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Renormalized Entropy Solutions of Scalar Conservation Laws

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Abstract. We introduce the new notion of renormalized entropy solution for a scalar conservation law $u_t + \mathrm{div}\ \Phi(u) = f$. Existence and uniqueness of renormalized entropy solutions is established in the general L^1-setting.

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1. – Introduction

Consider the Cauchy problem

$$(\mathrm{CP})(u_0, f) \quad \begin{cases} u_t + \mathrm{div}\ \Phi(u) = f & \text{on } Q =]0, T[\times\mathbb{R}^N \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R}^N \end{cases}$$

where $\Phi : \mathbb{R} \to \mathbb{R}^N$ is locally Lipschitz continuous, $T > 0$, $N \geq 1$. It is well-known (cf. [17], [18]) that, if $u_0 \in L^\infty(\mathbb{R}^N)$, $f \in L^\infty(Q)$, there exists a unique bounded entropy solution u of $(\mathrm{CP})(u_0, f)$, i.e. a function $u \in L^\infty(Q)$ satisfying

$$(1) \quad |u - k|_t + \mathrm{div}\ [\mathrm{sign}_0(u - k)(\Phi(u) - \Phi(k))] \leq \mathrm{sign}_0(u - k) f \qquad \text{in } \mathcal{D}'(Q)$$

for all $k \in \mathbb{R}$, and such that

$$(2) \qquad u(t, \cdot) \to u_0 \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } t \to 0 \text{ essentially.}$$

Using nonlinear semigroup theory (cf. [6], [12]) a generalized (mild) solution u of $(\mathrm{CP})(u_0, f)$ has been constructed for any $u_0 \in L^1(\mathbb{R}^N)$, $f \in L^1(Q)$ (cf. [4] and [11]). If u_0, f are bounded, the mild solution is also the unique entropy solution of $(\mathrm{CP})(u_0, f)$. However, it is not clear in which sense the mild solution "solves" the differential equation in the general L^1-setting. Note that if u_0, f

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are unbounded, in general, the mild solution u is unbounded and, as no growth condition is assumed on the flux Φ, the function $\Phi(u)$ may fail to be locally integrable. Consequently, the entropy condition (1) does not make sense and, in general, u can not be an entropy solution and not even a solution in the sense of distributions.

In this paper we propose the new notion of renormalized entropy solution of (CP)(u_0, f). The notion of renormalized solution has been intoduced in the last decade for different problems (Boltzmann equation, elliptic and parabolic problems in L^1 ...) and various existence and uniqueness results have been obtained (see e.g. [3], [7], [8], [13], [14], [15], ...). It is our aim in this paper to show that the idea of renormalization also applies to scalar conservation laws. We define a renormalized entropy solution of (CP)(u_0, f) (see Definition 1 below). This notion is shown to generalize the classical notion of entropy solution of (CP) (see Proposition 2.4). Existence and uniqueness of a renormalized entropy solution of (CP)(u_0, f) is established for any $(u_0, f) \in L^1(\mathbb{R}^N) \times L^1(Q)$ (see Proposition 3.1 and Proposition 3.2). Moreover, it is shown that the renormalized entropy solution is always the unique mild solution of (CP)(u_0, f).

2. – Renormalized entropy solutions

Let $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$, $f \in L^1_{\text{loc}}(Q)$. Recall (cf. [1], [2], [17, [18]) that an entropy subsolution (resp. entropy supersolution) of (CP)(u_0, f) in the sense of Kruzhkov is a measurable function $u : Q \to \mathbb{R}$ with $u^+ \in L^\infty_{\text{loc}}(Q)$ (resp. $u^- \in L^\infty_{\text{loc}}(Q)$), satisfying, for all $k \in \mathbb{R}$,

$$(3) \qquad (u-k)^+_t + \operatorname{div}\,[\chi_{\{u>k\}}(\Phi(u) - \Phi(k))] - \chi_{\{u>k\}} f \leq 0 \quad \text{in } \mathcal{D}'(Q)$$

(resp. $(k-u)^+_t + \operatorname{div}\,[\chi_{\{u<k\}}(\Phi(k) - \Phi(u))] + \chi_{\{u<k\}} f \leq 0$ in $\mathcal{D}'(Q)$) and such that

$$(4) \qquad (u(t,\cdot) - u_0)^+ \to 0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N) \text{ as } t \to 0 \text{ essentially}$$

(resp. $(u_0 - u(t,\cdot))^+ \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $t \to 0$ essentially). Obviously, a function u is an entropy solution of (CP)(u_0, f) (i.e. $u \in L^\infty_{\text{loc}}(Q)$ and satisfies (1) and (2) if and only if u is an entropy subsolution and an entropy supersolution of (CP)(u_0, f). For L^1-data u_0, f, in general, (locally) bounded entropy solutions do not exist. We propose the following notion of generalized entropy solution (for $r, s \in \mathbb{R}$, $r \wedge s = \min(r,s)$, $r \vee s = \max(r,s)$):

Definition 2.1.

(i) A **renormalized entropy subsolution** of (CP)(u_0, f) is a measurable function $u : Q \to \mathbb{R}$ such that

$$(5) \quad \forall\ \text{k,l} \in \mathbb{R},\ \mu_{k,l} = (u \wedge l - k)^+_t + \operatorname{div}\,[\chi_{\{u\wedge l>k\}}(\Phi(u \wedge l) - \Phi(k))] - \chi_{\{u \wedge l > k\}} f$$

is a Radon measure on Q,

(6) $$\forall\ \mathrm{k} \in \mathbb{R}, \quad \lim_{l\to\infty} \mu^+_{k,l}(Q) = 0$$

and, moreover,

(7) $\forall\ \mathrm{l} \in \mathbb{R}$, $(u(t,\cdot) \wedge l - u_0 \wedge l)^+ \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \to 0$ essentially.

(ii) A **renormalized entropy supersolution** of (CP)(u_0, f) is a measurable function $u : Q \to \mathbb{R}$ such that

(8) $$\forall\ \mathrm{k,l} \in \mathbb{R},:\ \nu_{k,l} = (k-u\vee l)^+_t + \mathrm{div}\ [\chi_{\{u\vee l<k\}}(\Phi(k)-\Phi(u\vee l))] + \chi_{\{u\vee l<k\}} f$$

is a Radon measure on Q,

(9) $$\forall\ \mathrm{k} \in \mathbb{R}, \quad \lim_{l\to-\infty} \nu^+_{k,l}(Q) = 0$$

and, moreover,

(10) $\forall\ \mathrm{l} \in \mathbb{R}$, $(u_0 \vee l - u(t,\cdot) \vee l)^+ \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \to 0$ essentially.

(iii) A measurable function $u : Q \to \mathbb{R}$ is a **renormalized entropy solution** of (CP)(u_0, f) if u is a renormalized entropy subsolution and a renormalized entropy supersolution of (CP)(u_0, f).

REMARK 2.2. Note that $\mu_{k,l}, \nu_{k,l}$ are well-defined in the sense of distributions for any measurable function u. If u is a renormalized entropy solution of (CP)(u_0, f), then, by definition, $\mu_{k,l}, \nu_{k,l}$ are Radon measures on Q. Note that $\mu_{k,l}, \nu_{k,l}$ are not supposed to be negative (compare with the classical entropy condition (3)). In fact, in general, the measures $\mu_{k,l}, \nu_{k,l}$ are not negative, even if u is a classical entropy solution in the sense of Kruzhkov. The generalized entropy condition satisfied by a renormalized entropy solution are the "limit entropy" conditions (6) and (9).

REMARK 2.3. The concept of renormalized entropy solution as defined above is an extension to the setting of scalar conservation laws of the notion of renormalized solution as it is known for elliptic equations. In fact, consider the elliptic problem in L^1: (E) $-\mathrm{div}\ a(x, \nabla v) = f$ on Ω, $v = 0$ on $\partial\Omega$ where Ω is an open bounded set in $\mathbb{R}^N$, $v \mapsto -\ \mathrm{div}\ a(x, \nabla v)$ is a monotone operator of Leray-Lions type from $W^{1,p}_0(\Omega)$ into $W^{-1,p'}(\Omega)$, $f \in L^1(\Omega)$, $p > 1$. The classical definition of a renormalized solution of (E) reads as follows (cf. [13], [14], [19], ...): a measurable function $u : \Omega \to \mathbb{R}$ is a renormalized solution of (E) if $T_k(u) = (u \vee (-k)) \wedge k \in W^{1,p}_0(\Omega)$ for any $k > 0$,

$$\int_\Omega a(x, \nabla u) \cdot \nabla(h(u)\phi) = \int_\Omega f h(u)\phi \qquad \text{for any } h \in C^1_c(\mathbb{R}), \phi \in C^\infty_c(\Omega)$$

and, moreover,

$$\frac{1}{n}\int_{\{n<|u|<2n\}} a(x,\nabla u)\cdot\nabla u \longrightarrow 0 \qquad \text{as } n\to\infty.$$

In [14], several equivalent definitions are given. In particular, it is shown that the following definition is equivalent with the classical definition of a renormalized solution: $u:\Omega\to\mathbb{R}^N$ is a measurable function with $T_k(u)\in W_0^{1,p}(\Omega)$ for any $k>0$,

$$\forall l\in\mathbb{R},\qquad \mu_l=-\mathrm{div}\ (a(x,\nabla T_l(u))-\chi_{\{|u|<l\}}f \quad \text{is a Radon measure on } \Omega,$$

and

$$\mu_l\to 0 \qquad \text{in } \mathcal{M}_b(\Omega) \text{ strongly as } l\to\infty. \tag{11}$$

Note that our definition of renormalized entropy solution is the extension of this last definition of renormalized solution taking into account the necessity of introducing entropy conditions in the setting of scalar conservation laws.

The notion of renormalized entropy solution is a generalization of the classical notion of bounded entropy solution. In fact, we have

PROPOSITION 2.4. *If $u^+\in L^\infty(Q)$, then u is a renormalized entropy subsolution of* (CP)(u_0,f) *if and only if u is an entropy subsolution of* (CP)(u_0,f) *in the sense of Kruzhkov.*

REMARK 2.5 Note that if u is a (renormalized) entropy subsolution of (CP)(u_0,f), then $-u$ is a (renormalized) entropy supersolution of the Cauchy problem $v_t+\mathrm{div}\ \Psi(v)=-f$ on Q, $v(0,\cdot)=-u_0$ on $\mathbb{R}^N$, where $\Psi(r)=-\Phi(-r)$. As a consequence, the corresponding result of Proposition 2.4 holds for (renormalized) entropy supersolutions. Therefore, in the class of bounded functions, the concepts of renormalized entropy solution and of entropy solutions in the sense of Kruzhkov coincide.

In the proof of Proposition 2.4, we use the following simple decomposition lemma:

LEMMA 2.6. *Let $k,l\in\mathbb{R}$, $k<l$. Then, for any $u\in\mathbb{R}$,*

$$\chi_{\{u\wedge l>k\}}(\Phi(u\wedge l)-\Phi(k))=\chi_{\{u>k\}}(\Phi(u)-\Phi(k))-\chi_{\{u>l\}}(\Phi(u)-\Phi(l)). \tag{12}$$

PROOF. As $l>k$, for any $u\in\mathbb{R}$, we have

$$\begin{aligned}\chi_{\{u\wedge l>k\}}(\Phi(u\wedge l)-\Phi(k)) &= \chi_{\{u>k\}}(\Phi(u\wedge l)-\Phi(k))\\ &= \chi_{\{v>k\}}(\Phi(u)-\Phi(k))-\chi_{\{u>k\}}(\Phi(u)-\Phi(u\wedge l))\\ &= \chi_{\{u>k\}}(\Phi(u)-\Phi(k))-\chi_{\{u>l\}}(\Phi(u)-\Phi(l))\end{aligned}$$

where the last equality holds as $\chi_{\{k<u<l\}}(\Phi(u)-\Phi(u\wedge l))=\chi_{\{k<u<l\}}(\Phi(u)-\Phi(u))=0$. □

PROOF OF PROPOSITION 2.4. Let u be an entropy subsolution in the sense of Kruzhkov with $u^+ \in L^\infty(Q)$. Let $k, l \in \mathbb{R}$. Note that we may always assume that $l > k$. In fact, if $l \leq k$, $\mu_{k,l} = 0$. Using the fact that, for $l > k$, $(u \wedge l - k)^+ = (u-k)^+ - (u-l)^+$, according to the preceding lemma,

$$\begin{aligned} \mu_{k,l} &= (u \wedge l - k)_t^+ + \operatorname{div}\ (\chi_{\{u\wedge l > k\}}(\Phi(u\wedge l) - \Phi(k))) - \chi_{\{u\wedge l>k\}} f \\ &= [(u-k)_t^+ + \operatorname{div}\ (\chi_{\{u>k\}}(\Phi(u) - \Phi(k))) - \chi_{\{u>k\}} f] \\ &\quad - [(u-l)_t^+ + \operatorname{div}\ (\chi_{\{u>l\}}(\Phi(u) - \Phi(l))) - \chi_{\{u<l\}} f] \\ &\quad - \chi_{\{u<l\}} f. \end{aligned} \tag{13}$$

As u is an entropy subsolution, according to (3), the expressions in the first two brackets of the right hand side in the above equality are non-positive Radon measures. Consequently, $\mu_{k,l}$ is a Radon measure and

$$\mu_{k,l}^+ \leq -(u-l)_t^+ - \operatorname{div}\ [\chi_{\{u>l\}}(\Phi(u) - \Phi(l))] + \chi_{\{u<l\}} f^+. \tag{14}$$

As $u^+ \in L^\infty(Q)$, it follows that $\mu_{k,l}^+(Q) = 0$ for any $k \in \mathbb{R}$ and $l \geq l_0 := \operatorname{ess-sup}_Q u^+$, hence (6) holds. Moreover, by (4), $(u(t,\cdot) - u_0)^+ \to 0$ as $t \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ essentially. Note that, for any $l \in \mathbb{R}$, $(u(t,\cdot) \wedge l - u_0 \wedge l)^+ \leq (u(t,\cdot) - u_0)^+$. As a consequence, (7) holds, and thus u is a renormalized entropy subsolution of (CP)(u_0, f).

The converse implication is immediate since $\mu_{k,l} = (u-k)_t^+ + \operatorname{div}(\chi_{\{u>k\}}(\Phi(u) - \Phi(k)) - \chi_{\{u>k\}} f$ for any $l > \operatorname{ess-sup}_Q u^+$. □

In the following proposition we make precise some properties satisfied by a renormalized entropy subsolution and the associated measures $\mu_{k,l}$. The result will be used in the next section to prove uniqueness of renormalized entropy solutions.

PROPOSITION 2.7. *Let u be a renormalized entropy subsolution of* (CP)(u_0, f), $u_0 \in L^1(\mathbb{R}^N)$, $f \in L^1(Q)$. *Then*

(i) $\forall k \in \mathbb{R}$, $\limsup_{l\to\infty} |\mu_{k,l}|(Q) < \infty$

(ii) $\forall k \in \mathbb{R}$, $\operatorname{ess-sup}_{(0,T)} \int_{\mathbb{R}^N} (u(t,\cdot) - k)^+ \leq \int_{\mathbb{R}^N} (u_0 - k)^+ + \int_Q |f|$

(iii) $\exists\, (\mu_k)_k \subset \mathcal{M}_b(Q)$, $\mu_k \leq 0$, *such that, for any* $k, l \in \mathbb{R}$, $k < l$,

$$\mu_{k,l} = \mu_k - \mu_l - \chi_{\{u>l\}} f \leq -\mu_l - \chi_{\{u>l\}} f, \tag{15}$$

and, moreover,

$$\mu_l(Q) \to 0 \quad \textit{as } l \to \infty. \tag{16}$$

REMARK 2.8. Recall that if v is a renormalized entropy supersolution of (CP)(v_0, g), then $u = -v$ is a renormalized entropy subsolution of $u_t + \operatorname{div}\Psi(u) = -g$, $u(0) = -v_0$, where $\Psi(r) = -\Phi(-r)$. Moreover, for any

$k, l \in \mathbb{R}$, $\nu_{k,l}(v) = \mu_{-k,-l}(u)$ where $\nu_{k,l}(v)$ is the measure corresponding to the supersolution v defined by (8), $\mu_{k,l}(u)$ the measure associated with the subsolution $u = -v$ defined by (5). Consequently, the corresponding results of Proposition 2.7 hold for a renormalized entropy supersolution v of (CP)(v_0, g):

(i) $\forall k \in \mathbb{R}$, $\limsup_{l\to\infty} |\nu_{k,-l}|(Q) < \infty$

(ii) $\forall k \in \mathbb{R}$, ess-sup$_{(0,T)} \int_{\mathbb{R}^N} (k - v(t,\cdot))^+ \leq \int_{\mathbb{R}^N} (k - v_0)^+ + \int_Q |g|$

(iii) $\exists\, (\nu_k)_k \subset \mathcal{M}_b(Q)$, $\nu_k \leq 0$, such that, for any $k, l \in \mathbb{R}$, $-l < k$,

$$\nu_{k,-l} = \nu_k - \nu_{-l} + \chi_{\{v<-l\}} g \leq -\nu_{-l} + \chi_{\{v<-l\}} g,$$

and, moreover,

$$\nu_{-l}(Q) \to 0 \quad \text{as } l \to \infty.$$

Proof of Proposition 2.7. First note that $|\mu_{k,l}| = 2\mu_{k,l}^+ - \mu_{k,l}$. Thus, for any $\xi \in C_c^\infty(Q)$, $0 \leq \xi \leq 1$,

$$\begin{aligned}\int_Q \xi d|\mu_{k,l}| \leq 2\mu_{k,l}^+(Q) + \int_Q (u \wedge l - k)^+ \xi_t \\ + \int_Q \operatorname{sign}^+(u \wedge l - k)(\Phi(u \wedge l) - \Phi(k)) \cdot \nabla\xi + \int_Q |f|.\end{aligned} \tag{17}$$

Using the initial condition (7) and the fact that Φ is locally Lipschitz continuous, passing to the limit with $\xi \to \chi_{]0,t[\times\mathbb{R}^N}$, we obtain

$$\begin{aligned}|\mu_{k,l}|(]0,t[\times\mathbb{R}^N) + \int_{\mathbb{R}^N} (u(t,\cdot) \wedge l - k)^+ \\ \leq 2\mu_{k,l}^+(Q) + \int_{\mathbb{R}^N} (u_0 \wedge l - k)^+ + \int_Q |f|\end{aligned} \tag{18}$$

for a.e. $0 < t < T$. In fact, let us recall the classical argument. Choose $\sigma \in C_c^\infty(]0,T[)$ with $0 \leq \sigma \leq 1$, $\rho \in C_c^\infty(\mathbb{R})$ with $\rho \geq 0$, $\int_{\mathbb{R}} \rho = 1$. Let $L = L(l)$ denote the Lipschitz constant of Φ on the set $\{|r| \leq l\}$. For $m \in \mathbb{N}$, $R > 0$, define

$$\psi_m(t,x) = \int_{m(|x|+Lt-R)}^{\infty} \rho(s)ds \tag{19}$$

and apply (17) with $\xi = \sigma\psi_m$. Due to the fact that

$$\begin{aligned}&-\sigma[(u \wedge l - k)^+(\psi_m)_t + \chi_{\{u\wedge l > k\}}[\Phi(u \wedge l) - \Phi(k)] \cdot \nabla_x \psi_m) \\ &= m\sigma\rho(m(|x| + Lt - R))(u \wedge l - k)^+ \left(L + \frac{\Phi(u \wedge l) - \Phi(k)}{(u \wedge l - k)^+} \cdot \nabla_x |x|\right) \\ &\geq 0\end{aligned} \tag{20}$$

we obtain

$$\int_Q \sigma \psi_m d|\mu_{k,l}| \le 2\mu_{k,l}^+(Q) + \int_Q \psi_m (u \wedge l - k)^+ \sigma_t + \int_Q |f|.$$

As $\psi_m \to \chi_{\{(t,x)\in Q; |x|\le R-Lt\}}$ as $m \to \infty$, passing to the limit with $m \to \infty$ yields

$$\begin{aligned}&\int_Q \sigma \chi_{\{x;|x|\le R-Lt\}} d|\mu_{k,l}|\\ &\le 2\mu_{k,l}^+(Q) + \int_0^T \int_{\{x;|x|\le R-Lt\}} (u \wedge l - k)^+ \sigma_t + \int_Q |f|.\end{aligned}$$

for any $\sigma \in C_c^\infty]0, T[$, $\sigma \ge 0$ which, of course, is equivalent to

$$\begin{aligned}&\int_\tau^t \int_{\{x;|x|\le R-Ls\}} d|\mu_{k,l}|(s,x) + \int_{\{x;|x|\le R-Lt\}} (u(t,x) \wedge l - k)^+ dx\\ &\le 2\mu_{k,l}^+(Q) + \int_{\{x;|x|\le R-L\tau\}} (u(\tau,x) \wedge l - k)^+ dx + \int_Q |f| d(t,x)\end{aligned}$$

for a.e. $0 < \tau < t < T$. According to (7), using the fact that, for any $k \in \mathbb{R}$, $((u(t,\cdot) \wedge l - k)^+ - (u_0 \wedge l - k)^+)^+ \le (u(t,\cdot) \wedge l - u_0 \wedge l)^+$, we have $(u(\tau) \wedge l - k)^+ \to (u_0 \wedge l - k)^+$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ essentially as $\tau \to 0$. Consequently, passing to the limit successively with $\tau \to 0$ and then $R \to \infty$, we obtain (18).

According to (6), passing in (18) with $l \to \infty$ yields

$$\limsup_{l\to\infty} |\mu_{k,l}|(Q) \le \int_{\mathbb{R}^N} (u_0 - k)^+ + \int_Q |f|,$$

i.e. (i) holds. Note that at the same time we also obtain the estimate

$$\operatorname*{ess\text{-}sup}_{(0,T)} \int_{\mathbb{R}^N} (u(t,\cdot) - k)^+ \le \int_{\mathbb{R}^N} (u_0 - k)^+ + \int_Q |f|,$$

i.e. (ii) holds. In particular, $u^+ \in L^\infty(0,T; L^1(\mathbb{R}^N))$.

Let $k_0 \in \mathbb{R}$ be fixed in the following. According to (i), $(\mu_{k_0,l})_l$ is bounded, hence weak*-relatively compact in $\mathcal{M}_b(Q)$, the space of bounded Radon measures on Q. Thus there exists a sequence $l_j \to +\infty$ such that μ_{k_0,l_j} converges weak* to some limit $\mu_{k_0} \in \mathcal{M}_b(Q)$. Note that, by (6), $\mu_{k_0} \le 0$. Next note that, for any $k, l \in \mathbb{R}$ with $k < l$,

$$\mu_{k,l} = \begin{matrix} \mu_{k,k_0} + \mu_{k_0,l} + \chi_{\{u>k_0\}} f \ \text{ if } k < k_0 < l \\ \mu_{k_0,l} - \mu_{k_0,k} - \chi_{\{u>k\}} f \ \text{ if } k_0 < k \end{matrix} \tag{21}$$

In fact, according to Lemma 2.6, for any $k, l, m \in \mathbb{R}$ with $k < l < m$, we have

$$\begin{aligned}&\chi_{\{u\wedge l>k\}}(\Phi(u \wedge l) - \Phi(k))\\ &\qquad = \chi_{\{(u\wedge m)\wedge l>k\}}(\Phi((u \wedge m) \wedge l) - \Phi(k))\\ &\qquad = \chi_{\{u\wedge m>k\}}(\Phi(u \wedge m) - \Phi(k)) - \chi_{\{u\wedge m>l\}}(\Phi(u \wedge m) - \Phi(l)).\end{aligned}$$

Moreover, recall that, for any $k < l$, $(u \wedge l - k)^+ = (u-k)^+ - (u-l)^+$, and then (21) follows.

As a consequence of (21), for any $k \in \mathbb{R}$, the sequence $(\mu_{k,l_j})_j$ converges weak* in $\mathcal{M}_b(Q)$ and

$$\mu_k \stackrel{\text{def}}{=} \lim_{j\to\infty} \mu_{k,l_j} = \begin{matrix} \mu_{k,k_0} + \mu_{k_0} + \chi_{\{u>k_0\}} f & \text{if } k < k_0 \\ \mu_{k_0} - \mu_{k_0,k} - \chi_{\{u>k\}} f & \text{if } k_0 < k. \end{matrix}$$

According to (6), we also have $\mu_k \leq 0$ for any $k \in \mathbb{R}$.

Using the same arguments as above, we finally obtain the following decomposition of the measure $\mu_{k,l}$:

$$\mu_{k,l} = \mu_{k,l_j} - \mu_{l,l_j} - \chi_{\{u>l\}} f \qquad \text{for all } j \text{ with } l_j > l.$$

Consequently, as $j \to \infty$,

$$\mu_{k,l} = \mu_k - \mu_l - \chi_{\{u>l\}} f \leq -\mu_l - \chi_{\{u>l\}} f,$$

and (15) holds. In particular, for $j \in \mathbb{N}$ with $l_j > k_0$,

$$0 \leq -\mu_{l_j} = \mu_{k_0,l_j} - \mu_{k_0} + \chi_{\{u>l_j\}} f. \tag{22}$$

By (6) and as $\mu_{k_0,l_j} \to \mu_{k_0}$ weak* in $\mathcal{M}_b(Q)$, we have

$$\begin{aligned} -\mu_{k_0}(Q) &\leq \liminf_{j\to\infty} \mu^-_{k_0,l_j}(Q) \\ &= \liminf_{j\to\infty} -\mu_{k_0,l_j}(Q), \end{aligned}$$

hence, by (22),

$$0 \leq \limsup_{j\to\infty} -\mu_{l_j}(Q) \leq \limsup_{j\to\infty} \mu_{k_0,l_j}(Q) - \mu_{k_0}(Q) + \limsup_{j\to\infty} \chi_{\{u>l_j\}} f \leq 0,$$

i.e.

$$\mu_{l_j}(Q) \to 0 \quad \text{as } j \to \infty. \tag{23}$$

Finally, let us prove that, for any sequence $l'_n \to \infty$, there is a subsequence (which we still denote l'_n) such that μ_{k_0,l'_n} converges weak-* to μ_{k_0} in $\mathcal{M}_b(Q)$ as $n \to \infty$. Then, by (21), μ_{k,l'_n} converges weak-* to μ_k as $n \to \infty$, and following the same arguments as before, we conclude that $\mu_{l'_n}(Q) \to 0$ as $n \to \infty$, i.e., as $\mu_{l'_n} \leq 0$, $\mu_{l'_n}$ converges to 0 strongly in $\mathcal{M}_b(Q)$. As a consequence, μ_l converges to 0 in $\mathcal{M}_b(Q)$ strongly as $l \to \infty$, i.e. (16) holds.

In order to complete the proof, let $(l'_n)_n$ be an arbitrary sequence with $l'_n \to \infty$. According to (i), for some subsequence, which we still denote l'_n for simplicity, we may assume that μ_{k_0,l'_n} converges to some $\mu'_{k_0} \in \mathcal{M}_b(Q)$. Note that by (21), for any $j, n \in \mathbb{N}$ with $k_0 < l'_n < l_j$,

$$\mu_{l'_n,l_j} = \mu_{k_0,l_j} - \mu_{k_0,l'_n} - \chi_{\{u>l'_n\}} f.$$

As $j \to \infty$, according to (6), we obtain

$$0 \geq \mu_{k_0} - \mu_{k_0,l'_n} - \chi_{\{u>l'_n\}} f.$$

Passing to the limit with $n \to \infty$ yields

$$0 \geq \mu_{k_0} - \mu'_{k_0},$$

i.e. $\mu_{k_0} \leq \mu'_{k_0}$. Next note that, again by (21), for any $j, n \in \mathbb{N}$ with $k_0 < l_j < l'_n$, we have

$$\mu_{l_j,l'_n} = \mu_{k_0,l'_n} - \mu_{k_0,l_j} - \chi_{\{u>l_j\}} f.$$

Passing to the limit successively now with first $n \to \infty$ and then $j \to \infty$, we obtain $0 \geq \mu'_{k_0} - \mu_{k_0}$, i.e. $\mu'_{k_0} \leq \mu_{k_0}$. We conclude that $\mu_{k_0} = \mu'_{k_0}$ which completes the proof of Proposition 2.7. □

3. – Existence and uniqueness

Uniqueness of a renormalized entropy solution of (CP)(u_0, f) is an immediate consequence of the following more general comparison principle (sign^+ denotes the multi-valued function defined by $\text{sign}^+(r) = 0$ if $r < 0$, $\text{sign}^+(0) = [0, 1]$, $\text{sign}^+(r) = 1$ if $r > 0$):

PROPOSITION 3.1. *Let u be a renormalized entropy subsolution of* (CP)(u_0, f), *v be a renormalized entropy supersolution of* (CP)(v_0, g), (u_0, f), $(v_0, g) \in L^1(\mathbb{R}^N) \times L^1(Q)$. *Then there exists $\alpha \in L^\infty(Q)$, $\alpha \in \text{sign}^+(u - v)$ a.e. on Q, such that, for a.e. $0 < t < T$,*

$$\int_{\mathbb{R}^N} (u(t) - v(t))^+ \leq \int_{\mathbb{R}^N} (u_0 - v_0)^+ + \int_0^t \int_{\mathbb{R}^N} \alpha(f - g). \tag{24}$$

PROOF. We use Kruzhkov's method of doubling variables: we consider two pairs of variables (t, x), (s, y) in Q and consider u as a function of (t, x), v as a function on (s, y). As u is a renormalized entropy subsolution of (CP)(u_0, f), for any $l \in \mathbb{R}$, for a.e. $(s, y) \in Q$,

$$\begin{aligned}
&\frac{\partial}{\partial t}(u \wedge l - (v(s,y) \vee (-l)))^+ + \text{div}_x[\chi_{\{u\wedge l > v(s,y)\vee(-l)\}}(\Phi(u \wedge l) - \Phi(v(s, y) \vee (-l)))] \\
&\quad - \chi_{\{u \wedge l > v(s,y)\vee(-l)\}} f \\
&= \mu_{v(s,y)\vee(-l),l} \\
&\leq -\mu_l + \chi_{\{u>l\}} |f|
\end{aligned}$$

where the last inequality follows from the decomposition result of Proposition 2.7 (iii) (which holds for $k < l$, but $\mu_{k,l} = 0$ otherwise). In the same way, as v is a renormalized entropy supersolution of (CP)(v_0, g), using Remark 2.8 (iii), for any $l \in \mathbb{R}$, for a.e. $(t,x) \in Q$,

$$\begin{aligned}
&\frac{\partial}{\partial s}(u(t,x) \wedge l-(v\vee(-l)))^{+}+\mathrm{div}_y[\chi_{\{u(t,x)\wedge l>v\vee(-l)\}}(\Phi(u(t,x) \wedge l)-\Phi(v\vee(-l)))]\\
&\quad + \chi_{\{u(t,x)\wedge l>v\vee(-l)\}}g\\
&\quad = \nu_{u(t,x)\wedge l,-l}\\
&\quad \leq -\nu_{-l} + \chi_{\{v<-l\}}|g|.
\end{aligned}$$

As a consequence we obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)(u \wedge l - (v \vee (-l)))^{+}\\
&\quad + (\mathrm{div}_x + \mathrm{div}_y)[\chi_{\{u\wedge l>v\vee(-l)\}}(\Phi(u \wedge l) - \Phi(v \vee (-l)))]\\
&\quad - \chi_{\{u\wedge l>v\vee(-l)\}}(f-g)\\
&\quad \leq -\mu_l - \nu_{-l} + \chi_{\{u>l\}}|f| + \chi_{\{v<-l\}}|g|
\end{aligned} \tag{25}$$

in $\mathcal{D}'(Q \times Q)$. Now let $(\varrho_n)_n$ be a classical sequence of mollifiers in $C_c^\infty(\mathbb{R}^N)$, $(\rho_n)_n$ a sequence of mollifiers in $C_c^\infty(\mathbb{R})$, $\xi \in C_c^\infty(Q)$, $0 \leq \xi \leq 1$. Define

$$\zeta_n(t,x,s,y) = \xi(\frac{t+s}{2}, \frac{x+y}{2})\varrho_n(x-y)\rho_n(t-s).$$

Note that, for n sufficiently large, $\zeta_n \in C_c^\infty(Q \times Q)$. Using ζ_n as a test function in (25) and passing to the limit with $n \to \infty$ yields

$$\begin{aligned}
&-\int_Q (u \wedge l - (v \vee (-l)))^{+}\xi_t - \int_{\{u\wedge l>v\vee(-l)\}} (\Phi(u \wedge l) - \Phi(v \vee (-l)) \cdot \nabla_x \xi\\
&\leq \int_Q \alpha_l(f-g)\xi - \mu_l(Q) - \nu_{-l}(Q) + \int_{\{u>l\}} |f| + \int_{\{v<-l\}} |g|
\end{aligned} \tag{26}$$

for some $\alpha_l \in L^\infty(Q)$, $\alpha_l(t,x) \in \mathrm{sign}^{+}(u(t,x) \wedge l - (v(t,x) \vee (-l))$ a.e. $(t,x) \in Q$. In fact, we have $(\frac{\partial}{\partial t} + \frac{\partial}{\partial s})\rho_n = 0 = (\nabla_x + \nabla_y)\varrho_n$ and, moreover,

$$\begin{aligned}
&\int_Q \int_Q \zeta_n(t,x,s,y)d\mu_l(t,x)d(s,y)\\
&\leq \int_Q \int_Q \varrho_n(x-y)\rho_n(t-s)d(s,y)d\mu_l(t,x)\\
&= \mu_l(Q),
\end{aligned}$$

respectively the corresponding estimate for ν_l, and (26) follows.

Applying (26) with $\xi(t,x) = \sigma(t)\psi_m(t,x)$ where $\sigma \in C_c^\infty(]0,T[$, $0 \le \sigma \le 1$, and ψ_m as defined in (19), using (20), passing to the limit with $m \to \infty$, we obtain, for a.e. $0 < \tau < t < T$,

$$\begin{aligned}
&\int_{\{x;|x|<R-Lt\}} (u(t,x) \wedge l - (v(t,x) \vee (-l)))^+ dx \\
&\le \int_{\{x;|x|<R-L\tau\}} (u(\tau,x) \wedge l - (v(\tau,\cdot) \vee (-l)))^+ dx \\
&\quad + \int_\tau^t \int_{\{x;|x|<R-Ls\}} \alpha_l(s,x)(f(s,x) - g(s,x))dx\,ds \\
&\quad - \mu_l(Q) - \nu_l(Q) + \int_{\{u>l\}} |f| + \int_{\{v<-l\}} |g|.
\end{aligned}$$

As $(u(\tau,x) \wedge l - (v(\tau,\cdot) \vee (-l)))^+ \le (u(\tau,x) \wedge l - u_0 \wedge l)^+ + (u_0 \wedge l - v_0 \vee (-l)))^+ + (v_0 \vee (-l) - v(\tau,\cdot) \vee (-l)))^+$, according to (7) and (10), passing to the limit first with $\tau \to 0$ and then $R \to \infty$ in the preceding inequality yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} (u(t,x) \wedge l - (v(t,x) \vee (-l)))^+ dx \\
&\le \int_{\mathbb{R}^N} (u_0 \wedge l - (v_0 \vee (-l)))^+ dx + \int_Q \alpha_l(s,x)(f(s,x) - g(s,x))d(s,x) \\
&\quad - \mu_l(Q) - \nu_l(Q) + \int_{\{u>l\}} |f| + \int_{\{v<-l\}} |g|.
\end{aligned}$$

Passing to the limit with $l \to \infty$, using (16) of Proposition 2.7 (iii) and Remark 2.8 (iii) and the fact that (a subsequence of) α_l converges weak* in $L^\infty(Q)$ to some $\alpha \in L^\infty(Q)$ with $\alpha(t,x) \in \operatorname{sign}^+(u(t,x) - (v(t,x))$ a.e. $(t,x) \in Q$, passing to the limit we obtain (24) and the proof is complete. □

The Cauchy problem (CP)(u_0, f) in L^1 is well-posed in the class of renormalized entropy solutions:

PROPOSITION 3.2. *For any $u_0 \in L^1(\mathbb{R}^N)$, $f \in L^1(Q)$, there exists a unique generalized entropy solution of* (CP)(u_0, f).

PROOF. As a renormalized entropy solution of (CP)$(u_0, f)$is a renormalized entropy subsolution and a renormalized entropy supersolution of (CP)(u_0, f) at the same time, uniqueness of renormalized entropy solutions follows at once from the comparison result, Proposition 3.1. Therefore it only remains to prove existence. To this end, for $n \in \mathbb{N}$, let u_n be the unique (bounded) entropy solution in the sense of Kruzhkov of (CP)(u_{0n}, f_n) where $u_{0n} = (u_0 \wedge n) \vee (-n)$, $f_n = (f \wedge n) \vee (-n)$. According to Proposition 2.4 and Proposition 3.1, u_n is also the unique renormalized entropy solution of (CP)(u_{0n}, f_n). In particular, for any $n \in \mathbb{N}$, for all $k, l \in \mathbb{R}$,

$$\mu_{k,l}^n = (u_n \wedge l - k)_t^+ + \operatorname{div}\ [\chi_{\{u_n \wedge l > k\}}(\Phi(u_n \wedge l) - \Phi(k))] - \chi_{\{u_n \wedge l > k\}} f_n$$

is a Radon measure on Q. Next, recall that u_n is also the unique mild solution in the sense of nonlinear semigroups of (CP)(u_{0n}, f_n). More precisely, u_n is the unique mild solution of the Cauchy problem (CP) $\frac{du}{dt} + Au \ni f_n, u(0) = u_{0n}$ where A is the m-accretive operator in $L^1(\mathbb{R}^N)$ defined as the closure in $L^1(\mathbb{R}^N)$ of the operator$A_0 = \{(v, w) \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \times L^1(\mathbb{R}^N); \int_{\mathbb{R}^N} \mathrm{sign}_0(v - k)(\Phi(v) - \Phi(k)) \cdot \nabla\xi + w\xi \geq 0$ for any $\xi \in C_c^\infty(\mathbb{R}^N), \xi \geq 0, k \in \mathbb{R}\}$ (cf. e.g. [4], [11]). By the general theory of nonlinear semigroups (cf. e.g. [6]), $u_n \to u$ in $C([0,T]; L^1(\mathbb{R}^N))$ where u is the unique mild solution of (CP) $\frac{du}{dt} + Au \ni f$, $u(0) = u_0$. Consequently, $\mu_{k,l}^n$ converges to $\mu_{k,l} = (u \wedge l - k)_t^+ + \mathrm{div}\ [\chi_{\{u\wedge l>k\}}(\Phi(u \wedge l) - \Phi(k))] - \chi_{\{u\wedge l>k\}} f$ in the sense of distributions as $n \to \infty$. Now let us recall (see the proof of Proposition 2.4) that the measure $\mu_{k,l}^n$ may be decomposed as follows:

$$\begin{aligned}\mu_{k,l}^n &= [(u_n - k)_t^+ + \mathrm{div}\ [\chi_{\{u_n>k\}}(\Phi(u_n) - \Phi(k))] - \chi_{\{u_n>k\}} f_n] \\ &\quad - [(u_n - l)_t^+ + \mathrm{div}\ [\chi_{\{u_n>l\}}(\Phi(u_n) - \Phi(l))] - \chi_{\{u_n>l\}} f_n] \\ &\quad - \chi_{\{u_n>l\}} f_n.\end{aligned}$$

As u_n satisfies the classical entropy condition, this decomposition implies

$$\begin{aligned}(\mu_{k,l}^n)^+ &\leq -(u_n - l)_t^+ - \mathrm{div}\ [\chi_{\{u_n>l\}}(\Phi(u_n) - \Phi(l))] + \chi_{\{u_n>l\}} f_n^+, \\ (\mu_{k,l}^n)^- &\leq -(u_n - k)_t^+ - \mathrm{div}\ [\chi_{\{u_n>k\}}(\Phi(u_n) - \Phi(k))] + \chi_{\{u_n>k\}} f_n + \chi_{\{u_n>l\}} f_n^+.\end{aligned}$$

As Φ is locally Lipschitz continuous, we obtain

$$(\mu_{k,l}^n)^+(Q) \leq \int_{\mathbb{R}^N} (u_{0n} - l)^+ + \int_Q \chi_{\{u_n>l\}} f_n^+ \leq \int_{\mathbb{R}^N} (u_0 - l)^+ + \int_Q \chi_{\{u_n>l\}} f^+ \tag{27}$$

and

$$\begin{aligned}(\mu_{k,l}^n)^-(Q) &\leq \int_{\mathbb{R}^N} (u_{0n} - k)^+ + (\chi_{\{u_n>k\}} + \chi_{\{u_n>l\}}) f_n^+ \\ &\leq \int_{\mathbb{R}^N} (u_0 - k)^+ + 2\chi_{\{u_n>k\}} |f|.\end{aligned} \tag{28}$$

Thus, for any $k, l \in \mathbb{R}$, $(\mu_{k,l}^n)_n$ is bounded in $\mathcal{M}_b(Q)$, and we may conclude that $\mu_{k,l}$ is a Radon measure on Q and $\mu_{k,l}^n$ converges to $\mu_{k,l}$ weak-* in $\mathcal{M}_b(Q)$ as $n \to \infty$. According to (27), we also obtain

$$\mu_{k,l}^+(Q) \leq \int_{\mathbb{R}^N} (u_0 - l)^+ + \int_Q \chi\{u \geq l\} f^+$$

for any $k, l \in \mathbb{R}$. As a consequence, (6) holds. The corresponding results for the measures $\nu_{k,l}$ can be obtained by using similar arguments, or by using the fact that, if u_n is an entropy solution of (CP)(u_{0n}, f_n), then $v_n = -u_n$ is an entropy solution of $(v_n)_t + \mathrm{div}\Psi(v_n) = -f_n$, $v_n(0) = -u_{0n}$ with $\Psi(r) = -\Phi(-r)$, and

$\nu_{k,l}(u_n) = \mu_{-k,-l}(v_n)$ for any $k, l \in \mathbb{R}$. Finally, recall that $u \in C([0, T[; L^1(\mathbb{R}^N))$ and $u(0) = u_0$, hence the initial conditions (7) and (10) are obviously satisfied, and the proof is complete. □

REMARK 3.3. Note that a corresponding existence and comparison result can be obtained for renormalized entropy solutions of the stationnary problem: (P) $u + \operatorname{div} \Phi(u) = f$ on $\mathbb{R}^N$, $f \in L^1(\mathbb{R}^N)$ where a renormalized entropy subsolution of (P) is defined to be a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ such that, for all $k, l \in \mathbb{R}$, $\mu_{k,l} = \chi_{\{u\wedge l>k\}}(u \wedge l - f) + \operatorname{div}(\chi_{\{u\wedge l>k\}}(\Phi(u \wedge l) - \Phi(k))$ is a Radon measure on $\mathbb{R}^N$, satisfying, for all $k \in \mathbb{R}$: $\lim_{l\to\infty} \mu^+_{k,l}(\mathbb{R}^N) = 0$. In the corresponding way, we may define a renormalized entropy supersolution and then a renormalized entropy solution of (P). Note that, using the concept of renormalized entropy solutions of (P), we may also completely characterize the closure A of the accretive operator A_0 as defined above (see proof of Proposition 3.2).

4. – Extensions and remarks

The assumption of local Lipschitz continuity of the flux Φ was made for simplicity and is not essential. In fact it is sufficient to assume that $\Phi : \mathbb{R} \to \mathbb{R}^N$ satisfies the more general uniqueness conditions of [5].

The "limit entropy" condition (6) may be slightly weakened. In fact, going through the proof of Proposition 3.1, one checks that it is actually sufficient to assume that

(i) $\forall k, l \in \mathbb{R}$, $\mu^+_{k,l}(Q) < \infty$
(ii) $\forall k \in \mathbb{R}$, $\mu^+_{k,l} \to 0$ in $\mathcal{D}'(Q)$ as $l \to \infty$
(iii) $\exists k_0 \in \mathbb{R}$, $\liminf_{l\to\infty} \mu^+_{k_0,l}(Q) = 0$.

Note, however, that a global condition on the measure of type (6) (or the more general conditions (i), (ii) and (ii) above) has to be imposed. The comparison and uniqueness result does not hold if we only assume a local limit entropy condition of type

$$\text{(29)} \qquad \forall k \in \mathbb{R}, \lim_{l\to\infty} \mu^+_{k,l}(K) = 0 \qquad \text{for all compact sets } K \subset [0, T[\times\mathbb{R}^N.$$

In fact, if u is an entropy subsolution in the sense of Kruzhkov, then, according to the proof of Proposition 2.4, $\mu_{k,l}$ defined by (5) are Radon measures satisfying conditions (i) and (ii) above and even (29). If the comparison result was satisfied with only these local conditions, then one would have uniqueness of a locally bounded entropy solution. But this is not true, according to the example of a non zero locally bounded entropy solution of $u_t + (u^3/3)_x = 0$ on $(0, T) \times \mathbb{R}$ with $u(0, \cdot) = 0$ on $\mathbb{R}$ constructed by E. Yu. Panov and A. Yu. Goritsky (cf. [16]).

It is also possible to define a renormalized entropy solution for scalar conservation laws on bounded domains with boundary conditions and prove their existence and uniqueness in the L^1-setting (see [9]). Finally, let us mention that the concept of renormalized entropy solutions is the appropriate solution concept for general quasi-linear second order differential equations in divergence form: $b(v)_t - \operatorname{div} a(v, \nabla\psi(v)) = f$ where $b, \psi : \mathbb{R} \to \mathbb{R}$ are continuous nondecreasing functions and $a : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a vector field of Leray-Lions type (cf. [10]).

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