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A Descendent Relation in Genus 2

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Abstract. A new codimension 2 relation among descendent strata in the moduli space of stable 3-pointed genus 2 curves is found. The space of pointed admissible double covers is used in the calculation. The resulting differential equations satisfied by the genus 2 gravitational potentials of varieties in Gromov-Witten theory are described. These are analogous to the WDVV-equations in genus 0 and Getzler's equations in genus 1. As an application, genus 2 descendent invariants of the projective plane are determined, including the classical genus 2 Severi degrees.

Mathematics Subject Classification (1991): 14H10 (primary), 14D20, 14D22, 14N10 (secondary).

0. – Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of Deligne-Mumford stable n-pointed genus g complex algebraic curves. There is an algebraic stratification of $\overline{\mathcal{M}}_{g,n}$ by the underlying topological type of the pointed curve. A relation among the cycle classes (or homological classes) of the closures of these strata directly yields differential equations satisfied by generating functions of Gromov-Witten invariants of algebraic varieties. The translation from a relation to differential equations is obtained by the splitting axiom of Gromov-Witten theory ([RT1], [KM1], [BM]). In genus 0, all strata relations are obtained from the basic linear equivalence of the three boundary strata in $\overline{\mathcal{M}}_{0,4}$ ([KM2]). The corresponding differential equation is the Witten-Dijkgraaf-Verlinde-Verlinde equation. In genus 1, Getzler has found a codimension 2 relation in $\overline{\mathcal{M}}_{1,4}$ ([G1]). The resulting differential equation has been used to calculate elliptic Gromov-Witten invariants ([G1]) and to prove a genus 1 prediction of the Virasoro conjecture for $\mathbf{P}^2$ ([P]). Getzler's equation has been studied in the context of semi-simple Frobenius manifolds in [KK] and [DZ].

Let X be a nonsingular complex projective variety. Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the moduli space of stable maps representing the class $\beta \in H_2(X,\mathbb{Z})$. Evaluating maps at the marked points yields n evaluation morphisms $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$. The *descendent invariants* are the integrals

$$(1)\quad \langle(\gamma_1\psi_1^{\alpha_1})\cdots(\gamma_n\psi_n^{\alpha_n})\rangle_{g,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \mathrm{ev}_1^*(\gamma_1)\cup\psi_1^{\alpha_1}\cup\cdots\cup\mathrm{ev}_n^*(\gamma_n)\cup\psi_n^{\alpha_n},$$

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where $\gamma_i \in H^*(X, \mathbb{Q})$ and ψ_i is the first Chern class of the cotangent line on (the stack) $\overline{\mathcal{M}}_{g,n}(X, \beta)$ corresponding to the i-th marked point. These invariants play a central role in Gromov-Witten theory. For example, they arise naturally in expressions for flat sections in the Dubrovin formalism ([D], [Gi]), in the virtual normal bundle terms in torus localization formulas ([Ko], [Gi], [GP1]), and in the Virasoro conjecture [EHX]. A geometric interpretation of certain low genus descendent invariants of $\mathbf{P}^2$ in terms of the classical characteristic numbers of plane curves is given in [GP2]. Some foundational issues concerning descendent integrals are treated in [RT2], [KM3], and [G2].

In genus 0 and 1, the classes of the cotangent lines on $\overline{\mathcal{M}}_{g,n}$ may be expressed as sums of boundary divisor classes. Such expressions yield topological recursion relations among the descendent invariants and may be used to prove that in genus 0 and 1 the descendents are determined by Gromov-Witten invariants. For genus $g \geq 2$, divisorial topological recursion relations do not exist: the classes of the boundary divisors and the cotangent lines are independent in $A^1(\overline{\mathcal{M}}_{g,n})$. In genus 2, Getzler has determined weaker topological recursion relations from boundary expressions for ψ_1^2 and $\psi_1\psi_2$ ([G2]).

The topological type strata of $\overline{\mathcal{M}}_{g,n}$ are naturally indexed by *stable dual graphs* of genus g and valence n: the vertices, edges, and markings of the graph correspond to the components, nodes, and marked points of the curve whose moduli point lies in the stratum. Let V denote the set of vertices of the graph Γ. For every vertex $v \in V$, let $g(v) \in \mathbb{Z}_+$ be the genus of v — the geometric genus of the corresponding component of the stable curve. The valence $n(v) \in \mathbb{Z}_+$ of the vertex is the number of incident flags (markings or half-edges). The valence of the graph is defined as the number of markings on it — the number of marked points on the corresponding stable curve. The dual graph of a stratum is equivalent data to the topological type and marking distribution of the curves it parametrizes.

Let $\mathcal{M}_\Gamma$ denote the stratum in $\overline{\mathcal{M}}_{g,n}$ corresponding to the dual graph Γ. We then have

$$\overline{\mathcal{M}}_{g,n} = \coprod_{\Gamma} \mathcal{M}_\Gamma,$$

where the union is taken over all stable graphs of genus g and valence n. The strata $\mathcal{M}_\Gamma$ are irreducible and locally closed. Let $\overline{\mathcal{M}}_\Gamma$ denote the closure of $\mathcal{M}_\Gamma$ in $\overline{\mathcal{M}}_{g,n}$. The natural morphism

$$\tau : \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)} \longrightarrow \overline{\mathcal{M}}_\Gamma$$

is the quotient map modulo the finite group of automorphisms of Γ (up to normalization). Let f be a flag of Γ incident to the vertex v_f, and let ψ_f denote the corresponding cotangent line class on $\prod \overline{\mathcal{M}}_{g(v),n(v)}$ obtained from the factor $\overline{\mathcal{M}}_{g(v_f),n(v_f)}$. A *descendent stratum class* in $A^*(\overline{\mathcal{M}}_{g,n})$ is

$$\frac{1}{|\mathrm{Aut}(\Gamma)|} \iota_* \tau_*(m)$$

where m is a monomial in the cotangent line classes ψ_f corresponding to flags f of Γ and $\iota : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ is the inclusion map. In particular, the usual boundary stratum class $[\overline{\mathcal{M}}_\Gamma] \in A^*(\overline{\mathcal{M}}_{g,n})$ is obtained by pushing forward the trivial monomial.

Relations among the descendent stratum classes yield differential equations for the generating functions of descendent invariants of algebraic varieties. In physics, this generating function F_X for a nonsingular projective variety X is called the full gravitational potential function. In particular, the Virasoro conjecture states that $\exp(F_X)$ is annihilated by an explicit representation ρ_X of the affine Virasoro algebra in an algebra of differential operators ([EHX]).

In this paper we present a new genus 2 relation among codimension 2 descendent stratum classes in $\overline{\mathcal{M}}_{2,3}$. Getzler has computed that $h^4(\overline{\mathcal{M}}_{2,3}) = 44$ via a subtle method using mixed Hodge theory and modular operads ([G2]). The number of descendent stratum classes in $A^2(\overline{\mathcal{M}}_{2,3})$ is 47. Exactly 2 relations among them are obtained from the basic genus 0 linear equivalence on $\overline{\mathcal{M}}_{0,4}$. Therefore, there must exist a new relation, at least in homology. We find an algebraic relation in $A^2(\overline{\mathcal{M}}_{2,3})$ via the admissible cover technique introduced in [P]. In fact, the relation lies in the $\mathbb{S}_3$-invariant subspace of $A^2(\overline{\mathcal{M}}_{2,3})$. As an application, we show that the resulting differential equations together with the known topological recursion relations are strong enough to determine all the genus 2 descendent integrals for $\mathbf{P}^2$.

The plan of the paper is as follows. The admissible double cover construction is reviewed in Section 1. In Section 2, this construction is used to calculate the new relation in $A^2(\overline{\mathcal{M}}_{2,3})$. Formulas expressing the cycle classes of Weierstrass loci in the moduli spaces of pointed curves of genus 1 and 2 in terms of descendent stratum classes are required for the computation of the new relation. These formulas are also obtained in Section 2 via the space of admissible double covers. The application to the descendent integrals of $\mathbf{P}^2$ appears in Section 3.

Our greatest mathematical debt is to E. Getzler for informing us of his homological computation. His work provided the motivation for our calculation. Thanks are also due to C. Faber and T. Graber for related conversations. The second author was partially supported by a National Science Foundation post-doctoral fellowship.

1. – Admissible double covers

A genus g *admissible cover* with n marked points and b branch points consists of a morphism $\pi : C \to D$ of pointed curves

$$(C,\ P_1, \dots, P_n), \quad (D,\ p_1, \dots, p_n, q_1, \dots, q_b)$$

satisfying the following conditions.

(1) C is a connected, reduced, nodal curve of arithmetic genus g.
(2) The markings P_i lie in the nonsingular locus C_{ns}.
(3) $\pi(P_i) = p_i$.
(4) $(D, \ p_1, \dots, p_n, q_1, \dots, q_b)$ is an $(n+b)$-pointed stable curve of genus 0.
(5) $\pi^{-1}(D_{\mathrm{sing}}) = C_{\mathrm{sing}}$.
(6) $\pi|_{C_{\mathrm{ns}}}$ is étale except over the points q_j where π is simply ramified.
(7) If $x \in C_{\mathrm{sing}}$, then
   (a) $x \in C_1 \cap C_2$, where C_1 and C_2 are distinct components of C,
   (b) $\pi(C_1)$ and $\pi(C_2)$ are distinct components of D,
   (c) the ramification numbers at x of the two morphisms

$$\pi : C_1 \to \pi(C_1) \quad \text{and} \quad \pi : C_2 \to \pi(C_2)$$

   are equal.

These conditions imply that the map $\pi : C \to D$ is of uniform degree d, where

$$-2d + b = 2g - 2.$$

Let $\overline{\mathcal{H}}_{d,g,n}$ be the space of n-pointed genus g admissible covers of $\mathbf{P}^1$ branched at b points.

Only the admissible double cover case in genus 1 and 2 will be considered in this paper. The space $\overline{\mathcal{H}}_{2,g,n}$ is an irreducible variety. There are natural morphisms

$$\lambda : \overline{\mathcal{H}}_{2,g,n} \to \overline{\mathcal{M}}_{g,n}, \quad \pi : \overline{\mathcal{H}}_{2,g,n} \to \overline{\mathcal{M}}_{0,n+2g+2} \tag{2}$$

obtained from the domain and range of the admissible cover respectively. In genus 1 and 2, λ is clearly surjective. The projection π is a finite map to $\overline{\mathcal{M}}_{0,n+2g+2}$. For each marking $i \in \{1, \dots, n\}$, there is a natural $\mathbb{Z}/2\mathbb{Z}$-action on $\overline{\mathcal{H}}_{2,g,n}$ given by switching the sheet of the i-th marking of C. These actions induce a product action in which the diagonal Δ acts trivially. Define the group $\mathbf{G}$ by:

$$\mathbf{G} = (\mathbb{Z}/2\mathbb{Z})^n/\Delta.$$

The action of $\mathbf{G}$ on $\overline{\mathcal{H}}_{2,g,n}$ is generically free and commutes with the projection π. Therefore, the quotient $\overline{\mathcal{H}}/\mathbf{G}$ naturally maps to $\overline{\mathcal{M}}_{0,n+2g+2}$. In fact, since the morphism

$$\overline{\mathcal{H}}/\mathbf{G} \to \overline{\mathcal{M}}_{0,n+2g+2}$$

is finite and birational, it is an isomorphism.

Spaces of admissible covers were defined in [HM]. The methods there may be used to construct spaces of pointed admissible covers. An alternative construction of the space of pointed admissible covers via Kontsevich's space of stable maps is given in [P]. A foundational treatment of the moduli problem of admissible covers is developed in [AV].

Our method of obtaining a codimension 2 relation in $\overline{\mathcal{M}}_{2,3}$ is the following. Consider the diagram:

$$\begin{tikzcd} \overline{\mathcal{H}}_{2,2,3} \arrow[r, "\lambda"] \arrow[d, "\pi"'] & \overline{\mathcal{M}}_{2,3} \\ \overline{\mathcal{M}}_{0,3+6} & \end{tikzcd} \tag{3}$$

There are relations in $A^2(\overline{\mathcal{M}}_{0,3+6})$ among the classes of codimension 2 strata. Such relations yield cycle relations in $A^2(\overline{\mathcal{M}}_{2,3})$ via pull-back by π and push-forward by λ. The resulting cycles in $\overline{\mathcal{M}}_{2,3}$ include Weierstrass loci. By further expressing the classes of these Weierstrass loci in terms of descendent stratum classes, a new nontrivial relation is obtained.

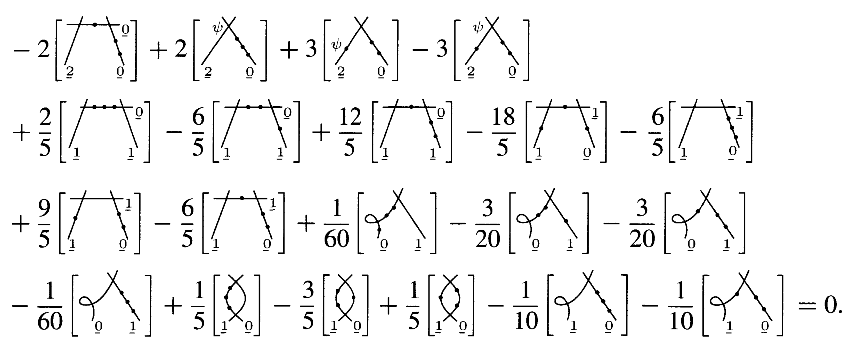
2. – The relation computation

2.1. – The new relation

The main result of this paper is the following theorem.

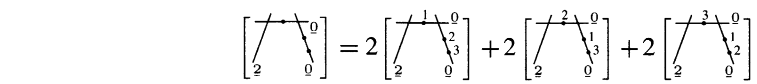
THEOREM 1. *The codimension 2 descendent stratum classes in $\overline{\mathcal{M}}_{2,3}$ satisfy a nontrivial rational equivalence:*

(4)



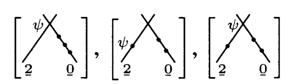
We begin by explaining our notation. All classes considered are *stack* fundamental classes. The stack class is equal to the ordinary (coarse) fundamental class divided by the order of the automorphism group of the generic moduli point. Chow groups are taken with rational coefficients. A stratum $\overline{\mathcal{M}}_{\Gamma}$ is denoted by the topological type of the stable curve with dual graph Γ. In the diagrams, the geometric genera of the components are underlined.

A descendent stratum class in $\overline{\mathcal{M}}_{2,3}$ with *unassigned* markings denotes the sum of descendent stratum classes over the 3! marking assignment choices. For example,



The main feature of this notation is that an unsymmetrized equation among descendent stratum classes may be symmetrized by simply erasing the markings. While our main relation (4) is symmetric, most of the auxiliary formulas expressing classes of geometric loci in terms of descendent stratum classes are not. Since many of these formulas are of independent interest, we present them in the unsymmetrized form.

There are 3 stratum classes in equation (4) with cotangent lines:



The cotangent line class is always on the genus 2 component. In the first and third classes above, the cotangent line is taken at the node; in the second class, it is taken at the marked point.

The new relation will be found via an admissible double cover construction. Let $\overline{\mathcal{M}}_{0,3+6}$ be the moduli space of stable genus zero curves with the marking set

$$\{p_1, p_2, p_3, b_1, \ldots, b_6\}.$$

The 9-pointed genus zero curve will be the base of the admissible cover: the points p_i correspond to the images downstairs of the marked points of the cover and the points b_i correspond to the branch points.

REMARK. Let λ and π be the morphisms from diagram (3). Let $\mathbb{S}_6$ act on $\overline{\mathcal{M}}_{0,3+6}$ by permuting the branch points b_i. Then the homomorphism

$$\lambda_* \pi^* : A^2(\overline{\mathcal{M}}_{0,3+6}) \to A^2(\overline{\mathcal{M}}_{2,3})$$

is $\mathbb{S}_6$-invariant.

2.2. – The first equation

Let D denote the boundary divisor $(p_1 p_2 | p_3 b_1 \ldots b_6)$ in $\overline{\mathcal{M}}_{0,3+6}$. The generic point of D parametrises a reducible genus zero curve with two components and marking splitting $\{p_1, p_2\} \cup \{p_3, b_1, \ldots, b_6\}$. As a variety D is isomorphic to $\overline{\mathcal{M}}_{0,A}$ with the marking set

$$A = \{\star, p_3, b_1, \ldots, b_6\}.$$

Here $\star$ denotes the node on the stable curve corresponding to the generic point of D. Consider the standard ([Ke],[FP]) 4-point linear equivalence of divisors on $\overline{\mathcal{M}}_{0,A}$ obtained by pullback via the forgetful morphism of the boundary equivalence on $\overline{\mathcal{M}}_{0,\{\star,p_3,b_1,b_2\}}$:

$$(b_1 b_2 | \star p_3) \sim (b_1 p_3 | \star b_2).$$

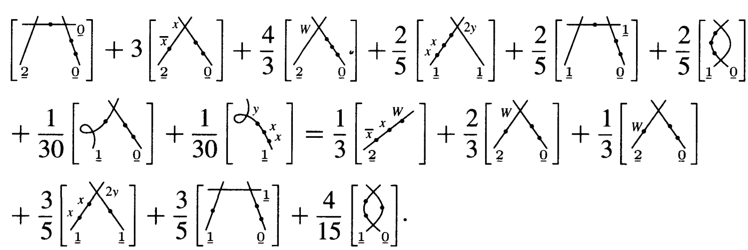
Let Λ_D denote the corresponding relation in $A^2(\overline{\mathcal{M}}_{0,3+6})$. Symmetrization with respect to the natural $\mathbb{S}_3$-action on $\overline{\mathcal{M}}_{0,3+6}$ permuting the points p_1, p_2, p_3 yields the relation

$$\sum_{\sigma \in \mathbb{S}_3} \sigma^* \Lambda_D \tag{5}$$

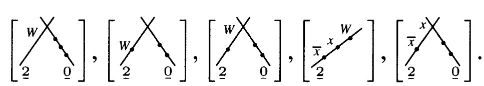
in $A^2(\overline{\mathcal{M}}_{0,3+6})$.

LEMMA 1. *The application of* $\lambda_* \pi^*$ *to relation* (5) *yields* (6! *times*) *the following rational equivalence in* $A^2(\overline{\mathcal{M}}_{2,3})$*:*

(6)

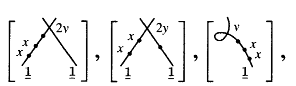


Five new stratum classes with smooth genus 2 components appear in this relation:



The condition to be a Weierstrass point is denoted by a W on the node or marking (and is always on the genus 2 component). The letters $x, \overline{x}$ designate the condition of being a hyperelliptic conjugate pair — they are *not* marking labels.

In the new classes with elliptic components,

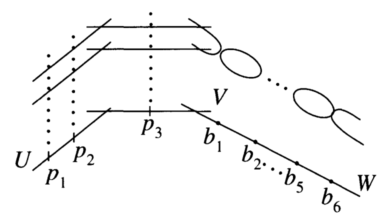


the letters x, y designate an imposed linear equivalence on the markings and nodes: the divisor sum of the points lettered x must be equivalent to the divisor sum of the points lettered y. In the first two classes, the sum of the points lettered x must be equivalent to twice the node point on the left component. The third class has an imposed linear equivalence on the normalization: the sum of the two preimages of the node must be equivalent to the sum of the points lettered x. Our previous convention regarding unassigned markings holds: the sum over the 3! marking choices is taken.

Theorem 1 will be obtained from Lemma 1 by expressing all occurring classes in terms of descendent stratum classes.

PPROOF OF LEMMA 1. The method of proof is by direct calculation of the map $\lambda_* \pi^*$ on each term of (5). The technique is identical to the elliptic calculations in [P]. We will give a representative example of the computation.

Let C be the codimension 2 stratum of $\overline{\mathcal{M}}_{0,3+6}$ occurring as the divisor $(\star p_3 | b_1 \dots b_6)$ in $\overline{\mathcal{M}}_{0,A}$. The generic point of C corresponds to a genus 0 stable curve which is a chain of three rational components U, V, W. The marking distribution is as follows: p_1 and p_2 on U, p_3 on V, and the six branch points $b_1, \dots, b_6$ on W. An admissible cover over such a curve consists of disjoint étale double covers of U and V and a genus 2 double cover of W branched over $b_1, \dots, b_6$.



The preimage $\pi^{-1}(C)$ has four irreducible components $\Sigma_1, \dots, \Sigma_4$ corresponding to four ways (up to isomorphism of the cover) of distributing the marked points among the sheets of the cover. The component Σ_4 parametrizes covers with all three marked points placed on the same sheet, whereas each of the components Σ_i, $1 \leq i \leq 3$ parametrizes covers with p_i placed on one sheet and the remaining two markings placed on the other sheet. Since the projection π is a finite group quotient, we can compute the pull-back of the class of C by π^* using the following lemma (see [V] for the proof).

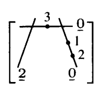
LEMMA 2. *Let* $\mathbf{G}$ *be a finite group, let* X *be an irreducible algebraic variety with a* $\mathbf{G}$*-action, and let* α *denote the quotient morphism* $\alpha : X \to X/\mathbf{G}$. *There exists a pull-back* $\alpha^* : A_*(X/\mathbf{G}) \to A_*(X)$ *defined by:*

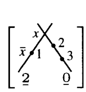
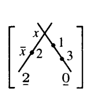
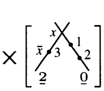
$$\alpha^*[V] = |Stab(V)| \cdot [\alpha^{-1}(V)_{\mathrm{red}}],$$

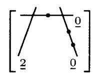
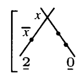
where V *is an irreducible subvariety of* $X/\mathbf{G}$, *the scheme* $\alpha^{-1}(V)_{\mathrm{red}}$ *is the reduced preimage of* V, *and* $|\mathrm{Stab}(V)|$ *is the size of the generic stabilizer of points over* V.

The stabilizer of points over C is trivial. Hence, by Lemma 2, we get

$$\pi^*[C] = [\Sigma_1] + [\Sigma_2] + [\Sigma_3] + [\Sigma_4].$$

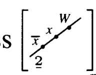
The λ push-forward of $[\Sigma_4]$ is $6! \times$ , whereas the push-forwards of $[\Sigma_1], [\Sigma_2], [\Sigma_3]$ are

$6! \times$ , $6! \times$ , and $6! \times$ , respectively. Let C_{sym} denote the cycle in $A^2(\overline{\mathcal{M}}_{0,3+6})$ obtained by symmetrizing C with respect to the $\mathbb{S}_3$-action: $C_{\mathrm{sym}} = \sum_{\sigma \in \mathbb{S}_3} \sigma^*(C)$. We obtain

$\lambda_* \pi^*[C_{\mathrm{sym}}] = 6! \times$  $+ 3 \cdot 6! \times$  .

The push-forwards of the other cycles are computed similarly. □

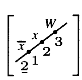
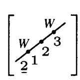
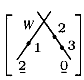
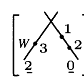
2.3. – The second equation

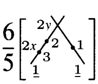
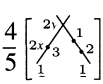
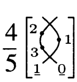
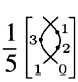
We find an expression for the codimension 2 class  appearing in (6). Consider the boundary divisor $D = (p_3 b_6 | p_1 p_2 b_1 \ldots b_5)$ in $\overline{\mathcal{M}}_{0,3+6}$. It is isomorphic to $\overline{\mathcal{M}}_{0,A}$ with the marking set $A = \{\star, p_1, p_2, b_1, \ldots, b_5\}$. Consider the following 4-point linear equivalence on $\overline{\mathcal{M}}_{0,A}$:

$$(p_1 p_2 | b_1 \star) \sim (p_1 b_1 | p_2 \star).$$

As in Section 2.2, let Λ_D be the corresponding relation in $A^2(\overline{\mathcal{M}}_{0,3+6})$.

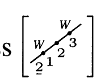
LEMMA 3. *The application of $\lambda_* \pi^*$ to relation Λ_D yields* $(2 \cdot 5!$ *times) the following rational equivalence in $A^2(\overline{\mathcal{M}}_{2,3})$:*

(7)  $= \frac{2}{5}$  $+ 2$  $-$ 

$+ \frac{6}{5}$  $- \frac{4}{5}$  $+ \frac{4}{5}$  $- \frac{1}{5}$  .

The method of proof is the same as of Lemma 1.

2.4. – The third equation

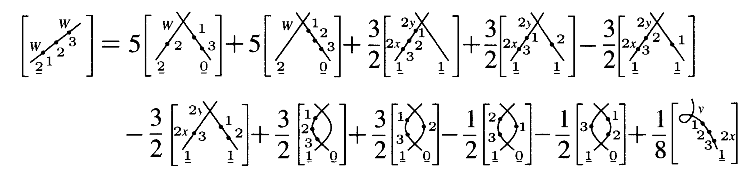
We find an expression for the codimension 2 class  appearing in (7). Consider the boundary divisor $D = (p_3 b_6 | p_1 p_2 b_1 \ldots b_5)$ in $\overline{\mathcal{M}}_{0,3+6}$. It is isomorphic to $\overline{\mathcal{M}}_{0,A}$ with the marking set $A = \{\star, p_1, p_2, b_1, \ldots, b_5\}$. Consider the following 4-point linear equivalence on $\overline{\mathcal{M}}_{0,A}$:

$$(p_1 b_1 | \star b_2) \sim (p_1 \star | b_1 b_2).$$

Let Λ_D be the corresponding relation in $A^2(\overline{\mathcal{M}}_{0,3+6})$.

LEMMA 4. *The application of $\lambda_* \pi^*$ to relation Λ_D yields* ($4 \cdot 4!$ *times*) *the following rational equivalence in $A^2(\overline{\mathcal{M}}_{2,3})$:*

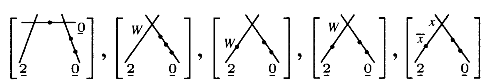
(8)



2.5. – Weierstrass to descendent stratum classes

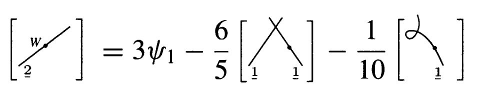
Combining relations (6), (7), and (8) yields a relation in $A^2(\overline{\mathcal{M}}_{2,3})$ in which the only strata with a smooth genus 2 component of the corresponding generic stable curve are:

(9)

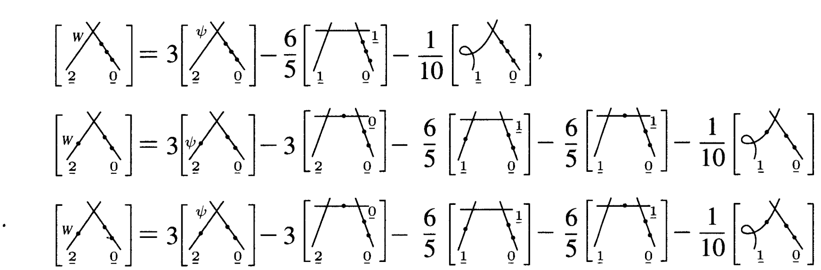


The first stratum is pure boundary. Expressions for the next 3 classes in terms of descendent stratum classes are obtained using the following result.

LEMMA 5. *The following linear equivalence holds in $A^1(\overline{\mathcal{M}}_{2,1})$:*



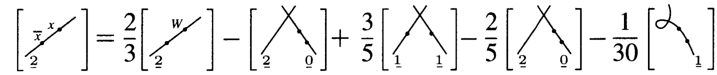
A proof can be found in [EH]. The relations obtained are:



Expression for the last class in (9) in terms of descendent stratum classes is obtained by combining the above formulas with the following Lemma.

LEMMA 6. *The following linear equivalence holds in $A^1(\overline{\mathcal{M}}_{2,2})$:*

(10)



PROOF. Consider the following 4-point linear equivalence on $\overline{\mathcal{M}}_{0,2+6}$:

(11) $$(p_1 p_2 | b_1 b_2) \sim (p_1 b_1 | p_2 b_2).$$

Following the notation in Section 2.1, the points p_1, p_2 correspond to the images downstairs of the marked points of the cover and the points $b_1, \ldots, b_6$ correspond to the branch points.

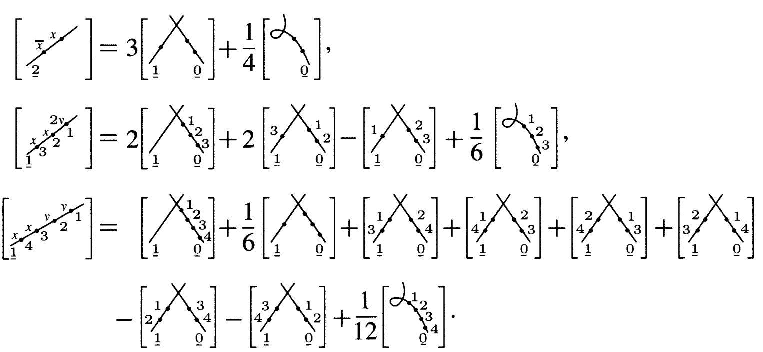
Consider the morphisms

$$\begin{array}{ccc} \overline{\mathcal{H}}_{2,2,2} & \xrightarrow{\lambda} & \overline{\mathcal{M}}_{2,2} \\ {\scriptstyle\pi}\downarrow & & \\ \overline{\mathcal{M}}_{0,2+6} & & \end{array}$$

The application of $\lambda_* \pi^*$ to relation (11) yields $3 \cdot 5!$ times relation (10). $\square$

2.6. – Genus 1

We list the relations in $A^1(\overline{\mathcal{M}}_{1,n})$ necessary to express the classes in (6), (7) and (8) with genus 1 components constrained by linear equivalence conditions in terms of the boundary classes. These relations are obtained similarly to the above relations, only using the space of elliptic admissible double covers (with 4 branch points). See [B] for details.



Lemmas 3-6 and the above equations express all the classes occurring in relation (6) in terms of descendent stratum classes. The resulting relation among descendent stratum classes is (4). The proof of Theorem 1 is complete.

3. – Descendent integrals on $\mathbf{P}^2$

3.1. – 1-cotangent line integrals

Let X be a nonsingular projective variety. Define the *1-cotangent line descendents* to be the integral invariants (1) of X with at most 1 cotangent line class. By Getzler's topological recursion relations in genus 2 ([G2]), the 1-cotangent line integrals determine all genus 2 descendent integrals. Relation (4) yields new universal equations satisfied by descendent invariants of X. It was initially hoped these equations would show that genus 2 descendents could be uniquely reconstructed from Gromov-Witten invariants for any space X. Such universal reconstruction results are known in genus 0 and 1. However, we found all of our reconstruction strategies thwarted by specific unexpected linear relations among the coefficients of relation (4). While universal descendent reconstruction may hold in genus 2, new ideas are required: either subtle strategies involving relation (4) or yet another genus 2 relation.

Much more can be said if attention is restricted to specific target spaces. In this section, relation (4) is shown to determine all 1-cotangent line descendents of $\mathbf{P}^2$ from degree 0 ones and the lower genus invariants. Hence, using Getzler's topological recursion relations, all genus 2 descendent invariants of $\mathbf{P}^2$ are determined via descendent stratum relations in $\overline{\mathcal{M}}_{2,n}$.

Let T_0, T_1, T_2 be the standard cohomology basis of $\mathbf{P}^2$ given by the fundamental, hyperplane, and point classes respectively. On $\mathbf{P}^2$, the basic 1-cotangent line descendent integrals assemble in 3 series:

$$
\begin{aligned}
N_d^{(2)} &= \langle T_2^{3d+1} \rangle_{2,d}, \\
H_d^{(2)} &= \langle (T_1\psi) \cdot T_2^{3d} \rangle_{2,d}, \\
P_d^{(2)} &= \langle (T_2\psi) \cdot T_2^{3d-1} \rangle_{2,d}.
\end{aligned}
\tag{12}
$$

All descendent integrals with at most 1 cotangent line class may be reduced to one of the basic integrals above via the string, dilaton, and divisor equations of Gromov-Witten theory (see [W]). The first two series are defined in degree 0. A direct virtual class computation yields:

$$
\begin{aligned}
N_0^{(2)} &= \langle T_2 \rangle_{2,0} = 0, \\
H_0^{(2)} &= \langle T_1\psi \rangle_{2,0} = -6 \cdot \int_{\overline{\mathcal{M}}_2} \lambda_1\lambda_2 = -\frac{1}{960}.
\end{aligned}
\tag{13}
$$

The integral of the Chern classes of the Hodge bundle over $\overline{\mathcal{M}}_2$ may be computed by a method due to Faber ([Fa]).

Following conventions, let the variable t_j^i correspond to the class $T_i\psi^j$, where $0 \leq i \leq 2$ and $j \geq 0$. Let t denote the variable set $\{t_j^i\}$, and let $\gamma = \sum t_j^i T_i \psi^j$ be the formal sum. Let $F_{2,\mathbf{P}^2}(t)$ denote the full genus 2 gravitational potential function:

$$
F_{2,\mathbf{P}^2}(t) = \sum_{d\geq 0}\sum_{n\geq 0} \frac{1}{n!}\langle \gamma^n \rangle_{2,d}. \tag{14}
$$

We will consider the cut-off of $F_{2,\mathbf{P}^2}(t)$ at 1-cotangent line:

$$
F^1_{2,\mathbf{P}^2}(t_0^0, t_0^1, t_0^2, t_1^0, t_1^1, t_1^2) = F_{2,\mathbf{P}^2}(t)|_{\{t_j^i=0 \ \forall j\geq 2,\ t_1^i t_1^j = 0 \ \forall i,j\}}.
$$

The function $F^1_{2,\mathbf{P}^2}$ may be written explicitly in terms of the 3 basic series:

$$\begin{aligned} F^1_{2,\mathbf{P}^2} = & \sum_{d\geq 0} N^{(2)}_d e^{dt^1_0} \frac{(t^2_0)^{3d+1}}{(3d+1)!} \\ & + \sum_{d\geq 0} N^{(2)}_d d\, t^0_0 t^1_1 e^{dt^1_0} \frac{(t^2_0)^{3d+1}}{(3d+1)!} + \sum_{d\geq 0} N^{(2)}_d t^0_0 t^2_1 e^{dt^1_0} \frac{(t^2_0)^{3d}}{(3d)!} \\ & + \sum_{d\geq 0} N^{(2)}_d (3d+3+d\, t^1_0)\; t^0_1\; e^{dt^1_0} \frac{(t^2_0)^{3d+1}}{(3d+1)!} \\ & + \sum_{d\geq 0} (H^{(2)}_d + N^{(2)}_d t^1_0)\; t^1_1\; e^{dt^1_0} \frac{(t^2_0)^{3d}}{(3d)!} \\ & + \sum_{d\geq 1} P^{(2)}_d\; t^2_1\; e^{dt^1_0} \frac{(t^2_0)^{3d-1}}{(3d-1)!}. \end{aligned}$$

Relation (4) will yield differential equations which determine $F^1_{2,\mathbf{P}^2}$ from degree 0 values (13) and the known elliptic and rational Gromov-Witten potentials.

The descendent integrals (1) are defined via the cotangent lines on the moduli space of maps. Let $2g-2+n>0$, and let

$$f : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$$

be the forgetful map. The following integrals are similar to the descendents:

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \mathrm{ev}_1^*(\gamma_1) \cup f^*(\psi_1^{\alpha_1}) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \cup f^*(\psi_n^{\alpha_n}). \tag{15}$$

The cotangent lines here are pulled back from the moduli space of stable pointed curves. The integral invariants (15) are naturally equivalent to the descendent invariants. The two sets of invariants are related by universal invertible transformations (see [KM3]). The differential equations obtained from relation (4) via the splitting formula in Gromov-Witten theory are most conveniently expressible in terms of generating functions for the invariants (15). However, in the restricted case of 1-cotangent line integrals on $\mathbf{P}^2$, the two sets of invariants are exactly equal.

LEMMA 7. *Let $g>0$ and $d\geq 0$. Let $\gamma_i \in H^*(\mathbf{P}^2,\mathbb{Q})$. An equality of 1-cotangent line integrals holds for $\mathbf{P}^2$:*

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbf{P}^2,d)]^{\mathrm{virt}}} f^*(\psi_1) \cup \prod_{i=1}^n ev_i^*(\gamma_i) = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbf{P}^2,d)]^{\mathrm{virt}}} \psi_1 \cup \prod_{i=1}^n ev_i^*(\gamma_i). \tag{16}$$

PROOF. Define the pairing matrix (g_{ij}) by

$$g_{ij} = \int_{\mathbf{P}^2} T_i \cup T_j.$$

Let $(g^{ij}) = (g_{ij})^{-1}$, so that $\sum g^{ij} T_i \boxtimes T_j$ is the class of the diagonal in $\mathbf{P}^2 \times \mathbf{P}^2$. By the formulas of [KM3], the difference between the integrals (16) is expressed in terms of pure Gromov-Witten invariants:

$$\sum \langle \gamma_1 \cdot T_i \rangle_{0,d_1} \; g^{ij} \; \langle T_j \cdot \gamma_2 \cdots \gamma_n \rangle_{g,d_2},$$

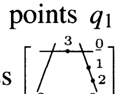
where the sum is over all degree splittings $d_1 + d_2 = d$ satisfying $d_1 > 0$ and the diagonal splitting. The only nonvanishing 2-point genus 0 Gromov-Witten invariant of positive degree is $\langle T_2 \cdot T_2 \rangle_{0,1}$. Hence, only the summands with $j = 0$ can contribute. However, Gromov-Witten invariants of genus $g > 0$ with an argument T_0 vanish by the axiom of the fundamental class. □

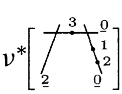
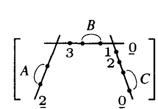
Let $\widetilde{F}_{2,\mathbf{P}^2}$ denote the full genus 2 potential function defined using the integrals (15) instead of the descendents, and let $\widetilde{F}^1_{2,\mathbf{P}^2}$ be the 1-cotangent line cut-off. Then, by Lemma 7, $F^1_{2,\mathbf{P}^2} = \widetilde{F}^1_{2,\mathbf{P}^2}$.

3.2. – Pull-backs of descendent stratum classes

In order to find differential equations via the splitting formula, the pull-backs of relation (4) to $\overline{\mathcal{M}}_{2,3+l}$ will be required. More generally, let ν denote the forgetful map

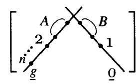
$$\nu : \overline{\mathcal{M}}_{g,n+l} \to \overline{\mathcal{M}}_{g,n}. \tag{17}$$

Let Γ be a stable dual graph of genus g and valence n, and let $\overline{\mathcal{M}}_\Gamma$ denote the corresponding closed boundary stratum of $\overline{\mathcal{M}}_{g,n}$ (as in Section 0). The class $[\overline{\mathcal{M}}_\Gamma]$ pulls back under the forgetful map ν to the sum of classes of boundary strata obtained by all possible distributions of the l extra points $q_1, \ldots, q_l$ on the vertices of Γ. For example, the pull back of the class  from $\overline{\mathcal{M}}_{2,3}$ to $\overline{\mathcal{M}}_{2,3+l}$ is given by:

(18) ν^*  $= \sum_{A \sqcup B \sqcup C = \{q_1, \ldots, q_l\}}$  .

While the pull-back of a descendent stratum class is a sum of descendent stratum classes, the pull-back is not obtained simply by distributing the extra markings as in (18). The additional complexity occurs because the pull-back of a cotangent line class via the forgetful map (17) is *not* the corresponding cotangent line class on the domain.

LEMMA 8. *There is an equality:*

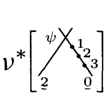
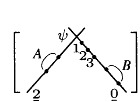
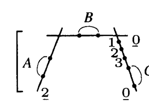
$$\nu^*(\psi_1) = \psi_1 - \sum_{A \sqcup B = \{q_1, \dots, q_l\}}$$  ,

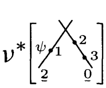
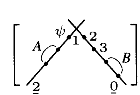
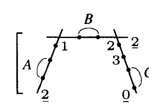
where the sum is over all <u>*stable*</u> *distributions of the l extra points.*

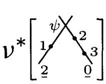
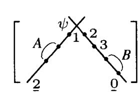
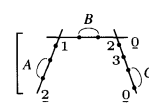
See [W] for the proof.

Lemma 8 easily implies the pull-back formulas for the three codimension 2 descendent stratum classes in $\overline{\mathcal{M}}_{2,3}$:

(19)

ν^*  $= \sum_{A \sqcup B = \{q_1, \dots, q_l\}}$  $- \sum_{\substack{A \sqcup B \sqcup C = \{q_1, \dots, q_l\} \\ B \neq \varnothing}}$  ,

ν^*  $= \sum_{A \sqcup B = \{q_1, \dots, q_l\}}$  $- \sum_{\substack{A \sqcup B \sqcup C = \{q_1, \dots, q_l\} \\ B \neq \varnothing}}$  ,

ν^*  $= \sum_{A \sqcup B = \{q_1, \dots, q_l\}}$  $- \sum_{\substack{A \sqcup B \sqcup C = \{q_1, \dots, q_l\} \\ B \neq \varnothing}}$  .

Using these formulas, the pull-backs of relation (4) to $\overline{\mathcal{M}}_{2,3+l}$ yield descendent stratum class relations.

3.3. – The splitting formula in Gromov-Witten theory

We review here the splitting formula following [KM1]. Let X be a nonsingular projective variety with cohomology basis $T_0, \dots, T_m$. Let $g_{ij} = \int T_i \cup T_j$, and let $(g^{ij}) = (g_{ij})^{-1}$. Let $2g - 2 + n > 0$. As in Section (3.1), let

$$f : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$$

be the forgetful map. Let $\gamma_i \in H^*(X, \mathbb{Q})$. A *Gromov-Witten class* $\langle \prod_{i=1}^n \gamma_i \rangle_{g,\beta}$ is an element of $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ defined via push-forward by f:

$$\langle \gamma_1 \cdots \gamma_n \rangle_{g,\beta} = f_*\Big([\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{virt}} \cap \prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i) \Big).$$

Let Γ be a stable dual graph of genus g and valence n. Let V and E denote the sets of vertices and edges of Γ. Let E_{flag} denote the set of half-edge flags (an edge is made up of 2 half-edge flags). For each vertex $v \in V$, let $I(v)$ and $E_{\mathrm{flag}}(v)$ be the sets of markings and half-edge flags incident at v. Recall that $n(v)$ and $g(v)$ denote the valence and the genus assignment of v. For an edge $e \in E$, let f_e and f_e^* denote the corresponding two half-edge flags. Let $\iota : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ denote the inclusion morphism. Let

$$\tau : \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_\Gamma$$

be the natural map. The splitting formula is:

$$\iota^* \langle \prod_{i=1}^{n} \gamma_i \rangle_{g,\beta} = \frac{1}{|\mathrm{Aut}(\Gamma)|} \tau_* \left(\sum_{\epsilon,\phi} \prod_{e \in E} g^{\epsilon(f_e)\epsilon(f_e^*)} \prod_{v \in V} \langle \prod_{i \in I(v)} \gamma_i \prod_{f \in E_{\mathrm{flag}}(v)} T_{\epsilon(f)} \rangle_{g(v),\phi(v)} \right).$$

The sum is over all functions $\epsilon : E_{\mathrm{flag}} \to \{0, \dots, m\}$ from the set of half-edge flags to the index set of the cohomology basis, and all functions $\phi : V \to H_2(X, \mathbb{Z})$ satisfying $\sum_{v \in V} \phi(v) = \beta$.

3.4. – Differential equations

Relation (4), the pull-back formulas of Section 3.2, and the splitting formula naturally yield differential equations for the potential function $\widetilde{F}_{2,X}$. Let $\widetilde{F}_{0,X}$ and $\widetilde{F}_{1,X}$ denote the full potential functions in genus 0 and 1 with respect to the integrals (15). Let $\tilde{\gamma} = \sum t_j^i T_i f^*(\psi^j)$ be the formal sum. Then

$$\widetilde{F}_{0,X}(t) = \sum_{d \geq 0} \sum_{n \geq 3} \frac{1}{n!} \langle \tilde{\gamma}^n \rangle_{0,d},$$

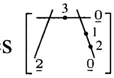
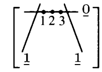
$$\widetilde{F}_{1,X}(t) = \sum_{d \geq 0} \sum_{n \geq 1} \frac{1}{n!} \langle \tilde{\gamma}^n \rangle_{1,d}.$$

These generating functions are sums over the stable range $\{2g - 2 + n > 0\}$ of n-pointed curves of genus g. Let $\widetilde{F}_{0,X}^0$ and $\widetilde{F}_{1,X}^0$ denote the restrictions to the small phase space $\{t_j^i = 0 \ \forall j \geq 1\}$. These are the 0-cotangent line cut-offs and involve only the Gromov-Witten invariants.

The differential equations are now described. An equation is obtained for every assignment of variables $\{t_j^i\}$ to the 3 markings of the marking set of $\overline{\mathcal{M}}_{2,3}$. Fix such an assignment $(t_{j_1}^{i_1}, t_{j_2}^{i_2}, t_{j_3}^{i_3})$. A differential equation

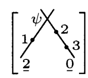
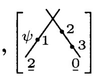
$$\mathcal{D}_{j_1 j_2 j_3}^{i_1 i_2 i_3}(\widetilde{F}_{0,X}, \widetilde{F}_{1,X}, \widetilde{F}_{2,X}) = 0 \tag{20}$$

is constructed from relation (4) in the following manner.

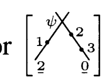
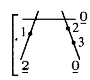
A pure boundary stratum $\overline{\mathcal{M}}_\Gamma \subset \overline{\mathcal{M}}_{2,3}$ naturally yields a differential expression: place potentials on the vertices of the dual graph, insert the 3 point conditions via differentiation, sum over diagonal splittings at the edges, and divide by the number of graph automorphisms. For example, the classes  and  yield the respective expressions

$$\frac{\partial \widetilde{F}_{2,X}}{\partial t_0^e} g^{ef} \frac{\partial^3 \widetilde{F}_{0,X}}{\partial t_0^f \partial t_{j_3}^{i_3} \partial t_0^k} g^{kl} \frac{\partial^3 \widetilde{F}_{0,X}}{\partial t_0^l \partial t_{j_1}^{i_1} \partial t_{j_2}^{i_2}},$$

$$\frac{1}{2} \cdot \frac{\partial \widetilde{F}_{1,X}}{\partial t_0^e} g^{ef} \frac{\partial^5 \widetilde{F}_{0,X}}{\partial t_0^f \partial t_{j_1}^{i_1} \partial t_{j_2}^{i_2} \partial t_{j_3}^{i_3} \partial t_0^k} g^{kl} \frac{\partial \widetilde{F}_{1,X}}{\partial t_0^l}.$$

Each descendent stratum class in relation (4) which is not a pure boundary class yields a two-term differential expression. The first term is again obtained by placing potentials on the vertices, inserting point conditions, and summing over diagonal splittings (no automorphisms occur for these graphs). For example, the first terms for the descendent stratum classes ,  are

$$\frac{\partial^2 \widetilde{F}_{2,X}}{\partial t_{j_1}^{i_1} \partial t_1^e} g^{ef} \frac{\partial^3 \widetilde{F}_{0,X}}{\partial t_0^f \partial t_{j_2}^{i_2} \partial t_{j_3}^{i_3}} \quad \text{and} \quad \frac{\partial^2 \widetilde{F}_{2,X}}{\partial t_{j_1+1}^{i_1} \partial t_0^e} g^{ef} \frac{\partial^3 \widetilde{F}_{0,X}}{\partial t_0^f \partial t_{j_2}^{i_2} \partial t_{j_3}^{i_3}}.$$

The second term is obtained from the correction graphs in the pull-back formulas (19). For , the correction graph is . The second term it yields is

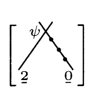
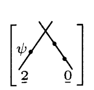
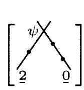
$$-\frac{\partial^2 \widetilde{F}_{2,X}}{\partial t_{j_1}^{i_1} \partial t_0^e} g^{ef} \frac{\partial^2 \widetilde{F}_{0,X}}{\partial t_0^f \partial t_0^k} g^{kl} \frac{\partial^3 \widetilde{F}_{0,X}}{\partial t_0^l \partial t_{j_2}^{i_2} \partial t_{j_3}^{i_3}}.$$

Equation (20) is constructed by replacing the classes in (4) by the corresponding differential expressions. Equation (20) is then easily proven by pulling-back (4) via the forgetful morphisms to the moduli spaces of pointed curves and applying the splitting formula.

Equations strictly among 1-cotangent line integrals of X may be obtained from the differential equations (20) by the following method. Let the variables t_j^i assigned to the 3 markings correspond to pure cohomology classes (i.e., $j = 0$). Restrict the left side of (20) to the small phase space:

$$\mathcal{D}_{000}^{i_1 i_2 i_3}(\widetilde{F}_{0,X}, \widetilde{F}_{1,X}, \widetilde{F}_{2,X}) \mid_{\{t_j^i = 0 \ \forall j \geq 1\}} = 0. \tag{21}$$

The derivatives $\partial/\partial t_j^i$ with $j \geq 1$ occur only in the terms of $\mathcal{D}_{000}^{i_1 i_2 i_3}$ obtained from the classes:

, , .

In each of these terms, the derivative appears only once, with $j = 1$. Moreover, the derivative acts on the genus 2 potential only. Hence, equation (21) implies

$$\mathcal{D}^{i_1 i_2 i_3}_{0\,0\,0}(\widetilde{F}^0_{0,X}, \widetilde{F}^0_{1,X}, \widetilde{F}^1_{2,X}) \mid_{\{t^i_j=0\ \forall j\geq 1\}} = 0. \tag{22}$$

The coefficient relations obtained from (22) are exactly among 1-cotangent line integrals of genus 2 and Gromov-Witten invariants of genus 0 and 1.

3.5. – Recursions for $\mathbf{P}^2$

The differential equations (22) yield recursive relations among the 3 basic series (12) of 1-cotangent line integrals of $\mathbf{P}^2$ (involving the lower genus Gromov-Witten invariants). Each choice of point assignment provides such recursions. In each degree $d \geq 1$, there are 3 basic 1-cotangent line integrals. Hence, 3 independent recursions are required. Four distinct equations of the form (22) are obtained by the following four marking assignments:

$$(t^1_0, t^1_0, t^1_0),$$
$$(t^1_0, t^1_0, t^2_0),$$
$$(t^1_0, t^2_0, t^2_0),$$
$$(t^2_0, t^2_0, t^2_0).$$

The resulting recursions are easily seen to determine the 3 series from the degree 0 values (13) and Gromov-Witten invariants of genus 0 and 1.

We include here the recursion obtained from the assignment (t^1_0, t^1_0, t^1_0):

$$\begin{aligned}
-3H^{(2)}_d + 3dP^{(2)}_d = &\sum_{\substack{d_1+d_2+d_3=d \\ d_i>0}} \left(p_{200}\, N^{(2)}_{d_1} N^{(0)}_{d_2} N^{(0)}_{d_3} + p_{110}\, N^{(1)}_{d_1} N^{(1)}_{d_2} N^{(0)}_{d_3}\right) \\
&+ \sum_{\substack{d_1+d_2=d \\ d_i>0}} \left(p_{20}\, N^{(2)}_{d_1} N^{(0)}_{d_2} + p_{h0}\, H^{(2)}_{d_1} N^{(0)}_{d_2} + p_{11}\, N^{(1)}_{d_1} N^{(1)}_{d_2} + p_{10}\, N^{(1)}_{d_1} N^{(0)}_{d_2}\right) \\
&- \tfrac{1}{960} d^4(d-1)(d-2) N^{(0)}_d + \tfrac{1}{40} d^2(5d-6) N^{(1)}_d.
\end{aligned}$$

The polynomial coefficients are defined by

$$\begin{aligned}
p_{200} &= -2\tbinom{3d-1}{3d_1+1,\ 3d_2-1,\ 3d_3-1} d_1 d_2^2 d_3^3 (d_2 + d_3), \\
p_{110} &= \tfrac{1}{5}\tbinom{3d-1}{3d_1,\ 3d_2,\ 3d_3-1} d_1 d_2 d_3^3 (-9d_1 d_2 + 6d_2^2 - 12 d_2 d_3 + d_3^2), \\
p_{20} &= \tbinom{3d-1}{3d_1+1} d_2 (3d_1^2 - 10 d_1 d_2 + 4 d_2^2) + \tbinom{3d-1}{3d_1} d_2^2 (3d_1 + 2d_2), \\
p_{h0} &= 2\tbinom{3d-1}{3d_1} d_2^4, \\
p_{11} &= \tfrac{3}{5}\tbinom{3d-1}{3d_1} d_1 (4d_1^2 - 9 d_1 d_2 + 2 d_2^2), \\
p_{10} &= \tfrac{1}{120}\tbinom{3d-1}{3d_1} d_1 d_2^3 (-18 d_1^2 + 36 d_1 d_2 - 6 d_2^2 + 5 d_1^3 - 33 d_1^2 d_2 + 3 d_1 d_2^2 + d_2^3).
\end{aligned}$$

A calculation of the first 10 values of the 3 series is tabulated below. There are at least four other mathematical methods to obtain the series $N_d^{(2)}$: the degenerations of [R] and [CH], the hyperelliptic methods of [Gr], and the virtual localization formula of [GP1]. In fact, virtual localization determines all gravitational descendents of $\mathbf{P}^n$. However, these four methods are computationally much more complex than the recursions obtained from (22).

Our recursions for $\mathbf{P}^2$ are most closely related to the Virasoro conjecture. In [EX], the authors use weak topological recursion relations in genus 2 together with the Virasoro conjecture for $\mathbf{P}^2$ to obtain recursions involving a fixed number of descendent integrals in each degree (and Gromov-Witten invariants of lower genus). The Virasoro conjecture generates enough relations to solve for these series. The numbers $N_d^{(2)}$ below agree with the values predicted in [EX]. If the method of [EX] is coupled with Getzler's stronger topological recursion relations ([G2]), then the Virasoro conjecture also yields recursions involving only the 3 basic series (12). It would be quite interesting to link our recursions to those predicted by the conjecture.

d	$N_d^{(2)}$	$H_d^{(2)}$	$P_d^{(2)}$
1	0	0	0
2	0	0	0
3	0	$-1/4$	$-1/12$
4	27	42	$25/4$
5	36855	130431	21119
6	58444767	239431851	33238513
7	122824720116	530315850624	63738316894
8	346860150644700	1532247146604636	161943939423280
9	1301798459308709880	5811753079971551880	547601957576517600
10	6383405726993645784000	28632855467501316224640	2432759415312389538720

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