This completes the description of the arithmetic structure associated to the Weeks manifold. The discussion above also shows that M_W covers $\mathbb{H}^3/G_4$ with degree 12, which completes the proof of Proposition 3.2 for the Weeks manifold.

We now deal with the Meyerhoff manifold. In what follows we shall denote the Meyerhoff manifold by M. From [Ch] M is arithmetic with invariant trace field and quaternion algebra being as described by the first case in Theorem 2.0.1. Furthermore, as is easily deduced from [Ch], M covers the orbifold $\mathbb{H}^3/G_1$(alternatively in the language of [MR1] M is derived from a quaternion algebra as $H_1(M, \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$). From the volume comparisons we see that the index is 24.   □

3.3. – Subgroup enumeration

The next step in the proof of Theorem 0.1 is to enumerate all the subgroups of G_i of the appropriate index j. Namely, by Theorem 2.0.1, j must be divisible by $12 = $ lcmtor(G_i) for $1 \leq i \leq 4$, by 4 for $5 \leq i \leq 8$, and also $j \cdot \text{Vol}(\mathbb{H}^3/G_i) \leq 1$.

We now give the list of subgroups obtained using the presentations in Proposition 3.1 and the Cayley/Magma group theory software. In the next section we give an alternative algorithm for ruling out torsion-free subgroups.

Case 1:

From Table 1, lcmtor(G_1) Vol$(\mathbb{H}^3/G_1) = 12 \cdot 0.04089 \cdots = 0.49068 \cdots$. There is a unique subgroup $\Gamma' \subset \Gamma_1$ of index 12, generators for which are:

$$< ba^{-1}, c, aca^{-1}ba > .$$

This contains c which has order 2, hence has torsion.

Magma can also be used to show there is a unique torsion-free subgroup $\Gamma \subset \Gamma_1$ of index 24 (there are 2 with torsion). We also find $\Gamma \subset \Gamma'$. By Proposition 3.2, $\mathbb{H}^3/\Gamma$ is isometric to the Meyerhoff manifold.

Case 2:

From Table 1, lcmtor(G_2) Vol$(\mathbb{H}^3/G_2) = 12 \cdot 0.05265 \cdots = 0.6318 \cdots$. As above, there is a unique subgroup of Γ_2 of index 12 generated by:

$$< ba^{-1}, caca^{-1}, cb^{-1}ac, a^{-1}cac > .$$

Notice that from the presentation given in Proposition 3.1, $a^{-1}b$ has order 2, hence ba^{-1} has order 2, and therefore this subgroup contains an element of finite order. Thus, there are no torsion-free subgroups of index 12.

Case 3:

From Table 1, $\mathrm{lcmtor}(G_3)\,\mathrm{Vol}(\mathbb{H}^3/G_3) = 12 \cdot 0.06596 \cdots = 0.7915 \cdots$. The subgroups of Γ_3 of index 12 are:

$$< a, b, cbababcbc >;$$

$$< a, c, bab, bcbcbcabcbcb >;$$

$$< a, bcb, cbcbc, bababab >;$$

$$< a, cbab, bcbcbcab >;$$

$$< b, acbca, abababa, cbababc >;$$

$$< b, (cba)^2, abcbacbca >;$$

$$< ca, (ba)^2, bcbcbcba >;$$

$$< ca, bcab, (ba)^4 >.$$

Since a, b and ca all have finite order, none of the groups are torsion-free and so there is no torsion-free subgroup of index 12.

Case 4:

From Table 1, $\mathrm{lcmtor}(G_4)\,\mathrm{Vol}(\mathbb{H}^3/G_4) = 12 \cdot 0.0785589 \cdots = 0.9427 \cdots$. There is a unique torsion-free subgroup Γ of index 12 in Γ_4. By Proposition 3.2, $\mathbb{H}^3/\Gamma$ is isometric to the Weeks manifold.

Case 5:

From Table 1, $\mathrm{lcmtor}(G_5)\,\mathrm{Vol}(\mathbb{H}^3/G_5) = 4 \cdot 0.11783 \cdots = 0.4713 \cdots$. The subgroups of Γ_5 of index 4 are:

$$< a, b, cac >;$$

$$< a, c, bab, bcacb >;$$

$$< a, bab, cac, (cb)^2 >.$$

Since a has finite order, none of these give torsion-free subgroups of index 4. The index 8 subgroups in Γ_5 are:

$$< a, bab, cac, bcacb >;$$

$$< b, aba, (ca)^2 >;$$

$$< c, aca, (ba)^2, bcacba >.$$

Again it is clear that each of these has an element of finite order, hence no torsion-free subgroup of index 8 exists. We have checked that the minimal index of a torsion-free subgroup of Γ_5 is 16. This seems interesting in light of the fact that for Fuchsian groups the minimal index of a torsion-free subgroup

is always either the least common multiple of the orders of the elements of finite order or twice this [EEK]. For more on this phenomenon, in particular, for construction of co-compact Kleinian groups where lcmtor is bounded but the minimal index of a torsion free subgroup gets arbitrarily large, see [JR1].

Case 6:
From Table 1, $\text{lcmtor}(G_6) \, \text{Vol}(\mathbb{H}^3/G_6) = 4 \cdot 0.15796 \cdots = 0.6318 \cdots$. The subgroups of Γ_6 of index 4 are:

$$< a, bab, cacb >;$$

$$< b, aba, (ca)^2 >;$$

$$< c, acba, (ba)^2 >.$$

Again all these groups have elements of finite order.

Case 7:
From Table 1, $\text{lcmtor}(G_7) \, \text{Vol}(\mathbb{H}^3/G_7) = 4 \cdot 0.22804 \cdots = 0.9121 \cdots$. Subgroups of Γ_7 of index 4 are listed below and again there are elements of finite order in all of them.

$$< a, bab, cac, (cb)^2 >;$$

$$< b, aba, (ca)^2, cbca >;$$

$$< c, aca, (ba)^2 >.$$

Case 8:
From Table 1, $\text{lcmtor}(G_8) \, \text{Vol}(\mathbb{H}^3/G_8) = 4 \cdot 0.24534 \cdots = 0.9813 \cdots$. There is a unique subgroup of index 4 in Γ_8. It is torsion-free and corresponds to the Meyerhoff manifold. This last claim can be justified as follows. G_8 is commensurable with G_1. In fact, by results in [Bo], $[G_1 : G_1 \cap G_8] = 12$ and $[G_8 : G_1 \cap G_8] = 2$. Thus, $G_1 \cap G_8 = \Gamma'$, the unique subgroup of G_1 of index 12. Γ', in turn, has a unique torsion-free subgroup of index 2, which corresponds to the Meyerhoff manifold (see Case (1) above). As G_8 has a unique subgroup of index 4, the two subgroups coincide.

It should be noted that the elements of finite order alluded to above are all nontrivial (and hence give rise to torsion) since in every case, adding that element as a relation changes the group in one of two easily verifiable ways: either changing $\Gamma_i/[\Gamma_i, \Gamma_i]$ or making Γ_i abelian (which is not possible for a finite covolume Kleinian group).

3.4. – The case of G_9

The one outstanding case to deal with is the case of the sextic field k of discriminant -215811 discussed in Section 2.5. Here, the algebra B is unramified at all finite places and there are two distinct types of maximal

orders. We shall show for the one type not handled in Section 2.5, that $\Gamma_{\mathfrak{D}}$ contains a copy of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

We first note that this algebra is isomorphic to the one with Hilbert symbol $\{-1, -1\}$ over the sextic field k. This is because 2 is inert in this field, and so $\{-1, -1\}$ is unramified at all finite places.

Firstly it is convenient to use the following representation of k, namely as $\mathbb{Q}(\alpha)$ where α satisfies $\alpha^6 + 5\alpha^5 + 8\alpha^4 - 12\alpha^2 - 8\alpha + 1 = 0$ (that this is the same field as the one described in Section 2.5 follows by uniqueness of the discriminant -215811 for sextic fields of one complex place). Then, using PARI, we find that $\mathcal{O}_k$, the algebraic integers in k is $\mathbb{Z}[\alpha]$. We will construct an explicit representation of B into $M(2, \mathbb{C})$ as follows.

As is well-known (see [CS3]) a pair of matrices $\{a, b\}$ generating a non-elementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ determines, and is completely determined up to conjugacy by, a triple of numbers $(\mathrm{tr}(a), \mathrm{tr}(b), \mathrm{tr}(ab))$. In view of this we define the following elements of $\mathrm{SL}(2, \mathbb{C})$: a has trace α, b has trace $\alpha^5 + 3\alpha^4 + 2\alpha^3 - 3\alpha^2 - 4\alpha$ and ab has trace $-\alpha^5 - 3\alpha^4 - 2\alpha^3 + 4\alpha^2 + 5\alpha - 1$. Note, by construction the field generated by traces of elements of the group $< a, b >$ is k. Define the k-subalgebra A of $M(2, \mathbb{C})$ to be $k[1, a, b, ab]$. This is a quaternion algebra over k. We claim it is isomorphic to B. To this end, let $\mathfrak{D} = \mathcal{O}_k[I, a, b, ab]$. It is an easy exercise to see that $\mathcal{O}_k[I, a, b, ab]$ always forms an order when a, b, and ab all have norm 1 and integral trace (cf. [GMMR]) Hence $\mathfrak{D}$ is an order in A. The Hilbert symbol for A can be computed directly as $\{\mathrm{tr}^2(a) - 4, \mathrm{tr}([a, b]) - 2\}$. We also remark that the discriminant of $\mathfrak{D}$ is the $\mathcal{O}_k$-ideal $< \mathrm{tr}([a, b]) - 2 >$. On performing this calculation, we get $\{\alpha^2 - 4, -\alpha^5 - 3\alpha^4 - \alpha^3 + 5\alpha^2 + 3\alpha - 4\}$ with the Hilbert symbol entries being units in $\mathcal{O}_k$. Thus we deduce that, since 2 is inert in k, together with the fact that both Hilbert symbol entries for A are negative at all real places (and so ramified at all real places, and unramified at all finite places) A is isomorphic to B, and $\mathfrak{D}$ is a maximal order of A. Thus it suffices to work with the representation A of B.

Now, a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ inside $k^*\mathfrak{D}^*/k^*$ is generated by the image of

$$(-\alpha^5 - 3\alpha^4 - 2\alpha^3 + 3\alpha^2 + 4\alpha)I + (-\alpha^5 - 3\alpha^4 - 2\alpha^3 + 4\alpha^2 + 5\alpha - 1)a + b + ab$$

and

$$(-\alpha^5 - 5\alpha^4 - 5\alpha^3 + 6\alpha^2 + 8\alpha - 1)I + (-\alpha^5 - 3\alpha^4 + 6\alpha^2 + \alpha - 5)a$$

$$+ (2\alpha^2 + 4\alpha + 1)b - (2\alpha + 3)ab.$$

One needs merely to check that the trace of each of these elements and the trace of their product is zero. Note that both of these elements have unit norm, and so are in the normalizer of $\mathfrak{D}$ (see [Bo]).

It remains now only to show that there is no 3-torsion in $\Gamma_{\mathfrak{D}}$. The simplest way to see this is by using the computer package as discussed in Section 3 to show that a and b generate a Kleinian group G of co-volume equal to 8

times the co-volume of the maximal group $\Gamma_{\mathfrak{D}}$. As discussed in the proof of Proposition 3.1, torsion elements produced by the program are readily observed from the presentation. In particular the group G can be seen to have no 3-torsion (see below). Hence there cannot be any 3-torsion in the maximal group, since the index of G in the maximal group is a power of 2.

For the sake of completeness, we indicate that the group G is determined by the computer package to have an unsimplified presentation

$$< a, b, c, d, e, f, g, h, i, j, k, l, m :$$
$$b = fk, c = ab = da = ke = g^{-2}, d = bj = ki = f^2, e = cj = ia = ad,$$
$$f = kj, g = dh, i = jf, l = eh = ag = hc, m = hj,$$
$$emb = bhe = cmf = blj = dlg = hic = 1 > .$$

In this computation, the maximum uncertainty was 3.48978×10^{-7} and the shortest geodesic has length 0.404575. There is no ortholength between two elliptic axes, since the group is found to be torsion free.

REMARK. In fact the group G discussed in the previous paragraphs is the fundamental group of $(1, 2)$-Dehn surgery on the complement of the knot 5_2. In particular G is torsion-free. This manifold (and a pair of 11-fold coverings) seems to be the only known examples of *integral* homology spheres which are arithmetic. Further details can be obtained from the authors.

3.5. – Completing the proof

PROOF OF THEOREM 0.1. We can now put together all the pieces of the proof. By Theorem 2.0.1, an arithmetic hyperbolic 3-manifold M_0 with $\mathrm{Vol}(M_0) \leq 1$ covers one of the nine orbifolds $\mathbb{H}^3/G_i$ described in Table 1. Section 3.4 rules out G_9. Furthermore, Proposition 3.1 gives presentations for the groups G_i for $i = 1, \ldots 8$ and the Cayley/Magma data given above show that only G_1, G_4 and G_8 have torsion-free subgroups giving rise to manifolds of volume ≤ 1. As remarked above, the uniqueness of each of these subgroups and Proposition 3.2 allow us to conclude that M_0 is isometric to the Weeks or Meyerhoff manifolds. In particular, these are also the unique arithmetic hyperbolic 3-manifolds attaining their respective volumes.   □

REMARK. A review of the proof shows that the main obstacle to extending Theorem 0.1, perhaps to listing volumes < 1.3, is the weak discriminant lower bound in degree 8, namely $d_k > 9.05^8$. The Generalized Riemann Hypothesis implies the far better bound $d_k > 9.26^8$ [Ma].

4. – Ruling out torsion-free subgroups

As an independent check of the results produced by Cayley/Magma, we decided to implement another method of ruling out the existence of torsion-free

subgroups of Γ_i of small index. We sketch two typical cases, one for subgroups of index 12 and another one for index 4. Consider first the case of Γ_1. From Theorem 2.0.1, the minimal index torsion-free subgroup is a multiple of 12.

LEMMA 4.1. Γ_1 *has no torsion-free subgroup of index* 12.

PROOF. If there were a torsion-free subgroup $\Gamma \subset \Gamma_1$ of index 12, there would be a map $\varphi : \Gamma_1 \to S_{12}$ such that $\ker(\varphi) \subset \Gamma$, $\varphi(a)$ and $\varphi(acbc)$ would be products of 4 disjoint 3-cycles and $\varphi(b)$, $\varphi(c)$, $\varphi(abc)$, and $\varphi(abcbcb)$ would be each a product of 6 disjoint 2-cycles (more generally, each element of finite order n must map to a product of $12/n$ n-cycles). We will show that there is no such map.

First, observe that $cbca$ and a generate an A_4 subgroup of Γ_1 ($cbca$ has order 3, a has order 3 and cbc has order 2). There is a unique map from A_4 to S_{12} (up to conjugacy) that has the correct cycle structure. Thus, without loss of generality, we may assume that

$$\varphi(cbca) = (1\,2\,3)(4\,5\,6)(7\,8\,9)(10\,11\,12)$$

$$\varphi(a) = (1\,7\,4)(2\,10\,9)(3\,5\,12)(6\,8\,11).$$

Next, observe that c and bca generate an S_3 subgroup of Γ_1. Note that the image of the product of these two was determined above. There are 27 possible ways to map S_3 to S_{12} with the correct cycle structure and so that a given element of order 3 has fixed image (these are enumerated below in Table 2 – one simply lists all the products of disjoint two-cycles that conjugate the fixed 3-cycle product to its inverse). Now, note that the complete map to S_{12} is determined, since a, c and $cbca$ form a generating set for Γ_1. Thus, for each of the 27 possibilities for $\varphi(c)$, we can calculate $\varphi(abcbcb)$ and show that in each case, it does not have the required cycle structure. The results of this calculation are summarized in Table 2 below and complete the proof that Γ_1 has no torsion-free subgroup of index 12.

Now consider the case of Γ_7. From Theorem 2.0.1, the minimal index torsion-free subgroup is a multiple of 4.

LEMMA 4.2. Γ_7 *has no torsion-free subgroups of index* 4.

PROOF. Suppose there exists a torsion-free subgroup Γ of index 4. Hence the permutation representation of Γ_7 on Γ yields a map φ into S_4. Since Γ is torsion-free, $\varphi(a)$, $\varphi(b)$ and $\varphi(c)$ must be products of two disjoint 2-cycles. There are only three such in S_4, and they lie in a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ subgroup. Hence the image of Γ_7 under φ is abelian. However, from Proposition 3.1 it is easy to observe that the element $cacababa$ lies in the kernel of the map induced by abelianizing, and has order 2. Hence this contradicts Γ being torsion-free. □

This method has been automated, and a computer program has been written which produces a detailed (and rather tedious) proof, based on a case-by-case examination of the possible representations into S_n, of the existence or nonexistence of torsion-free subgroups of index n, given a presentation of a group

Table 2

$\varphi(c)$	$\varphi(abcbcb)$
(1 4)(2 6)(3 5)(7 10)(8 12)(9 11)	(1 10 11 2 7)(5 8 9 6 12)
(1 4)(2 6)(3 5)(7 11)(8 10)(9 12)	(1 12 9)(2 8 4)(3 6 10)(5 7 11)
(1 4)(2 6)(3 5)(7 12)(8 11)(9 10)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 5)(2 4)(3 6)(7 10)(8 12)(9 11)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 5)(2 4)(3 6)(7 11)(8 10)(9 12)	(1 4 2 8 9)(3 6 10 11 5)
(1 5)(2 4)(3 6)(7 12)(8 11)(9 10)	(1 12 6 11 3)(2 9 4 5 7)
(1 6)(2 5)(3 4)(7 10)(8 12)(9 11)	(1 9 10 7 3)(4 5 11 8 12)
(1 6)(2 5)(3 4)(7 11)(8 10)(9 12)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 6)(2 5)(3 4)(7 12)(8 11)(9 10)	(1 9 6)(2 5 11)(3 10 7)(4 8 12)
(1 7)(2 9)(3 8)(4 10)(5 12)(6 11)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 7)(2 9)(3 8)(4 11)(5 10)(6 12)	(2 5 6 3 10)(4 7 11 12 8)
(1 7)(2 9)(3 8)(4 12)(5 11)(6 10)	(1 8 5)(2 4 12)(3 11 9)(6 7 10)
(1 8)(2 7)(3 9)(4 10)(5 12)(6 11)	(1 2 4 8 6)(3 12 9 7 10)
(1 8)(2 7)(3 9)(4 11)(5 10)(6 12)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 8)(2 7)(3 9)(4 12)(5 11)(6 10)	(1 8 5 6 7)(2 9 3 11 12)
(1 9)(2 8)(3 7)(4 10)(5 12)(6 11)	(1 5 10)(2 12 8)(3 7 6)(4 9 11)
(1 9)(2 8)(3 7)(4 11)(5 10)(6 12)	(1 2 12 5 10)(4 9 7 6 11)
(1 9)(2 8)(3 7)(4 12)(5 11)(6 10)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 10)(2 12)(3 11)(4 7)(5 9)(6 8)	(1 10 8)(2 7 5)(3 4 11)(6 9 12)
(1 10)(2 12)(3 11)(4 8)(5 7)(6 9)	(2 3 4 7 5)(6 9 12 10 8)
(1 10)(2 12)(3 11)(4 9)(5 8)(6 7)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 11)(2 10)(3 12)(4 7)(5 9)(6 8)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 11)(2 10)(3 12)(4 8)(5 7)(6 9)	(1 5 3 7 8)(4 11 9 10 6)
(1 11)(2 10)(3 12)(4 9)(5 8)(6 7)	(1 6 12)(2 11 7)(3 9 4)(5 8 10)
(1 12)(2 11)(3 10)(4 7)(5 9)(6 8)	(1 4 12 10 5)(2 3 9 11 8)
(1 12)(2 11)(3 10)(4 8)(5 7)(6 9)	(1 6 10)(2 8 5)(3 11 7)(4 9 12)
(1 12)(2 11)(3 10)(4 9)(5 8)(6 7)	(1 6 4 3 12)(2 11 7 8 10)

in which all conjugacy classes of elements of finite order correspond to proper power relators in the presentation. The presentations given in Proposition 3.1 are of this type (since they are derived from the face-pairings of an orbifold fundamental domain) and all eight presentations have been run through this machinery, confirming the Cayley/Magma results. The proofs generated by this method are available from the authors on request.

As an example of the sort of proof generated by the program, we present the computer-generated proof of Lemma 4.2 above.

ALTERNATE PROOF OF LEMMA 4.2. The proof proceeds by attempting to construct a representation $\varphi : \Gamma_7 \to S_4$ in which every element of order 2 maps to a product of 2 disjoint 2-cycles. After relabelling, we may assume that $\varphi(a)$ is $(0\,1)(2\,3)$. There are then three possibilities for $\varphi(b)(0)$: 0, 1, and 2 (the choice of 3 is conjugate to a choice of 2). 0 produces a fixed point for $\varphi(b)$ and is rejected. So, assume that $\varphi(b)(0) = 1$. Then, $\varphi(b)$ must also be $(0\,1)(2\,3)$. Consider the choices now for $\varphi(c)(0)$: the same three choices exist (0, 1 and 2) and again 0 produces a fixed point for $\varphi(c)$, so we first assume that $\varphi(c)(0) = 1$ which leads to a contradiction with the relation $\varphi(ababacbabcabacabacbab) = 1$. So, next we assume that $\varphi(c)(0) = 2$ which implies that $\varphi(c) = (0\,2)(1\,3)$. Again, the relation $\varphi(ababacbabcabacabacbab) = 1$ is not satisfied. So our assumption that $\varphi(b)(0) = 1$ is ruled out.

Next, we proceed to the final possibility for $\varphi(b)(0)$ and assume that $\varphi(b)(0) = 2$, leading to the deduction that $\varphi(b) = (0\,2)(1\,3)$. Now there are four distinct choices for $\varphi(c)(0)$, namely 0, 1, 2 and 3. Again $\varphi(c)(0) = 0$ leads to a fixed point for $\varphi(c)$, so we assume first that $\varphi(c)(0) = 1$ which implies that $\varphi(c) = (0\,1)(2\,3)$. This leads to a fixed point for $\varphi(cababa)$ which should be a product of 2 disjoint 2-cycles. Next, assume that $\varphi(c)(0) = 2$ implying that $\varphi(c) = (0\,2)(1\,3)$. In this case $\varphi(abcabacbab)$ has a fixed point. Our final possibility, then, is $\varphi(c)(0) = 3$ implying that $\varphi(c) = (0\,3)(1\,2)$. Here, the relation $\varphi(ababacbabcabacabacbab) = 1$ is again not satisfied, ruling out all possible cases. □

While this proof would certainly not be considered elegant, it is nevertheless effective and easy (though quite tedious in the case of the index 12 proofs) to verify by hand.

Appendix

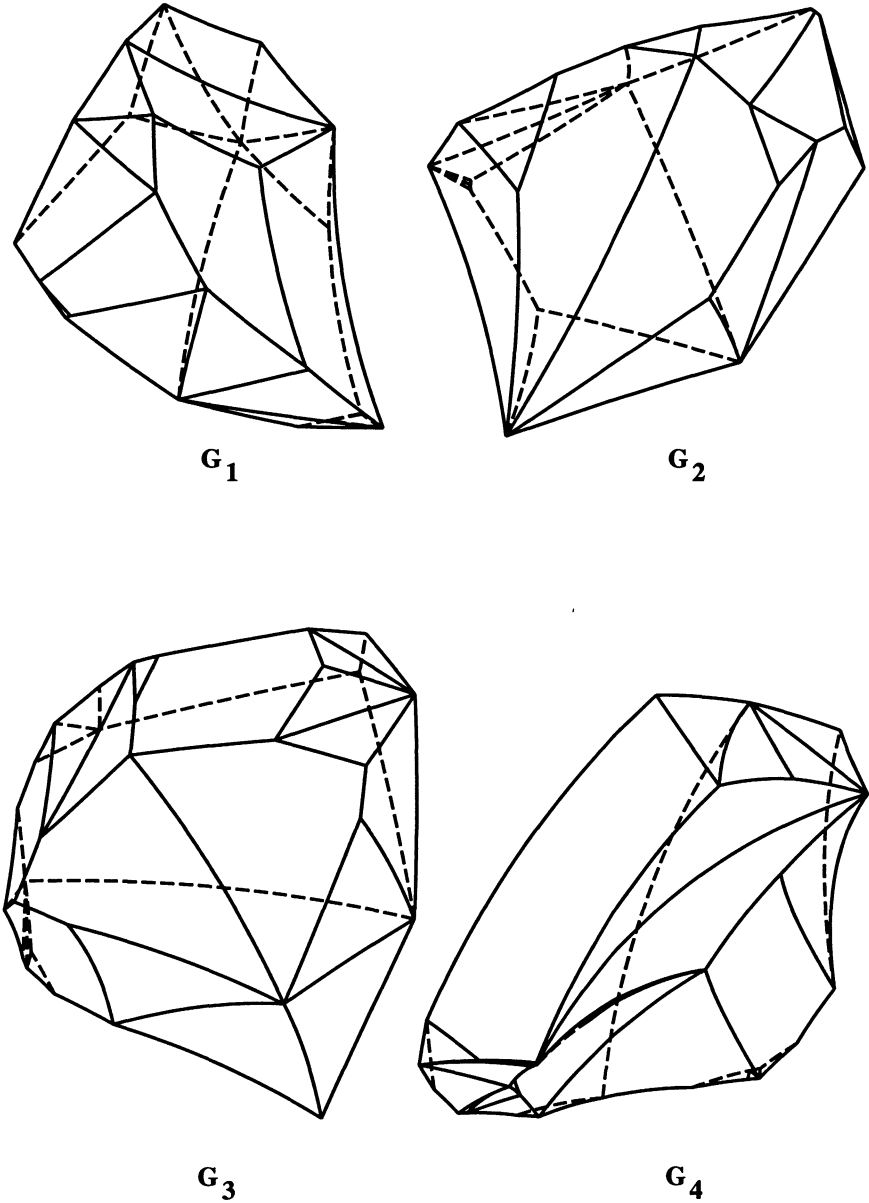**Conventions for Figure 2:** these polyhedra are drawn in the upper half-space model and are oriented (approximately) so that the viewer is beneath them, looking up.
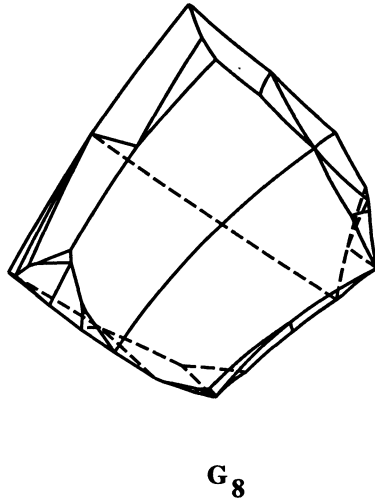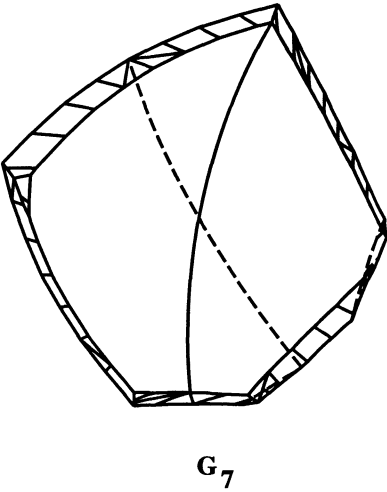

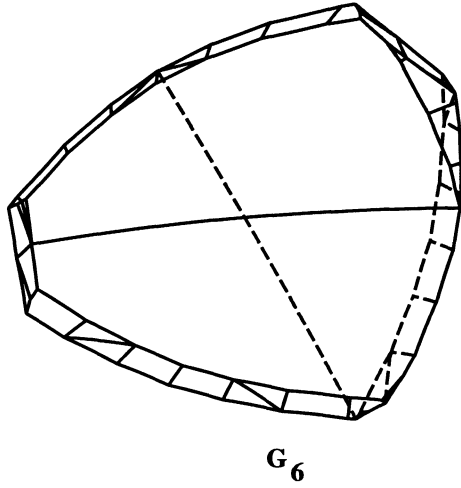
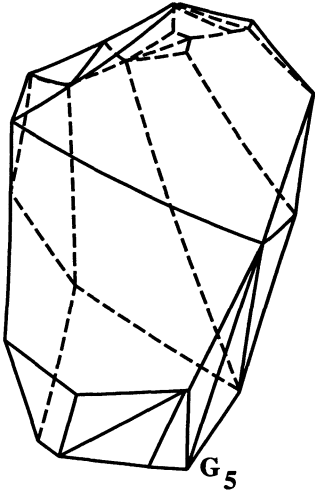Fig. 2. Fundamental Polyhedra

Fig. 2. Fundamental Polyhedra (cont.)

Table 3 - Presentation Details

Conventions for Table 3: The polyhedral presentations have generators a through l for G_1 and G_2, a through n for G_3, a through q for G_4, a through o for G_5, a through w for G_6, a through z and a_2 through d_2 for G_7, and a through r for G_8. The relations for the presentation of each group G are given in the second column of Table 3. Column 3 contains successive generator elimination equations which reduce each group to a 3-generator presentation. The correspondence between these generators and the generators given in Proposition 3.1 is then given in column 4. In most cases, these eliminations are sufficient to exactly reproduce the presentations of Proposition 3.1. G_2, G_5 and G_8 require some slight rewriting of relations to produce the indicated presentations.

G	Polyhedral Relations	Eliminations (in order)	Substitutions
G_1	$a^3, be^{-1}a, b^3, af^{-1}c$ $gj^{-1}d, ah^{-1}g, g^{-1}je, b^{-1}lg$ $c^{-1}kg, fih^{-1}, hld, di^{-1}h$ $f^{-1}kh, alj, c^2, d^2$ $e^2, f^2, g^2, i^2, k^2, l^2$	$j = la^{-1}, h = ga$ $c = fa^{-1}, e = ab$ $b = lg, g = ald$ $f = ldk, i = dld$ $k = dldldld$	$A = a^{-1}$ $B = d$ $C = l$
G_2	$b^3, cdb^{-1}, a^{-1}ec, c^3$ dfa, bfe, gic, dhg^{-1} $gk^{-1}a, ejg, fkh^{-1}, dk^{-1}h$ $eli^{-1}, e^{-1}ki, j^{-1}lc, d^2$ $e^2, f^2, g^2, j^2, k^2, l^2$	$i = k^{-1}e, h = kd^{-1}$ $a = kg^{-1}, l = jc^{-1}$ $f = b^{-1}e^{-1}, j = e^{-1}g^{-1}$ $d = c^{-1}b, k = ecg$ $e = b^{-1}cgc^{-1}bg^{-1}c^{-1}$	$A = b$ $B = c$ $C = g$
G_3	$a^3, bfd^{-1}, ag^{-1}d, d^{-1}fc$ $b^{-1}ge, efa, hie, h^{-1}kf$ $cih, ikd, ijf^{-1}, il^{-1}e$ $i^{-1}mf, alj, knf, mn^{-1}d$ $b^2, c^2, d^2, e^2, g^2, h^2$ j^2, l^2, m^2, n^2	$k = f^{-1}n^{-1}, m = d^{-1}n$ $l = a^{-1}j^{-1}, g = be^{-1}$ $b = df^{-1}, e = i^{-1}h^{-1}$ $n = dif^{-1}, a = f^{-1}hi$ $d = fh^{-1}i^{-1}, f = ij$ $i = c^{-1}h^{-1}$	$A = h$ $B = c$ $C = j$
G_4	a^3, b^3, bgf, chg^{-1} $ika^{-1}, i^{-1}oe, gni, b^{-1}ki$ $hmi^{-1}, cli, iq^{-1}c, ekj^{-1}$ $jnd^{-1}, a^{-1}oj, j^{-1}kh, fp^{-1}j$ $kn^{-1}f, kld^{-1}, d^{-1}pk, hnl^{-1}$ $mq^{-1}g, hq^{-1}m, e^2, f^2, g^2, h^2$ $i^2, j^2, l^2, o^2, p^2, q^2$	$a = ik, b = f^{-1}g^{-1}$ $c = gh^{-1}, d = jn$ $k = f^{-1}g^{-1}i^{-1}, n = g^{-1}i^{-1}$ $m = h^{-1}i, o = ie^{-1}$ $f = j^{-1}p, g = i^{-1}l^{-1}h$ $q = i^{-1}l^{-1}i, e = jl^{-1}hj^{-1}p$ $p = jh^{-1}lhj^{-1}, l = ih^{-1}i^{-1}hi^{-1}$	$A = i$ $B = j$ $C = h$
G_5	$b^{-1}kd, cj^{-1}d, dlc, efa^{-1}$ $bme, c^{-1}je, fnb^{-1}, fi^{-1}c$ $chf, gmb, amg, bo^{-1}h$ $i^{-1}ja, k^{-1}oc, lo^{-1}b, anm$ $con^{-1}, a^2, b^2, d^2, e^2, g^2$ $h^2, i^2, j^2, k^2, l^2, n^2$	$a = ef, c = dj$ $d = kb, f = bn$ $g = bm^{-1}, h = jkbnb$ $i = bkjbn, k = bjej$ $l = jejej, m = be$ $n = bebebeb, o = ejbebebeb$	$A = b$ $B = j$ $C = e$

Table 3 - Presentation Details (cont.)

G	Polyhedral Relations	Eliminations (in order)	Substitutions
G_6	$cda^{-1}, gj^{-1}e, iph, i^{-1}mc$	$a = cd, c = m^{-1}i$	$A = m$
	$jog^{-1}, dmj, km^{-1}d, k^{-1}ni$	$d = j^{-1}m^{-1}, g = e^{-1}j$	$B = n$
	$lmf, knl, mr^{-1}h, gqm$	$i = h^{-1}p^{-1}, j = m^{-1}km^{-1}$	$C = b$
	$mt^{-1}j, e^{-1}vm, lu^{-1}m, ms^{-1}b$	$k = l^{-1}n^{-1}, l = f^{-1}m^{-1}$	
	$mtk^{-1}, m^{-1}pi, nra, aq^{-1}n$	$p = nf^{-1}m^{-1}nh^{-1}$	
	$jv^{-1}n, np^{-1}d, nte^{-1}, ns^{-1}i$	$s = bm, t = fn^{-1}$	
	$nw^{-1}c, con, f^{-1}tn, bun$	$v = em^{-1}, e = fn^{-1}m^{-1}qm$	
	$nsl, oqd^{-1}, prc, qv^{-1}j$	$f = m^{-1}u^{-1}m, h = rm^{-1}$	
	mws^{-1}, b^2, e^2, f^2	$o = mnm^{-1}uq^{-1}$	
	h^2, m^2, n^2, o^2	$q = nm^{-1}n^{-1}u^{-1}mn^{-1}mnm^{-1}u$	
	q^2, r^2, u^2, w^2	$r = n^{-1}u^{-1}mn^{-1}m^{-1}nm^{-1}unm$	
		$u = b^{-1}n^{-1}, w = m^{-1}bm$	
G_7	$akc^{-1}, c^{-1}gf, amg^{-1}, gl^{-1}d$	$b = gl, c = ak$	$A = g$
	$dig, glb^{-1}, g^{-1}ib, gk^{-1}c$	$d = lg^{-1}, e = o^{-1}r$	$B = o$
	$gjf, hng^{-1}, g^{-1}kj, k^{-1}mf$	$f = j^{-1}g^{-1}, h = fn^{-1}$	$C = q$
	$oti, o^{-1}xj, ixo^{-1}, k^{-1}c_2o$	$i = t^{-1}o^{-1}, j = x^{-1}o$	
	$jto, dwo^{-1}, ovl^{-1}, bb_2o^{-1}$	$k = c_2o, l = ov$	
	$er^{-1}o, o^{-1}ul, b^{-1}d_2o, oa_2^{-1}a$	$m = a^{-1}g, n = oz^{-1}$	
	$osm^{-1}, kyo^{-1}, fp^{-1}o, os^{-1}h$	$p = xg^{-1}, r = g^{-1}x$	
	$nzo^{-1}, or^{-1}d, o^{-1}we, pyh$	$s = gz, t = o^{-1}xo^{-1}$	
	$gx^{-1}p, hqp, qxn^{-1}, gw^{-1}q$	$w = gv^{-1}, x = q^{-1}oz^{-1}$	
	$rx^{-1}g, rze^{-1}, n^{-1}wr, gzs^{-1}$	$a_2 = ao, d_2 = gu^{-1}$	
	$g^{-1}a_2s, fys, tc_2^{-1}g, gb_2t$	$b_2 = v^{-1}o^{-1}g^{-1}o, c_2 = oy^{-1}o^{-1}$	
	$g^{-1}d_2u, g^{-1}wv, a^2, g^2$	$a = gz^{-1}g^{-1}o^{-1}, u = ov^{-1}o^{-1}$	
	o^2, q^2, u^2, v^2, y^2	$v = g^{-1}q^{-1}g$	
	$z^2, a_2^2, b_2^2, c_2^2, d_2^2$	$y = zo^{-1}qog^{-1}o$	
		$z = o^{-1}qgo^{-1}g^{-1}q^{-1}o$	
G_8	$bfe^{-1}, e^{-1}fa, dje^{-1}, eic^{-1}$	$a = f^{-1}e, b = ef^{-1}, c = ei$	$A = e$
	$d^{-1}ge, e^{-1}kc, fjc^{-1}, d^{-1}kf$	$d = ej^{-1}, f = ln$	$B = l$
	$ghb, bk^{-1}g, ij^{-1}a, lnf^{-1}$	$m = i^{-1}r^{-1}, p = g^{-1}l$	$C = r$
	$f^{-1}rl, lo^{-1}b, gpl^{-1}, hp^{-1}l$	$q = j^{-1}l^{-1}, o = en^{-1}$	
	$jpl, am^{-1}l, mri, e^{-1}rm$	$h = l^{-1}g^{-1}l, k = ei^{-1}e^{-1}$	
	$iqm, e^{-1}on, op^{-1}h, e^2, g^2, h^2, i^2$	$g = e^{-1}je^{-1}, i = r^{-1}e^{-1}r$	
	$j^2, k^2, l^2, n^2, o^2, q^2, r^2$	$n = l^{-1}rl, j = l^{-1}r^{-1}er^{-1}e^{-1}r$	

Table 4 - Numerical Data

Group	Max. Uncertainty	Short Geodesic	Short Ortholength
G_1	3.6651655×10^{-8}	.215767	.107884
G_2	1.8146944×10^{-8}	.194764	.0973818
G_3	8.3967628×10^{-9}	.278891	.139445
G_4	1.8867982×10^{-6}	.292302	.146151
G_5	4.9626183×10^{-8}	.292302	.146151
G_6	2.6573629×10^{-4}	.287539	.143770
G_7	1.3768871×10^{-6}	.234004	.117002
G_8	2.1005956×10^{-4}	.289041	.144521

Recall from Section 3 that the uncertainties are compared to half the length of the shortest closed geodesic and half the shortest ortholength between non-intersecting elliptic axes (that is the length of the perpendicular bisector). The values in the third and fourth columns are lengths, so that the second column should be compared to half the third and fourth columns.

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