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Nonunique Continuation for Plane Uniformly Elliptic Equations in Sobolev Spaces

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Abstract. In the half plane $x \geq 0$, a Hölder continuous, non zero function $u(x, y)$, periodic in y is constructed: u has L^p $(1 < p < 2)$ second derivatives and it satisfies a.e. a second order, non variational, uniformly elliptic equation $Lu = 0$; moreover $u \equiv 0$ for x large enough.

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1. - Introduction

In dimension $n = 3$, for a second order, uniformly elliptic operator L, with Lipschitz continuous coefficients in a domain Ω, a unique continuation theorem holds (i.e., if $Lu = 0$ in Ω and $u \equiv 0$ in an open subset of Ω, then $u \equiv 0$ in Ω) (see e.g. Hörmander [6], [7], Miller [8]).

If the coefficients are merely Hölder continuous, there are examples of non unique continuations: the first one was constructed by Pliś in [10].

A beautiful and sharp example of non unique continuation is in Miller [8]: he constructs a solution u to a suitable elliptic equation: the solution is, for a certain x, the harmonic function: $e^{-N^6 x} \cos N^6 y$; for x somehow larger the solution is the harmonic function $e^{-(N+1)^6 x} \cos(N+1)^6 z$. Putting the pieces together, he is able to construct a C^∞ solution that, in a finite x-interval, becomes $\equiv 0$ and it solves an elliptic equation of the form $Lu = 0$, without zero order terms.

In dimension $n = 2$, the situation is completely different. There is a unique continuation theorem for uniformly elliptic equation merely with bounded measurable coefficients (see [3], [2]; more recent results are in [1], [13]).

However, a closer look shows that the unique continuation holds for solutions to $Lu = 0$, that have L^2 second derivatives. So it is natural to ask

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whether an example of non unique continuation for solutions to uniformly elliptic equations with L^p $(1 < p < 2)$ second derivatives could be found.

We asked ourselves how could be possible to imitate K. Miller approach. In dimension 3, K. Miller lets the solution quickly decay in x, cleverly working on the two remaining variables y and z, but in dimension 2 there is only one variable left.

In our case, one has, for a certain x, the function $u = e^{-x} \sin y$ (that in $\mathbf{R} \times [-\pi, \pi]$ has one "hump" and one "valley") and one would like to transform it, for larger x, into $u = e^{-2x} \sin 2y$ (that has in $\mathbf{R} \times [-\pi, \pi]$ two "humps" and two "valleys"), keeping u a solution to an elliptic equation. The authors' idea was to create humps and valleys by using the uniformly elliptic operator first introduced by Gilbarg and Serrin [5], that has tent-like solutions of the form $1 - (x^2 + y^2)^{\lambda/2}$ with L^p second derivatives (here $\lambda \in (0, 1)$, $1 < p < 2/(2 - \lambda) < 2$). The pieces were glued together by adapting a technique found in the beautiful example of Safonov [12].

Final problem: How to let the constructed function be a solution to an elliptic equation. The authors used a result of Pucci [11]: if u has negative Hessian, then u is a solution to an elliptic equation; this fact has been independently, cleverly and extensively used by Safonov in [12].

Eventually, in $\{(x, y) \in \mathbf{R}^2 : x \geq 0\}$, for every $p \in (1, 2)$, a function u, 2π periodic in y was constructed, identically zero for x sufficiently large, Hölder continuous with L^p second derivatives, satisfying a uniformly elliptic equation and of the form $e^{-x} \sin y$ in a neighbourhood of $x = 0$.The main result follows.

THEOREM. *Let:* $\mathbf{T} \sim (-\pi, \pi]$ *be the 1-dimensional torus,* $\Lambda := [0, +\infty) \times \mathbf{T}$, $1 < p < 2$. *There exists a uniformly elliptic equation in* Λ :

$$A_{11}(x, y)u_{xx} + 2A_{12}(x, y)u_{xy} + A_{22}(x, y)u_{yy} = 0$$

a positive constant X *and a function* $u \in W^{2,p}(\Lambda)$, *solution to the above equation, satisfying:*

(i) $u = e^{-x} \sin y$ *in a neighbourhood of* $x = 0$,
(ii) $u \equiv 0$ *for* $x \geq X$.

As a consequence, one can immediately construct non zero solutions with $L^p(1 < p < 2)$ first derivatives to second order uniformly elliptic variational equations (and to first order elliptic systems), that vanish in an open set.

The structure of the paper is the following. In Section 2, preliminary results are stated: Pucci's lemma, the gluing theorems and a suitable existence theorem for a Gilbarg-Serrin [5] type equation. In Section 3 a solution (periodic in y), in $[0, S] \times [-\pi, \pi]$, to an elliptic equation is constructed, that starts in a neighbourhood of $x = 0$ as $e^{-x} \sin y$ and becomes $ke^{-2x} \sin 2y$ near $x = S$. In Section 4 the example is constructed and in Section 5 there are remarks and applications.

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2. – Preliminary results

Throughout the paper, $W^{2,p}(G)$, $G \subset \mathbf{R}^2$, will be the space of functions in $L^p(G)$ with first and second derivatives in $L^p(G)$ $(1 < p < 2)$. "u is a solution to an elliptic equation" in G means that $u \in W^{2,p}(G)$ (at least) and a.e. in G there exist $a_{11}, a_{12}, a_{22} \in L^\infty(G)$, such that: (i) for $(x, y) \in G$, $(\lambda, \mu) \in \mathbf{R}^2$, $\lambda^2 + \mu^2 = 1$:

$$0 < \alpha \le a_{11}(x, y)\lambda^2 + 2a_{12}(x, y)\lambda\mu + a_{22}(x, y)\mu^2 \le \frac{1}{\alpha};$$

(ii) a.e. in G:

$$a_{11}(x, y)u_{xx} + 2a_{12}(x, y)u_{xy} + a_{22}(x, y)u_{yy} = 0. \tag{1}$$

In most of the paper, we will deal with functions of two variables x, y 2π–periodic in y. Let $\mathbf{T} \sim (-\pi, \pi]$ be the 1-dimensional torus. If a function ϕ is continuous on $\mathbf{T}$ minus a finite set of points, where it has removable singularities, we will sometimes write relations as "$\phi(y) \le K$, $y \in \mathbf{T}$" without mentioning the singularities.

$W^{2,p}((a, c) \times \mathbf{T})$ will be the space of functions $u(x, y)$, 2π-periodic in y, such that $u \in W^{2,p}((a, c) \times (-\pi, 2\pi))$.

$B_R = B_R(x_0, y_0)$ will be the open ball in $\mathbf{R}^2$ (or $\mathbf{R} \times \mathbf{T}$) centered in (x_0, y_0) with radius R.

If $u \in W^{2,p}$, $|Du| := |u_x| + |u_y|$, $|D^2u| := |u_{xx}| + |u_{yy}| + |u_{xy}|$, $H_u := u_{xx}u_{yy} - u_{xy}^2$.

The following lemma is a special case of a more general result by Pucci [11]. A 3 dimensional extension is in Safonov [12].

LEMMA 2.1 (Pucci's Lemma). *Let G be a bounded domain in $\mathbf{R} \times \mathbf{T}$ (or in $\mathbf{R}^2$) and let $u \in C^{1,1}(\overline{G})$. Let us assume that there exists a positive constant U such that:*

$$u_{xx}u_{yy} - u_{xy}^2 \le -U < 0 \quad a.e \text{ in } \overline{G}; \tag{2}$$

then, there exists in $\overline{G}$ a uniformly elliptic, second order operator $L := a_{11}(x, y)\frac{\partial^2}{\partial x^2} + 2a_{12}(x, y)\frac{\partial^2}{\partial x \partial y} + a_{22}(x, y)\frac{\partial^2}{\partial y^2}$, with bounded measurable coefficients, such that:

$$Lu = 0 \qquad a.e. \text{ in } G.$$

Next two propositions are "gluing theorems" that allow to patch different harmonic functions in such a way that the glued function satisfies an elliptic equation. The technique is a modification of the one in Safonov [12].

PROPOSITION 2.1. *Let:* $0 < \delta < R$, $B_{R-\delta}$, $B_{R+\delta}$ *open concentric balls and let* $G \subset \mathbf{R}^2$ *be the ring* $B_{R+\delta} \setminus \overline{B}_{R-\delta}$. *Let us define* $\Gamma_- = \partial B_{R-\delta}$, $\Gamma_+ = \partial B_{R+\delta}$, $\Gamma_0 = \partial B_R$; *let n be the outer normal to* Γ_0.

Assume that w_+, w_- *are harmonic functions in G, satisfying:*

$$w_+\Big|_{\Gamma_0} = w_-\Big|_{\Gamma_0}, \tag{3}$$

$$\frac{\partial w_+}{\partial n}\Big|_{\Gamma_0} > \frac{\partial w_-}{\partial n}\Big|_{\Gamma_0}, \tag{4}$$

$$\frac{\partial^2 w_+}{\partial n^2}\Big|_{\Gamma_0} > 0, \quad \frac{\partial^2 w_-}{\partial n^2}\Big|_{\Gamma_0} > 0. \tag{5}$$

Then, there exists $w \in C^{1,1}(G)$, *solution in G to a uniformly elliptic equation, such that:* $w = w_+$ *near* Γ_+, $w = w_-$ *near* Γ_-.

PROOF. Without loss of generality we can assume that $B_{R+\delta}, B_{R-\delta}$ are centered at $(0, 0)$ and use polar coordinates ρ, θ.

Then: $G = \{R - \delta < \rho < R + \delta\}$, $\Gamma_+ = \{\rho = R + \delta\}$, $\Gamma_- = \{\rho = R - \delta\}$, $\Gamma_0 = \{\rho = R\}$.

As a consequence of (4), (5), there exists $\delta_1 \in (0, \delta)$, $K_1 > 0$, such that, in $|\rho - R| < \delta_1$:

$$\frac{\partial}{\partial \rho}(w_+ - w_-) > 0, \tag{6}$$

$$\frac{\partial^2}{\partial \rho^2} w_+ \geq K_1 \quad \frac{\partial^2}{\partial \rho^2} w_- \geq K_1. \tag{7}$$

Notice that shrinking δ_1 does not change K_1.

As a consequence, we have:

$$w_+ > w_- \text{ in } R < \rho < R + \delta_1, \quad w_+ < w_- \text{ in } R - \delta_1 < \rho < R;$$

as w_+, w_- are harmonic, we also have:

$$-(w_-)_{\theta\theta}/\rho^2 - (w_-)_\rho/\rho \geq K_1, \tag{8}$$

$$-(w_+)_{\theta\theta}/\rho^2 - (w_+)_\rho/\rho \geq K_1 \tag{9}$$

in $|\rho - R| < \delta_1$.

Now let us notice that $(w_+)_\theta = (w_-)_\theta$ on $\rho = R$; then the function $\frac{1}{2}\frac{(w_{+\theta} - w_{-\theta})^2}{\rho^2(w_+ - w_-)}$ is defined in $0 < |\rho - R| < \delta_1$ and it can be extended to $\rho = R$ as a continuous function with value 0. By possibly shrinking δ_1, we may also assume that, in $|\rho - R| < \delta_1$:

$$\left|\frac{1}{2}\frac{(w_{+\theta} - w_{-\theta})^2}{\rho^2(w_+ - w_-)}\right| \leq K_1/4. \tag{10}$$

Let $\epsilon > 0$ so small that the set $D := \{z : |w_+(z) - w_-(z)| \leq \epsilon, |\rho - R| < \delta_1\}$ is a compact subset of $|\rho - R| < \delta_1$, it contains Γ_0 and it is the closure of a connected component of the open set: $\{z \in G : |w_+ - w_-| < \epsilon\}$.

Let us define:

$$w := w_+ \text{ in } \{R \leq \rho \leq R + \delta\} \backslash D,$$
$$w := w_- \text{ in } \{R - \delta \leq \rho \leq R\} \backslash D,$$
$$w := \frac{w_+ + w_-}{2} + \frac{\epsilon}{4} + \frac{(w_+ - w_-)^2}{4\epsilon} \text{ in } D.$$

Notice that $w \in C^1(G)$. We have also:

$$w_{\rho\rho} = w_{+\rho\rho} \quad \text{in } \{R < \rho < R + \delta\} \backslash D,$$
$$w_{\rho\rho} = w_{-\rho\rho} \quad \text{in } \{R - \delta < \rho < R\} \backslash D,$$
$$w_{\rho\rho} = \left(\frac{1}{2} + \frac{1}{2}\frac{w_+ - w_-}{\epsilon}\right) w_{+\rho\rho} + \left(\frac{1}{2} - \frac{1}{2}\frac{w_+ - w_-}{\epsilon}\right) w_{-\rho\rho}$$
$$+ \frac{1}{2\epsilon}(w_{+\rho} - w_{-\rho})^2 \quad \text{in } D \backslash \partial D,$$

so $w_{\rho\rho}$ is bounded and piecewise continuous in G; similar computation can be done for $w_{\rho\theta}, w_{\theta\theta}$; thus $w \in C^{1,1}(G)$ and it has piecewise continuous bounded second derivatives.

Clearly w is harmonic in $G \backslash D$; it remains to show that w satisfies an elliptic equation in D.

In $(D \backslash \partial D) \cap \{R < \rho < R + \delta\}$, we have $0 < w_+ - w_- < \epsilon$ and:

$$\frac{1}{2} + \frac{1}{2}\frac{w_+ - w_-}{\epsilon} \geq \frac{1}{2}, \quad \frac{1}{2} - \frac{1}{2}\frac{w_+ - w_-}{\epsilon} \geq 0;$$

thus, by (7):

$$w_1 := w_{\rho\rho} \geq \frac{1}{2} w_{+\rho\rho} \geq \frac{K_1}{2}; \tag{11}$$

and, by (8), (9) and (10): in $(D \backslash \partial D) \cap \{R < \rho < R + \delta\}$:

$$w_2 := -w_{\theta\theta}/\rho^2 - w_\rho/\rho \geq K_1/4. \tag{12}$$

In $(D \backslash \partial D) \cap \{R - \delta < \rho < R\}$, we have: $0 < w_- - w_+ < \epsilon$; one can also prove that (11) and (12) hold in $(D \backslash \partial D) \cap \{R < \rho < R + \delta\}$.

Thus, in $(D \backslash (\partial D \cup \Gamma_0))$, w satisfies the elliptic equation:

$$(w_2 - w_1) w_{\rho\rho} + w_1 \Delta w = 0.$$

The thesis follows. □

Next proposition is a gluing theorem where the derivative, normal to the interface, changes sign. For later use one has to be precise about the dependence of the bounds on the data.

PROPOSITION 2.2. *Let* $\Omega = (\alpha, \beta) \times \mathbf{T}$, $\Gamma = \{\frac{\alpha+\beta}{2}\} \times \mathbf{T}$, w_1, w_2 *odd and* 2π*-periodic in* y, *harmonic in* Ω *and such that* $w_1 = w_2 = \sin y$ *on* Γ. *Let us assume that there exist: a neighbourhood* $N := (\frac{\alpha+\beta}{2} - \beta_0, \frac{\alpha+\beta}{2} + \beta_0) \times \mathbf{T}$ *of* Γ *and positive constants* K_1, K_2 *such that, in* $N \backslash (\{y = 0\} \cup \{y = \pi\})$:

$$\frac{w_{2x} - w_{1x}}{\sin y} \geq K_1, \quad K_1 \leq \frac{-w_{jx}}{\sin y} \leq K_2, \quad (j = 1, 2); \tag{13}$$

$$\left|\frac{w_{jxx}}{\sin y}\right| \leq K_2 \quad \left|\left(\frac{w_{jx}}{\sin y}\right)_y\right| \leq K_2 \quad (j = 1, 2); \tag{14}$$

$$\left|\frac{w_{jxxx}}{\sin y}\right| \leq K_2 \qquad (j = 1, 2). \tag{15}$$

Then, there exist two open subsets O, O' *of* Ω, *such that* $\Omega = \overline{O} \cup O'$, $O \cap O' = \emptyset$ *and a function* $w \in C^{1,1}(\Omega) \cap C^2(O \cup O')$ *with bounded second derivatives, satisfying the properties:*

(i) $\Gamma \subset O \subset N$, $\partial O' \supset \partial \Omega$;

(ii) $w = w_1$ *in* $O' \cap \{x \leq \frac{\alpha+\beta}{2}\}$, $w = w_2$ *in* $O' \cap \{x \geq \frac{\alpha+\beta}{2}\}$;

(iii) w *is harmonic in* O';

(iv) *the bounds*:

$$|D^2 w| \leq K_3, \quad w_{xx} w_{yy} - w_{xy}^2 \leq -K_4 < 0$$

hold in O, *where* K_3, K_4 *depend on* β_0, K_1, K_2 *only.*

As a consequence of (iii), (iv) *and Pucci's lemma,* w *satisfies a.e. a uniformly elliptic equation* $Lw = 0$ *in* Ω.

PROOF. Without loss of generality, we may assume $\Omega = (-\alpha, \alpha) \times \mathbf{T}$, $\Gamma = \{0\} \times \mathbf{T}$, $0 < \beta_0 < 1$, $N = (-\beta_0, \beta_0) \times \mathbf{T}$.

Let us define:

$$v_j := \frac{w_j}{\sin y} \qquad (j = 1, 2);$$

v_1 and v_2 are $\equiv 1$ on $\Gamma \backslash (\{0\} \cup \{\pi\})$ and can be extended as smooth functions to N.

Moreover:

$$v_{jy}\,|_\Gamma = 0, \quad v_{jyy}|_\Gamma = 0, \quad v_{jxx}\,|_\Gamma = 1 \quad (j = 1, 2)\ . \tag{16}$$

Because of the assumptions (13), (14), (15), we have, in N:

$$v_{2x} - v_{1x} \geq K_1 \qquad K_1 \leq -v_{jx} \leq K_2 \qquad (j = 1, 2) \tag{17}$$

$$|v_{jxx}| \leq K_2 \qquad |v_{jxy}| \leq K_2 \qquad |v_{jxxx}| \leq K_2 \qquad (j = 1, 2). \tag{18}$$

We have also, in $N\backslash\Gamma$, by Cauchy theorem, (17), (18):

$$\left|\frac{v_{2y}-v_{1y}}{v_2-v_1}\right| \leq \frac{2K_2}{K_1}. \tag{19}$$

The inequalities (17), (18) imply that there exists $O_1 = (-\beta_1, \beta_1)\times\mathbf{T} \subset N$ (with β_1 depending on β_0, K_1, K_2 only) such that, in $O_1\backslash\Gamma$:

$$\frac{5}{4} \geq v_{jxx} \geq \frac{3}{4} \qquad (j=1,2), \tag{20}$$

$$\frac{1}{2}\min(v_{1xx}, v_{2xx}) - \frac{(v_{2y}-v_{1y})^2}{2|v_2-v_1|} \geq \frac{1}{4}. \tag{21}$$

Let us choose $\epsilon = \frac{\beta_1 K_1}{2}$. Then an open connected component O of the subset of $\{(x,y)\in\Omega : |v_2(x,y)-v_1(x,y)|<\epsilon\}$ satisfies $\overline{O}\subset O_1$; let us define also $O' = \Omega\backslash\overline{O}$.

The function w patching w_1 and w_2, can be defined as:

$$w(x,y) := w_2(x,y) \quad \text{in } \overline{O'}\cap\{x>0\};$$

$$w(x,y) := w_1(x,y) \quad \text{in } \overline{O'}\cap\{x<0\};$$

$$\begin{aligned} w(x,y) &:= \sin y\left[\frac{v_1(x,y)+v_2(x,y)}{2}+\frac{\epsilon}{4}+\frac{(v_2(x,y)-v_1(x,y))^2}{4\epsilon}\right] \\ &= \frac{w_1(x,y)+w_2(x,y)}{2}+\frac{\epsilon}{4}\sin y+\left(\frac{v_2(x,y)-v_1(x,y)}{4\epsilon}\right) \\ &\quad\cdot(w_2(x,y)-w_1(x,y)) \end{aligned}$$

in O.

Notice that $w(x,y) = w_2(x,y)$ if $v_2(x,y)-v_1(x,y)=\epsilon$, $w(x,y) = w_1(x,y)$ if $v_2(x,y)-v_1(x,y)=-\epsilon$, so w is $C^1(\Omega)$.

Let us evaluate the second derivatives of w in O: they turn out to be piecewise continuous and bounded; in O, we have:

$$\begin{aligned} w_{xx} &= \frac{w_{1xx}+w_{2xx}}{2}+\frac{v_2-v_1}{2\epsilon}(w_{2xx}-w_{1xx}) \\ &\quad+\frac{(v_{2x}-v_{1x})^2}{2\epsilon}\sin y, \end{aligned} \tag{22}$$

$$
\begin{aligned}
(23)\qquad w_{xy} = \sin y &\left[\frac{v_{1xy}+v_{2xy}}{2} + \frac{v_2-v_1}{2\epsilon}(v_{2xy}-v_{1xy})\right.\\
&+ \left.\frac{(v_{2x}-v_{1x})\cdot(v_{2y}-v_{1y})}{2\epsilon}\right]\\
&+ \cos y\left[\frac{v_{1x}+v_{2x}}{2} + \frac{v_2-v_1}{2\epsilon}(v_{2x}-v_{1x})\right],
\end{aligned}
$$

$$
\begin{aligned}
(24)\qquad w_{yy} = &\left[\frac{w_{1yy}+w_{2yy}}{2} + \frac{v_2-v_1}{2\epsilon}(w_{2yy}-w_{1yy})\right]\\
&+ \sin y\frac{(v_{2y}-v_{1y})^2}{2\epsilon} - \sin y\left[\frac{\epsilon}{4} + \frac{(v_2-v_1)^2}{4\epsilon}\right].
\end{aligned}
$$

Thus, $w \in C^{1,1}(\Omega)$, $w \in C^2(O)\cup C^2(O')$ and the second derivatives are bounded and discontinuous only on the set of measure zero: $|v_2(x,y) - v_1(x,y)| = \epsilon$.

Let us bound the second derivatives of w in O; we have, using (20):

$$
\frac{w_{xx}}{\sin y} \geq \frac{1}{2}\min(v_{2xx}, v_{1xx}) + \frac{(v_{2x}-v_{1x})^2}{2\epsilon} \geq \frac{3}{8};
$$

as w_1 and w_2 are harmonic, we also have, using (21):

$$
\begin{aligned}
\frac{-w_{yy}}{\sin y} &= \frac{v_{1xx}+v_{2xx}}{2} + \frac{v_2-v_1}{2\epsilon}(v_{2xx}-v_{1xx})\\
&\quad - \frac{(v_{2y}-v_{1y})^2}{2\epsilon} + \left[\frac{\epsilon}{4} + \frac{(v_2-v_1)^2}{4\epsilon}\right]\\
&\geq \frac{1}{2}\min(v_{1xx}, v_{2xx}) - \frac{(v_{2y}-v_{1y})^2}{2|v_2-v_1|} \geq \frac{1}{4}.
\end{aligned}
$$

Thus, in O:

$$
\frac{-w_{xx}w_{yy}}{(\sin y)^2} \geq \frac{3}{32}.
$$

By using (17), (19), we have in O: $-H_w \geq K_4$ where K_4 depends on K_1, K_2 only.

From (18): $|D_2 w| \leq 4K_2$ in O'; and (17), (18), (19), (22), (23), (24) in O, give (iv). The thesis follows. □

PROPOSITION 2.3. *Let $B_R \subset \mathbf{R}\times\mathbf{T}$ be a ball with center 0 and radius R; let $\Phi(x,y) = \varphi(\sqrt{x^2+y^2})$, where $\varphi \in C^\infty[0,R)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $[0, R/4]$, $\varphi \equiv 0$ in $[3R/4, R)$. Let $\bar{u}$ be a smooth function on ∂B, M a positive number, β a positive constant $\beta \in (0, 1/2)$. Let L be the uniformly elliptic operator:*

$$
\begin{aligned}
(25)\qquad L := &\ [\beta\Phi(x,y) + 1 - \Phi(x,y)]\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\\
&+ (1-2\beta)\Phi(x,y)\left(\frac{x^2}{x^2+y^2}\frac{\partial^2}{\partial x^2} + \frac{2xy}{x^2+y^2}\frac{\partial^2}{\partial x\partial y} + \frac{y^2}{x^2+y^2}\frac{\partial^2}{\partial y^2}\right).
\end{aligned}
$$

There exists $u \in W^{2,p}(B_R)$, $1 < p < 2(1-\beta)$, *such that:*

$$Lu = 0 \qquad \text{a.e. in } B_R\,, \tag{26}$$

$$u|_{\partial B_R} = \overline{u}\,; \tag{27}$$

and the (outer) normal derivatives of u satisfy:

$$\left.\frac{\partial u}{\partial n}\right|_{\partial B_R} < -M, \qquad \left.\frac{\partial^2 u}{\partial n^2}\right|_{\partial B_R} > M\,. \tag{28}$$

PROOF. The problem:

$$\begin{aligned} &Lu = 0 \quad \text{in } B_R \\ &u|_{\partial B_R} = \overline{u} \end{aligned}$$

has a unique solution $u_0 \in W^{2,2}(B_R)$ (see e.g. [9]), smooth in $\overline{B}_R \setminus \{0\}$; the function u_0 is harmonic in a neighbourhood of ∂B_R. Let us look for radial solutions $\widetilde{u}(x, y) = v(\sqrt{x^2 + y^2})$ to $Lu = 0$; v satisfies the O.D.E.:

$$(1 - \varphi(r) + \beta\varphi(r))\left(v''(r) + \frac{1}{r}v'(r)\right) + (1 - 2\beta)\varphi(r)v''(r) = 0\,;$$

the equation has two independent solutions $v_1 \equiv 1$ and v_2 such that in $(0, R/4)$:

$$v_2(r) = cr^{2-1/(1-\beta)} \qquad c \neq 0\,.$$

Let us choose $c > 0$, so that $v_2(R) = 1$; then: $v_2'(R) > 0$ and $v_2''(R) < 0$. Let us consider the function:

$$u(x, y) := u_0(x, y) + A(1 - v_2\left(\sqrt{x^2 + y^2}\right)\,,$$

where A is chosen so big that:

$$-Av_2'(R) + \sup_{\partial B_R}\left|\frac{\partial u_0}{\partial n}\right| < -M, \qquad -Av_2''(R) - \sup_{\partial B}\left|\frac{\partial^2 u_0}{\partial n^2}\right| > M\,. \tag{29}$$

It is not difficult to show that u satisfies (26), (27), (28). The thesis follows. □

3. – The shifting solution

The goal of this paragraph is to construct a solution to an elliptic equation that, as x increases in a finite interval, shifts from $e^{-x}\sin y$ to $c\cdot e^{-2x}\sin 2y$. More precisely, the following fact will be proved.

THEOREM 3.1. *Let* $1 < p < 2$. *There exist two constants* $S > 0$ *and* $0 < Q < \frac{1}{4}$, *a second order uniformly elliptic operator of the form:*

$$Lu := a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} \tag{30}$$

with coefficients $a_{11}, a_{12}, a_{22} \in L^\infty([0,S]\times\mathbf{T})$ *and a function* $u_0 \in W^{2,p}([0,S]\times\mathbf{T})$, *solution a.e. to the equation* $Lu = 0$ *in* $[0,S]\times\mathbf{T}$, *satisfying:*

- $u_0 = e^{-x}\sin y$ *in a neighbourhood of* $x = 0$;
- $u_0 = Qe^{-2(x-S)}\sin 2y$ *in a neighbourhood of* $x = S$.

The function u_0, *harmonic in a neighbourhood of* $x = 0$ *and* $x = S$, *is piecewise* C^2 *with bounded second derivatives, except arbitrarily small neighbourhoods of a finite set of points, where* u_0 *is Hölder continuous.*

Outline of the proof of Theorem 3.1

The proof will be carried on in several steps. The solution will be constructed as sum of two terms u_1 and u_2.

Most of the proof will be done in $[0, b+1/2]\times\mathbf{T}$, where $b \geq 5$ is a parameter that will be large and it will be fixed later.

Let us denote:

$$\Theta := \{0\}\times\mathbf{T}, \qquad \Gamma := \{b\}\times\mathbf{T}, \qquad \omega := \cosh b/\sinh b.$$

In what follows the calligraphic constants will be positive constants not depending on b.

Let us outline the strategy used.

STEP 1. A preliminary construction of u_1 is done. The function u_1 is defined odd and π-periodic in the y variable, harmonic in $[0, b+1/2]\times\mathbf{T}$, except four poles where it has a logarithmic singularity; $u_1 \equiv 0$ on Θ and its derivatives (of order ≤ 3) are $\leq \mathcal{L}e^{-b}$ there; $u_1 = \sin 2y$ on Γ; moreover, $u_1/\sin 2y$ and its derivatives (of order ≤ 3) can be extended as functions bounded (uniformly with respect to b) in $(b-\delta, b+\delta)\times\mathbf{T}$ (δ not depending on b) and $u_{1x}/\sin 2y \leq -10$ there.

STEP 2. The function u_1, defined in step 1, is changed in a neighbourhood O_1 of Γ and extended to $[0,+\infty)\times\mathbf{T}$. This is done by gluing, around $x = b$, u_1 (as defined in Step 1) with $e^{-2(x-b)}\sin 2y$, using Proposition 2.2.

The resulting function, again called u_1, is harmonic (with poles), except in O_1. In $O_1 \cup [b, b+1] \times \mathbf{T}$: $|D^2 u_1| \le \mathcal{D}$ and $(u_{1xy})^2 - u_{1xx}u_{1yy} \ge \mathcal{W} > 0$. The constants $\mathcal{D}$ and $\mathcal{W}$ do not depend on b. By Pucci's lemma, u_1 (outside of the poles) satisfies an elliptic equation in $[0, +\infty) \times \mathbf{T}$.

STEP 3. The construction of u_2 is done.

If b is large enough, a function u_2 can be defined in $(-\infty, b] \times \mathbf{T}$, $C^{1,1}$ with piecewise continuous bounded second derivatives, harmonic in $((-\infty, -\pi - 1) \cup (0, b)) \times \mathbf{T}$, satisfying an elliptic equation in $(-\infty, b)) \times \mathbf{T}$ and:

$$
\begin{array}{ll}
u_2 = e^{-x} \sin y & \text{in } [-\infty, -\pi - 1] \times \mathbf{T}, \\
H_{u_2} \le -1, \quad |D^2 u_2| \le 12 & \text{in } [-1, 0] \times \mathbf{T}; \\
u_2 = 0 & \text{on } \Gamma; \\
|Du_2| + |D^2 u_2| + |u_{2xxy}| \le 2/\sinh b & \text{on } \Gamma.
\end{array}
$$

STEP 4. Our goal will be to sum up u_1 and u_2: as they are defined in different sets, one may define u_1 in $x < 0$ and u_2 in $x > b$, so that:

(i) $u_1 \equiv 0$ in $(-\infty, -1] \times \mathbf{T}$; $|H_{u_1}| \le \mathcal{F}^2 \mathcal{L}^2 e^{-2b}$, $|D^2 u_1| \le \mathcal{L}\mathcal{F} e^{-b}$ in $[-1, 0] \times \mathbf{T}$,

(ii) $u_2 \equiv 0$ in $[b+1, +\infty) \times \mathbf{T}$; $|H_{u_2}| \le \mathcal{F}^2/(\sinh b)^2$, $|D^2 u_2| \le \mathcal{F}/(\sinh b)$ in $[b, b+1] \times \mathbf{T}$

(the constants $\mathcal{F}, \mathcal{L}$ do not depend on b).

The function $u_3 := u_1 + u_2$ equals $e^{-x} \sin y$ in $(-\infty, -\pi - 1] \times \mathbf{T}$ and $e^{2(x-b)} \sin 2y$ in $x > b + 1$ and it is solution to an elliptic equation in $E_1 := (-\infty, -1] \times \mathbf{T} \cup ([0, b] \times \mathbf{T} \setminus O_1) \cup [b+1, +\infty] \times \mathbf{T}$ (poles excluded).

Let us look at u_3 in $\mathbf{R} \times \mathbf{T} \setminus E_1$. To let u_3 be solution to an elliptic equation in $[-1, 0] \times \mathbf{T}$, let us notice that, in that set, $H_{u_2} \le -1$, $|H_{u_1}| \le \mathcal{F}^2 \mathcal{L}^2 e^{-2b}$, and $|D^2 u_2| \le 12$, $|D^2 u_1| \le \mathcal{L}\mathcal{F} e^{-b}$; then, if $b \ge b_2$, $H_{u_1+u_2} < 0$ in $[-1, 0] \times \mathbf{T}$, and u_3 is a solution to an elliptic equation there.

Similar procedure is done in O_1 and in $[b, b+1] \times \mathbf{T} \setminus O_1$, namely, choosing b sufficiently large , one can let $u_1 + u_2$ solve an elliptic equation there.

Then u_3 (outside of the poles) is $C^{1,1}$ (with piecewise bounded second derivatives) and has negative Hessian in $\mathbf{R} \times \mathbf{T} \setminus E_1$; then, it is a solution to an elliptic equation there, by Pucci's lemma.

STEP 5. In this step one takes care of the poles (using the first gluing theorem) by changing the solution, in a small ball around them, to a $W^{2,p}$ tent-like solution of a Gilbarg-Serrin type equation.

We get a function u_3 in $\mathbf{R} \times \mathbf{T}$, solution to an elliptic equation, with the properties: $u_3 = e^{-x} \sin y$ for $x << 0$, and $u_3 = c \cdot e^{-2x} \sin 2y$ for $x >> 1$, $u_3 \in W^{2,p}[a, c] \times \mathbf{T}$, $-\infty < a < c < +\infty$, $1 < p < 2$. Using this function, the theorem is easily proved.

Let us proceed to the detailed proof.

STEP 1. Preliminary construction of u_1 in $0 \le x \le b + 1/2$.

LEMMA 3.1. *For every $b \geq 5$, in $[0, b+1/2] \times \mathbf{T}$, there exists a function u_1 with the properties:*

(i) *u_1 is odd, π-periodic in y, harmonic in $E = [0, b+\frac{1}{2}] \times \mathbf{T} \setminus \{(b-1, -\frac{3}{4}\pi + k\pi/2), k = 0,1,2,3\}$, and; $u_1|_\Theta \equiv 0$, $u_1|_\Gamma \equiv \sin 2y$.*

(ii) *There exist constants $0 < \delta < 1/4$, $\mathcal{K} > 0$ (both not depending on b), such that, in $O_0 := (b-\delta, b+\delta) \times \mathbf{T}$, $u_1/\sin 2y$ and its derivatives with respect to x, y can be extended as continuous functions, satisfying:*

$$-\mathcal{K} \leq \frac{u_{1x}}{\sin 2y} \leq -10, \qquad \left|\frac{u_{1xx}}{\sin 2y}\right| \leq \mathcal{K},$$

$$\left|\left(\frac{u_{1x}}{\sin 2y}\right)_y\right| \leq \mathcal{K}, \qquad \left|\frac{u_{1xxx}}{\sin 2y}\right| \leq \mathcal{K}; \tag{31}$$

(iii) *There exists $\mathcal{L} > 0$ (not depending on b) such that:*

$$\begin{aligned} |u_{1x}|_\Theta &\leq \mathcal{L}e^{-b} \\ |u_{1xx}|_\Theta &\leq \mathcal{L}e^{-b} \\ |u_{1xy}|_\Theta &\leq \mathcal{L}e^{-b} \\ |u_{1xxx}|_\Theta &\leq \mathcal{L}e^{-b}. \end{aligned} \tag{32}$$

(iv) *Near $(b-1, -\frac{3\pi}{4} + k\frac{\pi}{2})$, $k = 0,1,2,3$ the function u_1 is of the form:*

$$u_1 = \widetilde{u}_1 + (-1)^{1+k} A \quad \log\left((x-b+1)^2 + \left(y + \frac{3}{4}\pi - k\frac{\pi}{2}\right)^2\right)$$

where $A > 0$ does not depend on b, $\widetilde{u}_1$ is harmonic in a neighbourhood of $(b-1, -\frac{3}{4}\pi + k\frac{\pi}{2})$ $k = 0,1,2,3$.

PROOF. As u_1 is odd and π-periodic in y, it is sufficient to define it in $E_0 := [0, b+1/2] \times [0, \pi/2] \setminus \{(b-1, \pi/4\}$.

Let us look for u_1 of the form:

$$u_1(x,y) = \frac{\sinh 2x}{\sinh 2b} \sin 2y + \mathcal{A}_1 g(x,y), \tag{33}$$

where $\mathcal{A}_1$ is a positive constant that it will be chosen later (not depending on b) and g is defined, for a moment, as follows: on $F_0 := [0,b] \times [0, \pi/2]$, g is the Green function, with pole $(b-1, \pi/4)$, to the problem $\Delta u = 0$ in F_0, $u|_{\partial F_0} = 0$.

Notice that g can be extended as an harmonic double periodic function, with periods $2b$ in x and π in y, odd with respect to x and y, to all the plane, except a countable grid of points; g can be written as $-\log|f(z)|$, where f is a meromorphic, double periodic, elliptic function on the complex plane (Courant-Hilbert [4], Vol. 1). The function u_1 is then defined in E_0 and satisfies (i) and (iv).

It remains to prove (ii) and (iii).

The proof of (ii) will be carried on in four remarks.

Checking $g(x, y)$ against the Green function for the half plane $x \le b$, and the Green function for the rectangle $[b-4, b] \times [0, \pi/2]$ with pole in the same point as $g(x, y)$, the remarks below can be proved.

REMARK 3.1. g and its derivatives of order ≤ 4 are bounded, in $[b-\frac{1}{4}, b+\frac{1}{4}] \times \mathbf{T}$, by a constant not depending on b.

REMARK 3.2. There exists a constant $\mathcal{K}_0$, not depending on b, for which:

$$\frac{g_x(b, y)}{\sin 2y} \le -\mathcal{K}_0 \qquad y \in \mathbf{T}. \tag{34}$$

From Remark 3.1, next remark follows.

REMARK 3.3. The following inequalities hold in $[b - 1/4, b + 1/4] \times \mathbf{T}$:

$$\begin{gathered} \left|\frac{g}{\sin 2y}\right| \le \mathcal{K}_1, \quad \left|\frac{g_x}{\sin 2y}\right| \le \mathcal{K}_1, \quad \left|\frac{g_{xx}}{\sin 2y}\right| \le \mathcal{K}_1, \\ \left|\left(\frac{g_x}{\sin 2y}\right)_y\right| \le \mathcal{K}_1, \quad \left|\frac{g_{xxx}}{\sin 2y}\right| \le \mathcal{K}_1; \end{gathered} \tag{35}$$

the constant $\mathcal{K}_1$ is positive and does not depend on b.

REMARK 3.4. Proof of (ii), Lemma 3.1.

We have, in $[0, b + 1/2] \times \mathbf{T}$:

$$\frac{u_1(x, y)}{\sin 2y} = \frac{\sinh 2x}{\sinh 2b} + \mathcal{A}_1 \frac{g(x, y)}{\sin 2y}; \tag{36}$$

and on Γ:

$$\frac{u_{1x}(b, y)}{\sin 2y} \le \frac{2\cosh 2b}{\sinh 2b} + \mathcal{A}_1 \frac{g_x(b, y)}{\sin 2y} \le 4 - \mathcal{A}_1\mathcal{K}_0.$$

Let us choose $\mathcal{A}_1$ such that: $4 - \mathcal{A}_1\mathcal{K}_0 = -10$. In $|x - b| < \delta := 2/(24 + \mathcal{A}_1\mathcal{K}_1)$:

$$\left|\frac{u_{1xx}}{\sin 2y}\right| \le 4\frac{\sinh(2b + 1/2)}{\sinh 2b} + \mathcal{A}_1 \left|\frac{g_{xx}}{\sin 2y}\right| \le 24 + \mathcal{A}_1\mathcal{K}_1,$$

$$\begin{aligned} \frac{u_{1x}(x, y)}{\sin 2y} &= \frac{u_{1x}(b, y)}{\sin 2y} + \frac{u_{1xx}(\xi, y)}{\sin 2y} \cdot (x - b) \\ &\le -10 + (24 + \mathcal{A}_1\mathcal{K}_1)\,\delta < -8. \end{aligned}$$

The first of the bound in (ii) follows. The remaining inequalites are consequences of the bounds (35) and the representation (36). The bounds (ii) are proved. □

The proof of (iii) will be carried on with two remarks.

REMARK 3.5. Let $0 \leq y \leq \pi/2$; then:

$$0 \leq g(b/2, y) \leq \mathcal{L}_1 \tag{37}$$

(where $\mathcal{L}_1$ is a positive constant not depending on b)

PROOF. Let $G(x, y)$ be the Green function for the ball centered in $(b-1, \pi/4)$ and radius the distance of the point from the left corners of F_0. Clearly $g(x, y) \leq G(x, y)$ in F_0. As $b \geq 5$ and:

$$g(b/2, y) \leq G(b/2, y) = -\frac{1}{2\pi} \log \frac{\sqrt{(b/2 - b + 1)^2 + (y - \pi/4)^2}}{\sqrt{(b-1)^2 + (\pi/4)^2}},$$

Remark 3.5 follows. □

REMARK 3.6 Proof of (iii).
In $[0, b/2] \times [0, \pi/2]$, u_1 is of the form:

$$u_1 = \sum_{n=1}^{\infty} \frac{\sinh 2nx}{\sinh nb} c_n \sin 2ny \tag{38}$$

where:

$$u_1(b/2, y) \sim \sum_n c_n \sin 2ny \, .$$

As a consequence of previous remark:

$$\begin{aligned} |c_n \sin 2ny| &\leq 2 \left[\mathcal{A}_1 \mathcal{L}_1 + \frac{\sinh b}{\sinh 2b} \right] \\ &\leq 2[\mathcal{A}_1 \mathcal{L}_1 + 1] \end{aligned}$$

(the constants on the right-hand side do not depend on b). Last inequality and (38), give (32). □

STEP 2. Construction of u_1 in $0 \leq x < +\infty$.

LEMMA 3.2. *There exists in* $[0, +\infty] \times \mathbf{T}$ *a function* (*again called*) u_1, *with the properties:*

(i) u_1 *is odd,* π*-periodic in* y;
(ii) u_1 *is harmonic in* $[0, +\infty] \times \mathbf{T}$ *minus a neighbourhood* O_1 *of* Γ *and the points* $\{(b-1, -\frac{3}{4}\pi + k\pi/2), k = 0, 1, 2, 3\}$; *in* $[b, +\infty) \times \mathbf{T} \setminus O_1$, $u_1(x, y) = e^{-2(x-b)} \sin 2y$;
(iii) *in* $O_1 \cup [b, b+1] \times \mathbf{T}$, u_1 *has piecewise continuous, bounded second derivatives and:*

$$|D^2 u_1| \leq \mathcal{D}, \quad u_{1xx} u_{1yy} - u_{1xy}^2 \leq -\mathcal{W}$$

where the positive constants $\mathcal{D}, \mathcal{W}$ *do not depend on b.*

(iv) u_1 *satisfies* (iii), (iv) *of Lemma* 3.1.
(v) u_1 (*outside of the poles*) *is a solution to an elliptic equation in* $[0, +\infty] \times \mathbf{T}$.

PROOF. Let u_1 be the function constructed in Step 1 (in $[0, b+1/2] \times \mathbf{T}$); let us make the change of variables $x' = 2x$, $y' = 2y$ and let us define:

$$w_1(x', y') := u_1(x'/2, y'/2);$$

w_1 is harmonic in $[2b-1/2, 2b+1/2] \times \mathbf{T}$, odd in y', 2π-periodic, $w_1(2b, y') = \sin y'$, and the bounds (31) give us in $\Omega := (2b - 2\delta, 2b + 2\delta) \times \mathbf{T}$:

$$-\mathcal{K}/2 \le \frac{w_{1x'}}{\sin y'} \le -5, \qquad \left|\frac{w_{1x'x'}}{\sin y'}\right| \le \mathcal{K}/4,$$

$$\left|\left(\frac{w_{1x'}}{\sin y'}\right)_y\right| \le \mathcal{K}/4, \qquad \left|\frac{w_{1x'x'x'}}{\sin y'}\right| \le \mathcal{K}/4;$$

Let us extend w_1, by gluing it to $w_2 := e^{-(x'-2b)} \sin y'$, across $\Gamma_0 = \{2b\} \times \mathbf{T}$ in Ω. Notice that w_1 and w_2 satisfy the hypothesis of Proposition 2.2, with *constants not depending on* b; to show this fact, it is sufficient to compute, in $|x' - 2b| < 2\delta$:

$$\frac{w_{2x'} - w_{1x'}}{\sin y'} \ge -e^{-(x'-2b)} + 5 \ge -e^{2\delta} + 5 \ge -e^{1/2} + 5 > 0$$

(as $0 < \delta < 1/4$).

By Proposition 2.2, there exist $O \subset\subset \Omega$, $O \supset \Gamma_0$, $O' = \Omega \setminus \overline{O}$ and $w \in C^{1,1}(\Omega)$, $w \in C^2(O \cup O')$(with bounded second derivatives), such that $w = w_1$ in $O' \cap \{x' < 2b\}$ and $w = w_2$ in $O' \cap \{x' > 2b\}$. In O:

$$|D^2 w| \le \mathcal{D}/4, \quad (w_{x'y'})^2 - w_{x'x'} w_{y'y'} \ge \mathcal{W}/16 \tag{39}$$

where the positive constants $\mathcal{D}, \mathcal{W}$ do not depend on b.

Let us change the variables back to $x = x'/2$, $y = y'/2$. Let $O_1 = \{(x, y) : (2x, 2y) \in O\}$, $O_1' = \{(x, y) : (2x, 2y) \in O'\}$.

Let us call again u_1 the new function, defined in $[0, +\infty) \times \mathbf{T}$ as:

$$\begin{aligned} u_1 &= u_1 \text{ (old)} && \text{in } [0, b] \times \mathbf{T} \cap O_1' \\ u_1 &= e^{-2(x-b)} \sin 2y && \text{in } [b, +\infty) \times \mathbf{T} \cap O_1' \\ u_1 &= w(2x, 2y) && \text{in } O_1 . \end{aligned}$$

The new function u_1 satisfies (i) and (ii) of the present lemma and (iii), (iv) of Lemma 3.1; in O_1:

$$\begin{aligned} &|D^2 u_1| \le \mathcal{D}, \\ &(u_{1xy})^2 - u_{1xx} u_{1yy} \ge \mathcal{W} > 0, \end{aligned} \tag{40}$$

with $\mathcal{D}, \mathcal{W}$ positive constants not depending on b. In $[b, b+1] \times \mathbf{T} \setminus O_1$: $|D_2 u_1| \le 4$, $u_{1xy}^2 - u_{1xx} u_{1yy} \ge e^{-4}$; so, by possibly changing $\mathcal{D}, \mathcal{W}$, one can assume that (40) holds true in $[b, b+1] \times \mathbf{T} \cup O_1$. Thus, u_1 satisfies (v). □

STEP 3. Construction of u_2 in $-\infty < x \le b$.

LEMMA 3.3. *There exists $b_1 \geq 5$, such that, if $b \geq b_1$, a function u_2 can be defined in $(-\infty, b] \times \mathbf{T}$, with the properties:*

(i) *$u_2 \in C^{1,1}$ with piecewise continuous bounded second derivatives, u_2 is harmonic in $((-\infty, -\pi - 1) \cup (0, b)) \times \mathbf{T}$, and it satisfies an elliptic equation in $(-\infty, b) \times \mathbf{T}$;*

(ii) *the following facts hold:*

$$\begin{array}{ll} u_2 = e^{-x} \sin y & \text{in } [-\infty, -\pi - 1] \times \mathbf{T}, \\ H_{u_2} \leq -1, \quad |D^2 u_2| \leq 12 & \text{in } [-1, 0] \times \mathbf{T}; \\ u_2 = 0 & \text{on } \Gamma; \\ |Du_2| + |D^2 u_2| + |u_{2xxy}| \leq \dfrac{2}{\sinh b} & \text{on } \Gamma. \end{array}$$

PROOF. Given $b \geq 5$, let us define, for a moment, u_2 in $x \in [-1, b]$ as follows:

$$\begin{array}{ll} u_2 = \dfrac{\sinh(b-x)}{\sinh b} \sin y & \text{in } [0, b] \times \mathbf{T}, \\ u_2 = e^{-\omega x} \sin y & \text{in } [-1, 0] \times \mathbf{T}; \end{array}$$

notice that:

$$\begin{gathered} u_2|_\Gamma = u_{2y}|_\Gamma = u_{2yy}|_\Gamma = u_{2xxy}|_\Gamma = 0, \\ u_{2x}|_\Gamma = -\sin y / \sinh b, \quad u_{2xy}|_\Gamma = -\cos y / \sinh b, \end{gathered}$$

so the last of (ii) holds.

On the other hand:

$$\begin{aligned} u_2|_\Theta &= \sin y \\ u_{2x}|_\Theta &= -\omega \sin y \end{aligned}$$

so $u_2 \in C^{1,1}$ across Θ, $u \in C^2$ in $[-1, b] \times \mathbf{T} \backslash \Theta$, with piecewise continuous and bounded second derivatives, u_2 is harmonic in $[0, b] \times \mathbf{T}$, u_2 satisfies the elliptic equation:

$$u_{xx} + \omega^2 u_{yy} = 0 \quad \text{in } [-1, 0] \times \mathbf{T}. \tag{41}$$

As $b \geq 5$, $1 < \omega < 1.0002$ is "almost 1", so u_2 is "almost" harmonic and:

$$|D^2 u_2| \leq 12, \; H_{u_2} \leq -1$$

in $[-1, 0] \times \mathbf{T}$.

Now, let us define u_2 in $x < -1$. Let:

$$\begin{aligned} &\sigma(t) = 1 \quad \text{if } t \leq 0 \\ &\sigma(t) = \omega \quad \text{if } t \geq \pi \\ &\sigma(t) = \frac{1+\omega}{2} + \frac{1-\omega}{2} \cos t \quad \text{in } 0 \leq t \leq \pi; \end{aligned} \tag{42}$$

the function u_2 can be defined, in $x < 0$, as:

$$u_2(x, y) := e^{-x\sigma(x+1+\pi)} \sin y .$$

Notice that:

$$u_2(x, y) = e^{-\omega x} \sin y \quad \text{in } -1 \le x \le 0 ,$$

$$u_2(x, y) = e^{-x} \sin y \quad \text{in } -\infty < x < -\pi - 1 ;$$

so the new definition matches with the previous one and (ii) is proved.

We have: $\dot\sigma(0) = \dot\sigma(\pi) = 0$, $\sigma(0) = 1$, $\sigma(\pi) = \omega$; then:

$$u_{2x}(x, y) = e^{-x\sigma(x+1+\pi)}(-\sigma - x\dot\sigma) \sin y$$

is continuous in $x \le 0$. Thus $u_2 \in C^{1,1}$, with piecewise continuous bounded second derivatives and it satisfies the second order partial differential equation:

$$0 = u_{2xx} + [(\sigma + x\dot\sigma)^2 - 2\dot\sigma - x\ddot\sigma]u_{yy} \tag{43}$$

Let us assume:

H1: 1st CONDITION ON b. Let $b_1 \ge 5$ so large that, for every $b \ge b_1$, ω is so close to 1, that:

$$[\sigma + (t - 1 - \pi)\dot\sigma]^2 - 2\dot\sigma - (t - 1 - \pi)\ddot\sigma \ge \frac{1}{2} \quad \text{in } [0, \pi] .$$

If we assume H1, then the partial differential equation (43) becomes elliptic and u_2 satisfies an elliptic equation (43), in $x < 0$: (i) is proved. □

The lemma is proved.

STEP 4. (Putting the pieces together)

LEMMA 3.4. *There esists a constant b_5 such that, if $b \ge b_5$, a function u_3 can be constructed, with the properties:*

(i) *outside of arbitrarily small balls centered in four points, $(b - 1, -3\pi/4 + k\pi/2)$, $k = 0, 1, 2, 3$, u_3 is $C^{1,1}[-\pi - 3, b+2] \times \mathbf{T}$; into the balls u is of the form given by* (iv) *of Lemma* 3.1;
(ii) $u_3 = e^{-x} \sin y$ *in* $(-\infty, -\pi - 2] \times \mathbf{T}$;
(iii) $u_3 = e^{-2(x-b)} \sin 2y$ *in* $(b + 1, +\infty] \times \mathbf{T}$
(iv) *u_3 is a solution to a elliptic equation in $[-\pi - 2, b+2] \times \mathbf{T}$ and u_3 is harmonic in a neighbourhood of $x = -\pi - 1$ and $x = b + 2$.*

PROOF. Our goal will be to define u_3 as $u_1 + u_2$, but first we have to define u_2 in $x < 0$ and u_1 in $x > b$.

Let $\varphi \in C_0^\infty(-1, 1)$, φ odd, $\varphi'(0) = 1$ and let

$$\mathcal{F} := 1000 \cdot \sup_{[-1,1]} |\varphi| + |\varphi'| + |\varphi''| .$$

Let us define u_1 in $x < 0$ as:

$$u_1(x, y) = \varphi(x) u_{1x}(0, y) ; \tag{44}$$

notice that u_1, u_{1y} are continuous across Θ; $u_{1x}(x, y)|_{x<0} = \varphi'(x) u_{1x}(0, y)$. Therefore, u_{1x} is continuous across Θ. In $-1 < x \leq 0$, we have:

$$|H_{u_1}| \leq \mathcal{L}^2 \mathcal{F}^2 e^{-2b}, \qquad |D_2 u_1| \leq \mathcal{L}\mathcal{F} e^{-b} \tag{45}$$

and

$$u_1 \equiv 0 \qquad \text{in } x \leq -1 .$$

Let us define u_2 in $x > b$, as:

$$u_2(x, y) = \varphi(x - b) u_{2x}(b, y) = -\frac{\varphi(x - b)}{\sinh b} \sin y .$$

Again u_2, u_{2y} are continuous across Γ; moreover $u_{2x} = -\frac{\varphi'(x-b)}{\sinh b} \sin y$ in $x > b$ matches with u_{2x} in $x \leq b$. Thus u_2 is $C^{1,1}$, $u_2 \equiv 0$ in $x \geq b + 1$ and in $b \leq x \leq b + 1$:

$$|H_{u_2}| \leq \frac{\mathcal{F}^2}{(\sinh b)^2} , \qquad |D_2 u_2| \leq \frac{\mathcal{F}}{\sinh b} . \tag{46}$$

Let us define now:

$$u_3 = u_1 + u_2 .$$

Let us show that, for a suitable choice of b, u_3 is the function we are looking for. Let us make four more assumptions.

(H2): 2nd CONDITION ON b. Let $b_2 \geq b_1$ so large that, for every $b \geq b_2$:

$$-\mathcal{N}_2 := -1 + \mathcal{F}^2 \mathcal{L}^2 e^{-2b} + \mathcal{F}\mathcal{L} e^{-b} < 0 .$$

(H3): 3nd CONDITION ON b. Let $b_3 \geq b_2$ so large that, for every $b \geq b_3$:

$$-\mathcal{N}_3 := -\mathcal{W} + \mathcal{D} \left(\frac{2\sinh(1/2) + 2\cosh(1/2)}{\sinh b} \right) < 0 .$$

(H4): 4nd CONDITION ON b. Let $b_4 \geq b_3$ so large that, for every $b \geq b_4$:

$$-\mathcal{N}_4 := -\mathcal{W} + \frac{\mathcal{F}^2}{(\sinh b)^2} + \frac{\mathcal{D}\mathcal{F}}{\sinh b} < 0 .$$

(H5): 5^{nd} CONDITION ON b. Let $b_5 \geq b_4$ so large that, for every $b \geq b_5$:

$$-\mathcal{N}_5 := -e^{-4} + \frac{\mathcal{F}^2}{(\sinh b)^2} + \frac{\mathcal{F}}{\sinh b} < 0 .$$

Now let us fix $b \geq b_5$ and let us check the properties of u_3.
(α) if $x < -1$, we have $u_3 = u_2$, u_3 is a solution of an elliptic equation there and $u_3 = e^{-x} \sin y$ in $x \leq -\pi - 1$.
(β) in $[-1, 0] \times \mathbf{T}$ $u_3 = u_1 + u_2 \in C^2([-1, 0] \times \mathbf{T})$; let us compute H_{u_3}: we have, by (ii) of Lemma 3.3, (45) and condition H2:

$$H_{u_3} = H_{u_1} + H_{u_2} + u_{1xx}u_{2yy} + u_{1yy}u_{2xx} - 2u_{1xy}u_{2xy} \leq -\mathcal{N}_2 < 0 ,$$

therefore, by Pucci's lemma, u_3 is solution to a uniformly elliptic equation in $[-1, 0] \times \mathbf{T}$.
(γ) in $[0, b] \times \mathbf{T} \backslash O_1$, u_3 is harmonic (but for the four poles, that will be fixed later).
(δ) In $O_1 \cap [b - \frac{1}{4}, b] \times \mathbf{T}$, $u_3 \in C^{1,1}$ and has piecewise continuous bounded second derivatives; let us compute H_{u_3}: we have by (iii) of Lemma 3.2 and (H3):

$$H_{u_3} = H_{u_1} + H_{u_2} + u_{1xx}u_{2yy} + u_{1yy}u_{2xx} - 2u_{1xy}u_{2xy} < -\mathcal{N}_3 < 0 ,$$

thus, by Pucci's lemma, u_3 is a solution to a uniformly elliptic equation in that set.
(ϵ) in $O_1 \cap [b, b + \frac{1}{4}] \times \mathbf{T}$, $u_3 \in C^{1,1}$ and has piecewise continuous bounded second derivatives; by (iii) of Lemma 3.2, (46) and (H4): $H_{u_3} < -\mathcal{N}_4 < 0$; again, by Pucci's lemma, u_3 is solution to a uniformly elliptic equation in that region.
(ζ) in $[b, b + 1] \times \mathbf{T} \setminus O_1$, $u_3 \in C^{1,1}$ and has piecewise continuous bounded second derivatives; by (46), (ii) of Lemma 3.3 and (H5): $H_{u_3} < -\mathcal{N}_5 < 0$, so u_3 is solution to an elliptic equation in that region.
(η) in $[b + 1, +\infty) \times \mathbf{T}$, $u_3 = u_1 = e^{-2(x-b)} \sin 2y$ is harmonic. □

STEP 5. Smoothing of u_3 and proof of Theorem 3.1.

The function u_3 constructed in Step 4 has all the properties we were looking for, except that it is discontinuous at $x = b - 1$, $y = \pm\pi/4, \pm(3/4)\pi$ where it has poles. Let us change u_3 in a neighbourhood of these points to make it a $W^{2,p}$ function (p arbitrary, $1 < p < 2$). It suffices to do this for $x = b - 1$, $y = \pi/4$.

Let us recall that, in a small neighbourhood of $(b - 1, \pi/4)$, u_3 is of the form:

$$u_3 = A \log(1/[(x - b + 1)^2 + (y - \pi/4)^2]) + \widetilde{u}_3(x, y)$$

where $A > 0$ and $\widetilde{u}_3(x, y)$ is harmonic.

Let B_R the ball of center $(b-1, \pi/4)$ and radius $0 < R < 1/8$. We have on ∂B_R:

$$u_3\Big|_{\partial B_R} = A \log\frac{1}{R} + \tilde{u}\Big|_{\partial B_R},$$

$$u_{3n}\Big|_{\partial B_R} = -\frac{A}{R} + \tilde{u}_n\Big|_{\partial B_R},$$

$$u_{3nn}\Big|_{\partial B_R} = \frac{A}{R^2} + \tilde{u}_{nn}\Big|_{\partial B_R}.$$

(n outer normal to ∂B_R). Let us choose R so small that $u_{3n} < 0$ and $u_{3nn} > 0$ on ∂B_R and $B_{2R} \subset F_0$.

Now let us use Proposition 2.3. In B_R there exists w_-, solution to $Lw_- = 0$ in B_R (where L, w_- satisfy (25), (26), (27), (28) with $\beta \in (0, 1 - p/2)$) such that $w_-|_{\partial B_R} = u_3|_{\partial B_R}$, $\max_{\partial B_R} \frac{\partial w_-}{\partial n}| < \min_{\partial B_R} \frac{\partial u_3}{\partial n}$, and $\frac{\partial^2 w_-}{\partial n^2}|_{\partial B_R} > 1$. As the difference $w_- - u_3$ is harmonic in $B_R \setminus B_{3R/4}$ and vanishes on ∂B_R, it can extended harmonic to $B_{5R/4}$.

Now Proposition 2.1 can be used with $0 < \delta < R/8$ and the extended w_- can be glued with $w_+ = u_3$ in $R - \delta < \rho < R + \delta$.

From now on let us call u_3 this new glued function. $u_3 \in W^{2,p}(F_0)$ and is the same as old u_3 outside of a small ball around $(b-1, \pi/4)$. Doing the same with the other poles we get a function, again called u_3, with the properties:

$u_3 = e^{-x} \sin y$ in $(-\infty, -\pi - 2] \times \mathbf{T}$
$u_3 = e^{-2(x-b)} \sin 2y$ in $(b+1, +\infty] \times \mathbf{T}$
u_3 is $W^{2,p}[-\pi - 2, b+2] \times \mathbf{T}$
u_3 is a solution to a elliptic equation in $(-\infty, +\infty) \times \mathbf{T}$ and u_3 is harmonic in a neighbourhood of $x = -\pi - 1$ and $x = b + 2$.

PROOF OF THEOREM 3.1. Define $S := b + \pi + 4$, $Q := e^{-\pi-6}$

$$u_0(x, y) := u_3(x - \pi - 2, y) e^{-(\pi+2)}$$

in $[0, S] \times \mathbf{T}$. Then $u_0 \in W^{2,p}([0, S] \times \mathbf{T})$, it is a solution to an elliptic equation in $[0, S] \times \mathbf{T}$, $u_0 = e^{-x} \sin y$ in a neighbourhood of $x = 0$, $u_0 = Q e^{-2(x-S)} \sin 2y$ in a neighbourhood of $x = S$. Moreover, $4Q < 1$. □

4. – The existence theorem

THEOREM 4.1. *Let* $\Lambda := [0, +\infty) \times \mathbf{T}$, $1 < p < 2$. *There exists a uniformly elliptic equation in* Λ:

(47) $$A_{11}(x, y) u_{xx} + 2A_{12}(x, y) u_{xy} + A_{22}(x, y) u_{yy} = 0$$

a positive constant X *and a function* $u \in W^{2,p}(\Lambda)$, *solution to* (47), *satisfying:*

(i) $u = e^{-x} \sin y$ *in a neighbourhood of* $x = 0$,
(ii) $u \equiv 0$ *for* $x \geq X$.

PROOF. Let u_0 be the function introduced in Theorem 3.1 and defined in $[0, S] \times \mathbf{T}$; let $0 < Q < 1/4$, $S > 0$ be the constants defined in that theorem. Let us define:

$$\begin{aligned}
J &:= \left\| D^2 u_0 \right\|_{L^p([0,S]\times\mathbf{T})} \\
s_0 &:= 0 \\
s_k &:= S\left(1 + \cdots + \frac{1}{2^{k-1}}\right) = s_{k-1} + S\frac{1}{2^{k-1}} \quad k = 1, 2, \ldots \\
X &:= \sum_{k=1}^{\infty}(s_k - s_{k-1}) = 2S.
\end{aligned}$$

Let $x \in [s_{k-1}, s_k]$ $(k \geq 1)$; $y \in \mathbf{T}$; let:

$$\xi := 2^{k-1}(x - s_{k-1}), \quad \eta := 2^{k-1} y; \tag{48}$$

let us define u in $[s_{k-1}, s_k] \times \mathbf{T}$, $k = 1, 2, \ldots$, as:

$$u(x, y) := Q^{k-1} u_0(\xi, \eta); \tag{49}$$

in $[0, s_1]$: $\quad u \equiv u_0$.

It is not difficult to see that u is defined in $[0, 2S) \times \mathbf{T}$, and harmonic in a neighbourhood of $x = s_k$ $(k = 1, 2, \ldots)$.

But for a countable set of points, $u \in C^{1,1}_{\mathrm{loc}}((0, 2S) \times \mathbf{T})$; let us prove that u is solution to an elliptic equation.

Let u_0 be solution to:

$$a_{11}(\xi, \eta) u_{0\xi\xi} + 2a_{12}(\xi, \eta) u_{0\xi\eta} + a_{22}(\xi, \eta) u_{0\epsilon ta\eta} = 0 \tag{50}$$

in $[0, S] \times \mathbf{T}$. Then, in $[s_{k-1}, s_k] \times \mathbf{T}$:

$$\begin{aligned}
u_{xx}(x, y) &= 2^{2(k-1)} Q^{k-1} u_{0\xi\xi}(\xi, \eta), \\
u_{xy}(x, y) &= 2^{2(k-1)} Q^{k-1} u_{0\xi\eta}(\xi, \eta), \\
u_{yy}(x, y) &= 2^{2(k-1)} Q^{k-1} u_{0\eta\eta}(\xi, \eta),
\end{aligned}$$

where ξ, η are given by (48). Then, in $[s_{k-1}, s_k] \times \mathbf{T}$, u satisfies the elliptic equation:

$$A_{11} u_{xx} + 2A_{12} u_{xy} + A_{22} u_{yy} = 0,$$

where:

$$A_{ij}(x, y) := a_{ij}\left(2^{k-1}(x - s_{k-1}), 2^{k-1} y\right) \quad 1 \leq i, j \leq 2.$$

From (49):

$$\max_{[s_{k-1}, s_k]\times\mathbf{T}} |u| = Q^{k-1} \max_{[0,S]\times\mathbf{T}} |u_0|,$$

then u is continuous in $[0, 2S] \times \mathbf{T}$ and $u \to 0$ uniformly as $x \to 2S$.

Let us prove that $u \in W^{2,p}([0,2S] \times \mathbf{T})$ $(1 < p < 2)$. We have:

$$\left\| D^2 u \right\|^p_{L^p([0,2S]\times\mathbf{T})} = \sum_{k=1}^{\infty} \int_{s_{k-1}}^{s_k} dx \int_0^{2\pi} |D^2 u(x,y)|^p \, dy$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} (4Q)^{p(k-1)} \int_0^S d\xi \int_0^{2\pi} |D^2 u_0(\xi,\eta)|^p \, d\eta$$

$$= J^p \sum_{k=1}^{\infty} \left(\frac{(4Q)^p}{2} \right)^{k-1} < \infty.$$

To prove that u can be extended as $\equiv 0$ in $x > X = 2S$, it is sufficient to show that u_x has trace a.e. zero on $x = 2S$. We have:

$$\int_0^{2\pi} |u_x(2S,y)|^p \, dy \le \int_0^{2\pi} |u_x(s_k,y)|^p \, dy + \int_{s_k}^{2S} dx \int_0^{2\pi} |D^2 u|^p \, dy;$$

and $|u_x(s_k,y)| = |(2Q)^k \sin 2^k y| \;\to\; 0$ as $k \to +\infty$, $s_k \to 2S$; then:

$$\int_0^{2\pi} |u_x(2S,y)|^p \, dy = 0;$$

therefore, u can be extended to zero in $[2S,+\infty)\times\mathbf{T}$ and the extended function is in $W^{2,p}([0,+\infty)\times\mathbf{T})$. □

5. – Applications

Let us make comments on the counterexample constructed.

Let us note that the summability exponent p can be chosen as close to 2 as one wants. Infact, the function constructed is "almost" $C^{1,1}$, but for a countable set of points. In small balls around these points, the choice of β in the Gilbarg-Serrin type operator depends upon the summability exponent p; if $p \nearrow 2$, then $\beta \searrow 0$; that means that, to get close to $p = 2$, one has to make the ellipticity constant small.

It is reasonable that, if one fixes $p_0 \in (1,2)$, then there could exist $\alpha(p_0)$, such that, if L is of the form (30), with ellipticity constant $\ge \alpha(p_0)$, then there could be unique continuation for solutions $W^{2,p}$ to $Lu = 0$, $(p_0 < p < 2)$.

Easy consequences of Theorem 4.1 are the following facts.

THEOREM 5.1. *The unique continuation property does not hold for solutions to elliptic systems that are* $W^{1,p}$, $1 < p < 2$.

PROOF. Let us use the notations and the results of previous section. Let u be the function, defined in Λ, introduced in Theorem 4.1. Let $v := u_x, w := -u_y, Z := v + iw$. Then: (i) $Z \in W^{1,p}(\Lambda)$; (ii) $Z \equiv 0$ in $x > X$; (iii) Z satisfies the elliptic system:

$$\begin{cases} v_x = \dfrac{2A_{12}}{A_{11}} w_x + \dfrac{A_{22}}{A_{11}} w_y \\ -v_y = w_x \,. \end{cases}$$

Then Z is a counterexample to the unique continuation property (thm. p. 261 in [2]). □

THEOREM 5.2. *Let Λ as in Theorem 4.1; there exist: a variational, second order, uniformly elliptic operator L_1, a positive number X and a function $w \in W^{2,p}(\Lambda)$, satisfying $L_1 w = 0$ and $w \equiv 0$ in $x > X$.*

PROOF. Let us use the notations and the results of previous section. Let u be the function, defined in Λ, introduced in Theorem 4.1. Let $\phi \in C^\infty(\Lambda)$, $\phi(0, y) = 0$, $y \in \mathbf{T}$. Then:

$$\begin{aligned} 0 &= \int \left(u_{xx} + \frac{2A_{12}}{A_{11}} u_{xy} + \frac{A_{22}}{A_{11}} u_{yy} \right) \phi_y \, dxdy \\ &= \int \left[u_{xy}\phi_x + \left(\frac{2A_{12}}{A_{11}} u_{xy} + \frac{A_{22}}{A_{11}} u_{yy} \right) \phi_y \right] dxdy \,. \end{aligned}$$

Let $w := -u_y$, then:

$$\int \left[w_x\phi_x + \left(\frac{2A_{12}}{A_{11}} w_x + \frac{A_{22}}{A_{11}} w_y \right) \phi_y \right] dxdy = 0$$

i.e. $w \in W^{1,p}(\Lambda), 1 < p < 2$ and it satisfies the second order, uniformly elliptic, variational equation:

$$w_{xx} + \left(\frac{2A_{12}}{A_{11}} w_x + \frac{A_{22}}{A_{11}} w_y \right)_y = 0$$

and $w \equiv 0$ in $x > X$. □

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