ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

GUSTAF GRIPENBERG
PHILIPPE CLÉMENT
STIG-OLOF LONDEN

Smoothness in fractional evolution equations and conservation laws

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 29, nº 1 (2000), p. 231-251
<http://www.numdam.org/item?id=ASNSP_2000_4_29_1_231_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXIX (2000), pp. 231-251

Smoothness in Fractional Evolution Equations and Conservation Laws

GUSTAF GRIPENBERG – PHILIPPE CLÉMENT – STIG-OLOF LONDEN

Abstract. The regularity of solutions of the equation

$$\big(D_t^\alpha(u-u_0)\big)(t,x)+\sigma(u)_x(t,x)=f(t,x), \quad t,\, x\geq 0,$$

where D_t^α denotes the fractional derivative, is studied in the case where $\sigma'>0$. It is also shown that the solution to the Riemann problem for the fractional Burgers equation (where $\sigma(\underline{r})=\frac{1}{2}\underline{r}^2$) is continuous and has compact support (in the x-direction). A result on the continuity of the interface is established. In order to prove these results it is first shown that if A is an m-accretive operator in a Banach space, k is log-convex with $\lim_{t\downarrow 0}k(t)=+\infty$, and if u is the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(t-s)\big(u(s)-y\big)\,\mathrm{d}s+A\big(u(t)\big)\ni f(t), \quad t>0, \quad u(0)=y,$$

then $A(u(t))$ is continuous when $t>0$.

Mathematics Subject Classification (1991): 35K99 (primary), 35L99, 45K05 (secondary.

1. – Introduction

Recently a new type of approximation of scalar conservation laws in several variables has been introduced in [3]. Rather than adding a viscosity term (for this appproach see, e.g., [8]), the order of derivation with respect to time is lowered, that is, the derivative is replaced by a fractional derivative of order $\alpha\in(0,1)$. Furthermore, instead of using the Crandall-Liggett theorem as is done in [4], another abstract result, [10], is employed to establish the existence of a *strong* solution. In [3] the convergence of these strong solutions as $\alpha\uparrow 1$ to the entropy solution of $u_t+\operatorname{div}\mathbf{g}(u)=0$ is proven and some estimates for the speed of convergence are established.

Pervenuto alla Redazione il 23 giugno 1999 e in forma definitiva il 21 ottobre 1999.

The aim of this paper is to investigate further these solutions in the one-dimensional case, i.e., we analyze the regularity of solutions of the nonlinear fractional conservation law

$$D_t^\alpha (u - u_0) + \sigma(u)_x = f. \tag{1}$$

Here D_t^α denotes the fractional derivative of order $\alpha \in (0,1)$, see [15, p. 133], i.e.,

$$(D_t^\alpha v)(t) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g_{1-\alpha}(t-s)v(s)\,\mathrm{d}s, \qquad t>0,$$
$$(D_t^\alpha v)(0) \stackrel{\text{def}}{=} \lim_{h\downarrow 0}\tfrac{1}{h}\int_0^h g_{1-\alpha}(h-s)v(s)\,\mathrm{d}s,$$

where

$$g_\beta(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\beta)}t^{\beta-1}, \quad t>0, \quad \beta>0,$$

and where v is (at least) continuous and satisfies $v(0)=0$.

As an important tool for studying this equation we consider the abstract fractional nonlinear evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(t-s)\big(u(s)-y\big)\,\mathrm{d}s + A\big(u(t)\big) \ni f(t), \quad t>0, \quad u(0)=y. \tag{2}$$

In (2), u is the unknown function with range in a Banach space X, $y \in X$ and $f : \mathbb{R}^+ \to X$ are given, and k is a locally integrable real-valued function with a singularity at the origin. The nonlinear operator A may be multivalued and maps $\mathcal{D}(A) \subset X$ into (subsets of) X. Our primary current interest concerns the continuity and boundedness of the function $A(u(\underline{t}))$.

In [10], the existence of a strong solution u of (2), satisfying $A(u) \in L^1_{\text{loc}}(\mathbb{R}^+; X)$, was obtained. Conditions implying that the solution u is continuous were given in [3].

In this paper we demonstrate that under rather weak hypotheses one has $A(u) \in \mathcal{C}((0,\infty); X)$. In addition this function is uniformly bounded on $(0,T]$ for each $T>0$. Subsequently, these facts are applied to examine the regularity of the solution of (1).

As a first application we get the continuity of the solution of the Riemann problem for the fractional Burgers equation, i.e., for equation (1) with $\sigma(\underline{u}) = \frac{1}{2}\underline{u}^2$ and $f=0$. This improves on a result of [11] concerning (1). (In [11] it was assumed that $\sigma'(\underline{u}) \geq c_0 > 0$; an assumption not satisfied by $\sigma(\underline{u}) = \frac{1}{2}\underline{u}^2$.) The special structure of the fractional Burgers equation implies that the solution vanishes when $x \geq \Gamma(1-\alpha)t^\alpha$, in contrast to the linear case where there is an infinite speed of propagation. We also establish a result on the continuity of the interface. Recall that the entropy solution to the Riemann problem for the (nonfractional) Burgers equation is 1 when $x < \frac{t}{2}$ and 0 when $x > \frac{t}{2}$.

A motivation for studying the Riemann problem is, of course, that it is the simplest case where one has a discontinuity. Recall also that many numerical methods use the solution to the Riemann problem (with other constant states than just 1 and 0) and that this problem provides all solutions to the Cauchy problem $u_t + \sigma(u)_x = 0$ which are invariant under the group of homotheties $(t, x) \mapsto (at, ax)$. This group leaves first order conservation laws invariant, see [14, p. 43].

Furthermore, in Theorem 3 the results obtained on (2) are combined with earlier Schauder estimates on linear equations, [2], to establish results on the smoothness of solutions of (1).

The regularity, both temporal and spatial, of solutions of equations involving fractional derivatives of order $\alpha \in (1, 2)$ have been studied in several papers; [5], [6], and [7]. See also the monograph [12] for further results and references.

2. – Statement of results

Our result on (2) is the following.

THEOREM 1. *Assume that X is a real Banach space and that*

(i) $k \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{R})$ *is positive and nonincreasing,* $\lim_{t\downarrow 0} k(t) = +\infty$, *and* $\log(k(\underline{t}))$ *is convex;*

(ii) *A is an m-accretive operator on X;*

(iii) $y \in \hat{\mathcal{D}}(A)$, *i.e.,* $y \in X$ *and* $\sup_{\lambda>0} \|A_\lambda y\|_X < \infty$;

(iv) $f \in \mathcal{C}(\mathbb{R}^+; X)$ *is such that* $\int_0^T \omega_{f,T}(s)|k'(s)|\,\mathrm{d}s < \infty$ *for each* $T > 0$ *where* $\omega_{f,T}$ *is the modulus of continuity of* f, *i.e.,* $\omega_{f,T}(\underline{s}) \overset{\text{def}}{=} \sup_{t_1,t_2\in[0,T],|t_1-t_2|\leq \underline{s}} \|f(t_1) - f(t_2)\|_X$.

Then there is a unique strong solution u of (2) *such that* $u \in \mathcal{C}(\mathbb{R}^+; X)$, $u(0) = y$, *and there is a function* $w \in \mathcal{C}((0,\infty); X)$ *such that* $\sup_{0<t<T} \|w(t)\|_X < \infty$ *for each* $T > 0$, $w(t) \in A(u(t))$ *for all* $t > 0$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t k(t-s)\big(u(s) - y\big)\,\mathrm{d}s + w(t) = f(t), \quad t > 0. \tag{3}$$

Moreover, if $0 \leq t < t + h \leq \tau$ *then*

$$\begin{aligned} \|u(t+h) - u(t)\|_X \leq &\int_0^t \|f(t+h-s) - f(t-s)\|_X r(s)\,\mathrm{d}s \\ &+ \left(\sup_{\tau\in[0,h]} \|f(\tau)\|_X + \sup_{\lambda>0}\|A_\lambda(y)\|_X\right) \int_t^{t+h} r(s)\,\mathrm{d}s, \end{aligned} \tag{4}$$

where r is the first kind resolvent of k, i.e.,

$$\int_{[0,t]} k(t-s)r(s)\,\mathrm{d}s = 1, \quad t \in (0, \tau]. \tag{5}$$

Here A_λ denotes the Yosida approximation of A, i.e., $A_\lambda \overset{\text{def}}{=} \frac{1}{\lambda}(I - J_\lambda)$ where $J_\lambda = (I + \lambda A)^{-1}$.

A function $u : \mathbb{R}^+ \to X$ is a strong solution of (2) if there exists a function $w \in L^1_{\text{loc}}(\mathbb{R}^+; X)$ such that $w(\underline{t}) \in A(u(\underline{t}))$ a.e. on $\mathbb{R}^+$ and $\int_0^t k(t-s)(u(s) - y)\,\mathrm{d}s = \int_0^t (f(s) - w(s))\,\mathrm{d}s$ for every $t \geq 0$.

Our next result concerns the homogeneous version of (1) with, essentially $\sigma(\underline{r}) = c\underline{r}^\gamma$, $\gamma > 1$. In particular, this includes the fractional Burgers equation.

THEOREM 2. *Assume that*

(i) $k \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R})$ *is positive and nonincreasing,* $\lim_{t\downarrow 0} k(t) = +\infty$, *and* $\log(k(\underline{t}))$ *is convex;*

(ii) $\sigma \in C^1(\mathbb{R}; \mathbb{R})$ *is strictly increasing on* $(0, 1)$ *and there are constants* C *and* $\gamma > 1$ *such that*
$$\frac{1}{C} r^\gamma \leq \sigma(r) \leq C r^\gamma, \quad r \in [0, 1].$$

Then there is a solution u *of the Riemann problem*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(t-s)\big(u(s,x) - \chi_{(-\infty,0]}(x)\big)\mathrm{d}s + \sigma(u)_x(t,x) = 0, \ t > 0, \ x \in \mathbb{R}, \\ &u(0,x) = \chi_{(-\infty,0]}(x), \quad x \in \mathbb{R}, \end{aligned} \tag{6}$$

which is continuous for $(t, x) \in \mathbb{R}^+ \times \mathbb{R} \setminus \{(0, 0)\}$ *and is such that for each* $t > 0$ *the function* $x \to u(t, x)$ *is absolutely continuous and nonincreasing, for each* $x \in \mathbb{R}$ *the function* $t \mapsto u(t, x)$ *is nondecreasing (so that the function* $t \mapsto \int_0^t k(t - s)(u(s, x) - \chi_{(-\infty,0]}(x))\,\mathrm{d}s$ *is locally absolutely continuous), and equation* (6) *holds a.e. on* $\mathbb{R}^+ \times \mathbb{R}$. *Moreover,*

$$u(t,x) = 0 \text{ when } x \geq \frac{1}{k(t)}\int_0^1 \frac{\sigma'(r)}{r}\,\mathrm{d}r, \quad t > 0, \tag{7}$$

and the function
$$\varphi(\underline{t}) \overset{\text{def}}{=} \inf\{x > 0 \mid u(\underline{t}, x) = 0\}$$
is continuous and strictly increasing.

Let X be a (complex) Banach space and let I be an interval. The Hölder spaces $C^{(\gamma)}(I; X)$, $\gamma \in [0, 1]$, are defined by
$$C^{(\gamma)}(I; X) \overset{\text{def}}{=} \left\{ f : I \to X \;\middle|\; \sup_{\substack{s,t\in I \\ s\neq t}} \frac{\|f(t) - f(s)\|_X}{|t-s|^\gamma} < \infty \right\},$$

with norm
$$\|f\|_{C^{(\gamma)}(I)} \overset{\text{def}}{=} \sup_{t\in I}\|f(t)\|_X + \sup_{\substack{s,t\in I \\ s\neq t}} \frac{\|f(t) - f(s)\|_X}{|t-s|^\gamma}.$$

If $\gamma \in (1,2]$, then $\mathcal{C}^{(\gamma)}(I;X) \stackrel{\text{def}}{=} \{ f \in \mathcal{C}^1(I;X) \mid f' \in \mathcal{C}^{(\gamma-1)}(I;X) \}$ with norm $\|f\|_{\mathcal{C}^{(\gamma)}(I)} \stackrel{\text{def}}{=} \sup_{t\in I}\|f(t)\|_X + \|f'\|_{\mathcal{C}^{(\gamma-1)}(I)}$. Observe that $\mathcal{C}^{(0)} \neq \mathcal{C}$ and $\mathcal{C}^{(1)} \neq \mathcal{C}^1$.

We consider a function of two variables to be a function of the first variable with values in a function space, that is, $f(\underline{t},\underline{x})$ is the function $t \mapsto (x \mapsto f(t,x))$.

THEOREM 3. *Assume that* $\alpha \in (0,1)$, $\tau > 0$, $\xi > 0$, $\mu \in (0,\alpha)$, *and that*

(i) $\sigma \in \mathcal{C}^{(2)}_{\text{loc}}(\mathbb{R};\mathbb{R})$ *and* $\sigma'(\underline{x}) > 0$;

(ii) $u_0 \in \mathcal{C}^{(1+\frac{\mu}{\alpha})}([0,\xi];\mathbb{R})$ *and* $u_0(0) = u_0'(0) = 0$;

(iii) $f \in \mathcal{C}^{(\mu)}([0,\tau],\mathcal{C}([0,\xi];\mathbb{R})) \cap \mathcal{C}^{(\alpha+\delta)}([0,\tau],L^1([0,\xi];\mathbb{R}))$ *where* $\delta > 0$, *and* $f(\underline{t},0) = 0$ *and* $f(0,\underline{x}) \in \mathcal{C}^{(\frac{\mu}{\alpha})}([0,\xi];\mathbb{R})$.

Then there is a unique solution u *of* (1) *on* $(0,\tau]\times(0,\xi]$ *with* $u(\underline{t},0) = 0$ *and* $u(0,\underline{x}) = u_0(\underline{x})$ *such that* $u_x \in \mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi];\mathbb{R}))$.

3. – Proofs

PROOF OF THEOREM 1. Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of functions that satisfy the assumption (i), except that $\lim_{t\downarrow 0} k_n(t) < \infty$, and are such that $\lim_{n\to\infty}\int_0^t k_n(s)\mathrm{d}s = \int_0^t k(s)\,\mathrm{d}s$, $\lim_{n\to\infty} k_n(t) = k(t)$, $\lim_{n\to\infty} k_n'(t) = k'(t)$, and $|k_n'(t)| \le |k'(t)|$ for all $t > 0$. We let ρ_n be the first kind resolvent associated with k_n (cf. (5)); thus ρ_n satisfies

$$\int_{[0,t]} k_n(t-s)\rho_n(\mathrm{d}s) = 1, \quad t \ge 0.$$

The measure ρ_n then has the pointmass $1/k_n(0)$ at 0 and is otherwise induced by an integrable function, that is

$$\rho_n([0,\underline{t}]) = \frac{1}{k_n(0)} + \int_0^{\underline{t}} r_n(s)\,\mathrm{d}s, \quad t \ge 0,$$

where r_n is nonnegative and nonincreasing, because k_n is log-convex, see [9, Lemma 2.1]. When k is replaced by k_n one can use (ii) and a standard fixed-point argument to show that there is a unique solution of (2); we denote this solution by u_n. It is a consequence of [3, Theorem 1] that u_n converges uniformly on compact subsets of $\mathbb{R}^+$ to a continuous function u. However, we need to know more. In particular our next purpose is to show that $w \in \mathcal{C}((0,\infty);X)$ where $w(\underline{t}) \in A(u(\underline{t}))$ is defined by (14).

By [3, formula (24)] we have for $0 \le t < t+h$

$$\begin{aligned}
\|u_n(t+h) - u_n(t)\|_X \le & \int_{[0,t]} \|f(t+h-s) - f(t-s)\|_X \rho_n(\mathrm{d}s) \\
& + \left(\sup_{\tau\in[0,h]} \|f(\tau)\|_X + \|A_{1/k_n(0)}(y)\|_X \right) \\
& \times \int_{[0,t]} \left(\int_{[0,h]} \big(k_n(t-s) - k_n(t-s+h-\sigma)\big)\rho_n(\mathrm{d}\sigma) \right) \rho_n(\mathrm{d}s).
\end{aligned} \tag{8}$$

Now a straightforward calculation using (5) (with k and r replaced by k_n and ρ_n, respectively) shows that

$$\begin{aligned}
& \int_{[0,t]} \left(\int_{[0,h]} \big(k_n(t-s) - k_n(t-s+h-\sigma)\big)\rho_n(\mathrm{d}\sigma) \right) \rho_n(\mathrm{d}s) \\
& \quad = \rho_n((t,t+h]) = \int_t^{t+h} r_n(s)\,\mathrm{d}s.
\end{aligned} \tag{9}$$

By [3, Theorem 1], (8), (9), and by the fact that $\lim_{n\to\infty} \rho_n([0,t]) = \int_0^t r(s)\,\mathrm{d}s$, we get (4).

By a change of variables,

$$\int_0^h \left(\int_{t-s}^t r_n(\sigma)\,\mathrm{d}\sigma \right) |k_n'|(s)\,\mathrm{d}s = \int_{t-h}^t \big(k_n(t-\sigma) - k_n(h)\big) r_n(\sigma)\,\mathrm{d}\sigma, \quad 0 < h \le t.$$

Since the functions r_n are nonincreasing, it follows that

$$\lim_{h\downarrow 0} \int_0^h \left(\int_{t-s}^t r_n(\sigma)\,\mathrm{d}\sigma \right) |k_n'|(s)\,\mathrm{d}s = 0, \tag{10}$$

uniformly for $n \ge 1$ and uniformly for t in a compact subset of $(0,\infty)$. Since $|k_n'(\underline{t})| \le |k'(\underline{t})|$ we deduce from (iv) that

$$\lim_{h\downarrow 0} \int_0^h \omega_{f,\tau}(s)|k_n'(s)|\,\mathrm{d}s = 0 \text{ uniformly in } n. \tag{11}$$

Use (9) in (8), replace $t+h$ and t by t and $t-s$, respectively, multiply by $|k_n'(s)|$, integrate with respect to s over $[0,h]$ and let $h \downarrow 0$. This gives, by (10) and (11),

$$\lim_{h\downarrow 0} \int_0^h \|u_n(t-s) - u_n(t)\|_X |k_n'(s)|\,\mathrm{d}s = 0, \tag{12}$$

uniformly for $n \ge 1$ and uniformly for t in a compact subset of $(0,\infty)$.

Now we can rewrite (2) (with k replaced by k_n) for each $t \geq 0$ as

$$k_n(t)\big(u_n(t) - y\big) + \int_0^t \big(u_n(t-s) - u_n(t)\big)k_n'(s)\,\mathrm{d}s + A(u_n(t)) \ni f(t). \tag{13}$$

By (12), and as u_n converges uniformly on compact subsets of $\mathbb{R}^+$ to the continuous function u, it follows that $k_n(t)\big(u_n(t) - y\big) + \int_0^t \big(u_n(t-s) - u_n(t)\big)k_n'(s)\,\mathrm{d}s$ converges uniformly on each compact subset of $(0,\infty)$ to $k(t)\big(u(t) - y\big) + \int_0^t \big(u(t-s) - u(t)\big)k'(s)\,\mathrm{d}s$ which must then be a continuous function on $(0,\infty)$. Let

$$w(t) \stackrel{\text{def}}{=} f(t) - k(t)\big(u(t) - y\big) - \int_0^t \big(u(t-s) - u(t)\big)k'(s)\,\mathrm{d}s, \tag{14}$$

so that (3) holds with $w \in C((0,\infty), X)$. Since A is m-accretive it is also closed and therefore we have by (13) and by the convergence results that $w(t) \in A(u(t))$ for all $t > 0$.

It remains to show that w is bounded on $(0, T]$ for each $T > 0$. Since $u(0) = y$ we get from (4), when we take $t = 0$, that

$$\|u(h) - y\|_X \leq \left(\sup_{\tau\in[0,h]} \|f(\tau)\|_X + \sup_{\lambda>0} \|A_\lambda(y)\|_X \right) \int_0^h r(s)\,\mathrm{d}s, \qquad h > 0.$$

Because k is nonincreasing there follows by (5) that $k(\underline{t}) \int_0^t r(s)\,\mathrm{d}s \leq 1$ so that

$$\|k(t)(u(t) - y)\|_X \leq \left(\sup_{\tau\in[0,t]} \|f(\tau)\|_X + \sup_{\lambda>0} \|A_\lambda(y)\|_X \right).$$

Similarly, replace t and $t+h$ in (4) by $t-s$ and t, respectively, multiply by $|k'(s)|$ and integrate over $[0,t]$ to obtain

$$\begin{aligned}\left\|\int_0^t \big(u(t-s) - u(t)\big)k'(s)\,\mathrm{d}s\right\| &\leq \int_0^t \omega_{f,\tau}(s) \int_0^{t-s} r(\sigma)\,\mathrm{d}\sigma\, |k'(s)|\,\mathrm{d}s \\ &+ \left(\sup_{\tau\in[0,t]} \|f(\tau)\|_X + \sup_{\lambda>0} \|A_\lambda(y)\|_X \right) \int_0^t \left(\int_{t-s}^t r(\sigma)\,\mathrm{d}\sigma \right) |k'(s)|\,\mathrm{d}s.\end{aligned}$$

Moreover, by (5),

$$\int_0^t \left(\int_{t-s}^t r(\sigma)\,\mathrm{d}\sigma \right) |k'(s)|\,\mathrm{d}s = \int_0^t \big(k(t-\sigma) - k(t)\big) r(\sigma)\,\mathrm{d}\sigma \leq 1,$$

and so by the fact that k and r are nonnegative and by (iii) and (iv) we get the desired conclusion. □

PROOF OF THEOREM 2. Since we will show that the solution takes its values in the interval [0, 1] we may without loss of generality assume that $\sigma \in C^1(\mathbb{R};\mathbb{R})$ is strictly increasing on $\mathbb{R}$.

We easily see that by taking $u(t,x) = 1$ for $x \le 0$ and $t \ge 0$ we have a solution in that region and we are left with the equation

$$(15)\qquad \begin{aligned} &\frac{\partial}{\partial t}\int_0^t k(t-s)u(s,x)\,\mathrm{d}s + \sigma(u)_x(t,x) = 0, \quad t>0, \quad x>0,\\ &u(t,0) = 1, \quad t>0,\\ &u(0,x) = 0, \quad x>0, \end{aligned}$$

In [11, Lemma 3] it is shown that if one lets $\mathcal{D}(A) = \{ u \in L^1(\mathbb{R}^+;\mathbb{R}) \mid \sigma(u) \in AC(\mathbb{R}^+;\mathbb{R}), u(0) = 1, \ \sigma(u)' \in L^1(\mathbb{R}^+;\mathbb{R}) \}$, and defines $A(u) = \sigma(u)'$, $u \in D(A)$, then A is a closed, m-accretive operator in $L^1(\mathbb{R}^+;\mathbb{R})$. By [11, Theorem 5] there exists a solution u of (15), which is nonincreasing in the x-variable and nondecreasing in the t-variable, such that the function $x \mapsto \sigma(u(t,x))$ is absolutely continuous for almost every $t > 0$, and such that the function $t \mapsto \int_0^t k(t-s)u(s,x)\,\mathrm{d}s$ is locally absolutely continuous for every $x \ge 0$, and (15) holds almost everywhere.

By Theorem 1 we know that the function $t \mapsto \sigma(u(t,\underline{x}))_x \in L^1(\mathbb{R}^+;\mathbb{R})$ is continuous on $(0,\infty)$ and that (15) holds in $L^1(\mathbb{R}^+;\mathbb{R})$ for all $t > 0$. Since $\sigma(u(t,0)) = \sigma(1)$ for all $t > 0$ and $\sigma(u(t,x)) = \int_0^x \sigma(u(t,y))_x\,\mathrm{d}y + \sigma(u(t,0))$ it follows that $\sigma(u)$ is continuous in $(0,\infty)\times\mathbb{R}^+$ and since σ is strictly increasing the same result holds for u. By Theorem 1 we also know that $u(t,\underline{x}) \to 0$ in $L^1(\mathbb{R}^+;\mathbb{R})$ as $t \downarrow 0$ and from the monotonicity properties of u we can therefore conclude that u is continuous in $\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0,0)\}$.

Next we derive an inequality that we will use repeatedly below. Assume that $x_0 \overset{\text{def}}{=} \varphi(t_0) < \infty$ and that $x_0 < x_1 \le \varphi(t_1)$ where $t_1 > t_0 \ge 0$. From the proof of Theorem 1 we know that for each $t > 0$ we have

$$\frac{\partial}{\partial t}\int_0^t k(t-s)u(s,x)\,\mathrm{d}s \overset{\text{a.e.}}{=} k(t)u(t,x) + \int_0^t \big(u(t-s,x) - u(t,x)\big)k'(s)\,\mathrm{d}s, \quad x>0,$$

(where the derivative with respect to t is a function with values in $L^1(\mathbb{R}^+;\mathbb{R})$). Since $u(s,x) = 0$ when $s \le t_0$ and $x > x_0$ (by the monotonicity properties of u), we can rewrite this equality for $t > t_0$ as

$$\begin{aligned} &\frac{\partial}{\partial t}\int_0^t k(t-s)u(s,x)\,\mathrm{d}s\\ &\overset{\text{a.e.}}{=} k(t-t_0)u(t,x) + \int_0^{t-t_0} \big(u(t-s,x) - u(t,x)\big)k'(s)\,\mathrm{d}s, \quad x > x_0. \end{aligned}$$

Because k is nonincreasing and u is nondecreasing in its first variable, it follows from the fact that (1) (or equivalently (3)) holds that for each $t > t_0$ we get

$$k(t-t_0)u(t,x) + \sigma'(u(t,x))u_x(t,x) \overset{\text{a.e.}}{\le} 0, \quad x > x_0.$$

In particular, if we choose $t = t_1$, then we know that $u(t,x) > 0$ for $x_0 < x < x_1$ and it follows by the continuity of u that

$$k(t_1 - t_0)(x_1 - x_0) \le \int_{u(t_1,x_1)}^{u(t_1,x_0)} \frac{\sigma'(r)}{r}\,\mathrm{d}r. \tag{16}$$

Since clearly $\varphi(0) = 0$ we may take $t_0 = 0$. Because the function $\frac{\sigma'(r)}{r}$ is integrable on $[0,1]$ and $k(t) > 0$, we see from (16) that we have $\varphi(t_1) < \infty$ and that (7) holds.

The monotonicity properties of u imply that φ is nondecreasing. By the continuity of the function u it follows that φ is continuous from the left, so in order to establish the claim about continuity we suppose to the contrary that there is a point $t_0 \ge 0$ such that $\lim_{t\downarrow t_0} \varphi(t) = \varphi(t_0) + \delta$ for some $\delta > 0$. If we choose $x_0 \stackrel{\text{def}}{=} \varphi(t_0)$ and $x_1 = x_0 + \delta$, then $x_1 \le \varphi(t_1)$ for each $t_1 > t_0$ and we get a contradiction from (16) if we let $t_1 \downarrow t_0$. Thus we have established the continuity of φ.

It remains to prove that φ is strictly increasing. Suppose that this is not the case but that there are two points $t_1 < t_2$ such that $\varphi(t_1) = \varphi(t_2)$. By the continuity of u we know that $t_1 > 0$ and that we can choose t_1 such that $\varphi(t) < \varphi(t_1)$ when $0 \le t < t_1$. We define $x_1 = \varphi(t_1)$.

We shall derive a contradiction and first we show that

$$\lim_{x\uparrow x_1} \sigma\bigl(u(t_1,x)\bigr)(x_1 - x)^{-\frac{\gamma}{\gamma-1}} = \infty. \tag{17}$$

Write $\frac{\sigma'(r)}{r} = \frac{\sigma(r)}{r^2} + \frac{\mathrm{d}}{\mathrm{d}r}\bigl(\frac{\sigma(r)}{r}\bigr)$, use the inequalities in (ii), and the facts that $\gamma > 1$ and $\sigma(u(t_1,x_1)) \ge 0$, to conclude from (16) that when $0 < t_0 < t_1$ and $x_0 = \varphi(t_0)$ we have

$$k(t_1 - t_0)(x_1 - x_0) \le \tfrac{\gamma}{\gamma-1} C^{\frac{2\gamma-1}{\gamma}} \sigma\bigl(u(t_1,x_0)\bigr)^{\frac{\gamma-1}{\gamma}}.$$

Since $\varphi(t) < \varphi(t_1)$ when $0 \le t < t_1$ it follows that $t_0 \uparrow t_1$, and hence $k(t_1 - t_0) \uparrow \infty$, when $x_0 \uparrow x_1$. By the above inequality we therefore obtain (17).

Next, let y be some small positive number and integrate both sides of equation (15) over $(x_1 - y, x_1)$. Then we get, because $u(t,x_1) = 0$ for all $t \in (0,t_2]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t k(t-s)\int_{x_1-y}^{x_1} u(s,v)\,\mathrm{d}v\,\mathrm{d}s = \sigma\bigl(u(t,x_1-y)\bigr), \quad t \in (0,t_2]. \tag{18}$$

We let r be the resolvent of first kind of k, that is, r satisfies (5). Our assumptions on k guarantee that such a resolvent exists and that it is positive and nonincreasing, see [9, Lemma 2.1]. Take the convolution (with respect to t)

of both sides of (18) with the function $p(t) \stackrel{\text{def}}{=} \int_0^t r(t-s)s^\alpha \mathrm{d}s$ where $\alpha > \frac{2-\gamma}{\gamma-1}$. By (5),

$$\int_0^{t_2} (t_2-s)^\alpha \int_{x_1-y}^{x_1} u(s,v)\,\mathrm{d}v\,\mathrm{d}s = \int_0^{t_2} p(t_2-s)\sigma\big(u(s,x_1-y)\big)\,\mathrm{d}s. \tag{19}$$

Using Hölder's inequality twice to estimate the left hand side of (19), we obtain

$$\begin{aligned}
&\int_0^{t_2} (t_2-s)^\alpha \int_{x_1-y}^{x_1} u(s,v)\,\mathrm{d}v\,\mathrm{d}s \\
&\leq \int_0^{t_2} (t_2-s)^\alpha \left(\int_{x_1-y}^{x_1} u(s,v)^\gamma\,\mathrm{d}v\right)^{\frac{1}{\gamma}} \mathrm{d}s\, y^{\frac{\gamma-1}{\gamma}} \\
&\leq \left(\int_0^{t_2} p(t_2-s)\int_{x_1-y}^{x_1} u(s,v)^\gamma\,\mathrm{d}v\,\mathrm{d}s\right)^{\frac{1}{\gamma}} y^{\frac{\gamma-1}{\gamma}} \left(\int_0^{t_2} \frac{s^{\frac{\alpha\gamma}{\gamma-1}}}{p(s)^{\frac{1}{\gamma-1}}}\,\mathrm{d}s\right)^{\frac{\gamma-1}{\gamma}}.
\end{aligned} \tag{20}$$

Since r is nonincreasing and not identically zero there exists a constant c_1 such that $p(t) \geq c_1 t^{\alpha+1}$ when $t \in [0,t_2]$ and therefore it follows from our choice of α that

$$\int_0^{t_2} \frac{s^{\frac{\alpha\gamma}{\gamma-1}}}{p(s)^{\frac{1}{\gamma-1}}}\,\mathrm{d}s < \infty. \tag{21}$$

If we now let

$$w(y) \stackrel{\text{def}}{=} \int_0^{t_2} p(t_2-s)\int_{x_1-y}^{x_1} \sigma\big(u(s,v)\big)\,\mathrm{d}v\,\mathrm{d}s,$$

then the right hand side of (19) equals $w'(y)$, and so by (ii), (20), and by (21) there is a constant c_2 such that

$$w'(y) \leq c_2 y^{\frac{\gamma-1}{\gamma}} w(y)^{\frac{1}{\gamma}}.$$

Since $w(0)=0$ and $w(y)>0$ for $y>0$ we get

$$w(y) \leq \left(c_2\tfrac{\gamma-1}{2\gamma-1}\right)^{\frac{\gamma}{\gamma-1}} y^{\frac{2\gamma-1}{\gamma-1}},$$

and we conclude that there is a constant c_3 such that

$$w'(y) \leq c_3 y^{\frac{\gamma}{\gamma-1}}. \tag{22}$$

But from the definition of w, from the fact that u is nondecreasing in its first variable, and by the monotonicity of σ it follows that

$$w'(y) \geq \int_0^{t_2-t_1} p(s)\,\mathrm{d}s\,\sigma\big(u(t_1,x_1-y)\big).$$

When this inequality is combined with (17) (where we take $x = x_1 - y$) and (22), a contradiction follows. This completes the proof. □

PROOF OF THEOREM 3. The idea of the proof is roughly as follows: First we show that if one has a solution for t on some interval $[0, T]$ (one clearly has such a solution when $T = 0$), then it can be extended to a slightly larger interval. From the proof of this fact one sees that if this extension procedure does not give a solution on the entire interval $[0, \tau]$ then there is some maximal interval $[0, \hat{\tau})$ on which there is a solution and which is such that $\sup_{T<\hat{\tau}} \|\sigma'(u)\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} = \infty$. In order to show that this last fact leads to a contradiction we then apply the same argument as when establishing the existence of a local solution, but we derive estimates for $\|u_x\|_{\mathcal{C}^{(\mu)}([0,T];L^1([0,\mathfrak{X}]))}$ instead of estimating $\|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\mathfrak{X}]))}$. It is of crucial importance for this part of the proof that we derive these estimates for all $\mathfrak{X} \in [0, \xi]$. In this connection, the use of Theorem 1 is essential.

First we show that we may, without loss of generality, assume that there are positive constants c_0, c_1, and c_2 such that

$$(23) \qquad 0 < c_0 \le \sigma'(\underline{r}) \le c_1 < \infty \text{ and } \sup_{r\neq s} \frac{|\sigma'(r) - \sigma'(s)|}{|r - s|} \le c_2 < \infty.$$

By (i) it is sufficient to show that there is an apriori bound for the solution. In analogy with the proof of Theorem 2 we let

$$(24) \qquad \mathcal{D}(A) = \left\{ v \in L^1([0, \xi]; \mathbb{R}) \;\middle|\; \sigma(v) \in AC([0, \xi]; \mathbb{R}),\ v(0) = 0 \right\},$$

and

$$(25) \qquad A(v) = \sigma(v)', \quad v \in \mathcal{D}(A).$$

Then one can easily show (cf. the proof of [11, Lemma 3]) that A is a closed, m-accretive operator in $L^1([0, \xi]; \mathbb{R})$ and that $\|(I + \lambda A)^{-1} v\|_{L^\infty([0,\xi])} \le \|v\|_{L^\infty([0,\xi])}$ for all $v \in L^\infty([0, \xi]; \mathbb{R})$ when $\lambda > 0$. Then it follows from [3, Theorem 4.(a), Prop. 5] that if we find a solution u of (1), then it must satisfy $\sup_{x\in[0,\xi]} |u(t, x)| \le \sup_{x\in[0,\xi]} |u_0(x)| + \int_0^t g_\alpha(t - s) \sup_{x\in[0,\xi]} |f(s, x)|\, ds$ and this is the desired apriori bound. Thus we shall for the rest of the proof assume that (23) holds.

Suppose next that there is a number $T \in [0, \tau)$ such that there is a solution $u \in \mathcal{C}([0, T] \times [0, \xi]; \mathbb{R})$ of (1) on $(0, T] \times (0, \xi]$ such that $u_x \in \mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]; \mathbb{R}))$, $u(0, \underline{x}) = u_0(\underline{x})$ and $u(\underline{t}, 0) = 0$; if $T = 0$ this solution is taken to be $u(0, \underline{x}) = u_0(\underline{x})$ (so that this hypothesis holds at least with $T = 0$).

We intend to show that this solution can be continued to $[0, \hat{T}] \times [0, \xi]$ where $\hat{T} > T$ and $\hat{T} - T$ is sufficiently small. We do this in two steps. In the first step we solve (27) with c given; in the second step we find a fixed-point for the map $c \mapsto \sigma'(v)$ (where v is the solution of (27) obtained in the first step). This continuation procedure is concluded by formula (41).

Thus we first show (using the same argument as in the proof of [2, Theorem 1]) that there are constants δ and M_1 depending on α, μ, τ, ξ, c_0, and c_1 such that if $\hat{T} \in (T, \tau]$ and $c \in \mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]; \mathbb{R}))$ satisfy

$$
\begin{aligned}
&c_0 \le c(\underline{t}, \underline{x}) \le c_1, \\
&c(t, x) = \sigma'(u(t, x)), \quad (t, x) \in [0, T] \times [0, \xi], \\
&(\hat{T} - T)^{\mu} \|c\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))} \le \delta,
\end{aligned} \tag{26}
$$

then there exists a unique solution v of the equation

$$
\begin{aligned}
&\big(D_t^{\alpha}(v - u_0)\big)(t, x) + c(t, x) v_x(t, x) = f(t, x), \quad (t, x) \in (0, \hat{T}] \times (0, \xi], \\
&v(0, x) = u_0(x), \quad x \in [0, \xi], \\
&v(t, 0) = 0, \quad t \in [0, \hat{T}],
\end{aligned} \tag{27}
$$

such that (clearly $v(t, x) = u(t, x)$ for $(t, x) \in [0, T] \times [0, \xi]$)

$$
\begin{aligned}
&\|v_x\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{X}]))} \le M_1 \|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} \\
&+ M_1 \|f\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))} + M_1 \|\sigma'(u_0(\underline{x}))u_0'(\underline{x}) - f(0, \underline{x})\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0,\xi])} \\
&+ M_1 \|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} \|c\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{X}]))}, \quad \mathfrak{X} \in [0, \xi].
\end{aligned} \tag{28}
$$

Observe that the first and last term of this inequality are written in terms of the space variable $\mathfrak{X} \in [0, \xi]$. The proof of the existence of v satisfying (27) will be completed by the paragraph containing formula (39).

To solve (27), we begin by studying the following equation:

$$
\big(D_t^{\alpha}(v - u_0)\big)(t, x) + b(x) v_x(t, x) = g(t, x), \quad t \in (0, \tau], \quad x \in (0, \xi], \tag{29}
$$

with boundary condition $v(\underline{t}, 0) = 0$ and initial condition $v(0, \underline{x}) = u_0(\underline{x})$ under the following assumption on the function b:

$$
b \in \mathcal{C}(\mathbb{R}^+; \mathbb{R}) \text{ and } 0 < c_0 \le b(\underline{x}) \le c_1 < \infty. \tag{30}
$$

We denote by B_b the linear operator in $\mathcal{C}_{0\mapsto 0}([0, \xi]; \mathbb{C}) \overset{\text{def}}{=} \{ q \in \mathcal{C}([0, \xi]; \mathbb{C}) \mid q(0) = 0 \}$ with domain

$$
\mathcal{D}(B_b) = \{ q \in \mathcal{C}^1([0, \xi]; \mathbb{C}) \mid q(0) = q'(0) = 0 \}
$$

and defined by

$$
(B_b q)(x) = b(x) q'(x), \quad x \in [0, \xi], \quad q \in \mathcal{D}(B_b).
$$

We denote by B the corresponding operator with $b(\underline{x}) = 1$ and ξ replaced by $\xi_0 = \xi / c_0$.

Thus (29) can be written as

$$D_t^{\alpha}(v - u_0) + B_b v = g. \tag{31}$$

Next, perform a change of variable $y = \int_0^x \frac{1}{b(s)}\,\mathrm{d}s$, so that equation (31) is replaced by

$$D_t^{\alpha}(v^b - u_0^b) + Bv^b = g^b, \tag{32}$$

where

$$\begin{aligned} g^b(\underline{t}, \underline{y}) &= g(\underline{t}, \rho(\underline{y})), \\ u_0^b(\underline{y}) &= u_0(\rho(\underline{y})), \end{aligned} \qquad y \in [0, \xi_b]$$

and

$$\begin{aligned} g^b(\underline{t}, \underline{y}) &= g(\underline{t}, \xi), \\ u_0^b(\underline{y}) &= u_0(\xi) + b(\xi)u_0'(\xi)(y - \xi_b), \end{aligned} \qquad y \in (\xi_b, \xi_0].$$

Here $\xi_b = \int_0^{\xi} \frac{1}{b(s)}\,\mathrm{d}s$ and ρ is the inverse of the function $x \mapsto \int_0^x \frac{1}{b(s)}\,\mathrm{d}s$. By [1, Theorem 6.(a)] equation (32) has a unique solution v^b which satisfies the bound

$$\begin{aligned} &\|Bv^b(\underline{t}) - g^b(0)\|_{C^{(\mu)}([0,\tau];C_{0\mapsto 0}([0,\xi_0]))} \\ &\quad \le M_2 \left(\|Bu_0^b - g^b(0)\|_{C^{(\frac{\mu}{\alpha})}([0,\xi_0])} + \|g^b(\underline{t}) - g^b(0)\|_{C^{(\mu)}([0,\tau];C_{0\mapsto 0}([0,\xi_0]))} \right), \end{aligned}$$

where M_2 depends on α, μ, τ and ξ_0. Now we change variables back again, that is, we define

$$v(\underline{t}, x) = v^b\left(\underline{t}, \int_0^x \frac{1}{b(s)}\,\mathrm{d}s\right), \qquad \text{for } x \in [0, \xi]. \tag{33}$$

We can therefore conclude that there is a unique solution v of (29) such that

$$\begin{aligned} \|v_x\|_{C^{(\mu)}([0,\tau];C([0,\xi];\mathbb{C}))} \le M_3 \Big(&\|b(\underline{x})u_0'(\underline{x}) - g(0, \underline{x})\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])} \\ &+ \|g\|_{C^{(\mu)}([0,\tau];C([0,\xi]))} \Big), \end{aligned} \tag{34}$$

where (with some crude estimates) $M_3 = \frac{1}{c_0}(M_2 \max\{2, c_1^{\frac{\mu}{\alpha}}\} + 1)$.

Our next claim is that (34) holds with τ replaced by an arbitrary $\hat{T} \in (0, \tau]$, ξ replaced by an arbitrary $\mathfrak{X} \in [0, \xi]$, and with M_3 unchanged. To see this, choose $\hat{T} \in (0, \tau]$, $\mathfrak{X} \in [0, \xi]$, and redefine b, u_0 and g as $b(x) = b(\mathfrak{X})$, $u_0(x) = u_0(\mathfrak{X}) + u_0'(\mathfrak{X})(x - \mathfrak{X})$, and $g(t, x) = g(t, \mathfrak{X})$ for $x \in (\mathfrak{X}, \xi]$ and $t \in [0, \hat{T}]$ and $g(t, x) = g(\hat{T}, x)$ for $x \in [0, \xi]$ and $t \in (\hat{T}, \tau]$. Then we can use the uniqueness of the solution and the definition of the Hölder norms to conclude that we in fact have our claim, i.e.,

$$\begin{aligned}\|v_x\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}];\mathbb{C}))} \leq M_3\Big(&\|b(\underline{x})u_0'(\underline{x}) - g(0,\underline{x})\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0,\mathfrak{x}])} \\ &+ \|g\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}\Big), \quad \hat{T}\in(0,\tau], \quad \mathfrak{x}\in[0,\xi].\end{aligned} \tag{35}$$

Choose

$$\delta = \frac{1}{4M_3}, \tag{36}$$

and $\hat{T} \in (T, \tau]$ such that the last part of (26) holds. Having a solution of (29) satisfying (35) and having chosen $\hat{T}$, we proceed to find a solution of (27).

Let P denote the set

$$P \stackrel{\text{def}}{=} \{\, p \in \mathcal{C}^{(\mu)}([0,\hat{T}]; \mathcal{C}([0,\xi];\mathbb{C})) \mid p(t,\underline{x}) = u_x(t,\underline{x}), \quad 0 \leq t \leq T \,\}.$$

For each $p \in P$ we have to find a solution w of the equation

$$D_t^\alpha(w-u_0)(\underline{t},\underline{x}) + c(T,\underline{x})w_x(\underline{t},\underline{x}) = f(\underline{t},\underline{x}) + \big(c(T,\underline{x}) - c(\underline{t},\underline{x})\big)p(\underline{t},\underline{x}), \tag{37}$$

on $[0,\hat{T}] \times [0,\xi]$ with boundary condition $w(\underline{t},0) = 0$ (and initial condition $w(0,\underline{x}) = u_0(\underline{x})$) and c as in (26). Note that this equation is of type (29). Observe also that the right-hand side of (37) evaluated at $t = 0$ is

$$f(0,x) + \big(c(T,x) - c(0,x)\big)u_0'(x),$$

and therefore the term $b(x)u_0'(\underline{x}) - g(0,\underline{x})$ appearing in (35) is now, when $b(\underline{x}) = c(T,\underline{x})$, equal to $c(0,x)u_0'(x) - f(0,x)$. Thus we conclude from (ii) and from the results above on (29) that we can find a solution w of (37) such that $w_x \in \mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi];\mathbb{C}))$. Moreover, the uniqueness guarantees that we have $w_x \in P$.

Let us denote the mapping $p \to w_x$ by $w_x = G(p)$. Using the linearity of equation (37), and (35) with $b(\underline{x}) = c(T,\underline{x})$ once more, we conclude that

$$\begin{aligned}&\|(G(p_1) - G(p_2))(\underline{t},\underline{x})\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))} \\ &\leq M_3\|(c(T,\underline{x}) - c(\underline{t},\underline{x}))(p_1-p_2)(\underline{t},\underline{x})\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}, \quad \mathfrak{x}\in[0,\xi].\end{aligned} \tag{38}$$

Let $p_\Delta = p_1 - p_2$ and $c_\Delta(\underline{t},\underline{x}) = c(T,\underline{x}) - c(\underline{t},\underline{x})$. Since p_1 and $p_2 \in P$ it follows that $p_\Delta(t,\underline{x}) = 0$ for $t \in [0,T]$ and therefore we can, when analyzing the term $(c(T,\underline{x}) - c(\underline{t},\underline{x}))(p_1-p_2)(\underline{t},\underline{x})$, assume that $c(t,\underline{x}) = c(T,\underline{x})$ for $t \in [0,T]$. Thus we conclude from the last part of (26) and from (36) that

$$\sup_{\substack{t\in[0,\hat{T}]\\ x\in[0,\mathfrak{x}]}} |c_\Delta(t,x)p_\Delta(t,x)| \leq \frac{1}{4M_3} \sup_{\substack{t\in[0,\hat{T}]\\ x\in[0,\mathfrak{x}]}} |p_\Delta(t,x)|, \quad \mathfrak{x}\in[0,\xi].$$

Furthermore, if we write $c_\Delta(t,x)p_\Delta(t,x)-c_\Delta(s,x)p_\Delta(s,x)=c_\Delta(t,x)(p_\Delta(t,x)-p_\Delta(s,x))+(c_\Delta(t,x)-c_\Delta(s,x))(p_\Delta(s,x)-p_\Delta(T,x))$ using the fact that $p_\Delta(T,\underline{x})=0$, and use (26) once again, then we conclude that

$$\begin{aligned}\|(c(T,x)-c(t,x))(p_1-p_2)(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}\\ \leq \frac{1}{2M_3}\|(p_1-p_2)(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))},\quad \mathfrak{X}\in[0,\xi].\end{aligned}$$

Hence we have, using (38), for every $\mathfrak{X}\in[0,\xi]$,

$$\|(G(p_1)-G(p_2))(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}\leq\frac{1}{2}\|(p_1-p_2)(t,x)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))},\tag{39}$$

and we see that the mapping G is a contraction and that there is a unique fixed-point, i.e., a function v such that $v_x=G(v_x)$. Thus we get a solution of (27) on the interval $[0,\hat{T}]$.

If we take $p_0\in P$ to be such that $p_0(t,\underline{x})=u_x(T,\underline{x})$ for $t\in[T,\hat{T}]$ then $\|p_0\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}=\|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\mathfrak{x}]))}$. Using inequality (35) to estimate $\|G(p_0)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}$ and then (39) to estimate $\|G(v_x)-G(p_0)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}$, we conclude that (28) holds with

$$M_1=\max\{1+4M_3\|c\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))},2M_3\}.$$

With c fixed, the solution v of (27) can of course be continued to $[0,\tau]\times[0,\xi]$. However, our goal is to solve (1), i.e., (27) with $c(\underline{t},\underline{x})=\sigma'(v(\underline{t},\underline{x}))$. For this purpose we apply another fixed-point argument on $[0,\hat{T}]$ with $\hat{T}-T$ sufficiently small (and recall that we have a solution of (1) on $[0,T]$).

We let M_4 be the constant

$$\begin{aligned}M_4&\stackrel{\text{def}}{=}c_1+c_2\max\{1,\xi\}M_1\|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))}\\&\quad+\xi c_2M_1\|f\|_{\mathcal{C}^{(\mu)}([0,\tau];\mathcal{C}([0,\xi]))}+\xi c_2M_1\|\sigma'(u_0(\underline{x}))u_0'(\underline{x})-f(0,\underline{x})\|_{\mathcal{C}^{(\frac{\mu}{\alpha})}([0,\xi])},\end{aligned}$$

and choose $\hat{T}\in(T,\tau]$ such that

$$(\hat{T}-T)^\mu\leq\frac{\delta}{M_4e^{M_4\xi}}.\tag{40}$$

For our fixed-point argument we let

$$\begin{aligned}V=\Big\{c\in\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi];\mathbb{R}))\,\Big|\,&c(t,x)=\sigma'(u(t,x)),\ t\in[0,T],\ x\in[0,\xi],\\&c_0\leq c(t,x)\leq c_1,\quad t\in[T,\hat{T}],\quad x\in[0,\xi],\\&\|c\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))}\leq M_4e^{M_4\mathfrak{X}},\quad \mathfrak{X}\in[0,\xi]\Big\}.\end{aligned}$$

Note that V is convex and not empty. Now we define the function $F(c)$ for $c \in V$ by

$$F(c)(\underline{t}, \underline{x}) \stackrel{\text{def}}{=} \sigma'\big(v(\underline{t}, \underline{x})\big),$$

where v is the solution of (27). (By the definition of V and by (40) condition (26) is satisfied and hence such a (unique) solution exists.)

By the uniqueness we know that we have $F(c)(t, x) = \sigma'(u(t, x))$ for $t \in [0, T]$ and $x \in [0, \xi]$ and by (23) we also have $c_0 \le F(c)(\underline{t}, \underline{x}) \le c_1$. Finally we note that since $v(\underline{t}, 0) = 0$ we have

$$F(c)(\underline{t}, \underline{x}) = \sigma'\left(\int_0^{\underline{x}} v_x(\underline{t}, r)\, \mathrm{d}r\right),$$

and it follows that

$$\begin{aligned} \|F(c)\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\mathfrak{x}]))} &\le c_1 + c_2 \int_0^{\mathfrak{x}} \|v_x\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,r]))}\, \mathrm{d}r \\ &\le M_4 + M_4 \int_0^{\mathfrak{x}} \|c\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,r]))}\, \mathrm{d}r \le M_4 \mathrm{e}^{M_4 \mathfrak{x}}, \quad \mathfrak{x} \in [0, \xi], \end{aligned}$$

where the second inequality is a consequence of (28) and the definition of M_4, and where the last inequality follows because $c \in V$. This shows that $F(c) \in V$.

Finally we observe that by [2, Theorem 1 and (4)] the set of solutions of (27) one gets when $c \in V$ is contained in a bounded subset of $\mathcal{C}^{((\mu+\alpha)/2)}([0, \hat{T}]; \mathcal{C}^{(1/2)}([0, \xi]; \mathbb{R}))$ (for example) and therefore this set of solutions, and hence also $F(c) = \sigma'(v)$ for $c \in V$ is contained in a compact subset of $\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathcal{C}([0, \xi]; \mathbb{R}))$. (Note in particular that since our boundary condition is now $v(\underline{t}, 0) = 0$ we do not need the assumption that the function $x \mapsto c(\underline{t}, x)$ is a continuous function with values in $\mathcal{C}^{(\mu)}([0, \hat{T}]; \mathbb{R})$. Therefore the constant M appearing in [2, formula (4)] depends on $\|c\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];\mathcal{C}([0,\xi]))}$, c_0 and c_1, but not otherwise on c.) Thus we know by the Schauder fixed-point theorem that there is a function $c \in V$ such that $F(c) = c$ and the corresponding solution of (27) is then the unique solution of (1) on $[0, \hat{T}] \times [0, \xi]$.

If the claim of the theorem does not hold there is, by the continuation argument above, a maximal number $\hat{\tau} \in (0, \tau]$ such that there is a solution of (1) on $(0, \hat{\tau}) \times (0, \xi]$, and such that $u_x \in \mathcal{C}^{(\mu)}([0, T]; \mathcal{C}([0, \xi]; \mathbb{R}))$ for all $T \in (0, \hat{\tau})$. If $\sup_{T<\hat{\tau}} \|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} < \infty$, then this solution can be continued by the argument used above, and we get a contradiction. Furthermore, it also follows from the argument in the above that if $\sup_{T<\hat{\tau}} \|\sigma'(u)\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} < \infty$, then $\sup_{T<\hat{\tau}} \|u_x\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} < \infty$. Thus we assume that

$$\sup_{T<\hat{\tau}} \|\sigma'(u)\|_{\mathcal{C}^{(\mu)}([0,T];\mathcal{C}([0,\xi]))} = \infty, \tag{41}$$

and we will derive a contradiction from this.

We want to apply Theorem 1 and therefore we define the operator A by (24) and (25). It is straightforward to check that by (ii) $y = u_0$ belongs to $\mathcal{D}(A) \subset \hat{\mathcal{D}}(A)$ and that by (iii) the function $t \mapsto f(t, \underline{x}) \in L^1([0,\xi];\mathbb{R})$ satisfies the assumption (iv) of Theorem 1. Thus Theorem 1 may be applied to (1) and so we obtain the existence of a unique (strong) solution $u \in \mathcal{C}([0,\tau]; L^1([0,\xi];\mathbb{R}))$. By uniqueness, this solution coincides with the one constructed above on $[0,\hat{\tau})\times[0,\xi]$.

It follows from Theorem 1, together with the results on the local solution that we already have established, that the function

$$t \mapsto \sigma(u)_x(t,\underline{x}) \in L^1([0,\xi];\mathbb{R}) \text{ is uniformly continuous on } [0,\hat{\tau}).$$

An immediate consequence of this result, of (23), and of the fact that $u(\underline{t}, 0) = 0$, is that

$$u \text{ is uniformly continuous on } [0,\hat{\tau}) \times [0,\xi], \tag{42}$$

and hence we also conclude that

$$t \mapsto u_x(t,\underline{x}) \in L^1([0,\xi];\mathbb{R}) \text{ is uniformly continuous on } [0,\hat{\tau}). \tag{43}$$

In the above, the results of [1] were applied to the operator $u \mapsto u_x$ in the space of continuous functions. Now we shall do the same thing but with integrable functions instead. We let $\xi_0 = \xi/c_0$ and denote by B the linear operator in $L^1([0,\xi_0];\mathbb{C})$ with domain

$$\mathcal{D}(B) = \{\, v \in \mathcal{AC}([0,\xi_0];\mathbb{C}) \mid v(0) = 0 \,\}$$

and

$$(Bv)(x) = v'(x), \quad x \in [0,\xi_0], \quad v \in \mathcal{D}(B).$$

As the norm in $\mathcal{D}(B)$ we can take $\|w\|_{\mathcal{D}(B)} = \|w'\|_{L^1([0,\xi_0])}$.

If $b \in \mathcal{C}(\mathbb{R}^+;\mathbb{R})$ satisfies $c_0 \le b(\underline{x}) \le c_1$, then we can use an argument similar to the one employed when deriving (35) to conclude that it follows from [1, Theorem 6] that there is a constant M_5 (which depends on α, μ, τ, ξ, c_0 and c_1) and a unique solution v of (29) such that

$$\|v_x\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];L^1([0,\mathcal{X}]))} \le M_5\Big(\big\|\chi_{[0,\mathcal{X}]}(\rho(\underline{y}))h_0(\rho(\underline{y}))\big\|_{\mathcal{D}_B(\frac{\mu}{\alpha},\infty)} + \|g\|_{\mathcal{C}^{(\mu)}([0,\hat{T}];L^1([0,\mathcal{X}]))}\Big), \tag{44}$$

for all $\hat{T} \in [0,\tau]$ and $\mathcal{X} \in [0,\xi]$ where $h_0(\underline{x}) = b(\underline{x})u_0'(\underline{x}) - g(0,\underline{x})$, ρ is the inverse of the function $x \mapsto y = \int_0^x \frac{1}{b(s)}\,\mathrm{d}s$, and where $\mathcal{D}_B(\frac{\mu}{\alpha}) =$

$(L^1([0,\xi_0];\mathbb{C}),\mathcal{D}(B))_{\frac{\mu}{\alpha},\infty}$. In this argument one extends the functions as constants in the t-direction and as 0 in the x-direction (but u_0 is extended as a constant) and changes the x-variable to the new variable $y=\int_0^x \frac{1}{b(s)}\,\mathrm{d}s$.

Having (44), our next goal is to estimate the first term on the right hand side. We claim that if h is an arbitrary function in $C^{(\frac{\mu}{\alpha})}([0,\xi];\mathbb{R})$, which is extended as zero to (ξ,∞), then there is a constant $M_6 \overset{\text{def}}{=} 2c_1^{\frac{\mu}{\alpha}}\xi_0+4$, such that

$$\text{(45)}\qquad \big\|\chi_{[0,\mathfrak{X}]}(\rho(\underline{y}))h(\rho(\underline{y}))\big\|_{\mathcal{D}_B(\frac{\mu}{\alpha},\infty)} \le M_6\|h\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])},\qquad \mathfrak{X}\in[0,\xi].$$

To see this we argue as follows: Let $w(y)=\chi_{[0,\mathfrak{X}]}(\rho(\underline{y}))h(\rho(\underline{y}))$ and extend this function as 0 on $(-\infty,0)$ and let $t\in(0,1)$ be arbitrary. Now write $w=w_1+w_2$ where $w_1(\underline{y})=\int_0^\infty \frac{1}{t}\mathrm{e}^{-\frac{r}{t}}(w(\underline{y})-w(\underline{y}-r))\,\mathrm{d}r$ and where $w_2(\underline{y})=\int_0^{\underline{y}} \frac{1}{t}\mathrm{e}^{-\frac{r}{t}}w(\underline{y}-r)\,\mathrm{d}r$. We note that $w(y)=0$ when $y<0$ and when $y>\mathfrak{X}_\rho \overset{\text{def}}{=} \int_0^{\mathfrak{X}}\frac{1}{b(s)}\,\mathrm{d}s$. Because ρ is Lipschitz continuous with constant c_1 we know that $|w(y)-w(y-r)|\le c_1^{\frac{\mu}{\alpha}}r^{\frac{\mu}{\alpha}}\|h\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])}$ when $0\le r\le y\le \mathfrak{X}_\rho$. Furthermore, $|w(y)-w(y-r)|\le \|h\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])}$ when $0\le y<r$ or $\mathfrak{X}_\rho<y\le\mathfrak{X}_\rho+r$ (because then either $w(y)$ or $w(y-r)$ vanishes) and $|w(y)-w(y-r)|=0$ otherwise. It follows from these inequalities that $\|w_1\|_{L^1([0,\xi_0])}\le (t^{\frac{\mu}{\alpha}}c_1^{\frac{\mu}{\alpha}}\xi_0\Gamma(1+\frac{\mu}{\alpha})+2t)\|h\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])}$. Furthermore, $\|w_2\|_{\mathcal{D}(B)}=\|w_2'\|_{L^1([0,\xi_0])}=\frac{1}{t}\|w_1\|_{L^1([0,\xi_0])}$ because $w_2'(\underline{y})=\frac{1}{t}w_1(\underline{y})$. Thus we see that $t^{-\frac{\mu}{\alpha}}\|w_1\|_{L^1([0,\xi_0])}+t^{1-\frac{\mu}{\alpha}}\|w_2\|_{\mathcal{D}(B)}\le M_6\|h\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])}$ and by the definition of the interpolation space $\mathcal{D}_B(\frac{\mu}{\alpha},\infty)=(L^1([0,\xi_0];\mathbb{C}),\mathcal{D}(B))_{\frac{\mu}{\alpha},\infty}$ (see e.g., [13, Definition 1.2.2]), this is exactly what we need in order to get (45).

Using (45) we see that (44) implies that the function v that solves (29) satisfies

$$\text{(46)}\quad \|v_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))}\le M_5\left(M_6\|h\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])}+\|g\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))}\right),$$

for all $\hat{T}\in[0,\tau]$ and all $\mathfrak{X}\in[0,\xi]$.

Let $c(\underline{t},\underline{x})\overset{\text{def}}{=}\sigma'(u(\underline{t},\underline{x}))$. By (42) we can choose $T\in(0,\hat{\tau})$ such that

$$\text{(47)}\qquad \sup_{\substack{t,s\in[0,\hat{\tau})\\ |t-s|\le\hat{\tau}-T}}\ \sup_{x\in[0,\xi]}|c(t,x)-c(s,x)|\le\frac{1}{2M_5}.$$

Let $\hat{T}$ be some arbitrary number in $(T,\hat{\tau})$.

Now we rewrite (1) in the form

$$\big(D_t^\alpha(u-u_0)\big)(t,x)+c(T,x)u_x(t,x)=f(t,x)+\big(c(T,x)-c(t,x)\big)u_x(t,x)$$
$$\overset{\text{def}}{=} g(t,x),\quad t\in[0,\hat{T}],\quad x\in[0,\xi].$$

Note that this equation is of type (29); hence the estimate (46) may be applied to u with $b(\underline{x}) = c(T, \underline{x})$ (and b extended as a constant for $x > \xi$). Also observe that $c(T, \underline{x})u_0'(\underline{x}) - g(0, \underline{x}) = c(0, \underline{x})u_0'(\underline{x}) - f(0, \underline{x}) = \sigma'(u_0(\underline{x}))u_0'(\underline{x}) - f(0, \underline{x})$. Thus we see by (46) that

$$\begin{aligned}(48)\qquad &\|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))} \le M_7 \\ &\quad + M_5 \big\| \chi_{[T,\hat{T}]}(\underline{t})\big(c(T, \underline{x}) - c(\underline{t}, \underline{x})\big)u_x(\underline{t}, \underline{x})\big\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))},\end{aligned}$$

where M_7 is some constant such that

$$\begin{aligned}&M_5\|f\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))} + M_5M_6\|\sigma'(u_0(\underline{x}))u_0'(\underline{x}) - f(0, \underline{x})\|_{C^{(\frac{\mu}{\alpha})}([0,\xi])} \\ &\quad + M_5\big\|\chi_{[0,T)}(\underline{t})\big(c(T, \underline{x}) - c(\underline{t}, \underline{x})\big)u_x(\underline{t}, \underline{x})\big\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))} \le M_7,\end{aligned}$$

for all $\mathfrak{X} \in [0, \xi]$ and for all $\hat{T} \in (T, \hat{\tau})$. Now a simple calculation shows that

$$\begin{aligned}&\big\|\chi_{[T,\hat{T}]}(\underline{t})\big(c(T, \underline{x}) - c(\underline{t}, \underline{x})\big)u_x(\underline{t}, \underline{x})\big\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))} \\ &\quad \le \sup_{t\in[T,\hat{T}]}\ \sup_{x\in[0,\xi]} |c(T, x) - c(t, x)|\,\|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))} \\ &\qquad + \sup_{\substack{t,s\in[0,\hat{T}]\\ t\neq s}} \int_0^{\mathfrak{X}} \frac{|c(t, x) - c(s, x)|}{|t - s|^{\mu}} |u_x(s, x)|\,\mathrm{d}x.\end{aligned}$$

Invoking this inequality together with (47) in (48) we get

$$\begin{aligned}(49)\qquad &\|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))} \le 2M_7 \\ &\quad + 2M_5 \sup_{\substack{t,s\in[0,\hat{T}]\\ t\neq s}} \int_0^{\mathfrak{X}} \frac{|c(t, x) - c(s, x)|}{|t - s|^{\mu}} |u_x(s, x)|\,\mathrm{d}x \\ &\le 2M_7 + 2M_5 \sup_{s\in[0,\hat{T}]} \int_0^{\mathfrak{X}} \|c\|_{C^{(\mu)}([0,\hat{T}];C([0,x]))} |u_x(s, x)|\mathrm{d}x.\end{aligned}$$

Since $c(\underline{t}, \underline{x}) = \sigma'\left(\int_0^{\underline{x}} u_x(\underline{t}, r)\,\mathrm{d}r\right)$ it follows from (23) that

$$(50)\qquad \|c\|_{C^{(\mu)}([0,\hat{T}];C([0,x]))} \le c_1 + c_2\|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,x]))}, \quad x \in [0, \xi].$$

From (49) and (50) it follows that for each $\mathfrak{X} \in [0, \xi]$ there exists a number $s(\mathfrak{X}) \in [0, \hat{\tau})$ such that

$$\begin{aligned}(51)\qquad &\|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{X}]))} \le 1 + 2M_7 + 2c_1M_5 \sup_{s\in[0,\hat{\tau})} \|u_x(s, \underline{x})\|_{L^1([0,\xi])} \\ &\quad + 2M_5c_2 \int_0^{\mathfrak{X}} \|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,x]))} |u_x(s(\mathfrak{X})), x)|\mathrm{d}x.\end{aligned}$$

By (43) there is a finite set of points $\{t_j\}_{j=1}^n \subset [0,\hat{\tau})$ such that if $s \in [0,\hat{\tau})$ then there is an index $j(s) \in \{1,\dots,n\}$ such that

$$\|u_x(s,\underline{x}) - u_x(t_{j(s)},\underline{x})\|_{L^1([0,\xi]} \leq \frac{1}{4M_5c_2}. \tag{52}$$

Let $M_8 = \max\{4M_5c_2, 2+4M_7+4c_1M_5 \sup_{s\in[0,\hat{\tau})}\|u_x(s,\underline{x})\|_{L^1([0,\xi])}\}$, (by (43) $M_8 < \infty$). Then we conclude from (51) and (52) that we in fact have

$$\begin{aligned}\|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{x}]))} &\leq M_8 + M_8\int_0^{\mathfrak{x}} \|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,x]))}|u_x(t_{j(s(\mathfrak{x}))},x)|\,\mathrm{d}x\\ &\leq M_8 + M_8\int_0^{\mathfrak{x}} \|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,x]))}p(x)\,\mathrm{d}x,\end{aligned}$$

where $p(\underline{x}) = \max_{1\leq j\leq n}|u_x(t_j,\underline{x})|$ so that we have $p \in L^1([0,\xi];\mathbb{R})$. But now it follows from Gronwall's inequality that

$$\|u_x\|_{C^{(\mu)}([0,\hat{T}];L^1([0,\mathfrak{x}]))} \leq M_8\mathrm{e}^{M_8\int_0^{\mathfrak{x}} p(s)\,\mathrm{d}s} \leq M_8\mathrm{e}^{M_8\|p\|_{L^1([0,\xi])}}.$$

This inequality combined with (50) contradicts (41) and the proof is complete. □

REFERENCES

[1] PH. CLÉMENT – G. GRIPENBERG – S-O. LONDEN, *Schauder estimates for equations with fractional derivatives*, Trans. Amer. Math. Soc., to appear.

[2] PH. CLÉMENT – G. GRIPENBERG – S-O. LONDEN, *Hölder regularity for a linear fractional evolution equation*, In: "Topics in Nonlinear Analysis", J. Escher – G. Simonett (eds.), vol. 35, Birkhäuser, Basel, 1999.

[3] B. COCKBURN – G. GRIPENBERG – S-O. LONDEN, *On convergence to entropy solutions of a single conservation law*, J. Differential Equations **128** (1996), 206-251.

[4] M. G. CRANDALL, *The semigroup approach to first order quasilinear equations in several space variables*, Israel J. Math. **12** (1972), 108-132.

[5] G. DA PRATO – M. IANNELLI, *Linear integro-differential equations in Banach spaces*, Rend. Sem. Mat. Univ. Padova **62** (1980), 207-219.

[6] G. DA PRATO – M. IANNELLI – E. SINESTRARI, *Temporal regularity for a class of integrodifferential equations in Banach spaces*, Boll. Un. Mat. Ital. **2** (1983), 171-185.

[7] G. DA PRATO – M. IANNELLI – E. SINESTRARI, *Regularity of solutions of a class of linear integrodifferential equations in Banach spaces*, J. Integral Equations **8** (1985), 27-40.

[8] E. GODLEWSKI – P.-A. RAVIART, "Hyperbolic systems of conservation laws", Ellipses, Paris, 1991.

[9] G. GRIPENBERG, *An abstract nonlinear Volterra equation*, Israel J. Math. **34** (1979), 198-212.

[10] G. GRIPENBERG, *Volterra integro-differential equations with accretive nonlinearity*, J. Differential Equations **60** (1985), 57-79.

[11] G. GRIPENBERG – S-O. LONDEN, *Fractional derivatives and smoothing in nonlinear conservation laws*, Differential Integral Equations **8** (1995), 1961-1976.

[12] J. PRÜSS, "Evolutionary integral equations and applications", Birkhäuser, Basel, 1995.

[13] A. LUNARDI, "Analytic semigroups and optimal regularity in parabolic problems", Birkhäuser, Basel, 1995.

[14] D. SERRE, "Systems of conservation laws 1", Cambridge University Press, Cambridge, 1999.

[15] A. ZYGMUND, "Trigonometric series II", Cambridge University Press, Cambridge, 1959.

Institute of Mathematics
Helsinki University of Technology
P.O. Box 1100
FIN-02015 HUT, Finland
gustaf.gripenberg@hut.fi, www.math.hut.fi/~ggripenb

Faculty of Technical Mathematics, and Informatics
Delft University of Technology
P.O. Box 5031
2600 GA Delft, The Netherlands
clement@twi.tudelft.nl

Institute of Mathematics
Helsinki University of Technology, P.O. Box 1100
FIN-02015 HUT, Finland
stig-olof.londen@hut.fi