ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

MARCOS MONTENEGRO

The construction of principal spectral curves for Lane-Emden systems and applications

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 29, nº 1 (2000), p. 193-229

<http://www.numdam.org/item?id=ASNSP_2000_4_29_1_193_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXIX (2000), pp. 193-229

The Construction of Principal Spectral Curves for Lane-Emden Systems and Applications

MARCOS MONTENEGRO

Abstract. In this article we develop a detailed study about the existence of principal eigenvalues for the Lane-Emden system

$$\begin{cases} -\mathcal{L}_1 u = \lambda \rho(x)|v|^{\alpha-1}v & \text{in } \Omega \\ -\mathcal{L}_2 v = \mu \tau(x)|u|^{\beta-1}u & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

when Ω is a smooth bounded domain, α and β are positive numbers with $\alpha\beta = 1$, ρ and τ are non-negative functions on Ω and $\mathcal{L}_i$ is a general elliptic differential operator of second order. We show that the set of the principal eigenvalues of the system above determines a curve in the plane which satisfies several properties such as simplicity, isolation, continuity, asymptotic behaviour. We also furnish a min-max type characterization for this curve. Motivated by these discussions, we investigate the existence and uniqueness of positive solutions for some semilinear elliptic systems in bounded domains and in the whole space.

Mathematics Subject Classification (1991): 35P30 (primary), 35J45, 35J55 (secondary.

Contents

Pervenuto alla Redazione il 24 settembre 1998 e in forma definitiva il 24 agosto 1999.

1. – Introduction

Consider the following linear Dirichlet problem

$$\text{(1.1)} \qquad \begin{cases} -\mathcal{L}u = \lambda\rho(x)u & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, ρ is a not identically null real-valued function on Ω verifying certain integrability condition and $\mathcal{L}$ is a uniformly elliptic differential operator of second order with continuous coefficients and satisfying the strong maximum principle.

A real number λ is said to be a principal eigenvalue of the problem (1.1) if it admits a positive solution in Ω. During the past years, many mathematicians have studied the existence and uniqueness of principal eigenvalues. For instance, when $\mathcal{L}$ is a self-adjoint operator, the existence of a principal eigenvalue for problem (1.1) was first established by Manes and Michelet [28] by using the variational characterization of the eigenvalues. Later, Hess and Kato [21] extended the existence of principal eigenvalues for general uniformly elliptic operators of second order. The main argument used in [21] is based on Krein-Rutman's theorem for strictly positive compact linear operators in ordered Banach spaces, see [2] and [25]. In particular, it is well-known that if ρ is a non-negative function, then the problem (1.1) possesses a unique principal eigenvalue λ_1 which is positive, simple, right-side isolated and if λ is another eigenvalue, then $|\lambda| > \lambda_1$. Recently, Berestycki, Nirenberg and Varadhan [4] have shown the existence of principal eigenvalues for general uniformly elliptic operators of second order in general domains. Their procedure relies on an approximation of Ω by smooth subdomains, on the standard existence of a principal eigenvalue in each subdomain and on a Harnack inequality due to Krilov-Safonov in order to obtain the convergence of the sequence generated by the principal eigenvalues. More recently, López-Gómez [26] has provided some necessary and sufficient conditions on ρ for the existence of principal eigenvalues of the problem (1.1) when $\mathcal{L}$ does not satisfy the strong maximum principle. In this case, the existence is essentially associated to the measure of the set $\{x \in \Omega : \rho(x) \neq 0\}$. As other references related to this subject, we can cite the works [5], [9], [15], [16], [19] and [32]. Also, it has been discussed the existence of principal eigenvalues for some nonlinear elliptic operators of second order, see the papers [1], [3], [37] and references therein.

Other linear eigenvalue problem that has been extensively investigated, it is the following

$$\text{(1.2)} \qquad \begin{cases} -\vec{\mathcal{L}}U = \lambda A(x)U & \text{in} \quad \Omega \\ \quad U = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

where $A(x)$ is a real-valued matrix of order m whose entries satisfy certain integrability conditions, $U = (u_1, \ldots, u_m)$, $\vec{\mathcal{L}} = \operatorname{diag}(\mathcal{L}_1, \ldots, \mathcal{L}_m)$ and each $\mathcal{L}_i$ is a operator of the same kind that $\mathcal{L}$.

A real number λ is said to be a principal eigenvalue of the system (1.2) if it admits a positive solution, that is, each coordinate is a positive function in Ω. In order to generalize the results corresponding to the scalar case, some authors have studied the existence and uniqueness of principal eigenvalues for system (1.2) when the matrix $A(x)$ is cooperative for each x in Ω. In the paper [20], it has been given sufficient conditions for the existence of principal eigenvalues, see also [10], [27], [33] and [35] as other references. Recently, Birindelli, Mitidieri and Sweers [6] have obtained the existence of principal eigenvalues in general domains. The arguments used in these works are based on some versions of Krein-Rutman's theorem since system (1.2) is a linear problem, see [14].

Part of the present paper concerns the existence of principal eigenvalues for the following class of semilinear elliptic systems

$$\begin{cases} -\mathcal{L}_1 u = \lambda\rho(x)|v|^{\alpha-1}v & \text{in } \Omega \\ -\mathcal{L}_2 v = \mu\tau(x)|u|^{\beta-1}u & \\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where α and β are positive numbers, ρ and τ are non-negative not identically null functions on Ω and each operator $\mathcal{L}_i$ is of the same type that $\mathcal{L}$.

The system (1.3) is refered as Lane-Emden system because it is a natural extension of the famous Lane-Emden equation

$$\{ -\Delta u = |u|^{p-1}u \quad \text{in} \quad \Omega$$

arising in astronomy. During the last decade, the system (1.3) has been extensively studied in the case that the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are equal and self-adjoint. For instance, we can list the papers [11], [12], [13], [17], [22], [29], [30], [31] and [36], where several results on existence and non-existence of positive solutions have been derived when $\alpha\beta \neq 1$. As a consequence of these recent developments, it has been established the concept of sublinearity for problem (1.3) in the self-adjoint case. It is said to be sublinear if $\alpha\beta \leq 1$. When $\alpha\beta < 1$, Clément and van der Vorst [12], Felmer and Martínez [17] and Montenegro [29], [31] have obtained results on existence of non-trivial solutions for the system above. Due to the analogy between some results related to Lane-Emden equation and those associated to the system (1.3) in case $\alpha\beta \neq 1$, it is natural to introduce the concept of principal eigenvalue for Lane-Emden systems when $\alpha\beta = 1$. More precisely, we say that a couple (λ, μ) in $\mathbb{R}^2$ is an eigenvalue of the problem (1.3), if it admits a non-trivial solution and is a principal eigenvalue if it possesses a positive solution. If (λ, μ) is an eigenvalue of the system (1.3), then $(-\lambda, -\mu)$ is too. Note that system (1.3) extends the eigenvalue problem (1.1) and since (1.3) is a nonlinear problem, Krein-Rutman's theorem can not be applied, in contrast to the problems (1.1) and (1.2). Moreover, the existence of principal eigenvalues for Lane-Emden systems has not been investigated even when the operator $\mathcal{L}_i$ is self-adjoint. In Section 2 we study the existence of principal eigenvalues when ρ and τ

are non-negative functions belonging to $L^p(\Omega)$ with $p \geq N$ and each $\mathcal{L}_i$ is a general uniformly elliptic operator of second order. We show that the set of the principal eigenvalues of the problem (1.3) is non-empty and determines a continuous curve Λ_1 in $\mathbb{R}^2_+$ which satisfies some asymptotic properties and divides $\mathbb{R}^2_+$ into two unbounded open connected components $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathbb{R}_+$ is the open interval $(0, +\infty)$. Similarly to the scalar case, we prove that Λ_1 is simple and upper isolated in a certain sense. We also furnish characterizations of $\mathcal{C}_1$ and $\mathcal{C}_2$ via Fredholm alternative, we conclude that neither $\mathcal{C}_1$ nor $-\mathcal{C}_1$ contains eigenvalues of the system (1.3) and finally we present a min-max type characterization for the curve Λ_1. Our strategy to find the set of the principal eigenvalues is advantageous because it is constructive. Our arguments are based on Krasnoselskii's method, Leray-Schauder degree theory and sub-supersolution technique. As a consequence of this discussion, we extend the concept of sublinearity to general Lane-Emden systems. In Section 3, by using degree theory and a truncation procedure, we derive some results on existence of non-trivial non-negative solutions for systems with sublinear character of the form

$$(1.4) \qquad \begin{cases} -\mathcal{L}_1 u = f(x,u,v) & \text{in } \Omega \\ -\mathcal{L}_2 v = g(x,u,v) & \\ u = 0 = v & \text{on } \partial\Omega . \end{cases}$$

Indeed, we analyze systems subject to non-ressonance conditions associated to spectral curve introduced in the second section. The results obtained here in particular generalize and improve some ones presented in the paper [29]. Moreover, considering more general operators, we solve a conjecture stated in [29] for $m = 2$. More precisely, consider the problem

$$(1.5) \qquad \begin{cases} -\mathcal{L}_1 u = f(v) & \text{in } \Omega \\ -\mathcal{L}_2 v = g(u) & \\ u = 0 = v & \text{on } \partial\Omega , \end{cases}$$

where $f, g : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are continuous functions satisfying the conditions

$$\lim_{s\to 0^+} \frac{f(s)}{s^{\alpha_1}} = a_1 , \quad \lim_{s\to 0^+} \frac{g(s)}{s^{\beta_1}} = b_1 ,$$

$$\lim_{s\to +\infty} \frac{f(s)}{s^{\alpha_2}} = a_2 , \quad \lim_{s\to +\infty} \frac{g(s)}{s^{\beta_2}} = b_2 ,$$

where α_i and β_i are positive numbers such that $\alpha_i \beta_i = 1$ and a_i and b_i are non-negative numbers.

Theorem 3.2 of this work proves the following conjecture:

CONJECTURE 1.1. For each $i = 1, 2$, there is a curve $C(\alpha_i, \beta_i)$ that divides $\mathbb{R}^2_+$ into two unbounded open connected components $\mathcal{C}_{1i} = \mathcal{C}_1(\alpha_i, \beta_i)$ and $\mathcal{C}_{2i} = \mathcal{C}_2(\alpha_i, \beta_i)$ such that if (a_1, b_1) belongs to $\mathcal{C}_{11}$ and (a_2, b_2) belongs to $\mathcal{C}_{22}$, then the system (1.5) admits a positive solution.

In Section 4, based on Krasnoselskii's method, we consider some nonlinearities for which the system (1.4) admits, at most, one positive solution. Thus, we derive the existence and uniqueness of positive solutions for some classes of problems investigated in the third section. As an illustration, if α and β are positive numbers with $\alpha\beta < 1$ and ρ and τ are non-negative not identically null functions belonging to $L^p(\Omega)$ with $p \geq N$, we conclude that the Lane-Emden system

$$\begin{cases} -\mathcal{L}_1 u = \rho(x) v^\alpha & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = \tau(x) u^\beta & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

admits a unique positive solution, extending the paper [17] which concerns the self-adjoint case with τ being a constant function.

Finally, we turn our discussion to the system

$$\begin{cases} -\mathcal{L}_1 u = \rho(x) v^\alpha \\ -\mathcal{L}_2 v = \tau(x) u^\beta \end{cases} \text{in} \quad \mathbb{R}^N, \tag{1.6}$$

where α and β are positive numbers and ρ and τ are non-negative not identically null functions belonging to $L^p_{\text{loc}}(\mathbb{R}^N)$ with $p \geq N$.

Recently, Brézis and Kamin [8] have furnished necessary and sufficient conditions for the existence of bounded positive solutions of the equation

$$-\Delta u = \rho(x) u^\alpha \ \text{ in } \ \mathbb{R}^N \tag{1.7}$$

when $0 < \alpha < 1$. In truth, they have shown the existence of a bounded positive solution for problem (1.7) if and only if ρ verifies the property (H), that is, the non-homogeneous equation $-\Delta U = \rho(x)$ in $\mathbb{R}^N$ has a bounded positive solution. This result has motivated the search of similar conditions for system (1.6). In Section 5, by using some results of the previous sections, we obtain necessary and sufficient conditions for the existence of a bounded positive solution of the problem (1.6) when $\alpha\beta < 1$, which extend those related to the scalar case. In fact, as a particular case, if each operator $\mathcal{L}_i$ is the Laplacian and the functions ρ and τ coincide, we establish the existence of a bounded positive solution for system (1.6) if and only if the function ρ satisfies the property (H). The reasonings used in [8] to establish the necessity of (H) do not apply to systems, so we use a version of a reduction procedure which has been obtained in [30] in a more general form.

In order to become the paper more organized, we fix some notations. Throughout the paper $N \geq 2$ is a natural number, $p \geq N$ is a real number and the symbol $\varepsilon \sim 0$ means that ε is a positive number small enough.

The systems to be investigated in Sections 2, 3 and 4 can be put in the form (1.4), where Ω is a bounded domain of class $C^{1,1}$ in $\mathbb{R}^N$, $f, g : \Omega \times \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are Carathéodory functions verifying the growth conditions

$$f(x,t,s) \leq k(x)(t + s^{\alpha'} + 1), \quad g(x,t,s) \leq k(x)(t^{\beta'} + s + 1)$$

for x almost everywhere in Ω and $t, s \geq 0$, α' and β' are positive numbers such that $\alpha'\beta' = 1$, k is a non-negative function in $L^p(\Omega)$ and each $\mathcal{L}_i$ represents a uniformly elliptic differential operator on Ω of the form

$$\mathcal{L}_i \equiv \sum_{k,j=1}^{N} a_{kj}^i(x)\partial_{kj} + \sum_{k=1}^{N} b_k^i(x)\partial_k + c^i(x)\,, \tag{1.8}$$

with continuous coefficients in $\overline{\Omega}$ and satisfying the strong maximum principle, that is, if u is a function in $W^{2,p}(\Omega)$ verifying $\mathcal{L}_i u \leq 0$ almost everywhere in Ω and $u \geq 0$ on $\partial\Omega$, then either $u \equiv 0$ in Ω or $u > 0$ in Ω and in this case, we have $\frac{\partial u(x)}{\partial \nu} < 0$ for each x on $\partial\Omega$ such that $u(x) = 0$, where ν is the outward unit normal to Ω in x. Since $\mathcal{L}_i$ satisfies the strong maximum principle, for each f in $L^p(\Omega)$ and u_0 in $W^{2,p}(\Omega)$, the non-homogeneous Dirichlet problem

$$\begin{cases} \mathcal{L}_i u = f(x) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

possesses a unique solution u in $W^{2,p}(\Omega)$ with $u - u_0$ in $W_0^{1,p}(\Omega)$. Furthermore, there is a positive constant c such that $\|u\|_{W^{2,p}} \leq c(\|\mathcal{L}_i u\|_{L^p} + \|u_0\|_{W^{2,p}})$ for every u and u_0 in $W^{2,p}(\Omega)$ with $u - u_0$ in $W_0^{1,p}(\Omega)$, see chapter 9 in the book [18]. There are some well-known conditions under which $\mathcal{L}_i$ verifies the strong maximum principle, see [7] and [34]. For example, two of these conditions are given by

(A) $c^i \leq 0$ in Ω,

(B) $\mathcal{L}_i$ admits a positive supersolution in $\overline{\Omega}$.

To simplify the notation, we write (SMP) in place of strong maximum principle.

By a solution (supersolution, subsolution) of the system (1.4) we mean a couple (u, v) in $(W^{2,p}(\Omega))^2$ satisfying

$$\begin{cases} -\mathcal{L}_1 u = (\geq,\ \leq)\ f(x, u, v) \\ -\mathcal{L}_2 v = (\geq,\ \leq)\ g(x, u, v) \end{cases}$$

almost everywhere in Ω and $u = (\geq, \leq)\ 0 = (\leq, \geq)\ v$ on $\partial\Omega$. We say that (u, v) is non-negative (positive) in Ω if each coordinate is. Let X be the real vector space $\{(u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : \ u = 0 = v \text{ on } \partial\Omega\}$ endowed with the norm $\|(u, v)\|_X = \|u\|_{C(\overline{\Omega})} + \|v\|_{C(\overline{\Omega})}$. Denote by B_R the ball $\{(u, v) \in X : \|(u, v)\|_X < R\}$ and by C the positive cone of X given by $\{(u, v) \in X : \ (u, v)$ is non-negative in $\Omega\}$.

In Section 5 we discuss problems in $\mathbb{R}^N$. There $\mathcal{L}_i$ is a differential operator of the type (1.8) defined in $\mathbb{R}^N$ with continuous coefficients and uniformly elliptic in each compact subset of $\mathbb{R}^N$ and by a solution of the system

$$\begin{cases} -\mathcal{L}_1 u = f(x, u, v) \\ -\mathcal{L}_2 v = g(x, u, v) \end{cases} \quad \text{in } \mathbb{R}^N$$

we mean a couple (u, v) in $(W^{2,p}_{\text{loc}}(\mathbb{R}^N))^2$ verifying the system above almost everywhere in $\mathbb{R}^N$. From now on, we will omit the expression “almost everywhere”.

2. – Construction and properties

Let us begin introducing some definitions related to the Lane-Emden system

$$\text{(2.1)} \qquad \begin{cases} -\mathcal{L}_1 u = \lambda\rho(x)|v|^{\alpha-1}v & \text{in } \Omega \\ -\mathcal{L}_2 v = \mu\tau(x)|u|^{\beta-1}u & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

During this section we assume that α and β are positive numbers with $\alpha\beta = 1$ and ρ and τ are non-negative not identically null functions on Ω verifying certain integrability conditions to be mentioned next. The definition below is a natural extension of that related to the scalar case:

DEFINITION 2.1. A couple (λ, μ) in $\mathbb{R}^2$ is said to be an eigenvalue of the problem (2.1) if it admits a non-trivial solution (φ, ψ) which we call an eigenfunction associated to the eigenvalue (λ, μ). Furthermore, if the system (2.1) possesses a positive solution, we say that (λ, μ) is a principal eigenvalue. We denote the set of the principal eigenvalues of the system (2.1) by $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$.

One of the main aims of paper, it is to characterize completely the set $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ via a constructive approach. Our procedure is performed first for functions ρ and τ belonging to $L^\infty(\Omega)$ and next, by applying an approximation argument, we extend it for functions ρ and τ belonging to $L^p(\Omega)$. Our reasoning is based on the study of the possible non-negative solutions of the system

$$\text{(2.2)} \qquad \begin{cases} -\mathcal{L}_1 u = \lambda\rho(x)v^{\alpha} + f_0(x) & \text{in } \Omega \\ -\mathcal{L}_2 v = \mu\tau(x)u^{\beta} + g_0(x) & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

For that matter, consider the sets

$$G^{\rho,\tau}_{\alpha,\beta}(\Omega) := \{(\lambda,\mu) \in \mathbb{R}^2_+ \ : \text{for any } f_0, g_0 \text{ in } L^p(\Omega) \text{ with } f_0, g_0 \geq 0 \text{ in } \Omega, \text{ system (2.2) admits a non-negative solution}\},$$

$$O^{\rho,\tau}_{\alpha,\beta}(\Omega) := \{(\lambda,\mu) \in \mathbb{R}^2_+ \ : \text{for any } f_0, g_0 \text{ in } C(\overline{\Omega}) \text{ with } f_0, g_0 \geq 0 \text{ in } \Omega, \text{ system (2.2) admits a non-negative solution}\},$$

$$\tilde{O}^{\rho,\tau}_{\alpha,\beta}(\Omega) := \{(\lambda,\mu) \in \mathbb{R}^2_+ \ : \text{for some } f_0, g_0 \text{ in } C(\overline{\Omega}) \text{ with } f_0, g_0 > 0 \text{ in } \overline{\Omega}, \text{ system (2.2) admits a positive solution}\}.$$

By definition, we have $G^{\rho,\tau}_{\alpha,\beta}(\Omega) \subset O^{\rho,\tau}_{\alpha,\beta}(\Omega) \subset \tilde{O}^{\rho,\tau}_{\alpha,\beta}(\Omega)$. The next two lemmas are fundamental in the investigation of the sets introduced above. The first one establishes the existence of a solution for problem (1.4) provided a subsolution and a supersolution exist:

LEMMA 2.1. *Assume that there are a non-negative subsolution* $(\underline{u}, \underline{v})$ *and a non-negative supersolution* $(\overline{u}, \overline{v})$ *of the system* (1.4) *satisfying* $\underline{u} \leq \overline{u}$ *and* $\underline{v} \leq \overline{v}$ *in* Ω. *Set* $M = \max\{\|(\underline{u}, \underline{v})\|_X, \|(\overline{u}, \overline{v})\|_X\}$. *Let* f *and* g *be non-negative Carathéodory functions such that* $f(x, t_1, s_1) \leq f(x, t_2, s_2)$ *and* $g(x, t_1, s_1) \leq g(x, t_2, s_2)$ *for every* x *in* Ω, $0 \leq t_1 \leq t_2 \leq M$ *and* $0 \leq s_1 \leq s_2 \leq M$ *and* $\sup\{f(\cdot, t, s) : 0 \leq t, s \leq M\}$ *and* $\sup\{g(\cdot, t, s) : 0 \leq t, s \leq M\}$ *are in* $L^p(\Omega)$. *Then the problem* (1.4) *admits a solution* (u, v) *verifying* $\underline{u} \leq u \leq \overline{u}$ *and* $\underline{v} \leq v \leq \overline{v}$ *in* Ω.

PROOF. By L^p theory and the (SMP), for each (u, v) in C with $\|(u, v)\|_X \leq M$, the problem

$$\begin{cases} -\mathcal{L}_1 z = f(x, u, v) & \text{in } \Omega \\ -\mathcal{L}_2 w = g(x, u, v) & \\ z = 0 = w & \text{on } \partial\Omega \end{cases}$$

admits a unique solution (z, w) in $(W^{2,p}(\Omega))^2 \cap C$. Denote (z, w) by $T(u, v)$. Let (u_1, v_1) and (u_2, v_2) be functions in C such that $u_1 \leq u_2$ and $v_1 \leq v_2$ in Ω and $\|(u_1, v_1)\|_X, \|(u_2, v_2)\|_X \leq M$. We affirm that the inequality $T(u_1, v_1) \leq T(u_2, v_2)$ occurs. Indeed, since

$$\begin{cases} -\mathcal{L}_1[(T(u_1, v_1))_1 - (T(u_2, v_2))_1] = f(x, u_1, v_1) - f(x, u_2, v_2) \leq 0 & \text{in } \Omega \\ -\mathcal{L}_2[(T(u_1, v_1))_2 - (T(u_2, v_2))_2] = g(x, u_1, v_1) - g(x, u_2, v_2) \leq 0 & \\ T(u_1, v_1) - T(u_2, v_2) = 0 & \text{on } \partial\Omega, \end{cases}$$

from the (SMP), we justify the claim. Now consider the sequence $\{(u_n, v_n)\}_0^\infty \subset C$ defined inductively by $(u_0, v_0) = (\underline{u}, \underline{v})$ and $(u_{n+1}, v_{n+1}) = T(u_n, v_n)$ for $n \geq 0$. Let us show that this sequence is well-defined, $u_n \leq u_{n+1} \leq \overline{u}$ and $v_n \leq v_{n+1} \leq \overline{v}$ in Ω for $n \geq 0$. In fact, since

$$\begin{cases} -\mathcal{L}_1(u_0 - u_1) \leq f(x, \underline{u}, \underline{v}) - f(x, \underline{u}, \underline{v}) = 0 & \text{in } \Omega \\ -\mathcal{L}_2(v_0 - v_1) \leq g(x, \underline{u}, \underline{v}) - g(x, \underline{u}, \underline{v}) = 0 & \\ u_0 - u_1 = 0 = v_0 - v_1 & \text{on } \partial\Omega, \end{cases}$$

applying the (SMP), we obtain $u_0 \leq u_1$ and $v_0 \leq v_1$ in Ω. Proceeding in a similar manner, we get $T(\overline{u}, \overline{v}) \leq (\overline{u}, \overline{v})$. Using the monotonicity of the mapping T and finite induction, we prove the assertion. Therefore, there are measurable functions $u, v : \Omega \to [0, +\infty)$ defined by $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ for x in Ω. Applying the dominated convergence theorem, we conclude that the functions u and v are in $L^p(\Omega)$, $\|u_n - u\|_{L^p} \to 0$, $\|v_n - v\|_{L^p} \to 0$, $\|f(., u_n, v_n) - f(., u, v)\|_{L^p} \to 0$ and $\|g(., u_n, v_n) - g(., u, v)\|_{L^p} \to 0$. By L^p estimates, we derive the bounds

$$\|u_{m+1} - u_{n+1}\|_{W^{2,p}} \leq c\|f(., u_m, v_m) - f(., u_n, v_n)\|_{L^p},$$
$$\|v_{m+1} - v_{n+1}\|_{W^{2,p}} \leq c\|g(., u_m, v_m) - g(., u_n, v_n)\|_{L^p}.$$

So, it follows that (u, v) is in $(W^{2,p}(\Omega))^2 \cap C$, $\|u_n - u\|_{W^{2,p}} \to 0$ and $\|v_n - v\|_{W^{2,p}} \to 0$. Therefore, (u, v) is a solution of the system (1.4) satisfying $\underline{u} \leq u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$ in Ω. □

The second lemma furnishes a basic result on existence of non-negative solutions for system (1.4):

LEMMA 2.2. *There is a positive constant ε_0 such that for each non-negative function k in $L^p(\Omega)$ and every non-negative Carathéodory functions f and g satisfying the upper bounds $f(x,t,s) \leq \varepsilon_0 \rho(x) s^\alpha + k(x)$ and $g(x,t,s) \leq \varepsilon_0 \tau(x) t^\beta + k(x)$ for every x in Ω and $t, s \geq 0$, the system* (1.4) *admits a non-negative solution.*

PROOF. Consider the mapping $H_{f,g} : [0,1] \times C \to C$ defined by $H_{f,g}(s,u,v) = (z,w)$, where (z,w) verifies

$$(2.3) \qquad \begin{cases} -\mathcal{L}_1 z = sf(x,u,v) & \text{in } \Omega \\ -\mathcal{L}_2 w = sg(x,u,v) & \\ z = 0 = w & \text{on } \partial\Omega \end{cases}$$

If for each positive number K, $\sup\{f(\cdot,t,s) : 0 \leq t, s \leq K\}$ and $\sup\{g(\cdot,t,s) : 0 \leq t, s \leq K\}$ are in $L^p(\Omega)$, from L^p theory and the (SMP), we conclude that the mapping $H_{f,g}$ is well-defined, continuous and compact. We affirm that there is a positive constant ε_0 such that for each non-negative function k in $L^p(\Omega)$ and non-negative functions f and g fulfilling the growth conditions stated in the lemma, there is a constant $M = M(k) > 0$ not depending on f and g such that $\|(u,v)\|_X < M$ for every (u,v) in C with $H_{f,g}(s,u,v) = (u,v)$ for some $0 \leq s \leq 1$. Indeed, if $H_{f,g}(s,u,v) = (u,v)$ for some $0 \leq s \leq 1$, using L^p estimates in (2.3) and Sobolev immersion, we obtain the bounds $\|u\|_{C(\overline{\Omega})} \leq \varepsilon_0 c \|\rho\|_{L^p} \|v\|^\alpha_{C(\overline{\Omega})} + c\|k\|_{L^p}$ and $\|v\|_{C(\overline{\Omega})} \leq \varepsilon_0 c \|\tau\|_{L^p} \|u\|^\beta_{C(\overline{\Omega})} + c\|k\|_{L^p}$ for a positive constant c depending only on N, p, Ω, $\mathcal{L}_1$ and $\mathcal{L}_2$. Thus, combining these two relations and taking $\varepsilon_0 > 0$ small enough, we conclude that $\|(u,v)\|_X < M$ for some positive constant M depending on k. By the homotopic invariance property of the degree theory in cones, it follows that $\deg(Id - H_{f,g}(1,.), B_M \cap C, 0) = \deg(Id - H_{f,g}(0,.), B_M \cap C, 0) = 1$ and as a consequence, we derive the existence of a non-negative solution for system (1.4). □

It follows immediately from Lemma 2.2 that $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ is non-empty because it contains the set $\{(\lambda,\mu) : 0 < \lambda, \mu < \varepsilon_0\}$. As mentioned previously, first let us analyze the sets $O^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and $\tilde{O}^{\rho,\tau}_{\alpha,\beta}(\Omega)$ for functions ρ and τ belonging to $L^\infty(\Omega)$. Based on Lemma 2.1, we derive the equality of the sets $O^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and $\tilde{O}^{\rho,\tau}_{\alpha,\beta}(\Omega)$:

LEMMA 2.3. *The sets $O^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and $\tilde{O}^{\rho,\tau}_{\alpha,\beta}(\Omega)$ are equal.*

PROOF. If (λ,μ) belongs to $\tilde{O}^{\rho,\tau}_{\alpha,\beta}(\Omega)$, by definition, there are functions f_1 and g_1 in $C(\overline{\Omega})$ with $f_1, g_1 > 0$ in $\overline{\Omega}$ such that the system

$$\begin{cases} -\mathcal{L}_1 u = \lambda\rho(x)v^\alpha + f_1(x) & \text{in } \Omega \\ -\mathcal{L}_2 v = \mu\tau(x)u^\beta + g_1(x) & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

possesses a positive solution (u_1, v_1). Let f_0 and g_0 be functions belonging to $C(\overline{\Omega})$ verifying $0 \leq f_0 \leq f_1$ and $0 \leq g_0 \leq g_1$ in Ω. In this case, $(0, 0)$ is a subsolution and (u_1, v_1) is a positive supersolution of the problem (2.2). Thus, from Lemma 2.1, we conclude that the system (2.2) admits a non-negative solution. If f_0 and g_0 are arbitrary non-negative functions in $C(\overline{\Omega})$, we consider the problem

$$(2.4) \qquad \begin{cases} -\mathcal{L}_1 u_\varepsilon = \lambda\rho(x)v_\varepsilon^\alpha \; + \; \varepsilon^\alpha f_0(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v_\varepsilon = \mu\tau(x)u_\varepsilon^\beta \; + \; \varepsilon g_0(x) & \\ u_\varepsilon = 0 = v_\varepsilon & \text{on} \quad \partial\Omega \end{cases}$$

Taking $\varepsilon \sim 0$ such that $\varepsilon^\alpha f_0 \leq f_1$ and $\varepsilon g_0 \leq g_1$ in Ω, from the first part, we obtain a non-negative solution $(u_\varepsilon, v_\varepsilon)$ for problem (2.4). Defining $u = \frac{1}{\varepsilon^\alpha} u_\varepsilon$ and $v = \frac{1}{\varepsilon} v_\varepsilon$, we find a non-negative solution for system (2.2). Therefore, (λ, μ) belongs to $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$. □

As a consequence of Lemmas 2.1 and 2.3, we obtain the following topological result:

LEMMA 2.4. *The set $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$ is open.*

PROOF. Take a couple (λ_0, μ_0) in $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$. By definition, given functions f_0 and g_0 in $C(\overline{\Omega})$ with $f_0, g_0 > 0$ in $\overline{\Omega}$, the system

$$\begin{cases} -\mathcal{L}_1 u = \lambda_0\rho(x)v^\alpha \; + \; f_0(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = \mu_0\tau(x)u^\beta \; + \; g_0(x) & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

admits a positive solution (u_0, v_0). Choose a positive number η satisfying $f_0 - \eta$ and $g_0 - \eta > 0$ in $\overline{\Omega}$. It is easy to see that $(0, 0)$ is a subsolution and there is a positive number ε such that (u_0, v_0) is a positive supersolution of the problem

$$(2.5) \qquad \begin{cases} -\mathcal{L}_1 u = \lambda\rho(x)v^\alpha \; + \; f_0(x) \; - \; \eta & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = \mu\tau(x)u^\beta \; + \; g_0(x) \; - \; \eta & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

for every (λ, μ) in $\mathbb{R}_+^2$ with $|\lambda - \lambda_0| < \varepsilon$ and $|\mu - \mu_0| < \varepsilon$. Applying Lemma 2.1, we find a positive solution for system (2.5). Hence, from Lemma 2.3, it follows that the set $\{(\lambda, \mu) \in \mathbb{R}_+^2 : \; |\lambda - \lambda_0| < \varepsilon \; \text{ and } \; |\mu - \mu_0| < \varepsilon\}$ is contained in $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$. □

The next result is crucial for the construction of the set $\Lambda_{\alpha,\beta}^{\rho,\tau}(\Omega)$ and the proof is based on a procedure due to Krasnoselskii, see [24]:

PROPOSITION 2.1. *For each $a > 0$, there is a positive number $t_0 = t_0(a)$ such that the couple (t, at) does not belong to $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$ for every $t \geq t_0$.*

PROOF. Suppose on the contrary that there is a sequence $\{t_n\}_0^\infty$ such that $t_n \to +\infty$ and (t_n, at_n) is in $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$ for some $a > 0$. Take functions f_0 and g_0 in $C(\overline{\Omega})$ and positive in Ω. Then, the system

$$\text{(2.6)} \qquad \begin{cases} -\mathcal{L}_1 u = t_n \rho(x) v^\alpha \ + \ f_0(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = at_n \tau(x) u^\beta \ + \ g_0(x) & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

admits a positive solution (u_n, v_n). Since $\alpha\beta = 1$, we have $\alpha \leq 1$ or $\beta \leq 1$. Without loss of generality, assume that $\alpha \leq 1$. Denote by φ a positive eigenfunction corresponding to the unique positive principal eigenvalue λ_1 of the scalar problem

$$\text{(2.7)} \qquad \begin{cases} -\mathcal{L}_1 u = \lambda \rho(x) u & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Let ψ be the positive solution of the problem

$$\text{(2.8)} \qquad \begin{cases} -\mathcal{L}_2 v = a\tau(x)\varphi^\beta & \text{in} \quad \Omega \\ \quad v = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Define the set $S_n = \{s > 0 \ : \ u_n > s\varphi \text{ and } v_n > s^\beta \psi \text{ in } \Omega\}$. From the (SMP), it follows that S_n is non-empty. Let $s_n^* = \sup\ S_n$. Clearly, we have $u_n \geq s_n^* \varphi$ and $v_n \geq s_n^{*\beta}\psi$ in Ω. Since $\varphi^{\frac{1}{\alpha}}$ belongs to $C^1(\overline{\Omega})$, $\frac{\partial}{\partial\nu}\psi < 0$ on $\partial\Omega$ and $t_n \to +\infty$, we conclude that $t_{n_0}^{\frac{1}{\alpha}}\psi > \lambda_1^{\frac{1}{\alpha}} \varphi^{\frac{1}{\alpha}}$ in Ω and $t_{n_0} > 1$ for some $n_0 \geq 0$. Therefore, from (2.6), (2.7) and (2.8), we get

$$\begin{aligned} -\mathcal{L}_1(u_{n_0} - s_{n_0}^*\varphi) &= t_{n_0}\rho(x)v_{n_0}^\alpha + f_0(x) - s_{n_0}^*\lambda_1\rho(x)\varphi \\ &\geq s_{n_0}^*\rho(x)(t_{n_0}\psi^\alpha - \lambda_1\varphi) \ + \ f_0(x) > 0\,, \\ -\mathcal{L}_2(v_{n_0} - s_{n_0}^{*\beta}\psi) &= at_{n_0}\tau(x)u_{n_0}^\beta + g_0(x) - s_{n_0}^{*\beta}a\tau(x)\varphi^\beta \\ &\geq s_{n_0}^{*\beta}a(t_{n_0} - 1)\tau(x)\varphi^\beta + g_0(x) > 0\,. \end{aligned}$$

From the (SMP), we derive $u_{n_0} > s_{n_0}^*\varphi$, $v_{n_0} > s_{n_0}^{*\beta}\psi$ in Ω and $\frac{\partial}{\partial\nu}(u_{n_0} - s_{n_0}^*\varphi) < 0$, $\frac{\partial}{\partial\nu}(v_{n_0} - s_{n_0}^{*\beta}\psi) < 0$ on $\partial\Omega$. Therefore, we conclude that $u_{n_0} > (s_{n_0}^* + \varepsilon)\varphi$ and $v_{n_0} > (s_{n_0}^* + \varepsilon)^\beta\psi$ in Ω for $\varepsilon \sim 0$, contradicting the definition of $s_{n_0}^*$. □

According to Lemma 2.2 and Proposition 2.1, for each $a > 0$, we can define the positive number

$$\text{(2.9)} \qquad t_a^* = \sup\{t > 0 : (t, at) \text{ belongs to } O_{\alpha,\beta}^{\rho,\tau}(\Omega)\}\,.$$

Motivated by the definition of t_a^*, we prove below that the set $\Lambda_{\alpha,\beta}^{\rho,\tau}(\Omega)$ is non-empty:

THEOREM 2.1. *For each $a > 0$, the couple (t_a^*, at_a^*) belongs to $\Lambda_{\alpha,\beta}^{\rho,\tau}(\Omega)$.*

PROOF. Fix $a > 0$. By definition, we can take a sequence $\{t_n\}_0^\infty$ of positive numbers converging to t_a^* such that (t_n, at_n) belongs to $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$. Choosing functions f_0 and g_0 in $C(\overline{\Omega})$ with $f_0, g_0 > 0$ in $\overline{\Omega}$, the system (2.6) admits a positive solution (u_n, v_n). We claim that $\|(u_n, v_n)\|_X \to +\infty$. Otherwise, there is a positive constant c such that, up to a subsequence, the bounds $\|u_n\|_{C(\overline{\Omega})}, \|v_n\|_{C(\overline{\Omega})} \leq c$ hold. Applying L^p estimates in each equation constituting the problem (2.6), one sees that $\|u_n\|_{W^{2,p}}, \|v_n\|_{W^{2,p}} \leq c_1$ for some positive constant c_1 not depending on n. By Kondrachov's immersion theorem, up to a subsequence, we obtain the convergences $u_n \to u$ and $v_n \to v$ in $C(\overline{\Omega})$. Again, using L^p estimates in the equations of (2.6), we conclude that

$$\|u_m - u_n\|_{W^{2,p}} \leq c_2 \left(|t_m - t_n| + \|v_m^\alpha - v_n^\alpha\|_{L^p}\right),$$
$$\|v_m - v_n\|_{W^{2,p}} \leq c_2 \left(|t_m - t_n| + \|u_m^\beta - u_n^\beta\|_{L^p}\right)$$

for some positive constant c_2 not depending on n. So, we derive the convergences $u_n \to u$ and $v_n \to v$ in $W^{2,p}(\Omega)$. Therefore, (u, v) is a positive solution of the system

$$\begin{cases} -\mathcal{L}_1 u = t_a^* \rho(x) v^\alpha + f_0(x) & \text{in } \Omega \\ -\mathcal{L}_2 v = at_a^* \tau(x) u^\beta + g_0(x) & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

From Lemma 2.3, it follows that (t_a^*, at_a^*) is in $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$, and from Lemma 2.4, we conclude that $(t_a^* + \varepsilon, a(t_a^* + \varepsilon))$ belongs to $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$ for $\varepsilon \sim 0$, contradicting the definition of t_a^*. Thus, the claim is valid. Define the functions

$$\tilde{u}_n = \frac{u_n}{\|u_n\|_{C(\overline{\Omega})} + \|v_n\|_{C(\overline{\Omega})}^\alpha} \quad \text{and} \quad \tilde{v}_n = \frac{v_n}{(\|u_n\|_{C(\overline{\Omega})} + \|v_n\|_{C(\overline{\Omega})}^\alpha)^{\frac{1}{\alpha}}}.$$

Then, $\|\tilde{u}_n\|_{C(\overline{\Omega})} + \|\tilde{v}_n\|_{C(\overline{\Omega})}^\alpha = 1$ and $(\tilde{u}_n, \tilde{v}_n)$ satisfies

$$(2.10) \quad \begin{cases} -\mathcal{L}_1 \tilde{u}_n = t_n \rho(x) \tilde{v}_n^\alpha + f_0(x) \{\|u_n\|_{C(\overline{\Omega})} + \|v_n\|_{C(\overline{\Omega})}^\alpha\}^{-1} & \text{in } \Omega \\ -\mathcal{L}_2 \tilde{v}_n = at_n \tau(x) \tilde{u}_n^\beta + g_0(x) \{\|u_n\|_{C(\overline{\Omega})} + \|v_n\|_{C(\overline{\Omega})}^\alpha\}^{-\frac{1}{\alpha}} & \\ \tilde{u}_n = 0 = \tilde{v}_n & \text{on } \partial\Omega \end{cases}$$

Passing to a subsequence, if necessary, and proceeding as above, we establish the convergences $\tilde{u}_n \to \tilde{u}$ and $\tilde{v}_n \to \tilde{v}$ in $W^{2,p}(\Omega)$. Hence, $\|\tilde{u}\|_{C(\overline{\Omega})} + \|\tilde{v}\|_{C(\overline{\Omega})}^\alpha = 1$ and $(\tilde{u}, \tilde{v})$ is a non-negative solution of the system

$$\begin{cases} -\mathcal{L}_1 u = t_a^* \rho(x) v^\alpha & \text{in } \Omega \\ -\mathcal{L}_2 v = at_a^* \tau(x) u^\beta & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

From the (SMP), it follows that (t_a^*, at_a^*) is in $\Lambda_{\alpha,\beta}^{\rho,\tau}(\Omega)$. □

Now we are ready to investigate the sets $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ when the functions ρ and τ belong to $L^p(\Omega)$. The next result establishes that the set $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ is non-empty in this more general situation:

THEOREM 2.2. *For each $a > 0$, there is a positive number $\tilde{t}_a$ such that the couple $(\tilde{t}_a, a\tilde{t}_a)$ belongs to $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$.*

PROOF. Let $a > 0$. Take two monotonically increasing sequences $\{\rho_n\}_0^\infty$ and $\{\tau_n\}_0^\infty$ of non-negative functions in $L^\infty(\Omega)$ converging to ρ and τ in $L^p(\Omega)$, respectively. Denote by t_n^* the positive number defined in (2.9) and associated to $O^{\rho_n,\tau_n}_{\alpha,\beta}(\Omega)$. By Theorem 2.1, it follows that (t_n^*, at_n^*) is in $\Lambda^{\rho_n,\tau_n}_{\alpha,\beta}(\Omega)$. Now choose a positive eigenfunction (u_n, v_n) corresponding to (t_n^*, at_n^*) which we can assume that $\|(u_n, v_n)\|_X = 1$ by homogeneity. To prove that $\{t_n^*\}_0^\infty$ is a decreasing sequence, it is sufficient to show that $t_n^* \geq t_{n+1}^*$ for every $n \geq 0$. Suppose on the contrary that $t_{n_0}^* < t_{n_0+1}^*$ for some $n_0 \geq 0$. Define the set $S = \{s > 0 \ : \ u_{n_0+1} > su_{n_0}$ and $v_{n_0+1} > s^\beta v_{n_0}$ in $\Omega\}$ and put $s^* = \sup S$. Then, the inequalities

$$\begin{aligned}
-\mathcal{L}_1(u_{n_0+1} - s^* u_{n_0}) &= t_{n_0+1}^* \rho_{n_0+1}(x) v_{n_0+1}^\alpha - s^* t_{n_0}^* \rho_{n_0}(x) v_{n_0}^\alpha \\
&\geq (t_{n_0+1}^* - t_{n_0}^*) s^* \rho_{n_0}(x) v_{n_0}^\alpha\,, \\
-\mathcal{L}_2(v_{n_0+1} - s^{*\beta} v_{n_0}) &= a t_{n_0+1}^* \tau_{n_0+1}(x) u_{n_0+1}^\beta \ - \ s^{*\beta} a t_{n_0}^* \tau_{n_0}(x) u_{n_0}^\beta \\
&\geq a(t_{n_0+1}^* - t_{n_0}^*) s^{*\beta} \tau_{n_0}(x) u_{n_0}^\beta
\end{aligned}$$

joint with the (SMP) imply $u_{n_0+1} > (s^* + \varepsilon) u_{n_0}$ and $v_{n_0+1} > (s^* + \varepsilon)^\beta v_{n_0}$ in Ω for $\varepsilon \sim 0$, a contradiction. Since $\{t_n^*\}_0^\infty$ is a decreasing sequence in $\mathbb{R}_+$, it converges to some $\tilde{t}_a$ in $[0, +\infty)$. Using L^p estimates as in Theorem 2.1, up to a subsequence, it follows that $u_n \to u$ and $v_n \to v$ in $W^{2,p}(\Omega)$. So, $\|(u, v)\|_X = 1$ and (u, v) is a non-negative solution of the problem

$$\begin{cases}
-\mathcal{L}_1 u = \tilde{t}_a \rho(x) v^\alpha & \text{in } \Omega \\
-\mathcal{L}_2 v = a\tilde{t}_a \tau(x) u^\beta & \\
u = 0 = v & \text{on } \partial\Omega
\end{cases}$$

From the (SMP), we conclude that $\tilde{t}_a > 0$ and $(\tilde{t}_a, a\tilde{t}_a)$ belongs to $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$. □

The next proposition determines completely the set $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$:

PROPOSITION 2.2. *The sets $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and $\{(\tilde{t}_a, a\tilde{t}_a) : a > 0\}$ are equal, where $\tilde{t}_a$ is found in Theorem* 2.2.

PROOF. By the (SMP) and Theorem 2.2, clearly we have $\{(\tilde{t}_a, a\tilde{t}_a) : a > 0\} \subset \Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega) \subset \mathbb{R}^2_+$. Thus, it is sufficient to show that if (t, at) belongs to $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ for some $a > 0$, then $t = \tilde{t}_a$. Indeed, take a positive eigenfunction (u, v) associated to a principal eigenvalue (t, at) and a positive eigenfunction (u_a, v_a) corresponding to the principal eigenvalue $(\tilde{t}_a, a\tilde{t}_a)$. By contradiction, assume without loss of generality that $t > \tilde{t}_a$. Consider the auxiliar set $S = \{s > 0 \ : \ u > su_a \text{ and } v > s^\beta v_a \text{ in } \Omega\}$ and denote $s^* = \sup S$. Since the inequalities

$$-\mathcal{L}_1(u - s^* u_a) = t\rho(x)v^\alpha \ - \ s^*\tilde{t}_a\rho(x)v_a^\alpha \geq (t - \tilde{t}_a)s^*\rho(x)v_a^\alpha\,,$$
$$-\mathcal{L}_2(v - s^{*\beta} v_a) = at\tau(x)u^\beta \ - \ s^{*\beta}a\tilde{t}_a\tau(x)u_a^\beta \geq a(t - \tilde{t}_a)s^{*\beta}\tau(x)u_a^\beta$$

are valid, arguing in a similar manner to the proof of Theorem 2.2, we derive a contradiction. □

Let us focus the set $G_{\alpha,\beta}^{\rho,\tau}(\Omega)$. The next lemma will be useful in this investigation:

LEMMA 2.5. *The sets* $\Lambda_{\alpha,\beta}^{\rho,\tau}(\Omega)$ *and* $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$ *are disjoint.*

PROOF. Suppose on the contrary that there is a couple (λ, μ) in $\Lambda_{\alpha,\beta}^{\rho,\tau}(\Omega) \cap O_{\alpha,\beta}^{\rho,\tau}(\Omega)$. Then, given functions f_0 and g_0 in $C(\overline{\Omega})$ and positive in Ω, by definition, the system (2.2) admits a positive solution (u, v). Take a positive eigenfunction (φ, ψ) associated to the principal eigenvalue (λ, μ). Introduce the standard set $S = \{s > 0 \ : \ u > s\varphi \text{ and } v > s^\beta\psi \text{ in } \Omega\}$ and set $s^* = \sup S$. Similarly to the proof of Proposition 2.1, using the relations

$$-\mathcal{L}_1(u - s^*\varphi) = \lambda\rho(x)v^\alpha \ + \ f_0(x) \ - \ s^*\lambda\rho(x)\psi^\alpha \ \geq \ f_0(x)\,,$$
$$-\mathcal{L}_2(v - s^{*\beta}\psi) = \mu\tau(x)u^\beta \ + \ g_0(x) \ - \ s^{*\beta}\mu\tau(x)\varphi^\beta \ \geq \ g_0(x)$$

and the (SMP), we arrive at an absurd. □

The first step in the study of the set $G_{\alpha,\beta}^{\rho,\tau}(\Omega)$ is provided in the lemma below:

LEMMA 2.6. *The sets* $G_{\alpha,\beta}^{\rho,\tau}(\Omega)$ *and* $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$ *are equal.*

PROOF. We already know that $G_{\alpha,\beta}^{\rho,\tau}(\Omega)$ is contained in $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$. Conversely, take a couple (λ, μ) in $O_{\alpha,\beta}^{\rho,\tau}(\Omega)$ and arbitrary non-negative functions f_0 and g_0 in $L^p(\Omega)$. Choose sequences of non-negative functions $\{f_n\}_0^\infty$ and $\{g_n\}_0^\infty$ in $C(\overline{\Omega})$ such that $f_n \to f_0$ and $g_n \to g_0$ in $L^p(\Omega)$. By definition, the system

$$\begin{cases} -\mathcal{L}_1 u = \lambda\rho(x)v^\alpha \ + \ f_n(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = \mu\tau(x)u^\beta \ + \ g_n(x) & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

possesses a non-negative solution (u_n, v_n). We claim that the sequences $\{u_n\}_0^\infty$ and $\{v_n\}_0^\infty$ are bounded in $C(\overline{\Omega})$. In fact, if for some subsequence the limit $\|(u_n, v_n)\|_X \to +\infty$ occurs, proceeding as in the proof of Theorem 2.1, we conclude that (λ, μ) belongs to $\Lambda_{\alpha,\beta}^{\rho,\tau}(\Omega)$, contradicting Lemma 2.5. Now applying L^p estimates as in Theorem 2.1, it follows that $u_n \to u$ and $v_n \to v$ in $W^{2,p}(\Omega)$. Hence, (u, v) is a non-negative solution of the problem (2.2), implying that (λ, μ) belongs to $G_{\alpha,\beta}^{\rho,\tau}(\Omega)$. □

The next step in the determination of the set $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ is a consequence of the Leray-Schauder degree theory:

PROPOSITION 2.3. *The set* $\{(t, at) : a > 0 \text{ and } 0 < t < \tilde{t}_a\}$ *is contained in* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$, *where* $\tilde{t}_a$ *is obtained in Theorem* 2.2.

PROOF. Fix $0 < t < \tilde{t}_a$ and arbitrary non-negative functions f_0 and g_0 in $L^p(\Omega)$. Consider the mapping $H : [0,1] \times C \to C$ defined by $H(s,u,v) = (z,w)$, where (z,w) satisfies

$$\begin{cases} -\mathcal{L}_1 z = st\rho(x)v^\alpha \ + \ sf_0(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 w = ast\tau(x)u^\beta \ + \ sg_0(x) & \\ z = 0 = w & \text{on} \quad \partial\Omega \end{cases}$$

By the (SMP) and L^p theory, it follows that the mapping H is well-defined, continuous and compact. We affirm that there is a positive constant M such that $\|(u,v)\|_X < M$ for every (u,v) in C verifying $H(s,u,v) = (u,v)$ for some $0 \leq s \leq 1$. Indeed, if there are sequences $\{s_n\}_0^\infty$ of numbers in $[0,1]$ and $\{u_n\}_0^\infty$ and $\{v_n\}_0^\infty$ of functions in C satisfying $H(s_n, u_n, v_n) = (u_n, v_n)$ and $\|(u_n, v_n)\|_X \to +\infty$, denoting by $\tilde{u}_n$ and $\tilde{v}_n$ the normalized functions as in the proof of Theorem 2.1 and performing a bootstrap procedure, we conclude that (st, ast) belongs to $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ for some $0 \leq s \leq 1$, contradicting Proposition 2.2. Thus, the claimed constant M exists. By the homotopic invariance property of the degree theory in cones, we obtain $\deg(Id - H(1,\cdot), B_M \cap C, 0) = \deg(Id - H(0,\cdot), B_M \cap C, 0) = 1$. Therefore, the system

$$\begin{cases} -\mathcal{L}_1 u = t\rho(x)v^\alpha \ + \ f_0(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = at\tau(x)u^\beta \ + \ g_0(x) & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases} \tag{2.11}$$

possesses a non-negative solution, implying that (t, at) is in $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$. □

The last step to characterize the set $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ is given in the lemma below:

LEMMA 2.7. *If* (t_0, at_0) *belongs to* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ *for some* $a > 0$ *and* $t_0 > 0$, *then* (t, at) *belongs to* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ *for every* $0 < t < t_0$.

PROOF. Take two arbitrary non-negative functions f_0 and g_0 in $L^p(\Omega)$. Replacing t by t_0 in the problem (2.11), by definition, one sees that this system admits a non-negative solution (u,v). For each positive number $t < t_0$, it follows that $(0,0)$ is a subsolution and (u,v) is a non-negative supersolution of the system (2.11). Hence, we conclude from Lemma 2.1 that (t, at) belongs to $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$. □

Now we are ready to list some consequences of the previous results. The first one furnishes a characterization of the set $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$:

COROLLARY 2.1. *The sets* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ *and* $\{(t, at) : a > 0 \text{ and } 0 < t < \tilde{t}_a\}$ *are equal, where* $\tilde{t}_a$ *is given in Theorem* 2.2.

PROOF. It follows easily from Proposition 2.3 and Lemmas 2.5, 2.6 and 2.7. □

Clearly, Proposition 2.2 and Corollary 2.1 establish the relation between the sets $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$:

COROLLARY 2.2. *The sets* $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ *and* $\partial G^{\rho,\tau}_{\alpha,\beta}(\Omega) \cap \mathbb{R}^2_+$ *are equal.*

The number $\tilde{t}_a$ found in Theorem 2.2 coincides with that defined in (2.9):

COROLLARY 2.3. *For each* $a > 0$*, the numbers* $\tilde{t}_a$ *and* t^*_a *are equal.*

PROOF. It follows from Proposition 2.2 and Lemmas 2.5, 2.6 and 2.7 that $\tilde{t}_a \geq t^*_a$. In addition, from Proposition 2.3, we conclude that $\tilde{t}_a \leq t^*_a$. So, we are done. □

The next result states that each point of the set $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ leads to uniqueness of non-negative solution for system (2.2):

PROPOSITION 2.4. *For each couple* (λ, μ) *in* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ *and non-negative functions* f_0 *and* g_0 *in* $L^p(\Omega)$*, the system* (2.2) *possesses a unique non-negative solution.*

PROOF. Take (λ, μ) in $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$. We separate the proof of uniqueness of solution in two cases. Assume first that $(0, 0)$ is a solution of the problem (2.2). If the system (2.2) possesses a positive solution, then the couple (λ, μ) belongs to $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$, contradicting Lemma 2.5. Now suppose that $(0, 0)$ is not a solution of (2.2) and assume on the contrary that the system (2.2) admits two positive solutions (u_1, v_1) and (u_2, v_2). Consider the standard set $S = \{s > 0 \, : \, u_1 > su_2$ and $v_1 > s^\beta v_2$ in $\Omega\}$ and put $s^* = \sup S$. Permuting (u_1, v_1) and (u_2, v_2) in the definition of S, if necessary, we can assume that $s^* \leq 1$. We claim that the identities $u_1 \equiv s^* u_2$ and $v_1 \equiv s^{*\beta} v_2$ are valid in Ω. By contradiction, if $u_1 \not\equiv s^* u_2$ or $v_1 \not\equiv s^{*\beta} v_2$ in Ω, using the relations

$$-\mathcal{L}_1(u_1 - s^* u_2) = \lambda\rho(x)v_1^\alpha - s^*\lambda\rho(x)v_2^\alpha + (1 - s^*)f_0(x),$$
$$-\mathcal{L}_2(v_1 - s^{*\beta} v_2) = \mu\tau(x)u_1^\beta - s^{*\beta}\mu\tau(x)u_2^\beta + (1 - s^{*\beta})g_0(x)$$

and the (SMP), we conclude that $u_1 > (s^* + \varepsilon)u_2$ or $v_1 > (s^* + \varepsilon)^\beta v_2$ in Ω for $\varepsilon \sim 0$. Again, using the expressions above, it is easy to see that $u_1 > (s^* + \varepsilon)u_2$ and $v_1 > (s^* + \varepsilon)^\beta v_2$ in Ω for $\varepsilon \sim 0$, a contradiction. To conclude the proof, it is sufficient to show that $s^* = 1$. Suppose on the contrary that $s^* < 1$. If $f_0 \not\equiv 0$ in Ω, the inequality $-\mathcal{L}_1(u_1 - s^* u_2) \geq (1 - s^*)f_0(x)$ and the (SMP) furnish a contradiction. If $g_0 \not\equiv 0$ in Ω, the reasoning is analogous. Hence, we derive $s^* = 1$. □

Let $\Lambda_1 : \mathbb{R}_+ \to \mathbb{R}^2_+$ be the mapping defined by $\Lambda_1(a) = (\lambda_1(a), \mu_1(a))$, where $\lambda_1(a) = \tilde{t}_a$ and $\mu_1(a) = a\tilde{t}_a$. By Proposition 2.2, the sets $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and $\{\Lambda_1(a) : \, a > 0\}$ are equal. The mapping Λ_1 will be refered as the principal spectral curve associated to the problem (2.1).

Let us investigate the question of simplicity of the curve Λ_1. For that subject, we introduce below the notion of simple eigenvalue for system (2.1):

DEFINITION 2.2. An eigenvalue (λ, μ) of the system (2.1) is said to be simple if for each pair of eigenfunctions (u_1, v_1) and (u_2, v_2) associated to (λ, μ) the identities $u_2 \equiv tu_1$ and $v_2 \equiv sv_1$ hold in Ω for some real numbers t and s. The curve Λ_1 is said to be simple if each point on the curve is a simple eigenvalue.

The next result is a natural extension to systems of the simplicity of principal eigenvalues known in the scalar case:

PROPOSITION 2.5. *The curve* Λ_1 *is simple.*

PROOF. Let (u, v) be a positive eigenfunction associated to a principal eigenvalue (λ, μ) on the curve Λ_1. To verify that (λ, μ) is a simple eigenvalue, it is sufficient to show that given another eigenfunction (u_1, v_1) corresponding to (λ, μ), we get $u_1 \equiv tu$ and $v_1 \equiv sv$ in Ω for some real numbers t and s. Taking $(-u_1, -v_1)$ in place of (u_1, v_1), if necessary, without loss of generality we can assume that the positive part of u_1 or v_1 is non-null. Thus, the auxiliar set $S = \{s > 0 \ : \ u > su_1 \ \text{ and } \ v > s^\beta v_1 \ \text{ in } \ \Omega\}$ is bounded. Define $s^* = \sup S$. We affirm that the identities $u \equiv s^* u_1$ and $v \equiv s^{*\beta} v_1$ occur in Ω. Without loss of generality, assume that $u \not\equiv s^* u_1$ in Ω. It follows from inequality

$$-\mathcal{L}_1(u - s^* u_1) = \lambda\rho(x)v^\alpha - s^*\lambda\rho(x)|v_1|^{\alpha-1}v_1 \ \geq \ 0$$

and the (SMP), that $u > (s^* + \varepsilon)u_1$ in Ω for $\varepsilon \sim 0$. Since the estimate

$$\begin{aligned} -\mathcal{L}_2(v - s^{*\beta} v_1) &= \mu\tau(x)u^\beta \ - \ s^{*\beta}\mu\tau(x)|u_1|^{\beta-1}u_1 \\ &\geq \begin{cases} \mu\tau(x)u^\beta & \text{if } \ u_1 \leq 0 \\ \mu[(s^* + \varepsilon)^\beta - s^{*\beta}]\tau(x)u_1^\beta & \text{if } \ u_1 > 0 \end{cases} \end{aligned}$$

is satisfied, we conclude from the (SMP) that $v > (s^* + \varepsilon)^\beta v_1$ in Ω for $\varepsilon \sim 0$. That is impossible. Therefore, the claim is justified. □

Let us turn our discussion to the question of isolation of the curve Λ_1. Before, it is necessary to introduce the following notions:

DEFINITION 2.3. The curve Λ_1 is said to be upper isolated (lower isolated) if for each $a > 0$, there is a positive number $\varepsilon = \varepsilon(a)$ such that the system (2.1) does not admit any eigenvalue in $B_\varepsilon(\Lambda_1(a)) \cap \overline{G^{\rho,\tau}_{\alpha,\beta}(\Omega)}^c$ $(B_\varepsilon(\Lambda_1(a)) \cap G^{\rho,\tau}_{\alpha,\beta}(\Omega))$.

The proposition below establishes the upper isolation of the curve Λ_1:

PROPOSITION 2.6. *The curve* Λ_1 *is upper isolated.*

PROOF. If the statement does not hold, there is a sequence of eigenvalues $\{(\lambda_n, \mu_n)\}_0^\infty$ contained in $B_{\varepsilon_n}(\lambda, \mu) \cap \overline{G^{\rho,\tau}_{\alpha,\beta}(\Omega)}^c$ for some principal eigenvalue (λ, μ) on the curve Λ_1 and some sequence $\{\varepsilon_n\}_0^\infty$ of positive numbers converging to zero. Take an eigenfunction (u_n, v_n) corresponding to (λ_n, μ_n) satisfying $\|(u_n, v_n)\|_X = 1$ by homogeneity. Applying L^p estimates to the system

$$\begin{cases} -\mathcal{L}_1 u_n = \lambda_n \rho(x)|v_n|^{\alpha-1} v_n & \text{in} \quad \Omega \\ -\mathcal{L}_2 v_n = \mu_n \tau(x)|u_n|^{\beta-1} u_n & \\ u_n = 0 = v_n & \text{on} \quad \partial\Omega \end{cases}$$

as in the proof of Theorem 2.1, up to a subsequence, we derive the convergences $u_n \to u$ and $v_n \to v$ in $W^{2,p}(\Omega)$. Thus, it follows that $\|(u, v)\|_X = 1$ and (u, v) is an eigenfunction associated to the principal eigenvalue (λ, μ). Choose a positive eigenfunction (φ, ψ) corresponding to (λ, μ). From the proof of Proposition 2.5, one sees that either $\varphi \equiv s^* u$ and $\psi \equiv s^{*\beta} v$ in Ω or $\varphi \equiv -s^* u$ and $\psi \equiv -s^{*\beta} v$ in Ω for some $s^* > 0$. Without loss of generality, suppose that $\varphi \equiv s^* u$ and $\psi \equiv s^{*\beta} v$ in Ω. From convergence in $W^{2,p}(\Omega)$ and Sobolev immersion, we get $u_n \to u$ and $v_n \to v$ in $C^1(\overline{\Omega})$. From the (SMP), we conclude that u_n and v_n are positive functions in Ω for n large enough. Hence, the eigenvalue (λ_n, μ_n) belongs to $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$, contradicting Corollary 2.2. □

The next proposition furnishes the lower isolation of the curve Λ_1:

PROPOSITION 2.7. *Neither $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ nor $-G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ contains eigenvalues of the system* (2.1). *In particular, the curve Λ_1 is lower isolated.*

PROOF. Suppose by contradiction that there is an eigenvalue (λ, μ) of the problem (2.1) belonging to $G^{\rho,\tau}_{\alpha,\beta}(\Omega) \cup -G^{\rho,\tau}_{\alpha,\beta}(\Omega)$. Denoting $a = \frac{\mu}{\lambda}$, we have $|\lambda| < \tilde{t}_a$. Consider a positive eigenfunction (u_a, v_a) associated to $(\tilde{t}_a, a\tilde{t}_a)$ and an eigenfunction (u, v) corresponding to (λ, μ). Without loss of generality, we can assume that (λ, μ) is in $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ and the positive part of u or v is non-null. Define the classical set $S = \{s > 0 \ : \ u_a > su \text{ and } v_a > s^\beta v \text{ in } \Omega\}$ and put $s^* = \sup S$. Proceeding as in the proof of Proposition 2.5, we conclude that the relations $u_a \equiv s^* u$ and $v_a \equiv s^{*\beta} v$ are satisfied in Ω. Therefore, it follows that (λ, μ) belongs to $\Lambda^{\rho,\tau}_{\alpha,\beta}(\Omega)$, contradicting Lemma 2.5. □

Now let us investigate the continuity of the curve Λ_1:

PROPOSITION 2.8. *The curve Λ_1 is a continuous function.*

PROOF. Suppose on the contrary that the curve Λ_1 is discontinuous in some point $a > 0$. Then, there are a positive number ε and a sequence $\{a_n\}_0^\infty$ converging to a in $\mathbb{R}_+$ such that $|\Lambda_1(a_n) - \Lambda_1(a)| \geq \varepsilon$. In particular, it follows that either $\tilde{t}_{a_n} < \tilde{t}_a$ and $a_n\tilde{t}_{a_n} < a\tilde{t}_a$ or $\tilde{t}_{a_n} > \tilde{t}_a$ and $a_n\tilde{t}_{a_n} > a\tilde{t}_a$ for n large enough. Without loss of generality, assume the first alternative. Take positive eigenfunctions (u_n, v_n) and (u, v) associated to the principal eigenvalues $(\tilde{t}_{a_n}, a_n\tilde{t}_{a_n})$ and $(\tilde{t}_a, a\tilde{t}_a)$, respectively. Consider the standard set $S = \{s > 0 \ : \ u > su_n \text{ and } v > s^\beta v_n \text{ in } \Omega\}$ and set $s^* = \sup S$. Proceeding as in the proof of Theorem 2.2, we conclude that the relations

$$-\mathcal{L}_1(u - s^* u_n) = \tilde{t}_a \rho(x) v^\alpha - s^* \tilde{t}_{a_n} \rho(x) v_n^\alpha \geq (\tilde{t}_a - \tilde{t}_{a_n}) s^* \rho(x) v_n^\alpha \,,$$

$$-\mathcal{L}_2(v - s^{*\beta} v_n) = a\tilde{t}_a \tau(x) u^\beta - s^{*\beta} a_n \tilde{t}_{a_n} \tau(x) u_n^\beta \geq (a\tilde{t}_a - a_n\tilde{t}_{a_n}) s^{*\beta} \tau(x) u_n^\beta$$

together with the (SMP) furnish a contradiction. □

Corollary 2.1 and Proposition 2.8 imply the following result:

COROLLARY 2.4. *The set* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ *is open.*

In order to study the asymptotic behaviour of the curve Λ_1, we need the following monotonicity result:

PROPOSITION 2.9. *The functions* $\lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $\mu_1 : \mathbb{R}_+ \to \mathbb{R}_+$ *are decreasing and increasing, respectively.*

PROOF. Since the proof of two assertions mentioned above are similar, it is sufficient to show only that λ_1 is a decreasing function. Suppose by contradiction that the inequality $\tilde{t}_{a_1} \leq \tilde{t}_{a_2}$ is valid for some $0 < a_1 < a_2$. Choose positive eigenfunctions (u_1, v_1) and (u_2, v_2) associated to the principal eigenvalues $(\tilde{t}_{a_1}, a_1\tilde{t}_{a_1})$ and $(\tilde{t}_{a_2}, a_2\tilde{t}_{a_2})$, respectively. Introduce the auxiliar set $S = \{s > 0 \ : \ u_2 > su_1 \text{ and } v_2 > s^\beta v_1 \text{ in } \Omega\}$ and define $s^* = \sup S$. Using the inequality

$$-\mathcal{L}_2(v_2 - s^{*\beta}v_1) = a_2\tilde{t}_{a_2}\tau(x)u_2^\beta - s^{*\beta}a_1\tilde{t}_{a_1}\tau(x)u_1^\beta \geq (a_2 - a_1)s^{*\beta}\tilde{t}_{a_1}\tau(x)u_1^\beta$$

and the (SMP), we get $v_2 > (s^* + \varepsilon)^\beta v_1$ in Ω for $\varepsilon \sim 0$. So, from the relation

$$-\mathcal{L}_1(u_2 - s^*u_1) = \tilde{t}_{a_2}\rho(x)v_2^\alpha - s^*\tilde{t}_{a_1}\rho(x)v_1^\alpha \geq \varepsilon\tilde{t}_{a_1}\rho(x)v_1^\alpha$$

and the (SMP), we obtain a contradiction. □

Based on Proposition 2.9, we separate the discussion of the asymptotic behaviour of the curve Λ_1 in two parts. The first one is presented in the proposition below:

PROPOSITION 2.10. *The limits* $\lambda_1(a) \to +\infty$ *as* $a \to 0^+$ *and* $\mu_1(a) \to +\infty$ *as* $a \to +\infty$ *hold.*

PROOF. By analogy of the proofs of two limits refered above, it is sufficient to show only the first one. Suppose on the contrary that the first limit does not occur. From Proposition 2.9, we get a sequence $\{a_n\}_0^\infty$ of positive numbers converging to zero such that the sequence $\{\tilde{t}_{a_n}\}_0^\infty$ converges to t in $\mathbb{R}_+$. In particular, the sequence $\{a_n\tilde{t}_{a_n}\}_0^\infty$ converges to zero. Let (u_n, v_n) be a positive eigenfunction associated to the principal eigenvalue $(\tilde{t}_{a_n}, a_n\tilde{t}_{a_n})$ satisfying $\|(u_n, v_n)\|_X = 1$. Arguing with L^p bounds as in the proof of Theorem 2.1, up to a subsequence, we obtain $u_n \to u$ and $v_n \to v$ in $W^{2,p}(\Omega)$. Therefore, it follows that $\|(u, v)\|_X = 1$ and (u, v) is a non-negative solution of the problem

$$\begin{cases} -\mathcal{L}_1 u = t\rho(x)v^\alpha & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = 0 & \\ u = 0 = v & \text{on} \quad \partial\Omega\,. \end{cases}$$

Using the (SMP), we arrive at a contradiction. □

The next result concerns the second part of the discussion about the asymptotic behaviour of the curve Λ_1:

PROPOSITION 2.11. *The limits* $\lambda_1(a) \to 0$ *as* $a \to +\infty$ *and* $\mu_1(a) \to 0$ *as* $a \to 0^+$ *hold.*

PROOF. By similarity of the proofs of two assertions stated above, it is sufficient to investigate only the first one. From Proposition 2.9, we conclude that $\tilde{t}_a$ converges to t in $[0, +\infty)$ as $a \to +\infty$. Fix $a > 0$ and denote by (u_a, v_a) a positive eigenfunction associated to the principal eigenvalue $(\tilde{t}_a, a\tilde{t}_a)$. Assume by contradiction that $t > 0$. Then, it follows from Proposition 2.9 and Corollary 2.1 that $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ contains the set $(0, t) \times \mathbb{R}_+$. Thus, we can take a couple (λ, μ) in $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ verifying $\lambda < t$ and $\mu \geq \frac{a\tilde{t}_a^{\beta+1}}{\lambda^\beta}$. Given functions f_0 and g_0 in $C(\overline{\Omega})$ with $f_0, g_0 > 0$ in Ω, by definition, the system (2.2) admits a positive solution (u, v). Putting $\tilde{u} = \frac{\tilde{t}_a}{\lambda} u$ and $\tilde{v} = v$, one sees that $(\tilde{u}, \tilde{v})$ verifies

$$\begin{cases} -\mathcal{L}_1 \tilde{u} = \tilde{t}_a \rho(x) \tilde{v}^\alpha + \dfrac{\tilde{t}_a}{\lambda} f_0(x) & \\ & \text{in } \Omega \\ -\mathcal{L}_2 \tilde{v} = \dfrac{\mu \lambda^\beta}{\tilde{t}_a^\beta} \tau(x) \tilde{u}^\beta + g_0(x) & \\ \tilde{u} = 0 = \tilde{v} & \text{on } \partial\Omega . \end{cases}$$

Define the set $S = \{s > 0 \ : \ \tilde{u} > s u_a \text{ and } \tilde{v} > s^\beta v_a \text{ in } \Omega\}$ and set $s^* = \sup S$. Using the inequalities

$$-\mathcal{L}_1(\tilde{u} - s^* u_a) = \tilde{t}_a \rho(x) \tilde{v}^\alpha + \frac{\tilde{t}_a}{\lambda} f_0(x) - s^* \tilde{t}_a \rho(x) v_a^\alpha \geq \frac{\tilde{t}_a}{\lambda} f_0(x),$$

$$\begin{aligned} -\mathcal{L}_2(\tilde{v} - s^{*\beta} v_a) &= \frac{\mu \lambda^\beta}{\tilde{t}_a^\beta} \tau(x) \tilde{u}^\beta + g_0(x) - s^{*\beta} a\tilde{t}_a \tau(x) u_a^\beta \\ &\geq \left(\frac{\mu \lambda^\beta}{\tilde{t}_a^\beta} - a\tilde{t}_a \right) s^{*\beta} \tau(x) u_a^\beta + g_0(x) \end{aligned}$$

and the (SMP), we find an absurd. □

Joining Corollary 2.1 and Propositions 2.8, 2.9, 2.10 and 2.11, we establish the following result:

COROLLARY 2.5. *The curve* Λ_1 *divides* $\mathbb{R}^2_+$ *into two unbounded open connected components* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$ *and* $\mathbb{R}^2_+ \setminus \overline{G^{\rho,\tau}_{\alpha,\beta}(\Omega)}$.

The next proposition investigates the non-existence of positive supersolutions and non-trivial non-negative subsolutions for system (2.1):

PROPOSITION 2.12. *The following assertions hold:*

(*i*) *If* (λ, μ) *is in* $\mathbb{R}^2_+ \setminus \overline{G^{\rho,\tau}_{\alpha,\beta}(\Omega)}$, *the system* (2.1) *does not admit any positive supersolution.*

(*ii*) *If* (λ, μ) *is in* $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$, *the system* (2.1) *does not admit any non-trivial non-negative subsolution.*

PROOF. Since the proofs of the sentences (i) and (ii) are similar, it is sufficient to show only the statement (i). Suppose on the contrary that the system (2.1) possesses a positive supersolution (u, v) for some couple (λ, μ) in $\mathbb{R}^2_+ \setminus \overline{G^{\rho,\tau}_{\alpha,\beta}(\Omega)}$. Taking $a = \frac{\mu}{\lambda}$, we get $\lambda > \tilde{t}_a$. Choose a positive eigenfunction (u_a, v_a) associated to the principal eigenvalue $(\tilde{t}_a, a\tilde{t}_a)$. Defining the auxiliar set $S = \{s > 0 \ : \ u > su_a$ and $v > s^\beta v_a$ in $\Omega\}$, setting $s^* = \sup S$ and using the standard procedure based on the inequalities

$$-\mathcal{L}_1(u - s^* u_a) \geq \lambda\rho(x)v^\alpha \ - \ s^*\tilde{t}_a\rho(x)v_a^\alpha \ \geq \ (\lambda - \tilde{t}_a)s^*\rho(x)v_a^\alpha\,,$$
$$-\mathcal{L}_2(v - s^{*\beta} v_a) \geq a\lambda\tau(x)u^\beta \ - \ s^{*\beta}a\tilde{t}_a\tau(x)u_a^\beta \ \geq \ a(\lambda - \tilde{t}_a)s^{*\beta}\tau(x)u_a^\beta$$

and on the (SMP), we obtain a contradiction. □

As a consequence of Proposition 2.12, we conclude that the curve Λ_1 is divisor for the existence of non-negative solutions of the problem (2.2):

COROLLARY 2.6. *For each* (λ, μ) *in* $\mathbb{R}^2_+ \setminus \overline{G^{\rho,\tau}_{\alpha,\beta}(\Omega)}$ *and non-negative functions* f_0 *and* g_0 *in* $L^p(\Omega)$, *the system* (2.2) *does not admit any positive solution.*

Finally, we finish this section with a min-max type characterization of the principal spectral curve Λ_1. Given $a > 0$, consider the system

$$\text{(2.12)} \qquad \begin{cases} -\mathcal{L}_1 u = t\rho(x)v^\alpha & \text{in } \ \Omega \\ -\mathcal{L}_2 v = at\tau(x)u^\beta & \\ u = 0 = v & \text{on } \ \partial\Omega \end{cases}$$

and define the sets $\mathcal{P}_1 = \{u \in W^{2,p}(\Omega) : u > 0$ in Ω and $\mathcal{L}_1 u < 0$ in $\Omega\}$ and $\mathcal{P}_2 = \{v \in W^{2,p}(\Omega) : v > 0$ in Ω and $\mathcal{L}_2 v < 0$ in $\Omega\}$.

The sets $\mathcal{P}_1$ and $\mathcal{P}_2$ allow us to establish the following representations for the functions λ_1 and μ_1:

THEOREM 2.3. *For each* $a > 0$, *we have*

$$\frac{1}{\lambda_1(a)^{\alpha+1}} = \inf_{u\in\mathcal{P}_1} \sup_{x\in\Omega} \left(-\frac{a^\alpha\rho(x)\{(-\mathcal{L}_2)^{-1}(\tau(x)u^\beta)\}^\alpha}{\mathcal{L}_1 u} \right),$$
$$\frac{1}{\mu_1(a)^{\beta+1}} = \inf_{v\in\mathcal{P}_2} \sup_{x\in\Omega} \left(-\frac{\tau(x)\{(-\mathcal{L}_1)^{-1}(\rho(x)v^\alpha)\}^\beta}{a^\beta\mathcal{L}_2 v} \right).$$

PROOF. By analogy of the proofs of two formulas written above, it is sufficient to show only the first one. Take $0 < t < \tilde{t}_a$. From Corollary 2.1, we see that (t, at) belongs to $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$. Thus, the system

$$\begin{cases} -\mathcal{L}_1 u = t\rho(x)v^\alpha \ + \ 1 & \text{in } \ \Omega \\ -\mathcal{L}_2 v = at\tau(x)u^\beta \ + \ 1 & \\ u = 0 = v & \text{on } \ \partial\Omega \end{cases}$$

admits a positive solution (φ, ψ). From the second equation of the system above and the (SMP), we obtain the inequality $\psi > at(-\mathcal{L}_2)^{-1}(\tau(x)\varphi^{\beta})$ in Ω which replacing in the first equation, we get $-\mathcal{L}_1\varphi > a^{\alpha}t^{\alpha+1}\rho(x)\{(-\mathcal{L}_2)^{-1}(\tau(x)\varphi^{\beta})\}^{\alpha}$ in Ω. Therefore, we derive

$$\frac{1}{t^{\alpha+1}} > \sup_{x\in\Omega}\left(-\frac{a^{\alpha}\rho(x)\{(-\mathcal{L}_2)^{-1}(\tau(x)\varphi^{\beta})\}^{\alpha}}{\mathcal{L}_1\varphi}\right)$$
$$\geq \inf_{u\in\mathcal{P}_1}\sup_{x\in\Omega}\left(-\frac{a^{\alpha}\rho(x)\{(-\mathcal{L}_2)^{-1}(\tau(x)u^{\beta})\}^{\alpha}}{\mathcal{L}_1 u}\right).$$

Since $0 < t < \tilde{t}_a$ is an arbitrary number, it follows that

$$\frac{1}{\lambda_1(a)^{\alpha+1}} \geq \inf_{u\in\mathcal{P}_1}\sup_{x\in\Omega}\left(-\frac{a^{\alpha}\rho(x)\{(-\mathcal{L}_2)^{-1}(\tau(x)u^{\beta})\}^{\alpha}}{\mathcal{L}_1 u}\right).$$

Now suppose by contradiction that the inequality above is strict. Then, by the definition of infimum, there are $t > \tilde{t}_a$ and φ in $\mathcal{P}_1$ satisfying

$$\frac{1}{t^{\alpha+1}} \geq \sup_{x\in\Omega}\left(-\frac{a^{\alpha}\rho(x)\{(-\mathcal{L}_2)^{-1}(\tau(x)\varphi^{\beta})\}^{\alpha}}{\mathcal{L}_1\varphi}\right).$$

Putting $\psi = at(-\mathcal{L}_2)^{-1}(\tau(x)\varphi^{\beta})$, we obtain positive functions φ and ψ in Ω verifying

$$\begin{cases} -\mathcal{L}_1\varphi & \geq t\rho(x)\psi^{\alpha} \\ -\mathcal{L}_2\psi & = at\tau(x)\varphi^{\beta} \end{cases} \quad \text{in } \Omega.$$

From (i) of Proposition 2.12, we establish a contradiction. Hence, we have

$$\frac{1}{\lambda_1(a)^{\alpha+1}} = \inf_{u\in\mathcal{P}_1}\sup_{x\in\Omega}\left(-\frac{a^{\alpha}\rho(x)\{(-\mathcal{L}_2)^{-1}(\tau(x)u^{\beta})\}^{\alpha}}{\mathcal{L}_1 u}\right). \qquad \square$$

REMARK 2.1. As a consequence of Theorem 2.3, there are positive constants γ and θ such that $\lambda_1(a) = \gamma a^{-\frac{\alpha}{\alpha+1}}$ and $\mu_1(a) = \theta a^{\frac{\beta}{\beta+1}}$. In particular, Theorem 2.3 provides an alternative proof of Propositions 2.8, 2.9, 2.10 and 2.11.

REMARK 2.2. It follows immediately from the proof of Theorem 2.3 that we can consider the space $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ instead of $W^{2,p}(\Omega)$ in the definitions of the sets $\mathcal{P}_1$ and $\mathcal{P}_2$.

3. – Existence for systems in bounded domains

In this section we discuss the existence of non-negative solutions for semilinear elliptic systems of the type (1.4) subject to certain non-ressonance conditions. At first, let us consider the following growth hypothesis:

$$
\begin{aligned}
f(x,t,s) &\le \varepsilon_0 \eta(x) t + \lambda \rho(x) s^{\alpha} + k(x),\\
g(x,t,s) &\le \mu \tau(x) t^{\beta} + \varepsilon_0 \xi(x) s + k(x)
\end{aligned}
\tag{3.1}
$$

for every x in Ω and $t, s \ge 0$, where α and β are positive numbers such that $\alpha\beta = 1$, k, η, ξ, ρ and τ are non-negative functions in $L^p(\Omega)$ with ρ and τ not being identically null in Ω and (λ, μ) belongs to $G^{\rho,\tau}_{\alpha,\beta}(\Omega)$.

The lemma below is fundamental for our purposes and it follows from the degree theory in cones:

LEMMA 3.1. *There is a positive constant ε_0 such that for each non-negative function k in $L^p(\Omega)$ and every non-negative Carathéodory functions f and g satisfying the condition* (3.1), *the system* (1.4) *admits a non-negative solution.*

PROOF. Consider the mapping $H_{f,g} : [0,1] \times C \to C$ defined in the proof of Lemma 2.2. From the condition (3.1), L^p theory and the (SMP), one sees that H is a well-defined, continuous and compact mapping. We affirm that there is a positive constant ε_0 such that for each non-negative function k in $L^p(\Omega)$ with $\|k\|_{L^p} \le 1$ and non-negative functions f and g satisfying (3.1), there is a positive constant M_0 not depending on k, f and g such that $\|(u,v)\|_X < M_0$ for every (u,v) in C verifying $H_{f,g}(s,u,v) = (u,v)$ for some $0 \le s \le 1$. Otherwise, there are sequences $\{s_n\}_0^{\infty}$ of real numbers in $[0,1]$, $\{(u_n, v_n)\}_0^{\infty}$ of couples in C, $\{\varepsilon_n\}_0^{\infty}$ of positive numbers, $\{k_n\}_0^{\infty}$ of non-negative functions in $L^p(\Omega)$ and $\{f_n\}_0^{\infty}$ and $\{g_n\}_0^{\infty}$ of non-negative functions fulfilling the assumption (3.1) with f_n, g_n, ε_n and k_n in place of f, g, ε_0 and k, respectively, such that $s_n \to s_0$, $\|(u_n, v_n)\|_X \to +\infty$, $\varepsilon_n \to 0$, $\|k_n\|_{L^p} \le 1$ and $H_{f_n,g_n}(s_n, u_n, v_n) = (u_n, v_n)$. Denote by $\tilde{u}_n$ and $\tilde{v}_n$ the normalized functions as in the proof of Theorem 2.1. Since $\|\tilde{u}_n\|_{C(\overline{\Omega})} + \|\tilde{v}_n\|^{\alpha}_{C(\overline{\Omega})} = 1$, using (3.1) and arguing as in the proof of Theorem 2.1, up to a subsequence, we conclude that $\tilde{u}_n \to \tilde{u}$ and $\tilde{v}_n \to \tilde{v}$ in $W^{2,p}(\Omega)$. So, it follows that $\|\tilde{u}\|_{C(\overline{\Omega})} + \|\tilde{v}\|^{\alpha}_{C(\overline{\Omega})} = 1$ and $(\tilde{u}, \tilde{v})$ satisfies

$$
\begin{cases}
-\mathcal{L}_1 \tilde{u} \le \lambda \rho(x) \tilde{v}^{\alpha} & \text{in } \Omega\\
-\mathcal{L}_2 \tilde{v} \le \mu \tau(x) \tilde{u}^{\beta} & \\
\tilde{u} = 0 = \tilde{v} & \text{on } \partial\Omega
\end{cases}
$$

Applying (ii) of Proposition 2.12, we obtain a contradiction. Thus, the claimed constants ε_0 and M_0 exist. Now consider an arbitrary non-negative function k in $L^p(\Omega)$ and non-negative functions f and g satisfying (3.1) with ε_0 provided above. We claim that there is a constant $M = M(k) > 0$ not depending on f and g such that $\|(u,v)\|_X < M$ for every (u,v) in C verifying $H_{f,g}(s,u,v) = (u,v)$ for some $0 \le s \le 1$. In fact, for each $\varepsilon > 0$, define the functions

$f_\varepsilon(x,t,s) = \varepsilon^\alpha f(x, \frac{t}{\varepsilon^\alpha}, \frac{s}{\varepsilon})$ and $g_\varepsilon(x,t,s) = \varepsilon g(x, \frac{t}{\varepsilon^\alpha}, \frac{s}{\varepsilon})$. Clearly, f_ε and g_ε fulfill (3.1) with $k_\varepsilon = \max\{\varepsilon^\alpha, \varepsilon\}k$ instead of k. Since the functions $u_\varepsilon = \varepsilon^\alpha u$ and $v_\varepsilon = \varepsilon v$ verify $H_{f_\varepsilon, g_\varepsilon}(s, u_\varepsilon, v_\varepsilon) = (u_\varepsilon, v_\varepsilon)$, taking ε small enough such that $\|k_\varepsilon\|_{L^p} \leq 1$, from the first part, it follows that $\|(u_\varepsilon, v_\varepsilon)\|_X < M_0$. Therefore, we conclude that $\|(u,v)\|_X < M$. Using the homotopic invariance property of the degree theory in cones, we finish the proof. □

As an application of Lemma 3.1, we furnish the following example:

EXAMPLE 3.1. If λ, μ, η, ξ, ρ and τ verify the conditions required in (3.1), then the system

$$\begin{cases} -\mathcal{L}_1 u = \varepsilon\eta(x)u + \lambda\rho(x)v^\alpha + k_1(x) & \text{in } \Omega \\ -\mathcal{L}_2 v = \mu\tau(x)u^\beta + \varepsilon\xi(x)v + k_2(x) & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

admits a non-negative solution for every $\varepsilon > 0$ small enough and every non-negative functions k_1 and k_2 belonging to $L^p(\Omega)$.

The previous lemma is not useful to seek non-trivial solutions when $f(x,0,0) = 0 = g(x,0,0)$ for every x in Ω. In order to deal with this case, we use a truncation argument together with the following conditions:

$$f(x,t,s) \geq \lambda_0\rho_0(x)s^{\alpha_0}, \quad g(x,t,s) \geq \mu_0\tau_0(x)t^{\beta_0} \tag{3.2}$$

for every x in Ω and $0 \leq t, s \leq \delta_0$, where α_0, β_0 and δ_0 are positive numbers, $\alpha_0\beta_0 = 1$, ρ_0 and τ_0 are non-negative not identically null functions in $L^p(\Omega)$ and (λ_0, μ_0) belongs to $\Lambda^{\rho_0,\tau_0}_{\alpha_0,\beta_0}(\Omega)$ and

$$f(x,t,s_1) \leq f(x,t,s_2), \quad g(x,t_1,s) \leq g(x,t_2,s) \tag{3.3}$$

for every x in Ω, $0 \leq t, s \leq \delta$, $0 \leq t_1 \leq t_2$ and $0 \leq s_1 \leq s_2$, where δ is a positive number.

Lemma 3.1 combined with the assumptions above establish the existence of a positive solution for system (1.4):

THEOREM 3.1. *Let f and g be non-negative Carathéodory functions satisfying the condition* (3.1) *with ε_0 provided in Lemma* 3.1. *Further if the assumptions* (3.2) *and* (3.3) *are fulfilled, then the system* (1.4) *possesses a positive solution.*

PROOF. At first, we show that the problem (1.4) has a positive subsolution. In fact, choose a positive eigenfunction (φ, ψ) associated to the principal eigenvalue (λ_0, μ_0) such that $\varphi, \psi \leq \min\{\delta_0, \delta\}$ in Ω. From the condition (3.2), it follows that (φ, ψ) is a positive subsolution of the system (1.4). Now, using the existence of a subsolution, we prove that the system (1.4) has a positive solution. Define the functions

$$F(x,t,s) = \begin{cases} f(x,t,s) & \text{if } t \geq \varphi(x) \text{ and } s \geq \psi(x), \\ f(x,t,\psi(x)) & \text{if } t \geq \varphi(x) \text{ and } s < \psi(x), \\ f(x,\varphi(x),s) & \text{if } t < \varphi(x) \text{ and } s \geq \psi(x), \\ f(x,\varphi(x),\psi(x)) & \text{if } t < \varphi(x) \text{ and } s < \psi(x), \end{cases}$$

$$G(x,t,s) = \begin{cases} g(x,t,s) & \text{if } t \geq \varphi(x) \text{ and } s \geq \psi(x), \\ g(x,t,\psi(x)) & \text{if } t \geq \varphi(x) \text{ and } s < \psi(x), \\ g(x,\varphi(x),s) & \text{if } t < \varphi(x) \text{ and } s \geq \psi(x), \\ g(x,\varphi(x),\psi(x)) & \text{if } t < \varphi(x) \text{ and } s < \psi(x) \end{cases}$$

Clearly, F and G are Carathéodory functions which satisfy the growth condition (3.1) with $\varepsilon_0 > 0$ given in Lemma 3.1 and some non-negative function k in $L^p(\Omega)$. By Lemma 3.1, the system

$$\begin{cases} -\mathcal{L}_1 u = F(x,u,v) & \text{in } \Omega \\ -\mathcal{L}_2 v = G(x,u,v) & \\ u = 0 = v & \text{on } \partial\Omega \end{cases}$$

admits a non-negative solution (u, v). We claim that $u \geq \varphi$ in Ω. Otherwise, the set $\Omega^- = \{x \in \Omega : u(x) < \varphi(x)\}$ is a non-empty open subset of Ω. Given x in Ω^-, if the inequality $v(x) \geq \psi(x)$ holds, by (3.3), it follows that $f(x,\varphi(x),\psi(x)) - F(x,u(x),v(x)) \leq 0$, and if $v(x) < \psi(x)$, by definition, we obtain $f(x,\varphi(x),\psi(x)) - F(x,u(x),v(x)) = 0$. So, we derive $\mathcal{L}_1(\varphi - u) \geq 0$ in Ω^- and $\varphi = u$ on $\partial\Omega^-$. Using the (SMP), we arrive at a contradiction. Therefore, we conclude that $u \geq \varphi$ in Ω and with similar ideas, we get $v \geq \psi$ in Ω. Finally, by the definition of F and G, one sees that (u, v) is a positive solution of the system (1.4). □

As consequences of Theorem 3.1, we present below some applications:

EXAMPLE 3.2. Consider the following systems

$$\begin{cases} -\mathcal{L}_1 u = \eta(x)u^\theta + \rho(x)v^\alpha & \text{in } \Omega \\ -\mathcal{L}_2 v = \tau(x)u^\beta + \xi(x)v^\gamma & \\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

$$\begin{cases} -\mathcal{L}_1 u = \rho_1(x)v^{\alpha_1} + \rho_2(x)v^{\alpha_2} & \text{in } \Omega \\ -\mathcal{L}_2 v = \tau_1(x)u^{\beta_1} + \tau_2(x)u^{\beta_2} & \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

Suppose that $\theta, \gamma, \alpha, \alpha_i, \beta, \beta_i > 0$, $\theta, \gamma < 1$, $\alpha\beta < 1$, $\max\{\alpha_1, \alpha_2\} \cdot \max\{\beta_1, \beta_2\} < 1$ and η, ξ, ρ, ρ_i, τ and τ_i are non-negative functions in $L^p(\Omega)$ such that ρ, τ, $\rho_1 + \rho_2$ and $\tau_1 + \tau_2$ are not identically null in Ω. Under these hypotheses, clearly the systems (3.4) and (3.5) verify the conditions required in Theorem 3.1. Therefore, these systems possesses a positive solution. In particular, for α, β, ρ and τ fulfilling the same conditions, we establish the existence of a positive solution for the problem

$$(3.6) \qquad \begin{cases} -\mathcal{L}_1 u = \rho(x)v^\alpha & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = \tau(x)u^\beta & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

REMARK 3.1. Several authors have obtained results on existence of non-trivial solutions for system (3.6) when $\mathcal{L}_i$ is the Laplacian, see [12], [17], [29] and [31]. The operators and nonlinearities studied here are more general than those investigated in the mentioned references.

Now we discuss the existence of solutions for systems subject to other non-ressonance conditions. Before stating the next theorem, we consider a classical result of the degree theory in cones due to Krasnoselskii, see [23]:

LEMMA 3.2. *Let C_0 be a cone in a Banach space X_0 and let $S : C_0 \to C_0$ be a compact mapping verifying $S(0) = 0$. Suppose that there are $s_0 > 0$ and $0 < r < R$ such that*

(i) $u \neq sSu$ for every u in C_0 with $\|u\|_{X_0} = r$ and $0 \leq s \leq 1$,

(ii) There is a compact mapping $H : [0, +\infty) \times C_0 \to C_0$ satisfying

(ii.1) $H(0, u) = Su$ for every u in C_0 with $\|u\|_{X_0} \leq R$,

(ii.2) $H(s, u) \neq u$ for every u in C_0 with $\|u\|_{X_0} \leq R$ and $s \geq s_0$,

(ii.3) $H(s, u) \neq u$ for every u in C_0 with $\|u\|_{X_0} = R$ and $s \geq 0$.

Then the mapping S admits a fixed point u in C_0 such that $r < \|u\|_{X_0} < R$.

Assume the following assumptions:

$$(3.7) \qquad f(x, t, s) \leq \lambda_0 \rho_0(x) s^{\alpha_0}, \quad g(x, t, s) \leq \mu_0 \tau_0(x) t^{\beta_0}$$

for every x in Ω and $0 \leq t, s \leq \delta_0$, where α_0, β_0 and δ_0 are positive numbers, $\alpha_0\beta_0 = 1$, ρ_0 and τ_0 are non-negative not identically null functions in $L^p(\Omega)$ and (λ_0, μ_0) belongs to $G^{\rho_0, \tau_0}_{\alpha_0, \beta_0}(\Omega)$,

$$(3.8) \qquad f(x, t, s) \leq k'(x)(t + s^\alpha + 1), \quad g(x, t, s) \leq k'(x)(t^\beta + s + 1)$$

for every x in Ω and $t, s \geq 0$, where α and β are positive numbers with $\alpha\beta = 1$ and k' is a non-negative function in $L^p(\Omega)$ and

$$(3.9) \qquad f(x, t, s) \geq \lambda\rho(x)s^\alpha - k(x), \quad g(x, t, s) \geq \mu\tau(x)t^\beta - k(x)$$

for every x in Ω and $t, s \geq 0$, where λ and μ are positive numbers, k, ρ and τ are non-negative not identically null functions in $L^p(\Omega)$ and (λ, μ) does not belong to $\overline{G^{\rho, \tau}_{\alpha, \beta}(\Omega)}$.

Note that the condition (3.7) implies $f(x,0,0) = 0 = g(x,0,0)$ for every x in Ω. However, the next result shows the existence of a non-trivial solution under the hypotheses stated above:

THEOREM 3.2. *If f and g are non-negative Carathéodory functions fulfilling the conditions* (3.7), (3.8) *and* (3.9), *then the system* (1.4) *possesses a non-trivial non-negative solution.*

PROOF. Consider the mapping $H : [0,+\infty)\times C \to C$ defined by $H(s,u,v) = (z,w)$, where (z,w) satisfies

$$\begin{cases} -\mathcal{L}_1 z = f(x,u,v) \ + \ sk(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 w = g(x,u,v) \ + \ sk(x) & \\ z = 0 = w & \text{on} \quad \partial\Omega \end{cases}$$

By the hypothesis (3.8), L^p theory and the (SMP), the mapping H is well-defined, continuous and compact. Also, it follows from (3.7) that $H(0,0,0) = 0$. Define $S : C \to C$ by $S(u,v) = H(0,u,v)$. At first, we prove that (i) of Lemma 3.2 is valid for $0 < r \le \delta_0$. Suppose that $(u,v) = sS(u,v)$ for some $0 \le s \le 1$ and (u,v) in C with $\|(u,v)\|_X = r$. Then, from (3.7), we get

$$\begin{cases} -\mathcal{L}_1 u = sf(x,u,v) & \le & \lambda_0\rho_0(x)v^{\alpha_0} & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = sg(x,u,v) & \le & \mu_0\tau_0(x)u^{\beta_0} & \\ u = 0 = v & & & \text{on} \quad \partial\Omega \end{cases}$$

Using (ii) of Proposition 2.12, we derive a contradiction. Now we show (ii) of Lemma 3.2. Assume that $H(s,u,v) = (u,v)$ for some s in $[0,+\infty)$ and (u,v) in C. By the definition of H, (u,v) verifies

$$\begin{cases} -\mathcal{L}_1 u = f(x,u,v) \ + \ sk(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = g(x,u,v) \ + \ sk(x) & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

Replacing the condition (3.9) in the system above, we obtain

$$\begin{cases} -\mathcal{L}_1 u \ge \lambda\rho(x)v^{\alpha} + sk(x) - k(x) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v \ge \mu\tau(x)u^{\beta} + sk(x) - k(x) & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases}$$

From the (SMP) and (i) of Proposition 2.12, one sees that $s \le 1$. We affirm that there is a constant $R > r$ such that $\|(u,v)\|_X < R$ for every (u,v) in C satisfying $H(s,u,v) = (u,v)$ for some $0 \le s \le 1$. Otherwise, there are sequences $\{s_n\}_0^\infty$ of real numbers in $[0,1]$ and $\{(u_n,v_n)\}_0^\infty$ of couples in C verifying $H(s_n,u_n,v_n) = (u_n,v_n)$ and $\|(u_n,v_n)\|_X \to +\infty$. Let $\tilde{u}_n$ and $\tilde{v}_n$ be the normalized functions as in the proof of Theorem 2.1. Since $\|\tilde{u}_n\|_{C(\overline{\Omega})} + \|\tilde{v}_n\|^{\alpha}_{C(\overline{\Omega})} = 1$, applying L^p estimates to the system satisfied by $(\tilde{u}_n,\tilde{v}_n)$ and using the condition (3.8), we conclude that $\tilde{u}_n \to \tilde{u}$ and $\tilde{v}_n \to \tilde{v}$ in $W^{2,p}(\Omega)$. Thus, it follows that $\|\tilde{u}\|_{C(\overline{\Omega})} + \|\tilde{v}\|^{\alpha}_{C(\overline{\Omega})} = 1$ and, from (3.9), we see that

$$\begin{cases} -\mathcal{L}_1\tilde{u} \geq \lambda\rho(x)\tilde{v}^{\alpha} & \text{in} \quad \Omega \\ -\mathcal{L}_2\tilde{v} \geq \mu\tau(x)\tilde{u}^{\beta} & \\ \tilde{u} = 0 = \tilde{v} & \text{on} \quad \partial\Omega \end{cases}$$

Again, from the (SMP) and (i) of Proposition 2.12, we obtain a contradiction. Applying Lemma 3.2, we conclude the proof. □

We give below an application from Theorem 3.2:

EXAMPLE 3.3. If f and g are non-negative Carathéodory functions verifying

$$\limsup_{t\to 0+}\frac{f(x,t)}{\rho_0(x)t^{\alpha_0}} < \lambda_0\,, \quad \limsup_{t\to 0+}\frac{g(x,t)}{\tau_0(x)t^{\beta_0}} < \mu_0\,,$$

$$\liminf_{t\to+\infty}\frac{f(x,t)}{\rho(x)t^{\alpha}} > \lambda\,, \quad \liminf_{t\to+\infty}\frac{g(x,t)}{\tau(x)t^{\beta}} > \mu$$

uniformly for x in Ω and

$$f(x,t) \leq k'(x)(t^{\alpha}+1)\,, \quad g(x,t) \leq k'(x)(t^{\beta}+1)$$

for every x in Ω and $t \geq 0$, where α_0, β_0, α, β, ρ_0, τ_0, ρ, τ, k', λ_0, μ_0, λ and μ are provided in the assumptions (3.7), (3.8) and (3.9), then Theorem 3.2 shows the existence of a positive solution for the system

$$\begin{cases} -\mathcal{L}_1 u = f(x,v) & \text{in} \quad \Omega \\ -\mathcal{L}_2 v = g(x,u) & \\ u = 0 = v & \text{on} \quad \partial\Omega \end{cases} \tag{3.10}$$

Further if the conditions $\rho_0 \equiv \rho$ and $\tau_0 \equiv \tau$ in Ω, $\alpha_0 \neq \alpha$ and $\beta_0 \neq \beta$ are assumed, then the functions

$$f(x,t) = \begin{cases} \lambda_0\rho(x)t^{\alpha_0} & \text{if } 0 \leq t \leq \left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{\alpha-\alpha_0}}, \\ \lambda\rho(x)t^{\alpha} & \text{if } t > \left(\frac{\lambda_0}{\lambda}\right)^{\frac{1}{\alpha-\alpha_0}}, \end{cases}$$

$$g(x,t) = \begin{cases} \mu_0\tau(x)t^{\beta_0} & \text{if } 0 \leq t \leq \left(\frac{\mu_0}{\mu}\right)^{\frac{1}{\beta-\beta_0}}, \\ \mu\tau(x)t^{\beta} & \text{if } t > \left(\frac{\mu_0}{\mu}\right)^{\frac{1}{\beta-\beta_0}} \end{cases}$$

satisfy the conditions required in Theorem 3.2.

REMARK 3.2. In the paper [29], we have stated a conjecture which in the case $m = 2$ coincides with Conjecture 1.1 of the introduction. According to the results developped in Section 2, we see that Theorem 3.2 proves this conjecture.

4. – Uniqueness for systems in bounded domains

In this section we investigate the uniqueness of positive solution for certain systems with strictly sublinear character. Besides, we analyze the relation of positive solutions of two different sublinear systems. In what follows, consider the conditions:

$$f(x, t_1, s_1) \le f(x, t_2, s_2), \quad g(x, t_1, s_1) \le g(x, t_2, s_2) \tag{4.1}$$

for every x in Ω, $0 < t_1 \le t_2$ and $0 < s_1 \le s_2$ and

$$\begin{aligned} f(x, s^{\frac{\beta+1}{\beta}} t_1, s^{\frac{\alpha+1}{\alpha}} t_2) &\ge s^{\alpha+1} f(x, t_1, t_2)\,, \\ g(x, s^{\frac{\beta+1}{\beta}} t_1, s^{\frac{\alpha+1}{\alpha}} t_2) &\ge s^{\beta+1} g(x, t_1, t_2) \end{aligned} \tag{4.2}$$

for every x in Ω, $0 < s \le 1$ and $t_1, t_2 > 0$, where α and β are positive numbers.

In order to show the first uniqueness result of positive solution for strictly sublinear systems, also assume that

$$\begin{aligned} f(x_0, t_1, s_1) < f(x_0, t_2, s_2) \quad &\text{for every} \quad 0 < t_1 \le t_2 \quad \text{and} \quad 0 < s_1 < s_2\,, \\ g(x_1, t_1, s_1) < g(x_1, t_2, s_2) \quad &\text{for every} \quad 0 < t_1 < t_2 \quad \text{and} \quad 0 < s_1 \le s_2 \end{aligned} \tag{4.3}$$

for some x_0 and x_1 in Ω.

Under the conditions above, we derive the following uniqueness result:

THEOREM 4.1. *Assume $\alpha\beta < 1$. If f and g are non-negative functions satisfying the conditions* (4.1), (4.2) *and* (4.3), *then the system* (1.4) *admits, at most, one positive solution.*

PROOF. Let (u_1, v_1) and (u_2, v_2) be positive solutions of the problem (1.4). Consider the auxiliar set $S = \{s > 0 \ : \ u_2 > s^{\frac{\beta+1}{\beta}} u_1$ and $v_2 > s^{\frac{\alpha+1}{\alpha}} v_1$ in $\Omega\}$ and denote $s^* = \sup S$. Permuting (u_1, v_1) and (u_2, v_2), if necessary, without loss of generality, we can assume that $s^* \le 1$. If $s^* < 1$, from assumptions (4.1), (4.2) and (4.3), we obtain the relations

$$\begin{aligned} -\mathcal{L}_1\Big(u_2 - s^{*\frac{\beta+1}{\beta}} u_1\Big) &= f(x, u_2, v_2) - s^{*\frac{\beta+1}{\beta}} f(x, u_1, v_1) \\ &\ge f(x, u_2, v_2) - s^{*\frac{\beta+1}{\beta}} f\Big(x, \frac{u_2}{s^{*\frac{\beta+1}{\beta}}}, \frac{v_2}{s^{*\frac{\alpha+1}{\alpha}}}\Big) \\ &\ge \Big(1 - s^{*\frac{1-\alpha\beta}{\beta}}\Big) f(x, u_2, v_2) \ge 0\,, \\ -\mathcal{L}_1\Big(u_2 - s^{*\frac{\beta+1}{\beta}} u_1\Big)(x_0) &\ge \Big(1 - s^{*\frac{1-\alpha\beta}{\beta}}\Big) f(x_0, u_2(x_0), v_2(x_0)) > 0 \end{aligned}$$

and in a similar manner, we get

$$-\mathcal{L}_2\Big(v_2 - s^{*\frac{\alpha+1}{\alpha}} v_1\Big) \geq \Big(1 - s^{*\frac{1-\alpha\beta}{\alpha}}\Big) g(x, u_2, v_2) \geq 0\,,$$
$$-\mathcal{L}_2\Big(v_2 - s^{*\frac{\alpha+1}{\alpha}} v_1\Big)(x_1) \geq \Big(1 - s^{*\frac{1-\alpha\beta}{\alpha}}\Big) g(x_1, u_2(x_1), v_2(x_1)) > 0\,.$$

Using the (SMP), we arrive at a contradiction. So, it follows that $s^* = 1$. We affirm that the identities $u_2 \equiv u_1$ and $v_2 \equiv v_1$ hold in Ω. By contradiction, if $u_2 \not\equiv u_1$ or $v_2 \not\equiv v_1$ in Ω, using the inequalities

$$-\mathcal{L}_1(u_2 - u_1) = f(x, u_2, v_2) - f(x, u_1, v_1) \geq 0\,,$$
$$-\mathcal{L}_2(v_2 - v_1) = g(x, u_2, v_2) - g(x, u_1, v_1) \geq 0$$

and the (SMP), we conclude that $u_2 > (1+\varepsilon)^{\frac{\beta+1}{\beta}} u_1$ or $v_2 > (1+\varepsilon)^{\frac{\alpha+1}{\alpha}} v_1$ in Ω for $\varepsilon \sim 0$. Now, using the condition (4.3) together with the relations above, it is easy to see that $u_2 > (1+\varepsilon)^{\frac{\beta+1}{\beta}} u_1$ and $v_2 > (1+\varepsilon)^{\frac{\alpha+1}{\alpha}} v_1$ in Ω for $\varepsilon \sim 0$, a contradiction. Hence, we finish the proof. □

When $f(x,t,s) = f(x,s)$ and $g(x,t,s) = g(x,t)$, we can weaken the condition (4.3). Indeed, suppose that

(4.4) $\quad f(x_0,\cdot)$ and $g(x_1,\cdot)$ are positive functions on $\mathbb{R}_+$

for some x_0 and x_1 in Ω.

If the condition (4.4) is assumed instead of (4.3), we also obtain the uniqueness of positive solution for system (3.10):

COROLLARY 4.1. *Assume $\alpha\beta < 1$. If f and g are non-negative functions verifying the conditions* (4.1), (4.2) *and* (4.4), *then the system* (3.10) *possesses, at most, one positive solution.*

PROOF. Let S and s^* be as in the proof of Theorem 4.1. Arguing exactly as in the previous theorem, we conclude that $s^* = 1$. On the other hand, since the functions f and g depend only on two variables, it follows that $u_2 \equiv u_1$ in Ω if and only if $v_2 \equiv v_1$ in Ω. Therefore, assuming on the contrary that $u_2 \not\equiv u_1$ and $v_2 \not\equiv v_1$ in Ω and using only the assumption (4.1) together with the (SMP), we derive an absurd. □

REMARK 4.1. Under the conditions stated in Example 3.2, we see that the systems (3.5) and (3.6) satisfy the hypotheses required in Theorem 4.1. So, in this case, we establish the existence and uniqueness of positive solutions for systems (3.5) and (3.6) by Theorems 3.1 and 4.1. Further if the restrictions $\theta \leq \frac{\beta(\alpha+1)}{\beta+1}$ and $\gamma \leq \frac{\alpha(\beta+1)}{\alpha+1}$ are imposed, it follows that the system (3.4) also verifies the assumptions of Theorem 4.1.

Now let us discuss the relation of positive solutions of two different systems. Let $\tilde{f}$ and $\tilde{g}$ be non-negative functions satisfying the conditions

$$\tilde{f}(x, t_1, s_1) \leq f(x, t_2, s_2), \quad \tilde{g}(x, t_1, s_1) \leq g(x, t_2, s_2) \tag{4.5}$$

for every x in Ω, $0 < t_1 \leq t_2$ and $0 < s_1 \leq s_2$ and

$$\begin{aligned} &\tilde{f}(x_0, t_1, s_1) < f(x_0, t_2, s_2) \quad \text{for every } 0 < t_1 \leq t_2 \text{ and } 0 < s_1 < s_2, \\ &\tilde{g}(x_1, t_1, s_1) < g(x_1, t_2, s_2) \quad \text{for every } 0 < t_1 < t_2 \text{ and } 0 < s_1 \leq s_2 \end{aligned} \tag{4.6}$$

for some x_0 and x_1 in Ω.

Consider the system

$$\begin{cases} -\mathcal{L}_1 u = \tilde{f}(x, u, v) & \text{in } \Omega \\ -\mathcal{L}_2 v = \tilde{g}(x, u, v) & \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{4.7}$$

The next result shows in particular that the positive solutions of the systems (1.4) and (4.7) are related:

THEOREM 4.2. *Assume $\alpha\beta = 1$ and the conditions* (4.1), (4.2) *for $s > 0$,* (4.5) *and* (4.6) *are fulfilled. If (u_2, v_2) is a positive supersolution of the system* (1.4) *and (u_1, v_1) is a positive subsolution of the problem* (4.7), *then the identities $u_2 \equiv tu_1$ and $v_2 \equiv sv_1$ are valid in Ω for some positive numbers t and s.*

PROOF. Let S and s^* be as in the proof of Theorem 4.1. We claim that the relations $u_2 \equiv s^{*\frac{\beta+1}{\beta}} u_1$ and $v_2 \equiv s^{*\frac{\alpha+1}{\alpha}} v_1$ occur in Ω. By similarity of the reasonings, it is sufficient to prove only that $u_2 \equiv s^{*\frac{\beta+1}{\beta}} u_1$ in Ω. Otherwise, from the estimate

$$\begin{aligned} -\mathcal{L}_1\left(u_2 - s^{*\frac{\beta+1}{\beta}} u_1\right) &\geq f(x, u_2, v_2) - s^{*\frac{\beta+1}{\beta}} \tilde{f}(x, u_1, v_1) \\ &\geq f(x, u_2, v_2) - s^{*\frac{\beta+1}{\beta}} f(x, \frac{u_2}{s^{*\frac{\beta+1}{\beta}}}, \frac{v_2}{s^{*\frac{\alpha+1}{\alpha}}}) \geq 0, \end{aligned}$$

and the (SMP), it follows that $u_2 > (s^* + \varepsilon)^{\frac{\beta+1}{\beta}} u_1$ in Ω for $\varepsilon \sim 0$. Therefore, using the inequalities

$$\begin{aligned} -\mathcal{L}_2\left(v_2 - s^{*\frac{\alpha+1}{\alpha}} v_1\right) &\geq g(x, u_2, v_2) - s^{*\frac{\alpha+1}{\alpha}} \tilde{g}(x, u_1, v_1) \\ &\geq g(x, u_2, v_2) - s^{*\frac{\alpha+1}{\alpha}} g(x, \frac{u_2}{s^{*\frac{\beta+1}{\beta}}}, \frac{v_2}{s^{*\frac{\alpha+1}{\alpha}}}) \geq 0, \end{aligned}$$

$$\begin{aligned} -\mathcal{L}_2\left(v_2 - s^{*\frac{\alpha+1}{\alpha}} v_1\right)(x_1) &\geq g(x_1, u_2(x_1), v_2(x_1)) - s^{*\frac{\alpha+1}{\alpha}} \tilde{g}(x_1, u_1(x_1), v_1(x_1)) \\ &> g(x_1, u_2(x_1), v_2(x_1)) - s^{*\frac{\alpha+1}{\alpha}} g(x_1, \frac{u_2(x_1)}{s^{*\frac{\beta+1}{\beta}}}, \frac{v_2(x_1)}{s^{*\frac{\alpha+1}{\alpha}}}) \geq 0 \end{aligned}$$

together with the (SMP), we obtain a contradiction. □

As direct consequences of the proof of Theorem 4.2, we derive the following corollaries:

COROLLARY 4.2. *Assume* $\alpha\beta = 1$ *and the conditions* (4.1), (4.2) *and* (4.3) *are fulfilled. If* (u_1, v_1) *and* (u_2, v_2) *are two positive solutions of the system* (1.4), *then the identities* $u_2 \equiv tu_1$ *and* $v_2 \equiv sv_1$ *hold in* Ω *for some positive numbers* t *and* s.

COROLLARY 4.3. *Besides the mentioned hypotheses in Corollary* 4.2, *if the function* f *or* g *satisfies the condition* (4.2) *with strict inequality in some point* x_2 *in* Ω *and for every* $0 < s < 1$, *then the system* (1.4) *possesses, at most, one positive solution.*

5. – Existence for systems in the whole space

In this last section we search for necessary and sufficient conditions which guarantee the existence of a bounded positive solution for the system

$$\left\{ \begin{array}{ll} -\mathcal{L}_1 u = \rho(x) v^\alpha \\ -\mathcal{L}_2 v = \tau(x) u^\beta \end{array} \right. \quad \text{in} \quad \mathbb{R}^N, \tag{5.1}$$

where α and β are positive numbers with $\alpha\beta < 1$ and ρ and τ are non-negative not identically null functions in $L^p_{\text{loc}}(\mathbb{R}^N)$.

Let us begin introducing a definition which extends that well-known in the scalar case due to Brézis and Kamin, see [8]:

DEFINITION 5.1. The couple (ρ, τ) is said to verify the property (H) if the non-homogeneous problems

$$-\mathcal{L}_1 U = \rho(x) \quad \text{and} \quad -\mathcal{L}_2 V = \tau(x) \quad \text{in} \quad \mathbb{R}^N \tag{5.2}$$

admit bounded positive solutions.

Our first result furnishes a sufficient condition for the existence of a bounded positive solution for system (5.1):

THEOREM 5.1. *If the couple* (ρ, τ) *verifies the property* (H), *then the system* (5.1) *admits a bounded positive solution which is a minimal solution.*

PROOF. Choose $R_0 > 0$ such that the functions ρ and τ are not identically null in B_{R_0}. By Remark 4.1, for each $R > R_0$, the system

$$\left\{ \begin{array}{ll} -\mathcal{L}_1 u = \rho(x) v^\alpha & \text{in} \quad B_R \\ -\mathcal{L}_2 v = \tau(x) u^\beta & \\ u = 0 = v & \text{on} \quad \partial B_R \end{array} \right. \tag{5.3}$$

possesses a unique positive solution (u_R, v_R). Take $\alpha_0 > \alpha$ and $\beta_0 > \beta$ with $\alpha_0\beta_0 = 1$. Denote by (φ, ψ) a positive eigenfunction associated to some principal eigenvalue (λ_0, μ_0) of $\Lambda^{\rho,\tau}_{\alpha_0,\beta_0}(B_R)$. Putting $\underline{u} = t^{\alpha_0}\varphi$ and $\underline{v} = t\psi$, we conclude that $(\underline{u}, \underline{v})$ is a positive subsolution of the problem (5.3) for $t > 0$ small enough. Let $\tilde{R} > R$. Diminishing t, if necessary, it follows that $\underline{u} \le u_{\tilde{R}}$ and $\underline{v} \le v_{\tilde{R}}$ in B_R. Since $(u_{\tilde{R}}, v_{\tilde{R}})$ is a supersolution of (5.3), using Lemma 2.1 and the uniqueness of positive solution for system (5.3), we obtain $\underline{u} \le u_R \le u_{\tilde{R}}$ and $\underline{v} \le v_R \le v_{\tilde{R}}$ in B_R. Let U and V be bounded positive solutions of the problems in (5.2). We claim that $u_R \le c_1 U$ and $v_R \le c_2 V$ in B_R for some positive constants c_1 and c_2 not depending on R. In fact, take $d > 0$ such that $U^\beta, V^\alpha \le d$ in $\mathbb{R}^N$. Setting $c_1 = d^{\frac{1+\alpha}{1-\alpha\beta}}$ and $c_2 = d^{\frac{1+\beta}{1-\alpha\beta}}$, we derive

$$\begin{cases} -\mathcal{L}_1(c_1U) = c_1\rho(x) = c_2^\alpha d\rho(x) \ge \rho(x)(c_2V)^\alpha \\ -\mathcal{L}_2(c_2V) = c_2\tau(x) = c_1^\beta d\tau(x) \ge \tau(x)(c_1U)^\beta \end{cases} \quad \text{in} \quad \mathbb{R}^N.$$

Since $(0, 0)$ is a subsolution and (c_1U, c_2V) is a positive supersolution of the system (5.3), again applying Lemma 2.1 and the uniqueness of positive solution, we get $u_R \le c_1U$ and $v_R \le c_2V$ in B_R for every $R > R_0$. Therefore, the limits $u_R(x) \to u(x)$ and $v_R(x) \to v(x)$ as $R \to +\infty$ exist for every x in $\mathbb{R}^N$ and (u, v) is a positive solution of the system (5.1) satisfying the inequalities $u \le c_1U$ and $v \le c_2V$ in $\mathbb{R}^N$. Clearly, (u, v) is a minimal solution. Indeed, since each bounded positive solution $(\tilde{u}, \tilde{v})$ of the problem (5.1) is a supersolution of the problem (5.3), it follows that $u_R \le \tilde{u}$ and $v_R \le \tilde{v}$ in B_R for every $R > R_0$. Hence, letting $R \to +\infty$, we conclude that $u \le \tilde{u}$ and $v \le \tilde{v}$ in $\mathbb{R}^N$. □

The next result presents a necessary condition for the existence of bounded positive solutions for problem (5.1):

THEOREM 5.2. *Assume that $\mathcal{L}_i \equiv \Delta$ for $i = 1, 2$ and $k_1\rho(x) \le \tau(x) \le k_2\rho(x)$ for x in $\mathbb{R}^N$ and some positive constants k_1 and k_2. If the system* (5.1) *admits a bounded positive solution, then the couple (ρ, τ) satisfies the property (H).*

The proof of Theorem 5.2 depends on the following lemma whose proof of a more general version can be found in [30]:

LEMMA 5.1. *Assume that $\mathcal{L}_i \equiv \Delta$ for $i = 1, 2$. Let (u, v) be a positive supersolution of the problem* (5.1) *and define the function $w = uv$. Then w satisfies*

$$\Delta w + \rho(x)^{\frac{\beta+1}{\alpha+\beta+2}}\tau(x)^{\frac{\alpha+1}{\alpha+\beta+2}}w^{\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+2}} \le \frac{|\nabla w|^2}{2w} \quad \text{in} \quad \mathbb{R}^N.$$

PROOF OF LEMMA 5.1. Clearly, the relations

$$\frac{|\nabla w|^2}{w} = 2(\nabla u, \nabla v)_{\mathbb{R}^N} + \frac{v}{u}|\nabla u|^2 + \frac{u}{v}|\nabla v|^2, \tag{5.4}$$

$$2(\nabla u, \nabla v)_{\mathbb{R}^N} \le \frac{v}{u}|\nabla u|^2 + \frac{u}{v}|\nabla v|^2, \tag{5.5}$$

$$\Delta w = 2(\nabla u, \nabla v)_{\mathbb{R}^N} + v\Delta u + u\Delta v \tag{5.6}$$

are satisfied. Combining (5.4) and (5.5), it follows that $2(\nabla u, \nabla v)_{\mathbb{R}^N} \le \frac{|\nabla w|^2}{2w}$. So, since (u, v) is a positive supersolution of the system (5.1), from (5.6), we obtain

$$\Delta w \le \frac{|\nabla w|^2}{2w} - \rho(x) v^{\alpha+1} - \tau(x) u^{\beta+1}.$$

Finally, from Young's inequality, we conclude that

$$\Delta w + \rho(x)^{\frac{\beta+1}{\alpha+\beta+2}} \tau(x)^{\frac{\alpha+1}{\alpha+\beta+2}} w^{\frac{(\alpha+1)(\beta+1)}{\alpha+\beta+2}} \le \frac{|\nabla w|^2}{2w} + \left(\frac{\beta+1}{\alpha+\beta+2} - 1\right) \rho(x) v^{\alpha+1}$$
$$+ \left(\frac{\alpha+1}{\alpha+\beta+2} - 1\right) \tau(x) u^{\beta+1} \le \frac{|\nabla w|^2}{2w}. \qquad \square$$

PROOF OF THEOREM 5.2. Let (u, v) be a bounded positive solution of the system (5.1). Take $w = uv$ and $\gamma = \frac{(\alpha+1)(\beta+1)}{\alpha+\beta+2}$. Since $\alpha\beta < 1$, one sees that $\frac{1}{2} < \gamma < 1$. Define the function $U_0 = \frac{1}{1-\gamma} w^{1-\gamma}$. Using Lemma 5.1, we obtain the estimate

$$-\Delta U_0 \ge \gamma w^{-\gamma-1} |\nabla w|^2 - \frac{1}{2} w^{-\gamma-1} |\nabla w|^2 + \rho(x)^{\frac{\beta+1}{\alpha+\beta+2}} \tau(x)^{\frac{\alpha+1}{\alpha+\beta+2}}$$
$$\ge \rho(x)^{\frac{\beta+1}{\alpha+\beta+2}} \tau(x)^{\frac{\alpha+1}{\alpha+\beta+2}} \ge k_1^{\frac{\alpha+1}{\alpha+\beta+2}} \rho(x).$$

Therefore, we conclude that $\tilde{U} = k_1^{-\frac{\alpha+1}{\alpha+\beta+2}} U_0$ is a bounded positive function verifying $-\Delta \tilde{U} \ge \rho(x)$ in $\mathbb{R}^N$. Arguing as above, we can construct a bounded positive function $\tilde{V}$ satisfying $-\Delta \tilde{V} \ge \tau(x)$ in $\mathbb{R}^N$. Choose $R_0 > 0$ such that the functions ρ and τ are not identically null in B_{R_0}. For each $R > R_0$, denote by U_R and V_R the positive solutions of the problems

$$\text{(5.7)} \qquad \begin{cases} -\Delta U = \rho(x) & \text{in } B_R \\ U = 0 & \text{on } \partial B_R, \end{cases}$$

$$\text{(5.8)} \qquad \begin{cases} -\Delta V = \tau(x) & \text{in } B_R \\ V = 0 & \text{on } \partial B_R, \end{cases}$$

respectively. Since the functions $\tilde{U}$ and $\tilde{V}$ are positive supersolutions of the problems (5.7) and (5.8), from the (SMP), we conclude that $U_R \le \tilde{U}$ and $V_R \le \tilde{V}$ in B_R. In addition, with similar ideas, we derive the inequalities $U_R \le U_{\tilde{R}}$ and $V_R \le V_{\tilde{R}}$ in B_R for $\tilde{R} > R$. Therefore, the limits $U_R(x) \to U(x)$ and $V_R(x) \to V(x)$ as $R \to +\infty$ exist for every x in $\mathbb{R}^N$ and the functions U and V are bounded positive solutions of the equations in (5.2). $\square$

Joining Theorems 5.1 and 5.2, we derive the following result:

COROLLARY 5.1. *If* $\mathcal{L}_i \equiv \Delta$ *for* $i = 1, 2$ *and* $\rho \equiv \tau$ *in* $\mathbb{R}^N$, *then the system* (5.1) *possesses a bounded positive solution if and only if* ρ *satisfies the property* (H).

The example below furnishes an application of Corollary 5.1:

EXAMPLE 5.1. Take the function $\rho(x) = \frac{1}{1+|x|^s}$ with $s > 0$. It is well-known that ρ verifies the property (H) if and only if $s > 2$, see appendix of the paper [8]. Therefore, if $\alpha, \beta > 0$ and $\alpha\beta < 1$, we conclude from Corollary 5.1 that the system

$$\begin{cases} -\Delta u &= \rho(x) v^{\alpha} \\ -\Delta v &= \rho(x) u^{\beta} \end{cases} \quad \text{in} \quad \mathbb{R}^N$$

possesses a bounded positive solution if and only if $s > 2$.

REFERENCES

[1] A. ALVINO – V. FERONE – G. TROMBETTI, *On the properties of some nonlinear eigenvalues*, SIAM J. Math. Anal., vol. 29, n. 2 (1998), 437-451.

[2] H. AMANN, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620-709.

[3] A. ANANE, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, C. R. Acad. Sci. Paris **305** (1987), 725-728.

[4] H. BERESTYCKI – L. NIRENBERG – S. R. S. VARADHAN, *The principal eigenvalue and maximum principle for second order elliptic operators in general domains*, Comm. Pure Appl. Math. **XLVII** (1994), 47-92.

[5] I. BIRINDELLI, *Hopf's lemma and anti-maximum principle in general domains*, J. Differential Equations **119** (1995), 450-472.

[6] I. BIRINDELLI – E. MITIDIERI – G. SWEERS, *Existence of the principal eigenvalue for cooperative elliptic systems in a general domain*, preprint.

[7] J. M. BONY, *Principe du maximum dans les espaces de Sobolev*, C. R. Acad. Sci. Paris Série A **265** (1967), 333-336.

[8] H. BRÉZIS – S. KAMIN, *Sublinear elliptic equations in* $\mathbb{R}^N$, Manuscripta Math. **74** (1992), 87-106.

[9] K. J. BROWN – C. C. LIN, *On the existence of positive eigenfunctions for a eigenvalue problem with indefinite weight-function*, J. Math. Anal. Appl. **75** (1980), 112-120.

[10] R. S. CANTRELL – K. SCHMIDT, *On the eigenvalue problem for coupled elliptic systems*, SIAM J. Math. Anal. **17** (1986), 850-862.

[11] PH. CLÉMENT – D. G. DE FIGUEIREDO – E. MITIDIERI, *Positive solutions of semilinear elliptic systems*, Comm. Partial Differential Equations, vol. 17, n. 5 & 6 (1992), 923-940.

[12] PH. CLÉMENT – R. C. A. M. VAN DER VORST, *On a semilinear elliptic system*, Differential Integral Equations **8** (1995), 1317-1329.

[13] D. G. DE FIGUEIREDO – P. L. FELMER, *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. **343** (1994), 99-116.

[14] B. DE PAGTER, *Irreducible compact operators*, Math. Z. **192** (1986), 149-153.

[15] M. D. DONSKER – S. R. S. VARADHAN, *On a variational formula for the principal eigenvalue for operators with maximum principle*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), 780-783.

[16] M. D. DONSKER – S. R. S. VARADHAN, *On the principal eigenvalue of second order elliptic differential operators*, Comm. Pure Appl. Math. **29** (1976), 595-621.

[17] P. L. FELMER – S. MARTÍNEZ, *existence and uniqueness of positive solutions to certain differential systems*, Adv. Differential Equations, vol. 3, n. 4 (1998), 575-593.

[18] D. GILBARG – N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order", Springer-Verlag, 1983.

[19] J.-P. GOSSEZ – E. LAMI-DOZO, *On the principal eigenvalue of a second order linear elliptic problem*, Arch. Rational Mech. Anal. **89** (1985), 169-175.

[20] P. HESS, *On the eigenvalue problem for weakly coupled elliptic systems*, Arch. Rational Mech. Anal. **81** (1983), 151-159.

[21] P. HESS – T. KATO, *On some linear and nonlinear eigenvalue problems with an indefinite weight function*, Comm. Partial Differential Equations **5** (1980), 999-1030.

[22] J. HULSHOF – R. C. A. M. VAN DER VORST, *Differential systems with strongly indefinite variational structure*, J. Funct. Anal. **114** (1993), 32-58.

[23] M. A. KRASNOSELSKII, *Fixed points of cone-compressing or cone-extending operators*, Sov. Math. Dokl. **1** (1960), 1285-1288.

[24] M. A. KRASNOSELSKII, "Positive Solutions of Operator Equations", Noordhoff, 1964.

[25] M. G. KREIN – M. A. RUTMAN, *Linear operators leaving invariant a cone in a Banach space*, Amer. Math. Soc. Transl. **10** (1962), 199-325.

[26] J. LÓPEZ-GÓMEZ, *The maximum principle and the existence of principal eigenvalue for some linear weighted boundary value problems*, J. Differential Equations **127** (1996), 263-294.

[27] J. LÓPEZ-GÓMEZ – M. MOLINA-MEYER, *The maximum principle for cooperative weakly coupled elliptic systems and some applications*, Differential Integral Equations, vol. 7, n. 2 (1994), 383-398.

[28] A. MANES – A. M. MICHELETTI, *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine*, Boll. Un. Mat. Ital. **7** (1973), 285-301.

[29] M. MONTENEGRO, *Sublinearity for semilinear elliptic systems*, preprint.

[30] M. MONTENEGRO, *Liouville type theorems and blow-up for semilinear differential systems*, preprint.

[31] M. MONTENEGRO, *Existence of solutions for some semilinear elliptic systems with singular coefficients*, to appear in Nonlinear Analysis.

[32] R. D. NUSSBAUM – Y. PINCHOVER, *On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications*, J. Anal. Math. **59** (1992), 161-177.

[33] M. H. PROTTER, *The generalized spectrum of second order elliptic systems*, Rocky Mountain J. Math. **9** (1979), 503-518.

[34] M. H. PROTTER – H. F. WEINBERGER, "Maximum Principles in Differential Equations", Prentice Hall, New Jersey, 1967.

[35] G. SWEERS, *Strong positivity in $C(\overline{\Omega})$ for elliptic systems*, Math. Z. **209** (1992), 251-271.

[36] R. C. A. M. VAN DER VORST, *Variational identities and applications to differential systems*, Arch. Rational Mech. Anal. **116** (1991), 375-398.

[37] W. WALTER, *Sturm-Liouville theory for the radial Δ_p-operator*, Math. Z. **227**, (1998) 175-185.

Departamento de Matemática
Universidade Federal de São Carlos
Rodovia Washington Luiz, Km 235
13565-905, SP, Brazil
montene@dm.ufscar.br