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Renormalized solutions of elliptic equations with general measure data

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Renormalized Solutions of Elliptic Equations with General Measure Data

GIANNI DAL MASO – FRANÇOIS MURAT –
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Abstract. We study the nonlinear monotone elliptic problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = \mu & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\Omega \subset \mathbf{R}^N$, μ is a Radon measure with bounded total variation on Ω , $1 < p \leq N$, and $u \mapsto -\operatorname{div}(a(x, \nabla u))$ is a monotone operator acting on $W_0^{1,p}(\Omega)$. We introduce a new definition of solution (the renormalized solution) in four equivalent ways. We prove the existence of a renormalized solution by an approximation procedure, where the key point is a stability result (the strong convergence in $W_0^{1,p}(\Omega)$ of the truncates). We also prove partial uniqueness results.

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1. – Introduction

In this paper we introduce and study a new notion of solution (the renormalized solution) for the elliptic problem

$$(1.1) \quad \begin{cases} -\operatorname{div}(a(x, \nabla u)) = \mu & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of \mathbf{R}^N , $N \geq 2$, $u \mapsto -\operatorname{div}(a(x, \nabla u))$ is a monotone operator defined on $W_0^{1,p}(\Omega)$ with values in $W^{-1,p'}(\Omega)$, $p > 1$, $1/p + 1/p' = 1$, and μ is a Radon measure with bounded variation on Ω .

If p is greater than the dimension N of the ambient space, then it is easily seen, by Sobolev embedding and duality arguments, that the space of measures with bounded variation on Ω is a subset of $W^{-1,p'}(\Omega)$, so that existence and uniqueness of solutions in $W_0^{1,p}(\Omega)$ is a direct consequence of the theory of monotone operators (see, e.g., [21] and [22]). This framework is, however, not applicable if $p \leq N$, since in this case simple examples (the Laplace equation in a ball, with μ the Dirac mass at the center) show that the solution cannot be expected to belong to the energy space (that is, $W_0^{1,p}(\Omega)$). Thus, it is necessary to change the functional setting in order to prove existence results.

In the linear case, i.e., if $p = 2$ and $a(x, \nabla u) = A(x) \nabla u$, where A is a uniformly elliptic matrix with $L^\infty(\Omega)$ coefficients, this problem was studied by G. Stampacchia, who introduced and studied in [29] a notion of solution defined by duality. This allowed him to prove both existence and uniqueness results. The solution introduced in [29] satisfies in particular

$$(1.2) \quad \begin{cases} u \in W_0^{1,q}(\Omega) \quad \forall q < \frac{N}{N-1}, \\ \int_\Omega A(x) \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu \quad \text{for every } \varphi \in C_c^\infty(\Omega), \end{cases}$$

i.e., it is a solution of (1.1) which belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, and satisfies the equation in the distributional sense (note that, in contrast with (1.1), the space and the sense of the equation are now specified in (1.2)). Let us

emphasize that (1.2) is not enough to characterize the (unique) solution in the sense of [29] when the coefficients of the matrix A are discontinuous (see [28]).

Unfortunately, Stampacchia's framework, which heavily relies on a duality argument, cannot be extended to the general nonlinear case, except in the case $p = 2$, where Stampacchia's ideas continue to work if the operator is strongly monotone and Lipschitz continuous with respect to ∇u (see [26] and [2]).

If the operator is nonlinear, the first attempt to solve problem (1.1) was done by L. Boccardo and T. Gallouët, who proved in [3] and [4], under the hypothesis $p > 2 - \frac{1}{N}$, the existence of a solution of (1.1) which satisfies

$$(1.3) \quad \begin{cases} u \in W_0^{1,q}(\Omega) \quad \forall q < \frac{N(p-1)}{N-1}, \\ \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\mu \quad \text{for every } \varphi \in C_c^\infty(\Omega). \end{cases}$$

Note that this framework coincides with the framework given by (1.2) if $p = 2$. The hypothesis on p is motivated by the fact that, if $p \leq 2 - \frac{1}{N}$, then $\frac{N(p-1)}{N-1} \leq 1$; in (1.3), the exponent $\frac{N(p-1)}{N-1}$ is sharp (see Example 2.16, below). This implies that, in order to obtain the existence of a solution for p close to 1, it is necessary to go out of the framework of classical Sobolev spaces.

Let us now turn to uniqueness. While Meyers' type estimates allow one to prove that the solution of (1.2) is unique in the case $N = 2$ (see [15]), a counterexample by J. Serrin shows (see [28] and [27]) that the solution of (1.2) (and hence of (1.3)) is not unique whenever $N \geq 3$.

Thus, the definition (1.3) is not enough in order to ensure uniqueness. Indeed, Stampacchia's definition, given in [29], which implies uniqueness, requires stronger conditions on the solution, namely that the equation is satisfied for a larger space of test functions. The next step thus consisted in finding some extra conditions on the distributional solutions of (1.1) in order to ensure both existence and uniqueness. This was done independently by P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J.L. Vazquez, by A. Dall'Aglio, and by P.-L. Lions and F. Murat, who introduced, respectively, the notions of *entropy* solution in [1], of *SOLA* (Solution Obtained as Limit of Approximations) in [9], and of *renormalized* solution in [23] and [25]. These settings were, however, limited to the case of a measure in $L^1(\Omega)$ (or in $L^1(\Omega) + W^{-1,p'}(\Omega)$, see [5] and [23]), and did not cover the case of a general measure μ .

These three frameworks, which are actually equivalent, are successful since they allow one to prove existence, uniqueness and continuity of the solutions with respect to the datum μ (for this latter result in the case of entropy solutions, see [20]).

The main tool of the uniqueness proof was, in the case of entropy and renormalized solutions, the fact that the truncates of the solutions actually belong to the energy space $W_0^{1,p}(\Omega)$, as well as an estimate on the decay of the energy of the solution on the sets where the solution is large (see (2.26), below), which is true only if the datum μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ (see Remark 2.24, below).

In the present paper we extend the notion of renormalized solution to the case of general measures with bounded total variation on Ω , and we prove the existence of such a solution. For these solutions, we prove a stability result, namely the strong convergence of truncates in $W_0^{1,p}(\Omega)$. We also introduce other definitions, which we prove to be equivalent. We finally make some attempts towards uniqueness, proving in particular in the linear case that the renormalized solution is unique, and in the nonlinear case that if the difference of two renormalized solutions of the same equation satisfies an extra property (which is not enjoyed by each of the solutions themselves), then these two solutions coincide. But uniqueness *sensu proprio* still remains an open problem, except in the case $p = 2$ and $p = N$ (see [26] and [2] for $p = 2$, and [16], [13], [12], and Remark 10.8 below for $p = N$).

The outline of the paper is as follows. After giving the definition and some preliminary results on p -capacity, we recall in Section 2 a decomposition result for measures μ which will be crucial in our treatment of the problem: every Radon measure μ with bounded total variation on Ω can be written as $\mu = \mu_0 + \mu_s$, where μ_0 does not charge the sets of zero p -capacity (a fact which can be proved to be equivalent to be a measure in $L^1(\Omega) + W^{-1,p'}(\Omega)$), and μ_s is concentrated on a subset E of Ω with zero p -capacity. In Section 2 we also give the definition of renormalized solution, and then three other definitions, which will be proved to be equivalent. In Section 3 we state the existence result for renormalized solutions, as well as a stability theorem for renormalized solutions. Section 4 is devoted to the proof of the equivalence between the four definitions of renormalized solution. In Section 5 we begin the proof of the stability theorem, and we introduce a family of cut-off functions, built after the set E where the singular part μ_s of the measure μ is concentrated. We use these cut-off functions to study the behaviour of the sequence of solutions both “near to” and “far from” the set E in Section 6 and in Section 7 respectively. In Section 8 we conclude the proof of the stability theorem, and also of the existence result. In Section 9 we prove some results concerning the difference of two renormalized solutions. In the final Section 10 we state and prove the uniqueness of the renormalized solution in the linear case and some partial uniqueness results in the nonlinear case.

The results of the present paper have been announced in [11].

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2. – Assumptions and definitions

2.1. – Preliminaries about capacities

Let Ω be a bounded, open subset of \mathbf{R}^N , $N \geq 2$; no smoothness is assumed on $\partial\Omega$. Let p and p' be real numbers, with

$$(2.1) \quad 1 < p \leq N,$$

and p' the Hölder conjugate exponent of p , i.e., $1/p + 1/p' = 1$.

The p -capacity $\text{cap}_p(B, \Omega)$ of any set $B \subseteq \Omega$ with respect to Ω is defined in the following classical way. The p -capacity of any compact set $K \subset \Omega$ is first defined as

$$\text{cap}_p(K, \Omega) = \inf \left\{ \int_{\Omega} |\nabla\varphi|^p dx : \varphi \in C_c^\infty(\Omega), \varphi \geq \chi_K \right\},$$

where χ_K is the characteristic function of K ; we will use the convention that $\inf \emptyset = +\infty$. The p -capacity of any open subset $U \subseteq \Omega$ is then defined by

$$\text{cap}_p(U, \Omega) = \sup \left\{ \text{cap}_p(K, \Omega), K \text{ compact}, K \subseteq U \right\}.$$

Finally, the p -capacity of any subset $B \subseteq \Omega$ is defined by

$$\text{cap}_p(B, \Omega) = \inf \left\{ \text{cap}_p(U, \Omega), U \text{ open}, B \subseteq U \right\}.$$

A function u defined on Ω is said to be cap_p -quasi continuous if for every $\varepsilon > 0$ there exists $B \subseteq \Omega$ with $\text{cap}_p(B, \Omega) < \varepsilon$ such that the restriction of u to $\Omega \setminus B$ is continuous. It is well known that every function in $W^{1,p}(\Omega)$ has a cap_p -quasi continuous representative, whose values are defined cap_p -quasi everywhere in Ω , that is, up to a subset of Ω of zero p -capacity. When we are dealing with the pointwise values of a function $u \in W^{1,p}(\Omega)$, we always

identify u with its cap_p -quasi continuous representative. With this convention for every subset B of Ω we have

$$(2.2) \quad \text{cap}_p(B, \Omega) = \inf \left\{ \int_{\Omega} |\nabla v|^p dx \right\},$$

where the infimum is taken over all functions v in $W_0^{1,p}(\Omega)$ such that $v = 1$ cap_p -quasi everywhere on B , and $v \geq 0$ cap_p -quasi everywhere on Ω .

It is well known that, if u is a cap_p -quasi continuous function defined on Ω such that $u = 0$ almost everywhere on an open set $U \subseteq \Omega$, then $u = 0$ cap_p -quasi everywhere on U (see, e.g., [17], Theorem 4.12). The following proposition extends this results to more general sets U , and will be used several times throughout the paper.

PROPOSITION 2.1. *Let u and v be cap_p -quasi continuous functions defined on Ω . Suppose that $u = 0$ almost everywhere on the set $\{v > 0\}$. Then $u = 0$ cap_p -quasi everywhere on $\{v > 0\}$.*

PROOF. Under our assumptions we have $u v^+ = 0$ almost everywhere on Ω , where $v^+ = \max\{v, 0\}$ is the positive part of v . As the function $u v^+$ is cap_p -quasi continuous, we deduce that $u v^+ = 0$ cap_p -quasi everywhere on Ω , which implies that $u = 0$ cap_p -quasi everywhere on the set $\{v > 0\}$. \square

2.2. – Preliminaries about measures

We define $\mathcal{M}_b(\Omega)$ as the space of all Radon measures on Ω with bounded total variation, and $C_b^0(\Omega)$ as the space of all bounded, continuous functions on Ω , so that $\int_{\Omega} \varphi d\mu$ is defined for $\varphi \in C_b^0(\Omega)$ and $\mu \in \mathcal{M}_b(\Omega)$. The positive part, the negative part, and the total variation of a measure μ in $\mathcal{M}_b(\Omega)$ are denoted by μ^+ , μ^- , and $|\mu|$, respectively.

DEFINITION 2.2. We say that a sequence $\{\mu_n\}$ of measures in $\mathcal{M}_b(\Omega)$ converges in the narrow topology to a measure μ in $\mathcal{M}_b(\Omega)$ if

$$(2.3) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} \varphi d\mu_n = \int_{\Omega} \varphi d\mu$$

for every $\varphi \in C_b^0(\Omega)$.

REMARK 2.3. It is well known that, if μ_n is nonnegative, then $\{\mu_n\}$ converges to μ in the narrow topology of measures if and only if $\mu_n(\Omega)$ converges to $\mu(\Omega)$ and (2.3) holds for every $\varphi \in C_c^\infty(\Omega)$. In particular, if $\mu_n \geq 0$, $\{\mu_n\}$ converges to μ in the narrow topology of measures if and only if (2.3) holds for every $\varphi \in C^\infty(\overline{\Omega})$.

We recall that for a measure μ in $\mathcal{M}_b(\Omega)$, and a Borel set $E \subseteq \Omega$, the measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for any Borel set $B \subseteq \Omega$. If a measure μ in $\mathcal{M}_b(\Omega)$ is such that $\mu = \mu \llcorner E$ for a certain Borel set E ,

the measure μ is said to be *concentrated* on E . Recall that one cannot in general define a smallest set (in the sense of inclusion) where a given measure is concentrated.

We define $\mathcal{M}_0(\Omega)$ as the set of all measures μ in $\mathcal{M}_b(\Omega)$ which are “absolutely continuous” with respect to the p -capacity, i.e., which satisfy $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ such that $\text{cap}_p(B, \Omega) = 0$. We define $\mathcal{M}_s(\Omega)$ as the set of all measures μ in $\mathcal{M}_b(\Omega)$ which are “singular” with respect to the p -capacity, i.e., the measures for which there exists a Borel set $E \subset \Omega$, with $\text{cap}_p(E, \Omega) = 0$, such that $\mu = \mu \llcorner E$.

The following result is the analogue of the Lebesgue decomposition theorem, and can be proved in the same way.

PROPOSITION 2.4. *For every measure μ in $\mathcal{M}_b(\Omega)$ there exists a unique pair of measures (μ_0, μ_s) , with μ_0 in $\mathcal{M}_0(\Omega)$ and μ_s in $\mathcal{M}_s(\Omega)$, such that $\mu = \mu_0 + \mu_s$. If μ is nonnegative, so are μ_0 and μ_s .*

PROOF. See [14], Lemma 2.1. □

The measures μ_0 and μ_s will be called the *absolutely continuous* and the *singular* part of μ with respect to the p -capacity. To deal with μ_0 we need a further decomposition result.

PROPOSITION 2.5. *Let μ_0 be a measure in $\mathcal{M}_b(\Omega)$. Then μ_0 belongs to $\mathcal{M}_0(\Omega)$ if and only if it belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. Thus, if μ_0 belongs to $\mathcal{M}_0(\Omega)$, there exist f in $L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$, such that*

$$(2.4) \quad \mu_0 = f - \text{div}(g),$$

in the sense of distributions; moreover one has

$$\int_{\Omega} v d\mu_0 = \int_{\Omega} f v dx + \int_{\Omega} g \cdot \nabla v dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega).$$

Note that the decomposition (2.4) is not unique since $L^1(\Omega) \cap W^{-1,p'}(\Omega) \neq \{0\}$.

PROOF. See [5], Theorem 2.1. □

Putting together the results of Propositions 2.4 and 2.5, and the Hahn decomposition theorem, we obtain the following result.

PROPOSITION 2.6. *Every measure μ in $\mathcal{M}_b(\Omega)$ can be decomposed as follows*

$$(2.5) \quad \mu = \mu_0 + \mu_s = f - \text{div}(g) + \mu_s^+ - \mu_s^-,$$

where μ_0 is a measure in $\mathcal{M}_0(\Omega)$, and so can be written as $f - \text{div}(g)$, with f in $L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$, while μ_s^+ and μ_s^- (the positive and negative part of μ_s) are two nonnegative measures in $\mathcal{M}_b(\Omega)$ which are concentrated on two disjoint subsets E^+ and E^- of zero p -capacity. We set $E = E^+ \cup E^-$.

The following technical propositions will be used several times in what follows; the second one is a well known consequence of the Egorov theorem.

PROPOSITION 2.7. *Let μ_0 be a measure in $\mathcal{M}_0(\Omega)$, and let v be a function in $W_0^{1,p}(\Omega)$. Then (the cap_p -quasi continuous representative of) v is measurable with respect to μ_0 . If v further belongs to $L^\infty(\Omega)$, then (the cap_p -quasi continuous representative of) v belongs to $L^\infty(\Omega, \mu_0)$, hence to $L^1(\Omega, \mu_0)$.*

PROOF. Every cap_p -quasi continuous function coincides cap_p -quasi everywhere with a Borel function and is therefore measurable for any measure μ_0 in $\mathcal{M}_0(\Omega)$, since these measures do not charge sets of zero p -capacity. If v belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then there exists a constant k such that $|v| \leq k$ almost everywhere on Ω . Consequently the cap_p -quasi continuous representative of v satisfies $|v| \leq k$ cap_p -quasi everywhere on Ω (see [17], Theorem 4.12), and thus μ_0 -almost everywhere on Ω . \square

PROPOSITION 2.8. *Let Ω be a bounded, open subset of \mathbf{R}^N ; let ρ_ε be a sequence of $L^1(\Omega)$ functions that converges to ρ weakly in $L^1(\Omega)$, and let σ_ε be a sequence of functions in $L^\infty(\Omega)$ that is bounded in $L^\infty(\Omega)$ and converges to σ almost everywhere in Ω . Then*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_\varepsilon \sigma_\varepsilon dx = \int_{\Omega} \rho \sigma dx .$$

2.3. – Assumptions on the operator

Let $a : \Omega \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ be a Carathéodory function (that is, $a(\cdot, \xi)$ is measurable on Ω for every ξ in \mathbf{R}^N , and $a(x, \cdot)$ is continuous on \mathbf{R}^N for almost every x in Ω) which satisfies the following hypotheses:

$$(2.6) \quad a(x, \xi) \cdot \xi \geq \alpha |\xi|^p ,$$

for almost every x in Ω and for every ξ in \mathbf{R}^N , where $\alpha > 0$ is a constant;

$$(2.7) \quad |a(x, \xi)| \leq \gamma [b(x) + |\xi|]^{p-1} ,$$

for almost every x in Ω and for every ξ in \mathbf{R}^N , where $\gamma > 0$ is a constant and b is a nonnegative function in $L^p(\Omega)$;

$$(2.8) \quad (a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') > 0 ,$$

for almost every x in Ω and for every ξ, ξ' in \mathbf{R}^N , $\xi \neq \xi'$.

A consequence of (2.6), and of the continuity of a with respect to ξ , is that, for almost every x in Ω ,

$$a(x, 0) = 0 .$$

Thanks to hypotheses (2.6)-(2.8), the map $u \mapsto -\text{div}(a(x, \nabla u))$ is a coercive, continuous, bounded, and monotone operator defined on $W_0^{1,p}(\Omega)$ with

values in its dual space $W^{-1,p'}(\Omega)$; moreover, by the standard theory of monotone operators (see, e.g., [21] and [22]), for every T in $W^{-1,p'}(\Omega)$ there exists one and only one solution v of the problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla v)) = T & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$(2.9) \quad \begin{cases} v \in W_0^{1,p}(\Omega), \\ \int_{\Omega} a(x, \nabla v) \cdot \nabla \varphi \, dx = \langle T, \varphi \rangle, \quad \forall \varphi \in W_0^{1,p}(\Omega), \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. If $p > N$, then $\mathcal{M}_b(\Omega)$ is a subset of $W^{-1,p'}(\Omega)$, so that this classical result implies both existence and uniqueness of a solution of (1.1) for every measure μ in $\mathcal{M}_b(\Omega)$. This explains the restrictions $p \leq N$ and $N \geq 2$ that we have imposed.

2.4. – Definition of renormalized solution

We begin by introducing some of the tools which will be used to define renormalized solutions.

For $k > 0$ and for $s \in \mathbf{R}$ we define as usual $T_k(s) = \max(-k, \min(k, s))$ (see if necessary the figure after (4.2) in Section 4.1).

We begin (following [1]) with the definition of the gradient of a function whose truncatures belong to $W_0^{1,p}(\Omega)$.

DEFINITION 2.9. Let u be a measurable function defined on Ω which is finite almost everywhere, and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. Then there exists (see [1], Lemma 2.1) a measurable function $v : \Omega \rightarrow \mathbf{R}^N$ such that

$$(2.10) \quad \nabla T_k(u) = v \chi_{\{|u| \leq k\}} \quad \text{almost everywhere in } \Omega, \text{ for every } k > 0,$$

which is unique up to almost everywhere equivalence. We define the gradient ∇u of u as this function v , and denote $\nabla u = v$.

REMARK 2.10. We explicitly remark that the gradient defined in this way is not, in general, the gradient used in the definition of Sobolev spaces, since it is possible that u does not belong to $L^1_{\text{loc}}(\Omega)$ (and thus the gradient of u in distributional sense is not defined) or that v does not belong to $(L^1_{\text{loc}}(\Omega))^N$ (see Example 2.16 below). However, if v belongs to $(L^1_{\text{loc}}(\Omega))^N$, then u belongs to $W^{1,1}_{\text{loc}}(\Omega)$ and v is the distributional gradient of u . Indeed, let ω be a ball such that $\bar{\omega} \subset \Omega$. By (2.10), $\nabla T_k(u)$ is bounded in $(L^1(\omega))^N$ independently of k , and by Poincaré-Wirtinger inequality, the function z_k defined by

$$z_k = T_k(u) - \int_{\omega} T_k(u) \, dx$$

is bounded in $W^{1,1}(\omega)$, independently of k . Therefore, up to a subsequence still denoted by z_k , z_k converges almost everywhere to some z belonging to $W^{1,1}(\omega)$. Since both z and u are finite almost everywhere, then

$$\int_{\omega} T_k(u) dx$$

converges to some finite constant. This implies that $T_k(u)$ is bounded in $W^{1,1}(\omega)$. By Fatou lemma we then conclude that u belongs to $L^1(\omega)$. Since $\nabla T_k(u)$ converges to v strongly in $(L^1(\omega))^N$ by (2.10) and by Lebesgue dominated convergence theorem, u belongs to $W^{1,1}(\omega)$, and v is the usual distributional gradient of u . If v is moreover assumed to belong to $(L^q(\Omega))^N$ for some $1 \leq q \leq p$, then a similar proof implies that u belongs to $W_0^{1,q}(\Omega)$.

On the other hand, if u belongs to $L_{\text{loc}}^1(\Omega)$, the function v defined by (2.10) (which does not in general belong to $(L_{\text{loc}}^1(\Omega))^N$) does not in general coincide with the distributional gradient of u . Consider indeed the case where Ω is the unit ball of \mathbf{R}^N and where $u(x) = \frac{x_1}{|x|^N}$. The function u belongs to $L^q(\Omega)$, for every $q < \frac{N}{N-1}$, and one has

$$\frac{\partial T_k(u)}{\partial x_1} = \left\{ \frac{1}{|x|^N} - N \frac{x_1^2}{|x|^{N+2}} \right\} \chi_{\{|u| \leq k\}},$$

so that

$$v_1 = \frac{1}{|x|^N} - N \frac{x_1^2}{|x|^{N+2}}$$

does not belong to $L_{\text{loc}}^1(\Omega)$, which implies that v is not in $(L_{\text{loc}}^1(\Omega))^N$. On the other hand, we have in distributional sense

$$\frac{\partial u}{\partial x_1} = \mathbf{p}\mathbf{v} \left\{ \frac{1}{|x|^N} - N \frac{x_1^2}{|x|^{N+2}} \right\} + \frac{1}{N} \sigma_{N-1} \delta_0,$$

where $\mathbf{p}\mathbf{v}$ denotes the principal value, σ_{N-1} the $(N-1)$ -dimensional measure of the surface of the unit sphere of \mathbf{R}^N , and δ_0 is the Dirac mass at the origin.

REMARK 2.11. Under the assumptions made on u in the Definition 2.9, u has a cap_p -quasi continuous representative, which we still denote by u . Let us observe explicitly that this cap_p -quasi continuous representative can be infinite on a set of positive p -capacity, as shown for example by the function

$$u_0(x) = \frac{1}{|x_N|} (1 - |x|^2),$$

which satisfies the requirements of Definition 2.9 in the unit ball of \mathbf{R}^N .

If, in addition to these requirements, the function u is assumed to satisfy the estimate

$$(2.11) \quad \int_{\Omega} |\nabla T_k(u)|^p dx \leq c(k+1) \quad \text{for every } k > 0,$$

where c is independent of k , then the cap_p -quasi continuous representative is cap_p -quasi everywhere finite. Indeed from (2.11) and (2.2) we deduce, using $v = |T_k(u)|/k$, that

$$\text{cap}_p(\{|u| \geq k\}, \Omega) \leq \int_{\Omega} \frac{|\nabla T_k(u)|^p}{k^p} dx \leq \frac{c(k+1)}{k^p},$$

for every $k > 0$, which implies

$$\text{cap}_p(\{|u| = +\infty\}, \Omega) = 0.$$

As observed in [1], the set of functions u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$ is not a linear space. That is, if u and v are such that both $T_k(u)$ and $T_k(v)$ belong to $W_0^{1,p}(\Omega)$ for every $k > 0$ (so that ∇u and ∇v can be defined as in Definition 2.9), this may not be the case for (as an example) $u + v$, and so $\nabla(u + v)$ may not be defined. The following lemma proves that if u , v and $u + \lambda v$ have truncates in $W_0^{1,p}(\Omega)$ then we have the usual formula $\nabla(u + \lambda v) = \nabla u + \lambda \nabla v$.

LEMMA 2.12. *Let $\lambda \in \mathbf{R}$ and let u and \hat{u} be two measurable functions defined on Ω which are finite almost everywhere, and which are such that $T_k(u)$, $T_k(\hat{u})$ and $T_k(u + \lambda \hat{u})$ belong to $W_0^{1,p}(\Omega)$ for every $k > 0$. Then*

$$(2.12) \quad \nabla(u + \lambda \hat{u}) = \nabla u + \lambda \nabla \hat{u} \quad \text{almost everywhere on } \Omega,$$

where ∇u , $\nabla \hat{u}$ and $\nabla(u + \lambda \hat{u})$ are the gradients of u , \hat{u} and $u + \lambda \hat{u}$ introduced in Definition 2.9.

PROOF. Let $E_n = \{|u| < n\} \cap \{|\hat{u}| < n\}$. On E_n , we have $T_n(u) = u$ and $T_n(\hat{u}) = \hat{u}$, so that for every $k > 0$,

$$T_k(T_n(u) + \lambda T_n(\hat{u})) = T_k(u + \lambda \hat{u}) \quad \text{a.e. in } E_n,$$

and therefore, since both functions belong to $W_0^{1,p}(\Omega)$,

$$(2.13) \quad \nabla T_k(T_n(u) + \lambda T_n(\hat{u})) = \nabla T_k(u + \lambda \hat{u}) \quad \text{a.e. in } E_n.$$

Since $T_n(u)$ and $T_n(\hat{u})$ belong to $W_0^{1,p}(\Omega)$, we have, using a classical property of the truncates in $W_0^{1,p}(\Omega)$, and then the definition of ∇u and $\nabla \hat{u}$,

$$\begin{aligned} \nabla T_k(T_n(u) + \lambda T_n(\hat{u})) &= \chi_{\{|T_n(u) + \lambda T_n(\hat{u})| \leq k\}} (\nabla T_n(u) + \lambda \nabla T_n(\hat{u})) \\ &= \chi_{\{|T_n(u) + \lambda T_n(\hat{u})| \leq k\}} (\chi_{\{|u| \leq n\}} \nabla u + \lambda \chi_{\{|\hat{u}| \leq n\}} \nabla \hat{u}), \end{aligned}$$

almost everywhere in Ω . Therefore

$$(2.14) \quad \nabla T_k(T_n(u) + \lambda T_n(\hat{u})) = \chi_{\{|u+\lambda\hat{u}| \leq k\}}(\nabla u + \lambda \nabla \hat{u}) \quad \text{a.e. in } E_n.$$

On the other hand, by definition of $\nabla(u + \lambda \hat{u})$,

$$(2.15) \quad \nabla T_k(u + \lambda \hat{u}) = \chi_{\{|u+\lambda\hat{u}| \leq k\}} \nabla(u + \lambda \hat{u}) \quad \text{a.e. in } E_n.$$

Putting together (2.13), (2.14) and (2.15), we obtain

$$(2.16) \quad \chi_{\{|u+\lambda\hat{u}| \leq k\}}(\nabla u + \lambda \nabla \hat{u}) = \chi_{\{|u+\lambda\hat{u}| \leq k\}} \nabla(u + \lambda \hat{u}) \quad \text{a.e. in } E_n.$$

Since $\bigcup_{n \in \mathbb{N}} E_n$ at most differs from Ω by a set of zero Lebesgue measure (since u and \hat{u} are almost everywhere finite), (2.16) also holds almost everywhere in Ω . Since $\bigcup_{k \in \mathbb{N}} \{|u + \lambda \hat{u}| \leq k\}$ at most differs from Ω by a set of zero Lebesgue measure, we have proved (2.12). \square

We are now in a position to define the notion of renormalized solution. Other (equivalent) definitions will be given in Section 2.5. We recall that by Proposition 2.4 every measure μ in $\mathcal{M}_b(\Omega)$ can be written in a unique way as $\mu = \mu_0 + \mu_s$, with μ_0 in $\mathcal{M}_0(\Omega)$ and μ_s in $\mathcal{M}_s(\Omega)$.

DEFINITION 2.13. Assume that a satisfies (2.6)–(2.8), and let μ be a measure in $\mathcal{M}_b(\Omega)$. A function u is a *renormalized solution* of problem (1.1) if the following conditions hold:

- (a) the function u is measurable and finite almost everywhere, and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$;
- (b) the gradient ∇u , introduced in Definition 2.9, satisfies

$$(2.17) \quad |\nabla u|^{p-1} \text{ belongs to } L^q(\Omega), \text{ for every } q < \frac{N}{N-1};$$

- (c) if w belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exist $k > 0$, and $w^{+\infty}$ and $w^{-\infty}$ in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that

$$(2.18) \quad \begin{cases} w = w^{+\infty} & \text{almost everywhere on the set } \{u > k\}, \\ w = w^{-\infty} & \text{almost everywhere on the set } \{u < -k\}, \end{cases}$$

then

$$(2.19) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla w \, dx = \int_{\Omega} w \, d\mu_0 + \int_{\Omega} w^{+\infty} \, d\mu_s^+ - \int_{\Omega} w^{-\infty} \, d\mu_s^-.$$

REMARK 2.14. Every term in (2.19) is well defined. Indeed, the integral on the left hand side can be written as

$$\int_{\{u < -k\}} a(x, \nabla u) \cdot \nabla w \, dx + \int_{\{|u| \leq k\}} a(x, \nabla u) \cdot \nabla w \, dx + \int_{\{u > k\}} a(x, \nabla u) \cdot \nabla w \, dx,$$

where all three terms are well defined: actually, since $|\nabla u|^{p-1}$ belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-1}$, hypothesis (2.7) implies that $a(x, \nabla u)$ belongs to $(L^q(\Omega))^N$, for every $q < \frac{N}{N-1}$; on the other hand $\nabla w = \nabla w^{-\infty}$ almost everywhere on $\{u < -k\}$, so that ∇w belongs to $(L^r(\{u < -k\}))^N$ (with $r > N$) and, consequently, $a(x, \nabla u) \cdot \nabla w$ is integrable on $\{u < -k\}$; in the same way we prove that $a(x, \nabla u) \cdot \nabla w$ is integrable on $\{u > k\}$. Finally, as $\nabla u = \nabla T_k(u)$ almost everywhere on $\{|u| \leq k\}$, we have $a(x, \nabla u) \cdot \nabla w = a(x, \nabla T_k(u)) \cdot \nabla w$ almost everywhere on $\{|u| \leq k\}$, which implies that $a(x, \nabla u) \cdot \nabla w$ is integrable on $\{|u| \leq k\}$, since w belongs to $W_0^{1,p}(\Omega)$ and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ and, consequently, $a(x, \nabla T_k(u))$ belongs to $(L^{p'}(\Omega))^N$ by (2.7).

As for the right hand side, the terms

$$\int_{\Omega} w^{+\infty} d\mu_s^+ - \int_{\Omega} w^{-\infty} d\mu_s^-$$

are obviously well defined, as both $w^{+\infty}$ and $w^{-\infty}$ are continuous and bounded on Ω , while the term

$$\int_{\Omega} w d\mu_0$$

is well defined since w belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and so to $L^1(\Omega, \mu_0)$ (see Proposition 2.7).

REMARK 2.15. The first condition in (2.18) can be written as $w - w^{+\infty} = 0$ almost everywhere on $\{u > k\}$. By Proposition 2.1 this implies that $w - w^{+\infty} = 0$ cap_p -quasi everywhere on $\{u > k\}$, hence $w = w^{+\infty}$ cap_p -quasi everywhere on $\{u > k\}$. Similarly the second condition in (2.18) implies that $w = w^{-\infty}$ cap_p -quasi everywhere on $\{u < k\}$.

EXAMPLE 2.16. Observe that we did not assume that the function u belongs to some Lebesgue space $L^r(\Omega)$, with $r \geq 1$, but only that u is Lebesgue measurable, and finite almost everywhere. Indeed it is possible that the function u does not belong to $L^1_{\text{loc}}(\Omega)$, as it is shown in the following example.

Let $\Omega = B_1(0) = \{x \in \mathbf{R}^N : |x| < 1\}$, let σ_{N-1} be the $(N - 1)$ -dimensional measure of $\partial B_1(0)$, and let $\gamma = \frac{N-p}{p-1}$. Consider the function defined by

$$(2.20) \quad u(x) = \begin{cases} \frac{1}{\gamma} (|x|^{-\gamma} - 1), & \text{if } 1 < p < N, \\ -\log(|x|), & \text{if } p = N. \end{cases}$$

Note that u belongs to $L^1_{\text{loc}}(\Omega)$ if and only if $\gamma < N$, i.e., if and only if $p > \frac{2N}{N+1}$.

Let us show that u is a renormalized solution of the equation

$$(2.21) \quad \begin{cases} -\text{div} (|\nabla u|^{p-2} \nabla u) = \sigma_{N-1} \delta_0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_0 is the Dirac mass concentrated at the origin. Indeed, $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ (and is actually Lipschitz continuous and zero on the boundary of Ω). Thus, (a) is satisfied. Defining ∇u by Definition 2.9, we have

$$\nabla u = -\frac{x}{|x|^{\gamma+2}} \quad \text{for every } 1 < p \leq N,$$

which implies that $|\nabla u|^{p-1}$ is equal to $|x|^{1-N}$ and belongs to $L^q(\Omega)$ for every $q < \frac{N}{N-1}$. Thus (b) is satisfied. Note that if $p > 2 - \frac{1}{N}$, then $\frac{N(p-1)}{N-1} > 1$, so that u belongs to the Sobolev space $W_0^{1,q}(\Omega)$ (see Remark 2.10), for every $q < \frac{N(p-1)}{N-1}$, but not for $q = \frac{N(p-1)}{N-1}$.

For what concerns (c), consider a function w in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which belongs to $W^{1,r}(\omega) \cap L^\infty(\omega)$, with $r > N$, in a neighbourhood ω of the origin. Using the fact that

$$|\nabla u|^{p-2} \nabla u \cdot \nabla w = \frac{x \cdot \nabla w}{|x|^N}$$

belongs to $L^1(\Omega)$ since ∇w belongs to $(L^r(\omega))^N$, integrating by parts on $B_1(0) \setminus B_\varepsilon(0)$, using the fact that $\frac{x}{|x|^N}$ is a smooth function with $\operatorname{div}(\frac{x}{|x|^N}) = 0$, and finally using the continuity of $w^{+\infty}$ at the origin and the fact that $w = w^{+\infty}$ on ω , we obtain

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla w \, dx &= \lim_{\varepsilon \rightarrow 0^+} \int_{B_1(0) \setminus B_\varepsilon(0)} \frac{x \cdot \nabla w}{|x|^N} \, dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{\partial B_\varepsilon(0)} w \frac{x}{|x|^N} \cdot \frac{x}{|x|} \, d\sigma \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon^{N-1}} \int_{\partial B_\varepsilon(0)} w^{+\infty} \, d\sigma = \sigma_{N-1} w^{+\infty}(0) \\ &= \sigma_{N-1} \int_{\Omega} w^{+\infty} \, d\delta_0. \end{aligned}$$

In this example u does not belong to $L^1_{\text{loc}}(\Omega)$ if $1 < p \leq \frac{2N}{N+1}$, and in this case ∇u is not the distributional gradient of u . Note that the function u defined by (2.20) is the unique solution in the sense of distributions of (2.21) which belongs to $W^{1,p}(B_1(0) \setminus B_\varepsilon(0))$ for every $\varepsilon > 0$, and has the behaviour described by (2.20) near the origin (see [19] and [18]).

REMARK 2.17. Every function w in $C_c^1(\Omega)$ is an admissible test function in (c) of Definition 2.13. Thus, if u is a renormalized solution of (1.1) in the sense of Definition 2.13, and if $p > 2 - \frac{1}{N}$, then it is also a solution of (1.3).

But there are more admissible functions, built after u , such as $T_k(u)$ (choosing $w^{+\infty} \equiv k$ and $w^{-\infty} \equiv -k$).

If φ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and $k > 0$, then it is possible to choose in (2.19) the function $w = T_k(u - \varphi)$; indeed, setting $M = \|\varphi\|_{L^\infty(\Omega)}$,

then $w = T_k(T_{k+M}(u) - \varphi)$, so that w belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and we can choose $w^{+\infty} \equiv k$ and $w^{-\infty} \equiv -k$ on the sets $\{u > k + M\}$ and $\{u < -k - M\}$, respectively. Thus, a renormalized solution of (1.1) turns out to be an entropy solution of (1.1) (with equality sign) in the sense defined in [1]. Hence, if the measure μ does not charge the sets of zero p -capacity, i.e., is a measure in $\mathcal{M}_0(\Omega)$, there exists at most one renormalized solution of (1.1), due to the uniqueness result of [5]. Furthermore, note that the definition of entropy solution with datum a Dirac mass given in [5], Remark 3.4, does not imply that an entropy solution is a distributional solution. In contrast, we have seen above that a renormalized solution in the sense of Definition 2.13 is also a distributional solution, and this rules out the counterexample to uniqueness given in [5], Remark 3.4.

REMARK 2.18. Using $w = T_k(u)$ in (2.19) (with $w^{+\infty} = k$ and $w^{-\infty} = -k$), we obtain

$$\int_{\Omega} a(x, \nabla u) \nabla T_k(u) \, dx \leq k |\mu_0|(\Omega) + k \mu_s^+(\Omega) + k \mu_s^-(\Omega),$$

from which we deduce, with the help of (2.6), the estimate

$$(2.22) \quad \alpha \int_{\Omega} |\nabla T_k(u)|^p \, dx \leq k |\mu|(\Omega), \quad \text{for every } k > 0.$$

By Remark 2.11 this implies that (the cap_p -quasi continuous representative of) u is finite cap_p -quasi everywhere.

REMARK 2.19. Let $s^+ = \max(s, 0)$, and choose in (2.19) $w = T_k(u^+)$ (which is an admissible test function with $w^{+\infty} = k$ and $w^{-\infty} = 0$); in this case, we have

$$\int_{\Omega} w^{+\infty} \, d\mu_s^+ - \int_{\Omega} w^{-\infty} \, d\mu_s^- = k \mu_s^+(\Omega).$$

If we formally write

$$\int_{\Omega} w^{+\infty} \, d\mu_s^+ - \int_{\Omega} w^{-\infty} \, d\mu_s^- = \int_{\Omega} T_k(u^+) \, d\mu_s^+$$

(which is the idea of the formulation, since $w^{+\infty}$ and $w^{-\infty}$ represent the values of $w = T_k(u^+)$, but is not correct since $T_k(u^+)$, which belongs to $W_0^{1,p}(\Omega)$, may not be measurable for the measure μ_s^+ , which belongs to $\mathcal{M}_s(\Omega)$), we obtain that

$$\text{formally, } T_k(u^+) = k \quad \mu_s^+\text{-almost everywhere.}$$

Since k is arbitrary we formally have $u^+ = +\infty$ μ_s^+ -almost everywhere; analogously we have $u^- = -\infty$ μ_s^- -almost everywhere. This expresses the fact that

the function u is infinite on the points where the measure μ_s is “living”, even if μ_s is concentrated on a set of p -capacity zero, and if u is defined pointwise except on a set of p -capacity zero.

REMARK 2.20. A measure in $\mathcal{M}_b(\Omega)$ is not the most general possible datum for (1.1). Indeed, there exist elements in $W^{-1,p'}(\Omega)$ which are not measures, and data of the form

$$\mu - \operatorname{div}(F),$$

with μ in $\mathcal{M}_b(\Omega)$ and F in $(L^{p'}(\Omega))^N$ can be considered. However, the new term $-\operatorname{div}(F)$, due to its “regularity”, does not give any additional difficulty as far as the existence result is concerned. For the sake of simplicity, we will restrict ourselves to the case of a datum μ belonging to $\mathcal{M}_b(\Omega)$, but let us explicitly state that the existence result for renormalized solutions of the present paper holds if μ in $\mathcal{M}_b(\Omega)$ is replaced by

$$\mu - \operatorname{div}(F) \quad \text{with } \mu \text{ in } \mathcal{M}_b(\Omega) \text{ and } F \text{ in } (L^{p'}(\Omega))^N.$$

Let us however stress that, in this case, the three definitions of renormalized solutions given in Section 2.5 below are no longer equivalent to Definition 2.13, and thus have to be modified. The main difficulty lies in the fact that in such a case the “energy tail” which appears in (2.23) below (similar considerations apply to (2.24)) has to be replaced by

$$E(t, s) = \frac{1}{t - s} \int_{\{s \leq u \leq t\}} a(x, \nabla u) \cdot \nabla u \varphi \, dx,$$

and that it is only possible to prove that a subsequence $E(t_n, s_n)$ (for some conveniently chosen t_n and s_n) converges to

$$\int_{\Omega} \varphi \, d\mu_s^+.$$

See [10] for a detailed proof of this fact.

2.5. – Other definitions

Besides Definition 2.13, other definitions of renormalized solutions can be given. We give here three different formulations of renormalized solution, which will turn out to be equivalent to Definition 2.13 (see Theorem 2.33, proved in Section 4.3).

We recall that $W^{1,\infty}(\mathbf{R})$ is the space of all bounded Lipschitz continuous functions on \mathbf{R} .

DEFINITION 2.21. Assume that a satisfies (2.6)-(2.8), and let μ be a measure in $\mathcal{M}_b(\Omega)$. A function u is a renormalized solution of (1.1) if u satisfies (a) and (b) of Definition 2.13, and if the following conditions hold:

(d) for every φ in $C_b^0(\Omega)$ we have

$$(2.23) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{n \leq u < 2n\}} a(x, \nabla u) \cdot \nabla u \varphi \, dx = \int_{\Omega} \varphi \, d\mu_s^+,$$

and

$$(2.24) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{-2n < u \leq -n\}} a(x, \nabla u) \cdot \nabla u \varphi \, dx = \int_{\Omega} \varphi \, d\mu_s^-;$$

(e) for every h in $W^{1,\infty}(\mathbf{R})$ with compact support in \mathbf{R} we have

$$(2.25) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla u h'(u) \varphi \, dx + \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi h(u) \, dx = \int_{\Omega} h(u) \varphi \, d\mu_0,$$

for every φ in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $h(u) \varphi$ belongs to $W_0^{1,p}(\Omega)$.

REMARK 2.22. Observe that every term in (2.25) has a meaning: indeed, in the right hand side $h(u) \varphi$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and hence to $L^1(\Omega, \mu_0)$ by Proposition 2.7. On the other hand, since $\text{supp}(h) \subseteq [-M, M]$ for some $M > 0$, the left hand side can be written as

$$\int_{\Omega} a(x, \nabla T_M(u)) \cdot \nabla T_M(u) h'(u) \varphi \, dx + \int_{\Omega} a(x, \nabla T_M(u)) \cdot \nabla \varphi h(u) \, dx,$$

where both integrals are well defined in view of (2.7), since both φ and $T_M(u)$ belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Note also that the composite function $h'(u)$ is not defined on the set B of points x of Ω such that h is not differentiable at $u(x)$; but one has $\nabla T_M(u) = 0$ almost everywhere on B , and the product $\nabla T_M(u) h'(u)$ is well defined almost everywhere on Ω and coincides with the gradient of the composite function $h(u) = h(T_M(u))$ (see [24] and [6]).

REMARK 2.23. In Definition 2.21 we assume that (b) holds, but this requirement is not used to give a meaning to the various terms which enter in Definition 2.21 (confront this with (2.19) in Definition 2.13). Actually (b) is a consequence of (a), (d) and (e) (and therefore could be omitted in Definition 2.21). We however decided to put (b) among the requirements of Definition 2.21 in order to make simpler the proof of the equivalence between the various definitions.

REMARK 2.24. Definition 2.21 extends to the case of a general measure of $\mathcal{M}_b(\Omega)$ the definition of renormalized solution given in [23] (see [25] and [26]) when μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ (i.e. to $\mathcal{M}_0(\Omega)$, by Proposition 2.4). Indeed in these papers, the definition of renormalized solution included (e) as well as

$$(2.26) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{|n \leq |u| < 2n\}} a(x, \nabla u) \cdot \nabla u \, dx = 0,$$

which coincides with (2.23) and (2.24) in the case $\mu_s^+ = \mu_s^- = 0$. In the present paper (2.23) and (2.24) replace (2.26), and specify the behaviour of the energy of u on the set where u is very large; furthermore, they say that the sequence

$$\frac{1}{n} a(x, \nabla u) \cdot \nabla u \chi_{\{n \leq u < 2n\}}$$

converges to μ_s^+ in the narrow topology, as well as the sequence

$$\frac{1}{n} a(x, \nabla u) \cdot \nabla u \chi_{\{-2n < u \leq -n\}}$$

converges to μ_s^- in the narrow topology. Thus the measures μ_s^+ and μ_s^- can be, in some sense, “reconstructed” starting from the energy of the solution where it is infinite. This fact explains the link between the singular part of the measure μ and the set where u is infinite; see also Remark 2.19.

Conditions (2.23) and (2.24) can be removed if we enlarge the class of admissible test functions.

DEFINITION 2.25. Assume that a satisfies (2.6)-(2.8), and let μ be a measure in $\mathcal{M}_b(\Omega)$. A function u is a renormalized solution of (1.1) if u satisfies (a) and (b) of Definition 2.13, and if the following condition holds:

(f) for every h in $W^{1,\infty}(\mathbf{R})$ such that h' has compact support in \mathbf{R} , we have

$$\begin{aligned} (2.27) \quad & \int_{\Omega} a(x, \nabla u) \cdot \nabla u h'(u) \varphi \, dx + \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi h(u) \, dx \\ & = \int_{\Omega} h(u) \varphi \, d\mu_0 + h(+\infty) \int_{\Omega} \varphi \, d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi \, d\mu_s^-, \end{aligned}$$

for every φ in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that $h(u) \varphi$ belongs to $W_0^{1,p}(\Omega)$. Here $h(+\infty)$ and $h(-\infty)$ are the limits of $h(s)$ at $+\infty$ and $-\infty$ respectively (note that h is constant for $|s|$ large).

REMARK 2.26. As in (2.25), every term in (2.27) is well defined: this is clear for the right hand side since $h(u) \varphi$ belongs to $L^\infty(\Omega, \mu_0)$ (see Proposition 2.7), and thus to $L^1(\Omega, \mu_0)$. Since $\text{supp}(h') \subseteq [-M, M]$, for some M , the left hand side can be written as

$$\int_{\Omega} a(x, \nabla T_M(u)) \cdot \nabla T_M(u) h'(u) \varphi \, dx + \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi h(u) \, dx,$$

where both terms are finite: the first one in view of (2.7) since $T_M(u)$ belongs to $W_0^{1,p}(\Omega)$, and the second one since $a(x, \nabla u)$ belongs to $(L^q(\Omega))^N$ for every $q < \frac{N}{N-1}$ due to the hypothesis (b) on $|\nabla u|^{p-1}$ and to (2.7). For the definition of the product $\nabla T_M(u) h'(u)$ we refer to the final part of Remark 2.22.

REMARK 2.27. Definitions 2.21 and 2.25 state that any renormalized solution u is in some sense equal to $+\infty$ on the sets charged by μ_s^+ , and to $-\infty$ on the sets charged by μ_s^- ; in the case of Definition 2.25 this is clearly expressed by the presence of the two terms $h(+\infty)$ and $h(-\infty)$ in (2.27).

REMARK 2.28. By Theorem 1.2 (and Remark 3.6) of [10], every renormalized solution u of (1.1) in the sense of Definition 2.25 is a *reachable* solution of (1.1) in the sense of [10], i.e., there exists a sequence μ_n in $\mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$ which converges to μ weakly* in $\mathcal{M}_b(\Omega)$, such that the solution u_n of (2.9) with $T = \mu_n$ converges to u almost everywhere in Ω .

The following definition selects a particular approximating sequence which will turn out to be useful.

DEFINITION 2.29. Assume that a satisfies (2.6)-(2.8), and let μ be a measure in $\mathcal{M}_b(\Omega)$. A function u is a renormalized solution of (1.1) if u satisfies (a) and (b) of Definition 2.13, and if the following conditions hold:

- (g) for every $k > 0$ there exist two nonnegative measures in $\mathcal{M}_0(\Omega)$, λ_k^+ and λ_k^- , concentrated on the sets $\{u = k\}$ and $\{u = -k\}$, respectively, such that $\lambda_k^+ \rightarrow \mu_s^+$ and $\lambda_k^- \rightarrow \mu_s^-$ in the narrow topology of measures;
- (h) for every $k > 0$,

$$(2.28) \quad \int_{\{|u| < k\}} a(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\{|u| < k\}} \varphi \, d\mu_0 + \int_{\Omega} \varphi \, d\lambda_k^+ - \int_{\Omega} \varphi \, d\lambda_k^-,$$

for every φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

As above, every term in (2.28) is well defined due to the regularity of $T_k(u)$ and φ .

REMARK 2.30. Similarly to the observation made in Remark 2.23, requirement (b) can be omitted in Definition 2.29.

REMARK 2.31. If we use $\varphi = T_k(u)$ as test function in (2.28), arguing as in Remark 2.18 we obtain (2.22), so that (the cap_p -quasi continuous representative of) u is finite cap_p -quasi everywhere by Remark 2.11.

REMARK 2.32. Definition 2.29 makes explicit the equation solved by $T_k(u)$. Indeed, (2.28) is equivalent to

$$(2.29) \quad -\text{div}(a(x, \nabla T_k(u))) = \mu_0 \llcorner \{|u| < k\} + \lambda_k^+ - \lambda_k^- \quad \text{in } \mathcal{D}'(\Omega).$$

Since $-\text{div}(a(x, \nabla T_k(u)))$ belongs to $W^{-1,p'}(\Omega)$ (due to the hypotheses on $T_k(u)$ and to (2.7)), the measure $\mu_k = \mu_0 \llcorner \{|u| < k\} + \lambda_k^+ - \lambda_k^-$ belongs to $W^{-1,p'}(\Omega)$. Since $|u| < +\infty$ cap_p -quasi everywhere (see Remark 2.31), and hence μ_0 -almost everywhere, the sequence $\mu_0 \llcorner \{|u| < k\}$ converges to μ_0 in the narrow topology of measures as $k \rightarrow +\infty$. This implies that μ_k converges to μ in the narrow topology of measures. Thus u is a reachable solution in the

sense of [10]. Note however that u is not a SOLA in the sense of [9], since a stronger convergence of μ_k (in the weak topology of $L^1(\Omega)$) is required by that definition.

The following theorem will be proved in Section 4.3.

THEOREM 2.33. *Definitions 2.13, 2.21, 2.25, and 2.29 are equivalent.*

3. – Statement of the existence and stability results

The main result of the paper is the following existence theorem.

THEOREM 3.1. *Assume that a satisfies (2.6)–(2.8), and let μ be a measure in $\mathcal{M}_b(\Omega)$. Then there exists a renormalized solution of (1.1).*

We will obtain this existence result by an approximation process: let μ be a measure in $\mathcal{M}_b(\Omega)$, which is decomposed (see (2.5)) as

$$\mu = \mu_0 + \mu_s = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-,$$

with f in $L^1(\Omega)$, g in $(L^{p'}(\Omega))^N$, and μ_s^+ and μ_s^- nonnegative measures concentrated on two disjoint sets of zero p -capacity. We will approximate the measure μ by a sequence μ_ε defined as

$$\mu_\varepsilon = f_\varepsilon - \operatorname{div}(g_\varepsilon) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus,$$

where ε belongs to a sequence of positive numbers that converges to zero, and

$$(3.1) \quad \begin{cases} f_\varepsilon \text{ is a sequence of functions in } L^{p'}(\Omega) \\ \text{that converges to } f \text{ weakly in } L^1(\Omega); \end{cases}$$

$$(3.2) \quad \begin{cases} g_\varepsilon \text{ is a sequence of functions in } (L^{p'}(\Omega))^N \\ \text{that converges to } g \text{ strongly in } (L^{p'}(\Omega))^N, \\ \text{and such that } \operatorname{div}(g_\varepsilon) \text{ is bounded in } \mathcal{M}_b(\Omega); \end{cases}$$

$$(3.3) \quad \begin{cases} \lambda_\varepsilon^\oplus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that} \\ \text{converges to } \mu_s^+ \text{ in the narrow topology of measures;} \end{cases}$$

$$(3.4) \quad \begin{cases} \lambda_\varepsilon^\ominus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that} \\ \text{converges to } \mu_s^- \text{ in the narrow topology of measures.} \end{cases}$$

Such an approximation exists, as it is easily seen by separately approximating f , μ_s^+ , and μ_s^- by convolution and by taking $g_\varepsilon = g$. Note that μ_ε belongs to $W^{-1,p'}(\Omega)$ for every $\varepsilon > 0$.

Given such an approximation of μ , let u_ε be the unique solution of the following problem

$$(3.5) \quad \begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega), \\ \int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla \varphi \, dx = \langle \mu_\varepsilon, \varphi \rangle, \quad \forall \varphi \in W_0^{1,p}(\Omega), \end{cases}$$

We then have the following result.

THEOREM 3.2. *Suppose that a satisfies hypotheses (2.6)–(2.8), and let u_ε be the solution of (3.5), where $f_\varepsilon, g_\varepsilon, \lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ are sequences of functions that satisfy (3.1)–(3.4). Then there exists a subsequence of u_ε , still denoted by u_ε , which converges almost everywhere to a renormalized solution u of (1.1) with datum μ , in the sense of Definition 2.13. Moreover,*

$$(3.6) \quad T_k(u_\varepsilon) \rightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega),$$

for every $k > 0$.

REMARK 3.3. We explicitly remark that we require that the two sequences $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ are sequences of nonnegative functions which converge to two measures which are mutually singular, and singular with respect to the p -capacity. If we do not make these requirements, the result (3.6) of Theorem 3.2 may fail, see Examples 8.2 and 8.3 in Section 8.

If μ belongs to $\mathcal{M}_b(\Omega) \cap W^{-1,p'}(\Omega)$, then $\mu \in \mathcal{M}_0(\Omega)$, so that $\mu = \mu_0$ and $\mu_s = 0$. Consequently the (unique) solution of (1.1) in the sense of (2.9) is a renormalized solution in the sense of Definition 2.13. Therefore Theorem 3.2 is a consequence of Theorem 3.4 below, which deals with the stability of renormalized solutions with respect to the convergence of the right hand side.

We will indeed prove below that a sequence of renormalized solutions, whose right hand sides converge in the same way as that considered in the statement of the existence result, always has a subsequence which converges to a renormalized solution corresponding to the limit data.

More precisely, let μ_ε and μ be measures in $\mathcal{M}_b(\Omega)$ which can be decomposed as

$$(3.7) \quad \mu_\varepsilon = f_\varepsilon - \operatorname{div}(g_\varepsilon) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus,$$

$$(3.8) \quad \mu = f - \operatorname{div}(g) + \mu_s^+ - \mu_s^-,$$

where ε belongs to a sequence of positive numbers that converges to zero, and

$$(3.9) \quad \begin{cases} f \in L^1(\Omega), \quad g \in (L^{p'}(\Omega))^N, \\ \mu_s = \mu_s^+ - \mu_s^- \text{ is a measure in } \mathcal{M}_s(\Omega) \text{ with positive and} \\ \quad \text{negative parts } \mu_s^+ \text{ and } \mu_s^-, \text{ respectively;} \end{cases}$$

$$(3.10) \quad \begin{cases} f_\varepsilon \text{ is a sequence of } L^1(\Omega) \text{ functions} \\ \text{that converges to } f \text{ weakly in } L^1(\Omega); \end{cases}$$

$$(3.11) \quad \begin{cases} g_\varepsilon \text{ is a sequence of functions in } (L^{p'}(\Omega))^N \\ \text{that converges to } g \text{ strongly in } (L^{p'}(\Omega))^N, \\ \text{and such that } \operatorname{div}(g_\varepsilon) \text{ is bounded in } \mathcal{M}_b(\Omega); \end{cases}$$

$$(3.12) \quad \begin{cases} \lambda_\varepsilon^\oplus \text{ is a sequence of nonnegative measures in } \mathcal{M}_b(\Omega) \text{ that} \\ \text{converges to } \mu_s^+ \text{ in the narrow topology of measures;} \end{cases}$$

$$(3.13) \quad \begin{cases} \lambda_\varepsilon^\ominus \text{ is a sequence of nonnegative measures in } \mathcal{M}_b(\Omega) \text{ that} \\ \text{converges to } \mu_s^- \text{ in the narrow topology of measures.} \end{cases}$$

Then the following theorem holds.

THEOREM 3.4. *Assume that (3.9)–(3.13) hold, and let u_ε be a sequence of renormalized solutions of (1.1) with data μ_ε , in the sense of Definition 2.13. Then there exists a subsequence of u_ε , still denoted by u_ε , which converges almost everywhere to a renormalized solution u of (1.1) with datum μ , in the sense of Definition 2.13. Moreover,*

$$(3.14) \quad T_k(u_\varepsilon) \rightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega),$$

for every $k > 0$.

REMARK 3.5. We emphasize that we do not assume that $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ belong to $\mathcal{M}_s(\Omega)$, but only that they belong to $\mathcal{M}_b(\Omega)$, and that we do not assume that $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ are the positive and negative part of a given measure, but only that they are nonnegative. This is the reason why we use the unconventional notation $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$. Let us also note that in the approximations (3.3) and (3.4), $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ are functions in $L^{p'}(\Omega)$, and so, regarded as measures, they belong to $\mathcal{M}_0(\Omega)$.

In the general case of approximations satisfying (3.12) and (3.13), we have by Proposition 2.4

$$\lambda_\varepsilon^\oplus = \lambda_{\varepsilon,0}^\oplus + \lambda_{\varepsilon,s}^\oplus, \quad \lambda_\varepsilon^\ominus = \lambda_{\varepsilon,0}^\ominus + \lambda_{\varepsilon,s}^\ominus,$$

with $\lambda_{\varepsilon,0}^\oplus$ and $\lambda_{\varepsilon,0}^\ominus$ in $\mathcal{M}_0(\Omega)$, and $\lambda_{\varepsilon,s}^\oplus$ and $\lambda_{\varepsilon,s}^\ominus$ in $\mathcal{M}_s(\Omega)$. Note that

$$(3.15) \quad \lambda_{\varepsilon,0}^\oplus, \lambda_{\varepsilon,0}^\ominus, \lambda_{\varepsilon,s}^\oplus, \lambda_{\varepsilon,s}^\ominus \quad \text{are nonnegative.}$$

Using Propositions 2.4 and 2.5 we obtain

$$(3.16) \quad \begin{aligned} \mu_\varepsilon &= f_\varepsilon - \operatorname{div}(g_\varepsilon) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus \\ &= [f_\varepsilon - \operatorname{div}(g_\varepsilon) + \lambda_{\varepsilon,0}^\oplus - \lambda_{\varepsilon,0}^\ominus] + [\lambda_{\varepsilon,s}^\oplus - \lambda_{\varepsilon,s}^\ominus] \\ &= \mu_{\varepsilon,0} + \mu_{\varepsilon,s}, \end{aligned}$$

where

$$(3.17) \quad \begin{cases} \mu_{\varepsilon,0} = f_\varepsilon - \operatorname{div}(g_\varepsilon) + \lambda_{\varepsilon,0}^\oplus - \lambda_{\varepsilon,0}^\ominus, \\ \mu_{\varepsilon,s} = \lambda_{\varepsilon,s}^\oplus - \lambda_{\varepsilon,s}^\ominus, \end{cases}$$

with $\mu_{\varepsilon,0}$ and $\mu_{\varepsilon,s}$ the absolutely continuous and the singular part of μ_ε with respect to the p -capacity. In particular we have

$$(3.18) \quad 0 \leq \mu_{\varepsilon,s}^+ \leq \lambda_{\varepsilon,s}^\oplus, \quad 0 \leq \mu_{\varepsilon,s}^- \leq \lambda_{\varepsilon,s}^\ominus.$$

4. – Proof of the equivalence of the definitions

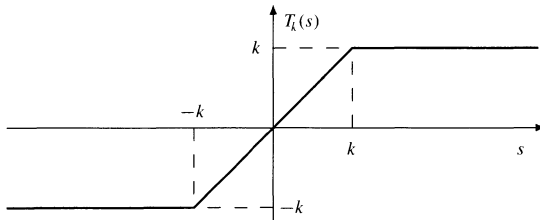
In this section we prove that all definitions of renormalized solution given in Sections 2.4 and 2.5 are equivalent. We begin by fixing some notation that will be used throughout the rest of the paper.

4.1. – Notation

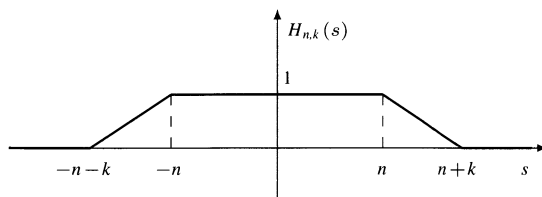
We will use the following functions of one real variable s , which may depend on one or more nonnegative real parameters such as k and n .

$$(4.1) \quad s^+ = \max(s, 0), \quad s^- = \max(-s, 0);$$

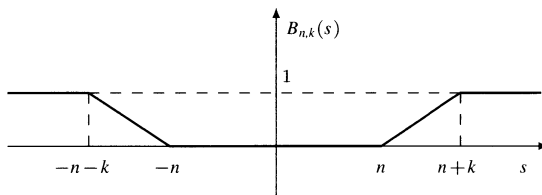
$$(4.2) \quad T_k(s) = \max(-k, \min(k, s));$$



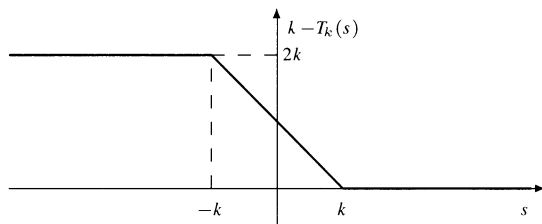
$$(4.3) \quad H_{n,k}(s) = \begin{cases} 0 & \text{if } s < -n - k, \\ \frac{s + n + k}{k} & \text{if } -n - k \leq s < -n, \\ 1 & \text{if } |s| \leq n, \\ \frac{n + k - s}{k} & \text{if } n < s \leq n + k, \\ 0 & \text{if } s > n + k; \end{cases}$$



(4.4)
$$B_{n,k}(s) = 1 - H_{n,k}(s).$$



We will also use the function $k - T_k(s)$ (and its companion $k + T_k(s)$):



Throughout the paper c will denote a generic constant, which may change from line to line. These constants c will depend on Ω , on the constants N, p, α, γ and on the function b which appear in Section 2.3, but will always be independent of other parameters (such as $\varepsilon, \delta, \eta, n, k$).

Moreover, if η, δ and ε are positive real numbers, and n is a positive integer, we will denote by $\omega(\eta, \delta, n, \varepsilon)$ any quantity such that

$$\lim_{\eta \rightarrow 0^+} \limsup_{\delta \rightarrow 0^+} \limsup_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0^+} |\omega(\eta, \delta, n, \varepsilon)| = 0.$$

If the order in which the limits are taken will be different, we will change the order of appearance of the parameters, from the last limit to be taken, to the first: for example, $\omega(\eta, n, \delta, \varepsilon)$ is any quantity whose absolute value converges to zero after taking the limits in ε, δ, n and η successively. If the quantity we consider does not depend on one parameter among η, δ, n and ε , we will omit the dependence on the corresponding parameter: as an example, $\omega(\eta, \varepsilon)$ is any quantity such that

$$\lim_{\eta \rightarrow 0^+} \limsup_{\varepsilon \rightarrow 0^+} |\omega(\eta, \varepsilon)| = 0.$$

Finally, we will denote (for example) by $\omega_{\eta,\delta}(n, \varepsilon)$ a quantity that depends on $\eta, \delta, n, \varepsilon$ and is such that

$$\lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0^+} |\omega_{\eta,\delta}(n, \varepsilon)| = 0,$$

for any fixed values of η and δ . As an example,

$$\frac{n}{\delta} \varepsilon + \frac{1}{\eta} \frac{1}{\delta n} + \frac{\delta}{\eta} + \eta = \omega(\eta, \delta, n, \varepsilon), \quad n \varepsilon = \omega_n(\varepsilon).$$

4.2. – Estimates on level sets

We prove now an estimate of the measure of the level sets of a renormalized solution of (1.1) and of its gradient.

THEOREM 4.1. *Let μ be a measure in $\mathcal{M}_b(\Omega)$ and let u be a renormalized solution of (1.1) according to Definition 2.13. Then the following inequality holds:*

$$(4.5) \quad \frac{1}{k} \int_{\{n \leq |u| < n+k\}} a(x, \nabla u) \cdot \nabla u \, dx \leq |\mu|(\Omega) \quad \forall n \geq 0, \forall k > 0.$$

If $p < N$ we have, for every $k > 0$,

$$(4.6) \quad \text{meas} (\{|u| > k\}) \leq c \frac{|\mu|(\Omega)^{\frac{N}{N-p}}}{k^{\frac{N(p-1)}{N-p}}},$$

$$(4.7) \quad \text{meas} (\{|\nabla u| > k\}) \leq c \frac{|\mu|(\Omega)^{\frac{N}{N-1}}}{k^{\frac{N(p-1)}{N-1}}},$$

for some positive constant c independent of u and μ . If $p = N$ we have, for every $k > 0$,

$$(4.8) \quad \text{meas} (\{|u| > k\}) \leq c_r \frac{|\mu|(\Omega)^r}{k^{r(p-1)}},$$

for every $r > 1$, for some positive constant c_r independent of u and μ , and

$$(4.9) \quad \text{meas} (\{|\nabla u| > k\}) \leq c_s \frac{|\mu|(\Omega)^{\frac{N}{N-1}}}{k^s},$$

for every $s < N$, for some positive constant c_s independent of u and μ .

PROOF. Once we have proved (4.5), inequalities (4.6), (4.7), (4.8) and (4.9) follow as in [1], Lemma 4.1 and Lemma 4.2. In order to prove (4.5) we choose $w = B_{n,k}(u^+)$ as test function in (2.19), where $B_{n,k}(s)$ is defined (4.4). We thus have

$$\frac{1}{k} \int_{\{n \leq u < n+k\}} a(x, \nabla u) \cdot \nabla u \, dx = \int_{\Omega} B_{n,k}(u^+) \, d\mu_0 + \mu_s^+(\Omega).$$

Since $B_{n,k}(u^+) \leq 1$ μ_0 -almost everywhere on Ω and $B_{n,k}(u^+) = 0$ μ_0 -almost everywhere on $\{u \leq 0\}$, we have

$$(4.10) \quad \frac{1}{k} \int_{\{n \leq u < n+k\}} a(x, \nabla u) \cdot \nabla u \, dx \leq |\mu_0|(\{u > 0\}) + \mu_s^+(\Omega).$$

Choosing $w = B_{n,k}(u^-)$ as test function in (2.19) and repeating the same argument, we get

$$(4.11) \quad \frac{1}{k} \int_{\{-n-k < u \leq -n\}} a(x, \nabla u) \cdot \nabla u \, dx \leq |\mu_0|(\{u < 0\}) + \mu_s^-(\Omega).$$

By adding (4.10) and (4.11) we obtain (4.5). □

4.3. – Proof of the equivalence

This section is devoted to the proof of the equivalence of the four definitions of renormalized solution.

PROOF OF THEOREM 2.33. We will prove the following facts:

$$2.13 \implies 2.21 \implies 2.25 \implies 2.29 \implies 2.13.$$

STEP 1: Definition 2.13 implies Definition 2.21.

If u is a renormalized solution according to Definition 2.13, and h and φ are as in condition (e) of Definition 2.21, then it is possible to choose $w = h(u) \varphi$ as test function in (2.19), with $w^{+\infty} = w^{-\infty} = 0$ and $k = M$, where M is such that $\text{supp}(h) \subseteq [-M, M]$. Using the final part of Remark 2.22 about the definition of $\nabla T_M(u) h'(u)$, we thus obtain (2.25).

In order to prove that (2.23) holds true, we choose $w = B_{n,n}(u^+) \varphi$, where $B_{n,n}$ is defined by (4.4), and φ belongs to $C^1(\overline{\Omega})$. We obtain, since $w^{+\infty} = \varphi$, $w^{-\infty} = 0$, and $k = 2n$,

$$\begin{aligned} & \frac{1}{n} \int_{\{n \leq u < 2n\}} a(x, \nabla u) \cdot \nabla u \varphi \, dx + \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi B_{n,n}(u^+) \, dx \\ &= \int_{\Omega} B_{n,n}(u^+) \varphi \, d\mu_0 + \int_{\Omega} \varphi \, d\mu_s^+. \end{aligned}$$

Since $B_{n,n}(s^+)$ decreases to zero in \mathbf{R} as $n \rightarrow +\infty$, it is easy to see that

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi B_{n,n}(u^+) dx = \omega(n),$$

and that (recall that $B_{n,n}(u^+) \varphi$ belongs to $L^1(\Omega, \mu_0)$ by Proposition 2.7)

$$\int_{\Omega} B_{n,n}(u^+) \varphi d\mu_0 = \omega(n),$$

so that

$$(4.12) \quad \frac{1}{n} \int_{\{n \leq u < 2n\}} a(x, \nabla u) \cdot \nabla u \varphi dx = \int_{\Omega} \varphi d\mu_s^+ + \omega(n),$$

which is (2.23) for $\varphi \in C^1(\overline{\Omega})$. Since the functions

$$\frac{1}{n} a(x, \nabla u) \cdot \nabla u \chi_{\{n \leq u < 2n\}}$$

are nonnegative by (2.6), (4.12) holds for every $\varphi \in C_b^0(\Omega)$ (see Remark 2.3) and (2.23) is then proved. Analogous calculations yield (2.24).

STEP 2: Definition 2.21 implies Definition 2.25.

Assume that u is a renormalized solution of (1.1) according to Definition 2.21.

If h is as in Definition 2.25, let M be such that $\text{supp}(h') \subseteq [-M, M]$. Let n be greater than M , and let $h_n(s)$ be the Lipschitz continuous function defined by

$$h_n(s) = \begin{cases} 0 & \text{if } s < -2n, \\ h(-\infty) \frac{s + 2n}{n} & \text{if } -2n \leq s < -n, \\ h(s) & \text{if } |s| \leq n, \\ h(+\infty) \frac{2n - s}{n} & \text{if } n < s \leq 2n, \\ 0 & \text{if } s > 2n. \end{cases}$$

Let φ be a function in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that $h(u) \varphi$ belongs to $W_0^{1,p}(\Omega)$. As $h_n(u) \varphi \in W^{1,p}(\Omega)$ and $|h_n(u) \varphi| \leq |h(u) \varphi|$ almost everywhere in Ω , we have $h_n(u) \varphi \in W_0^{1,p}(\Omega)$. Choosing $h_n(u) \varphi$ as test function in (2.25), we get

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla u h'_n(u) \varphi dx + \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi h_n(u) dx = \int_{\Omega} h_n(u) \varphi d\mu_0.$$

Since $h_n(s)$ converges to $h(s)$ as n tends to infinity, and $a(x, \nabla u)$ belongs to $(L^q(\Omega))^N$ for every $q < \frac{N}{N-1}$, it is easy to prove that

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi h_n(u) dx = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi h(u) dx + \omega(n),$$

and that (observe that $|h_n(u)| \leq |h(u)|$ and use Proposition 2.7)

$$\int_{\Omega} h_n(u) \varphi d\mu_0 = \int_{\Omega} h(u) \varphi d\mu_0 + \omega(n).$$

On the other hand, by (2.23) and (2.24) we have

$$\begin{aligned} & \int_{\Omega} a(x, \nabla u) \cdot \nabla u h'_n(u) \varphi dx \\ &= \int_{\{|u| \leq n\}} a(x, \nabla u) \cdot \nabla u h'(u) \varphi dx - \frac{h(+\infty)}{n} \int_{\{n \leq u < 2n\}} a(x, \nabla u) \cdot \nabla u \varphi dx \\ & \quad + \frac{h(-\infty)}{n} \int_{\{-2n < u \leq -n\}} a(x, \nabla u) \cdot \nabla u \varphi dx \\ &= \int_{\Omega} a(x, \nabla u) \cdot \nabla u h'(u) \varphi dx \\ & \quad - h(+\infty) \int_{\Omega} \varphi d\mu_s^+ + h(-\infty) \int_{\Omega} \varphi d\mu_s^- + \omega(n). \end{aligned}$$

Putting together the results, we have proved that (2.27) holds true.

STEP 3: Definition 2.25 implies Definition 2.29.

Let u be a renormalized solution of (1.1) according to Definition 2.25. Fix $k > 0$ and $\delta > 0$, and choose $h(s) = H_{k-\delta, \delta}(s)$ and φ in $C_c^1(\Omega)$ in (2.27), where $H_{k-\delta, \delta}$ is defined in (4.3). Observe that $h(u)\varphi$ belongs to $W_0^{1,p}(\Omega)$, so that this test function is admissible. We get

$$\begin{aligned} \text{(A)} \quad & -\frac{1}{\delta} \int_{\{k-\delta < u < k\}} a(x, \nabla u) \cdot \nabla u \varphi dx \\ \text{(B)} \quad & +\frac{1}{\delta} \int_{\{-k < u < -k+\delta\}} a(x, \nabla u) \cdot \nabla u \varphi dx \\ \text{(C)} \quad & + \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi H_{k-\delta, \delta}(u) dx \\ \text{(D)} \quad & = \int_{\Omega} \varphi H_{k-\delta, \delta}(u) d\mu_0. \end{aligned}$$

By Theorem 1.2 and Lemma 4.5 of [10] there exist two nonnegative measures in $\mathcal{M}_0(\Omega)$, λ_k^+ and λ_k^- , concentrated on $\{u = k\}$ and $\{u = -k\}$ respectively, such that

$$(4.13) \quad \text{(A)} = \int_{\Omega} \varphi d\lambda_k^+ + \omega(\delta),$$

$$(4.14) \quad \text{(B)} = \int_{\Omega} \varphi d\lambda_k^- + \omega(\delta),$$

for every $\varphi \in C^1(\overline{\Omega})$, and hence for every $\varphi \in C_b^0(\Omega)$ (see Remark 2.3). Since $H_{k-\delta,\delta}(u)$ converges to $\chi_{\{|u|<k\}}$ both in the weak* topology of $L^\infty(\Omega)$ and cap_p -quasi everywhere (hence μ_0 almost everywhere) as δ tends to zero, we have

$$(4.15) \quad (C) = \int_{\{|u|<k\}} a(x, \nabla u) \cdot \nabla \varphi \, dx + \omega(\delta),$$

and

$$(4.16) \quad (D) = \int_{\Omega} \varphi \chi_{\{|u|<k\}} \, d\mu_0 + \omega(\delta).$$

Putting together (4.13)-(4.16), we have for every φ in $C_c^1(\Omega)$

$$(4.17) \quad \int_{\{|u|<k\}} a(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\lambda_k^+ - \int_{\Omega} \varphi \, d\lambda_k^- + \int_{\Omega} \varphi \chi_{\{|u|<k\}} \, d\mu_0.$$

An easy approximation argument (recall that λ_k^+ and λ_k^- belong to $\mathcal{M}_0(\Omega)$) then shows that (4.17) holds for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, which gives (2.28).

It remains to prove the narrow convergence of λ_k^+ to μ_s^+ . We fix $k > 0$ and we write (2.28) with k replaced by $2k$. Then we choose $B_{k,k}(u^+) \varphi$ as test function, where $B_{k,k}(s)$ is defined in (4.4) and φ belongs to $C^1(\overline{\Omega})$. We obtain

$$\begin{aligned} (A) \quad & \frac{1}{k} \int_{\{k \leq u < 2k\}} a(x, \nabla u) \cdot \nabla u \, \varphi \, dx \\ (B) \quad & + \int_{\{0 < u < 2k\}} a(x, \nabla u) \cdot \nabla \varphi \, B_{k,k}(u^+) \, dx \\ (C) \quad & = \int_{\Omega} \varphi \, B_{k,k}(u^+) \chi_{\{|u|<2k\}} \, d\mu_0 \\ (D) \quad & + \int_{\Omega} B_{k,k}(u^+) \varphi \, d\lambda_{2k}^+ \\ (E) \quad & - \int_{\Omega} B_{k,k}(u^+) \varphi \, d\lambda_{2k}^- . \end{aligned}$$

Since λ_{2k}^+ is concentrated on the set $\{u = 2k\}$, and $B_{k,k}(u^+) = 1$ on this set, we have

$$(4.18) \quad (D) = \int_{\Omega} \varphi \, d\lambda_{2k}^+.$$

On the other hand, since λ_{2k}^- is concentrated on the set $\{u = -2k\}$, and $B_{k,k}(u^+) = 0$ on this set, we have

$$(4.19) \quad (E) = 0.$$

Since $B_{k,k}(u^+)$ converges to 0 as k tends to infinity, both in the weak* topology of $L^\infty(\Omega)$ and cap_p -quasi everywhere (hence μ_0 almost everywhere), we have

$$(4.20) \quad (\text{B}) = \omega(k),$$

and

$$(4.21) \quad (\text{C}) = \omega(k).$$

Choosing $h(s) = B_{k,k}(s^+)$ and φ in $C^1(\overline{\Omega})$ in (2.27), we obtain as in (4.12) (see Step 1) that

$$(\text{A}) = \int_{\Omega} \varphi d\mu_s^+ + \omega(k).$$

Putting together this latter fact and (4.18)-(4.21), we have

$$\int_{\Omega} \varphi d\lambda_{2k}^+ = \int_{\Omega} \varphi d\mu_s^+ + \omega(k),$$

for every φ in $C^1(\overline{\Omega})$, and by Remark 2.3 we conclude that $\{\lambda_{2k}^+\}$ converges to μ_s^+ in the narrow topology of measures.

Reasoning in the same way, we prove that λ_k^- converges to μ_s^- in the narrow topology of measures.

STEP 4: Definition 2.29 implies Definition 2.13.

Let u be a renormalized solution of (1.1) according to Definition 2.29. We choose w satisfying the hypotheses of Definition 2.13 as test function in (2.28) and we obtain

$$\int_{\{|u|<k\}} a(x, \nabla u) \cdot \nabla w dx = \int_{\{|u|<k\}} w d\mu_0 + \int_{\Omega} w d\lambda_k^+ - \int_{\Omega} w d\lambda_k^-,$$

for every $k > 0$. Let k_0 be such that $w = w^{+\infty}$ almost everywhere on the set $\{u > k_0\}$ and $w = w^{-\infty}$ almost everywhere on the set $\{u < -k_0\}$; then we have, for every $k > k_0$,

$$\begin{aligned} \int_{\{|u|<k\}} a(x, \nabla u) \cdot \nabla w dx &= \int_{\{|u|\leq k_0\}} a(x, \nabla u) \cdot \nabla w dx \\ &+ \int_{\{k_0 < u < k\}} a(x, \nabla u) \cdot \nabla w^{+\infty} dx \\ &+ \int_{\{-k < u < -k_0\}} a(x, \nabla u) \cdot \nabla w^{-\infty} dx. \end{aligned}$$

Passing to the limit in k , which is possible for the terms of the right hand side due to the regularity of u and w , we have

$$\int_{\{|u|<k\}} a(x, \nabla u) \cdot \nabla w dx = \int_{\Omega} a(x, \nabla u) \cdot \nabla w dx + \omega(k).$$

Since (the cap_p -quasi continuous representative of) u is finite cap_p -quasi everywhere (see Remark 2.31), the sequence $\chi_{\{|u|<k\}}$ converges to 1 cap_p -quasi everywhere as k tends to infinity, and so μ_0 almost everywhere. Therefore we have

$$\int_{\{|u|<k\}} w \, d\mu_0 = \int_{\Omega} w \, d\mu_0 + \omega(k).$$

Finally, recalling that λ_k^+ belongs to $\mathcal{M}_0(\Omega)$ and is concentrated on $\{u = k\}$, and that $w = w^{+\infty}$ cap_p -quasi everywhere on $\{u > k_0\}$ (see Proposition 2.1), we have for $k > k_0$,

$$\int_{\Omega} w \, d\lambda_k^+ = \int_{\Omega} w^{+\infty} \, d\lambda_k^+ = \int_{\Omega} w^{+\infty} \, d\mu_s^+ + \omega(k),$$

since $w^{+\infty}$ is continuous and bounded. The term $\int_{\Omega} w \, d\lambda_k^-$ can be treated in the same way. Putting together the results, we obtain (2.19). \square

5. – Proof of the stability result: first steps

In this section we begin the proof of the stability theorem (Theorem 3.4): we will perform Step 1 to Step 4. The remaining steps will be done in Section 8, after some additional results have been obtained.

5.1. – Proof of Theorem 3.4: beginning

In this section we will obtain some *a priori* estimates and convergence properties of the sequence u_ε . We remark that, if the u_ε are “classical” weak solutions of (3.5), with data belonging to $W^{-1,p'}(\Omega)$ and bounded in $\mathcal{M}_b(\Omega)$ as in the statement of Theorem 3.2, the results of the present section have already been obtained in the literature (see [3], [4], [7], [1]). We derive here these properties using only the fact that the u_ε are renormalized solutions (hence, solutions which may not belong to the energy space $W_0^{1,p}(\Omega)$) corresponding to measure data, and adapting the previous techniques to our case. We also explicitly remark that (for the results of this section only) we will not use the fact that μ_ε converges in the sense specified by (3.9)-(3.13), but only that μ_ε is bounded in $\mathcal{M}_b(\Omega)$.

STEP 1: a priori estimates.

By the assumptions on μ_ε , there exists a positive constant c such that $|\mu_\varepsilon|(\Omega) \leq c$ for every $\varepsilon > 0$. If $1 < p < N$ we thus have, from (4.5), (4.6), and (4.7) (see Theorem 4.1),

$$(5.1) \quad \frac{1}{k} \int_{\{n \leq |u_\varepsilon| < n+k\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx \leq c,$$

$$(5.2) \quad \text{meas}(\{|u_\varepsilon| > k\}) \leq \frac{c}{k \frac{N(p-1)}{N-p}}, \quad \text{meas}(\{|\nabla u_\varepsilon| > k\}) \leq \frac{c}{k \frac{N(p-1)}{N-1}},$$

for every $k > 0$ and for every $n \geq 0$. The first inequality implies that

$$(5.3) \quad \frac{1}{k} \int_{\{|n \leq |u_\varepsilon| < n+k\}} a(x, \nabla T_{n+k}(u_\varepsilon)) \cdot \nabla T_{n+k}(u_\varepsilon) dx \leq c,$$

where c is independent of k , n , and ε . Using (2.6), we obtain from (5.3)

$$(5.4) \quad \int_{\Omega} |\nabla B_{n,k}(u_\varepsilon)|^p dx = \frac{1}{k} \int_{\{|n \leq |u_\varepsilon| < n+k\}} |\nabla T_{n+k}(u_\varepsilon)|^p dx \leq c,$$

where $B_{n,k}$ is the function defined in (4.4). Taking $n = 0$, we get

$$(5.5) \quad \int_{\Omega} |\nabla T_k(u_\varepsilon)|^p dx \leq ck,$$

with c independent of k and ε .

To obtain the boundedness of $|\nabla u_\varepsilon|^{p-1}$ in $L^q(\Omega)$, for every $q < \frac{N}{N-1}$, we observe that, by the second inequality of (5.2), we have

$$\begin{aligned} \int_{\Omega} |\nabla u_\varepsilon|^{q(p-1)} dx &= \int_{\{|\nabla u_\varepsilon| \leq 1\}} |\nabla u_\varepsilon|^{q(p-1)} dx + \int_{\{|\nabla u_\varepsilon| > 1\}} |\nabla u_\varepsilon|^{q(p-1)} dx \\ &\leq \text{meas}(\Omega) + q(p-1) \int_1^{+\infty} t^{q(p-1)-1} \text{meas}(\{|\nabla u_\varepsilon| > t\}) dt \\ &\leq \text{meas}(\Omega) + q(p-1) \int_1^{+\infty} t^{q(p-1)-1} \frac{c}{t^{\frac{N(p-1)}{N-1}}} dt < +\infty, \end{aligned}$$

since $\frac{N(p-1)}{N-1} - q(p-1) + 1 > 1$ for every $q < \frac{N}{N-1}$. Thus

$$(5.6) \quad |\nabla u_\varepsilon|^{p-1} \text{ is bounded in } L^q(\Omega), \text{ for every } q < \frac{N}{N-1}.$$

If $p = N$ we obtain again (5.1), (5.2) (even if with different exponents and constants as in (4.8) and (4.9)), and (5.3), while (5.6) is obtained using (4.9) instead of (4.7); to derive it we proceed as before: the only change consists in observing that, once $q < \frac{N}{N-1}$ is fixed, it is then possible to choose $s < N$ such that $s > q(N-1)$, so that

$$\int_1^{+\infty} t^{q(N-1)-1} \frac{c_s}{t^s} dt < +\infty.$$

STEP 2: up to a subsequence, u_ε is a Cauchy sequence in measure.

Let $\sigma > 0$ and $\eta > 0$ be fixed. For every $k > 0$, and every ε and δ we have

$$\{|u_\varepsilon - u_\delta| > \sigma\} \subseteq \{|u_\varepsilon| > k\} \cup \{|u_\delta| > k\} \cup \{|T_k(u_\varepsilon) - T_k(u_\delta)| > \sigma\}.$$

Let $\eta > 0$ be fixed. By the first inequality of (5.2), there exists $k > 0$ such that, for every ε and δ ,

$$\text{meas}(\{|u_\varepsilon| > k\}) + \text{meas}(\{|u_\delta| > k\}) < \frac{\eta}{2}.$$

Once k is chosen, we deduce from (5.5) that $T_k(u_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$, and so, up to a subsequence still denoted by u_ε , $T_k(u_\varepsilon)$ is (strongly convergent in $L^p(\Omega)$ and hence) a Cauchy sequence in measure. Consequently, there exists ε_0 such that, for every ε and δ smaller than ε_0 , we have

$$\text{meas}(\{|T_k(u_\varepsilon) - T_k(u_\delta)| > \sigma\}) < \frac{\eta}{2}.$$

We have thus proved the claim: up to a subsequence, u_ε is a Cauchy sequence in measure. Therefore, there exist a further subsequence, still denoted by u_ε , and a measurable function u , which is finite almost everywhere, such that

$$(5.7) \quad u_\varepsilon \text{ converges to } u \text{ almost everywhere in } \Omega.$$

From now on, we will consider this particular subsequence and the limit function u .

STEP 3: weak convergence of truncates.

Since from (5.5) we obtain that $T_k(u_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$, for every fixed k there exists a subsequence (which may depend on k), and a function v_k in $W_0^{1,p}(\Omega)$, such that $T_k(u_\varepsilon)$ converges to v_k weakly in $W_0^{1,p}(\Omega)$. Since $T_k(s)$ is continuous, the result of Step 2 implies that $T_k(u_\varepsilon)$ converges to $T_k(u)$ almost everywhere. Thus we have, for every k , $v_k = T_k(u)$; moreover, since the limit is independent of the subsequence chosen, we have that for the whole sequence u_ε defined at the end of Step 2,

$$(5.8) \quad T_k(u_\varepsilon) \text{ converges to } T_k(u) \text{ weakly in } W_0^{1,p}(\Omega), \text{ for every } k > 0;$$

this implies, in particular, that

$$(5.9) \quad T_k(u) \text{ belongs to } W_0^{1,p}(\Omega) \text{ for every } k > 0,$$

so that the approximate gradient of u is defined according to Definition 2.9.

Using (5.4) and the weak convergence of $B_{n,k}(u_\varepsilon)$, we obtain by weak lower semicontinuity that u satisfies

$$(5.10) \quad \frac{1}{k} \int_{\{n \leq |u| < n+k\}} |\nabla T_{n+k}(u)|^p dx = \int_{\Omega} |\nabla B_{n,k}(u)|^p dx \leq c,$$

for every $k > 0, n \geq 0$, which implies, choosing $n = 0$,

$$(5.11) \quad \int_{\Omega} |\nabla T_k(u)|^p dx \leq c k,$$

for some positive constant c independent of n and k .

STEP 4: ∇u_{ε} is a Cauchy sequence in measure.

We first observe that u_{ε} , being a renormalized solution in the sense of Definition 2.13, is also a renormalized solution in the sense of Definition 2.29 (see Theorem 2.33). Thus, for every $k > 0$, there exist two nonnegative measures in $\mathcal{M}_0(\Omega)$, $\lambda_{\varepsilon,k}^+$ and $\lambda_{\varepsilon,k}^-$, concentrated on the sets $\{u_{\varepsilon} = k\}$ and $\{u_{\varepsilon} = -k\}$ respectively, such that

$$(5.12) \quad \int_{\{|u_{\varepsilon}| < k\}} a(x, \nabla u_{\varepsilon}) \cdot \nabla \varphi dx = \int_{\{|u_{\varepsilon}| < k\}} \varphi d\mu_{\varepsilon,0} + \int_{\Omega} \varphi d\lambda_{\varepsilon,k}^+ - \int_{\Omega} \varphi d\lambda_{\varepsilon,k}^-,$$

for every φ in $W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, where $\mu_{\varepsilon,0}$ is the absolutely continuous part of μ_{ε} with respect to the p -capacity as defined in (3.16). If we choose $\varphi = T_k(u_{\varepsilon})$ as test function, we have

$$\int_{\Omega} a(x, \nabla T_k(u_{\varepsilon})) \cdot \nabla T_k(u_{\varepsilon}) dx = \int_{\Omega} T_k(u_{\varepsilon}) \chi_{\{|u_{\varepsilon}| < k\}} d\mu_{\varepsilon,0} + k(\lambda_{\varepsilon,k}^+(\Omega) + \lambda_{\varepsilon,k}^-(\Omega)),$$

since $\lambda_{\varepsilon,k}^+$ is concentrated on the set $\{u_{\varepsilon} = k\}$ (where $T_k(u_{\varepsilon}) = k$) and $\lambda_{\varepsilon,k}^-$ is concentrated on the set $\{u_{\varepsilon} = -k\}$ (where $T_k(u_{\varepsilon}) = -k$). Thus we have, since $|T_k(u_{\varepsilon})| \leq k$ $\mu_{\varepsilon,0}$ -almost everywhere,

$$k(\lambda_{\varepsilon,k}^+(\Omega) + \lambda_{\varepsilon,k}^-(\Omega)) \leq \int_{\Omega} a(x, \nabla T_k(u_{\varepsilon})) \cdot \nabla T_k(u_{\varepsilon}) dx + k|\mu_{\varepsilon,0}|(\Omega),$$

so that from (5.5) it follows

$$\lambda_{\varepsilon,k}^+(\Omega) + \lambda_{\varepsilon,k}^-(\Omega) \leq c,$$

for every k and every ε . Thus the measures

$$\mu_{\varepsilon,k} = \mu_{\varepsilon,0} \llcorner \{|u_{\varepsilon}| < k\} + \lambda_{\varepsilon,k}^+ - \lambda_{\varepsilon,k}^-$$

are bounded in $\mathcal{M}_b(\Omega)$.

From (5.12) we deduce that $T_k(u_{\varepsilon})$ is a solution in $W_0^{1,p}(\Omega)$ of

$$-\operatorname{div}(a(x, \nabla T_k(u_{\varepsilon}))) = \mu_{\varepsilon,k} \quad \text{in } \mathcal{D}'(\Omega),$$

where the measures $\mu_{\varepsilon,k}$ are bounded in $\mathcal{M}_b(\Omega)$, and where $T_k(u_{\varepsilon})$ is bounded in $W_0^{1,p}(\Omega)$ independently of ε for every fixed k . Thus, by a result of [4] and [7], we have that $\nabla T_k(u_{\varepsilon})$ is a Cauchy sequence in measure. We can now

repeat the same proof of Step 2: if $\sigma > 0$ is fixed, we have, for every $k > 0$ and for every ε and δ ,

$$\{|\nabla u_\varepsilon - \nabla u_\delta| > \sigma\} \subseteq \{|u_\varepsilon| > k\} \cup \{|u_\delta| > k\} \cup \{|\nabla T_k(u_\varepsilon) - \nabla T_k(u_\delta)| > \sigma\}.$$

Thus, using the first inequality of (5.2) as in Step 2, we obtain that ∇u_ε is a Cauchy sequence in measure. Passing to a subsequence, we may assume that

$$(5.13) \quad \nabla u_\varepsilon \rightarrow \nabla u \quad \text{almost everywhere in } \Omega.$$

Thus, by the boundedness of $|\nabla u_\varepsilon|^{p-1}$ in $L^q(\Omega)$, for every $q < \frac{N}{N-1}$, it follows from Fatou lemma, (2.7) and Vitali theorem that,

$$(5.14) \quad |\nabla u|^{p-1} \text{ belongs to } L^q(\Omega), \text{ for every } q < \frac{N}{N-1},$$

$$(5.15) \quad a(x, \nabla u_\varepsilon) \rightarrow a(x, \nabla u) \text{ strongly in } (L^q(\Omega))^N, \text{ for every } q < \frac{N}{N-1}.$$

STEP 5: strong convergence of the truncates in $W_0^{1,p}(\Omega)$.

Since this result requires many preliminary estimates, which will be obtained in Sections 6 and 7, we prefer to interrupt the proof of Theorem 3.4 and to continue it in Section 8. □

5.2. – Definition of the cut-off functions

In order to prove the strong convergence of the truncates, we consider the set E where the measure μ_s is concentrated and study the behaviour of the solutions u_ε near E (Section 6) and far from E (Section 7). The precise meaning of “near E ” and “far from E ” is specified by means of two families of cut-off functions, ψ_δ^+ and ψ_δ^- , that we now introduce.

LEMMA 5.1. *Let μ_s be a measure in $\mathcal{M}_s(\Omega)$, decomposed as $\mu_s = \mu_s^+ - \mu_s^-$, with μ_s^+ and μ_s^- concentrated on two disjoint subsets E^+ and E^- of zero p -capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$, such that*

$$(5.16) \quad \mu_s^+(E^+ \setminus K_\delta^+) \leq \delta, \quad \mu_s^+(E^- \setminus K_\delta^-) \leq \delta,$$

and two functions ψ_δ^+ and ψ_δ^- in $C_c^\infty(\Omega)$, such that

$$(5.17) \quad 0 \leq \psi_\delta^+ \leq 1, \quad 0 \leq \psi_\delta^- \leq 1, \quad \text{on } \Omega,$$

$$(5.18) \quad \psi_\delta^+ \equiv 1 \quad \text{on } K_\delta^+, \quad \psi_\delta^- \equiv 1 \quad \text{on } K_\delta^-,$$

$$(5.19) \quad \text{supp}(\psi_\delta^+) \cap \text{supp}(\psi_\delta^-) = \emptyset,$$

$$(5.20) \quad \int_\Omega |\nabla \psi_\delta^+|^p dx \leq \delta, \quad \int_\Omega |\nabla \psi_\delta^-|^p dx \leq \delta.$$

Moreover, if $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ satisfy (3.12) and (3.13) respectively, we have

$$(5.21) \quad \int_{\Omega} \psi_\delta^- d\lambda_\varepsilon^\oplus = \omega(\delta, \varepsilon), \quad \int_{\Omega} \psi_\delta^- d\mu_s^+ \leq \delta,$$

$$(5.22) \quad \int_{\Omega} \psi_\delta^+ d\lambda_\varepsilon^\ominus = \omega(\delta, \varepsilon), \quad \int_{\Omega} \psi_\delta^+ d\mu_s^- \leq \delta,$$

$$(5.23) \quad \int_{\Omega} (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_\varepsilon^\oplus = \omega(\eta, \delta, \varepsilon), \quad \int_{\Omega} (1 - \psi_\delta^+ \psi_\eta^+) d\mu_s^+ \leq \delta + \eta,$$

$$(5.24) \quad \int_{\Omega} (1 - \psi_\delta^- \psi_\eta^-) d\lambda_\varepsilon^\ominus = \omega(\eta, \delta, \varepsilon), \quad \int_{\Omega} (1 - \psi_\delta^- \psi_\eta^-) d\mu_s^- \leq \delta + \eta.$$

PROOF. Let δ be a fixed positive number. Due to the regularity of the measures μ_s^+ and μ_s^- , there exist two compact subsets of Ω , K_δ^+ and K_δ^- , with

$$K_\delta^+ \subseteq E^+, \quad K_\delta^- \subseteq E^-,$$

such that (5.16) holds. Moreover, since $K_\delta^+ \cap K_\delta^- = \emptyset$ (because $E^+ \cap E^- = \emptyset$), there exist two open subsets of Ω , U_δ^+ and U_δ^- , such that

$$K_\delta^+ \subseteq U_\delta^+, \quad K_\delta^- \subseteq U_\delta^-, \quad U_\delta^+ \cap U_\delta^- = \emptyset.$$

Let us explicitly remark that it may happen that $U_\delta^+ \cap E^- \neq \emptyset$ and $U_\delta^- \cap E^+ \neq \emptyset$. Finally, since $\text{cap}_p(E^+, \Omega) = 0$, and $\text{cap}_p(E^-, \Omega) = 0$, we have

$$\text{cap}_p(K_\delta^+, \Omega) = 0, \quad \text{cap}_p(K_\delta^-, \Omega) = 0,$$

which implies in particular (see for example [17], Lemma 2.9),

$$\text{cap}_p(K_\delta^+, U_\delta^+) = 0, \quad \text{cap}_p(K_\delta^-, U_\delta^-) = 0.$$

Thus, by definition of the p -capacity of a compact set, there exist two functions ψ_δ^+ and ψ_δ^- such that (5.18) holds true and

$$\begin{aligned} \psi_\delta^+ &\in C_c^\infty(U_\delta^+), & \psi_\delta^- &\in C_c^\infty(U_\delta^-), \\ \int_{U_\delta^+} |\nabla \psi_\delta^+|^p dx &\leq \delta, & \int_{U_\delta^-} |\nabla \psi_\delta^-|^p dx &\leq \delta, \\ 0 \leq \psi_\delta^+ &\leq 1 \quad \text{on } U_\delta^+, & 0 \leq \psi_\delta^- &\leq 1 \quad \text{on } U_\delta^-. \end{aligned}$$

From now on, we will consider ψ_δ^+ and ψ_δ^- as functions in $C_c^\infty(\Omega)$, setting $\psi_\delta^+ \equiv 0$ and $\psi_\delta^- \equiv 0$ on $\Omega \setminus U_\delta^+$ and $\Omega \setminus U_\delta^-$ respectively, so that we have (5.17), (5.19) and (5.20).

If $\lambda_\varepsilon^\oplus$ is as in the statement, we have, for every $\delta > 0$,

$$0 \leq \int_\Omega \psi_\delta^- d\lambda_\varepsilon^\oplus = \int_\Omega \psi_\delta^- d\mu_s^+ + \omega_\delta(\varepsilon),$$

while recalling (5.16) we have

$$\begin{aligned} 0 &\leq \int_\Omega \psi_\delta^- d\mu_s^+ = \int_{U_\delta^-} \psi_\delta^- d\mu_s^+ \leq \mu_s^+(U_\delta^-) \leq \mu_s^+(\Omega \setminus U_\delta^+) \\ &\leq \mu_s^+(\Omega \setminus K_\delta^+) = \mu_s^+(E^+ \setminus K_\delta^+) \leq \delta. \end{aligned}$$

Therefore (5.21) is proved. The proof of (5.22) is analogous.

Let now δ and η be two fixed positive real numbers; we have

$$0 \leq \int_\Omega (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_\varepsilon^\oplus = \int_\Omega (1 - \psi_\delta^+ \psi_\eta^+) d\mu_s^+ + \omega_{\delta,\eta}(\varepsilon);$$

on the other hand, since $1 - \psi_\delta^+ \psi_\eta^+$ belongs to $C^\infty(\Omega)$, and is identically zero on $K_\delta^+ \cap K_\eta^+$, with $0 \leq 1 - \psi_\delta^+ \psi_\eta^+ \leq 1$ on Ω , we have, recalling (5.16),

$$\begin{aligned} 0 &\leq \int_\Omega (1 - \psi_\delta^+ \psi_\eta^+) d\mu_s^+ = \int_{\Omega \setminus (K_\delta^+ \cap K_\eta^+)} (1 - \psi_\delta^+ \psi_\eta^+) d\mu_s^+ \\ &\leq \mu_s^+(\Omega \setminus (K_\delta^+ \cap K_\eta^+)) \\ &\leq \mu_s^+(\Omega \setminus K_\delta^+) + \mu_s^+(\Omega \setminus K_\eta^+) \\ &\leq \delta + \eta. \end{aligned}$$

This proves (5.23). The proof of (5.24) is analogous. □

6. – Near E

In this section we study the behaviour of the approximate solutions u_ε near the set E where the measure μ_s is concentrated. We consider a renormalized solution u_ε of the equation

$$(6.1) \quad \begin{cases} -\operatorname{div}(a(x, \nabla u_\varepsilon)) = f_\varepsilon - \operatorname{div}(g_\varepsilon) + \lambda_\varepsilon^\oplus - \lambda_\varepsilon^\ominus & \text{on } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_\varepsilon, g_\varepsilon, \lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ satisfy (3.9)-(3.13), and we prove that, in some sense, the sequence u_ε tends to $+\infty$ on a neighbourhood of E^+ , and to $-\infty$ on a neighbourhood of E^- ; this reflects on the behaviour of the gradients of $T_k(u_\varepsilon)$.

According to Definition 2.13 and to (3.16) and (3.17) in Remark 3.5 u_ε satisfies

$$\begin{aligned}
 \int_{\Omega} a(x, \nabla u_\varepsilon) \cdot \nabla w \, dx &= \int_{\Omega} f_\varepsilon w \, dx + \langle -\operatorname{div}(g_\varepsilon), w \rangle \\
 (6.2) \qquad \qquad \qquad &+ \int_{\Omega} w \, d\lambda_{\varepsilon,0}^\oplus - \int_{\Omega} w \, d\lambda_{\varepsilon,0}^\ominus \\
 &+ \int_{\Omega} w^{+\infty} \, d\mu_{\varepsilon,s}^+ - \int_{\Omega} w^{-\infty} \, d\mu_{\varepsilon,s}^-,
 \end{aligned}$$

for every w in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for which there exist $k > 0$ and $w^{+\infty}$ and $w^{-\infty}$ in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that $w = w^{+\infty}$ almost everywhere on the set $\{u_\varepsilon > k\}$, and $w = w^{-\infty}$ almost everywhere on the set $\{u_\varepsilon < -k\}$.

We suppose that we have already extracted a subsequence, still denoted by u_ε , such that u_ε converges almost everywhere to a function u , and which satisfies (5.7), (5.8), (5.13), (5.15).

Our first result is the following.

LEMMA 6.1. *Let $f_\varepsilon, g_\varepsilon, \lambda_\varepsilon^\oplus$, and $\lambda_\varepsilon^\ominus$ be sequences which satisfy (3.9)-(3.13), and let u_ε be a sequence of renormalized solutions of (6.1) which satisfies (5.7), (5.8), (5.13), (5.15). Let η be a positive real number, and let ϕ_η^\oplus and ϕ_η^\ominus be functions in $W^{1,\infty}(\Omega)$ such that*

$$\begin{aligned}
 (6.3) \qquad \qquad \qquad &0 \leq \phi_\eta^\ominus \leq 1, \qquad 0 \leq \phi_\eta^\oplus \leq 1, \\
 &0 \leq \int_{\Omega} \phi_\eta^\ominus \, d\mu_s^+ \leq \eta, \qquad 0 \leq \int_{\Omega} \phi_\eta^\oplus \, d\mu_s^- \leq \eta.
 \end{aligned}$$

We then have

$$(6.4) \qquad \left\{ \begin{aligned} &\frac{1}{n} \int_{\{n \leq u_\varepsilon < 2n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \phi_\eta^\ominus \, dx \leq \omega_\eta(n, \varepsilon) + \eta, \\ &\int_{\{u_\varepsilon > 2n\}} \phi_\eta^\ominus \, d\lambda_{\varepsilon,0}^\ominus \leq \omega_\eta(n, \varepsilon) + \eta, \end{aligned} \right.$$

$$(6.5) \qquad \left\{ \begin{aligned} &\frac{1}{n} \int_{\{-2n < u_\varepsilon \leq -n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \phi_\eta^\oplus \, dx \leq \omega_\eta(n, \varepsilon) + \eta, \\ &\int_{\{u_\varepsilon < -2n\}} \phi_\eta^\oplus \, d\lambda_{\varepsilon,0}^\oplus \leq \omega_\eta(n, \varepsilon) + \eta. \end{aligned} \right.$$

REMARK 6.2. Let us make some comments on the results of this lemma. We will discuss only (6.4), since the same comments can be made for (6.5). Note that the only hypothesis on ϕ_η^\ominus is to be “close to 0” (in the sense of the first assertion of (6.3)) on the set where μ_s^+ is concentrated, and that there is no requirement on ϕ_η^\ominus with respect to μ_s^- and $\lambda_\varepsilon^\ominus$. Similar considerations hold for ϕ_η^\oplus .

The first inequality of (6.4) says, in some sense, that the energy of u_ε on the set $\{n \leq u_\varepsilon < 2n\}$, once divided by n (which is the “width” of the strip $\{n \leq u_\varepsilon < 2n\}$), vanishes as ε tends to zero, and then n tends to infinity, on the set where μ_s^+ is not concentrated (and where ϕ_η^\ominus can be close (or even equal) to 1). In contrast, note that using (5.1) with $k = n$, we have that

$$\frac{1}{n} \int_{\{n \leq u_\varepsilon < 2n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \, dx$$

is bounded for every n and ε , but does not converge to zero as ε tends to zero and then n tends to infinity, since its limit is $\mu_s^+(\Omega)$ (see (2.23)).

The second part of (6.4) describes a similar fact, for what concerns $\lambda_{\varepsilon,0}^\ominus$: even if ϕ_η^\ominus can be close (or even equal) to 1 on the set where $\lambda_{\varepsilon,0}^\ominus$ is concentrated, the fact that we restrict our attention to the set where u_ε is larger than $2n$ (that is, where u_ε is positive and very large) yields a quantity that converges to zero; see also Remarks 2.19 and 2.24.

PROOF OF LEMMA 6.1. Let $\beta_n(s) = B_{n,n}(s^+)$, where $B_{n,n}$ is defined by (4.4). By (5.4) we have

$$(6.6) \quad \int_\Omega |\nabla \beta_n(u_\varepsilon)|^p \, dx = \frac{1}{n^p} \int_{\{n \leq u_\varepsilon < 2n\}} |\nabla u_\varepsilon|^p \, dx \leq \frac{c}{n^{p-1}}.$$

Fix now n and let ε tend to zero. As $\beta_n(0) = 0$, the sequence $\beta_n(u_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$. Furthermore, since u_ε converges to u almost everywhere, and since $\beta_n(s)$ is continuous and bounded by 1, we have

$$(6.7) \quad \beta_n(u_\varepsilon) \rightarrow \beta_n(u) \quad \text{almost everywhere and weakly}^* \text{ in } L^\infty(\Omega).$$

This fact, together with (6.6), implies, as ε tends to 0^+ ,

$$(6.8) \quad \beta_n(u_\varepsilon) \rightarrow \beta_n(u) \quad \text{weakly in } W_0^{1,p}(\Omega).$$

Using the weak lower semicontinuity in (6.6), as in the proof of (5.10), we obtain

$$(6.9) \quad \frac{1}{n^p} \int_{\{n \leq u < 2n\}} |\nabla u|^p \, dx = \int_\Omega |\nabla \beta_n(u)|^p \, dx \leq \frac{c}{n^{p-1}}.$$

Since $\beta_n(0) = 0$, and since $\beta_n(s)$ is bounded by 1, (6.9) implies that, as n tends to infinity,

$$(6.10) \quad \beta_n(u) \rightarrow 0 \quad \text{almost everywhere and weakly}^* \text{ in } L^\infty(\Omega),$$

$$(6.11) \quad \beta_n(u) \rightarrow 0 \quad \text{strongly in } W_0^{1,p}(\Omega).$$

We now choose $w = \beta_n(u_\varepsilon) \phi_\eta^\ominus$ as test function in (6.2) (setting $w^{+\infty} = \phi_\eta^\ominus$ and $w^{-\infty} = 0$, this choice is admissible since $w = \phi_\eta^\ominus$ almost everywhere on $\{u_\varepsilon > 2n\}$ and $w = 0$ almost everywhere on $\{u_\varepsilon < -2n\}$). We obtain

$$\begin{aligned}
 \text{(A)} \quad & \frac{1}{n} \int_{\{n \leq u_\varepsilon < 2n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \phi_\eta^\ominus dx \\
 \text{(B)} \quad & + \int_\Omega a(x, \nabla u_\varepsilon) \cdot \nabla \phi_\eta^\ominus \beta_n(u_\varepsilon) dx \\
 \text{(C)} \quad & = \int_\Omega f_\varepsilon \beta_n(u_\varepsilon) \phi_\eta^\ominus dx \\
 \text{(D)} \quad & - \langle \operatorname{div}(g_\varepsilon), \beta_n(u_\varepsilon) \phi_\eta^\ominus \rangle \\
 \text{(E)} \quad & + \int_\Omega \beta_n(u_\varepsilon) \phi_\eta^\ominus d\lambda_{\varepsilon,0}^\oplus \\
 \text{(F)} \quad & - \int_\Omega \beta_n(u_\varepsilon) \phi_\eta^\ominus d\lambda_{\varepsilon,0}^\ominus \\
 \text{(G)} \quad & + \int_\Omega \phi_\eta^\ominus d\mu_{\varepsilon,s}^+.
 \end{aligned}$$

Recalling that $a(x, \nabla u_\varepsilon)$ converges to $a(x, \nabla u)$ strongly in $(L^q(\Omega))^N$ for every $q < \frac{N}{N-1}$ by (5.15), that ϕ_η^\ominus belongs to $W^{1,\infty}(\Omega)$, that (6.7) holds, and using (6.10), we obtain

$$(6.12) \quad \text{(B)} = \int_\Omega a(x, \nabla u) \cdot \nabla \phi_\eta^\ominus \beta_n(u) dx + \omega_{\eta,n}(\varepsilon) = \omega_\eta(n, \varepsilon).$$

Moreover, using Proposition 2.8, (6.7), the weak $L^1(\Omega)$ convergence of f_ε to f , and (6.10), we have

$$(6.13) \quad \text{(C)} = \int_\Omega f \beta_n(u) \phi_\eta^\ominus dx + \omega_{\eta,n}(\varepsilon) = \omega_\eta(n, \varepsilon).$$

Furthermore,

$$(6.14) \quad \text{(D)} = - \langle \operatorname{div}(g), \beta_n(u) \phi_\eta^\ominus \rangle + \omega_{\eta,n}(\varepsilon) = \omega_\eta(n, \varepsilon),$$

due to the strong convergence (3.11) of $-\operatorname{div}(g_\varepsilon)$ to $-\operatorname{div}(g)$ in $W^{-1,p'}(\Omega)$ and to (6.8) and (6.11). Finally, since $0 \leq \beta_n(u_\varepsilon) \leq 1$ and $0 \leq \mu_{\varepsilon,s}^+ \leq \lambda_{\varepsilon,s}^\oplus$ (see (3.18)), and since ϕ_η^\ominus is continuous, we have

$$(6.15) \quad \text{(E)} + \text{(G)} \leq \int_\Omega \phi_\eta^\ominus d\lambda_\varepsilon^\oplus = \int_\Omega \phi_\eta^\ominus d\mu_s^+ + \omega_\eta(\varepsilon) \leq \omega_\eta(\varepsilon) + \eta,$$

where we have used (6.3) in the last inequality. Thus, observing that

$$(6.16) \quad -\text{(F)} \geq \int_{\{u_\varepsilon > 2n\}} \phi_\eta^\ominus d\lambda_{\varepsilon,0}^\ominus,$$

we obtain, putting together (6.12)-(6.16),

$$\frac{1}{n} \int_{\{n \leq u_\varepsilon < 2n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \phi_\eta^\ominus dx + \int_{\{u_\varepsilon > 2n\}} \phi_\eta^\ominus d\lambda_{\varepsilon,0}^\ominus \leq \omega_\eta(n, \varepsilon) + \eta,$$

which gives (6.4) since both terms are nonnegative.

Estimate (6.5) can be obtained exactly in the same way, using $B_{n,n}(s^-)$ and ϕ_η^\oplus , as well as the second part of (6.3). □

LEMMA 6.3. *Let k be a positive real number. Let $f_\varepsilon, g_\varepsilon, \lambda_\varepsilon^\oplus$, and $\lambda_\varepsilon^\ominus$ be sequences which satisfy (3.9)-(3.13), and let u_ε be a sequence of renormalized solutions of (6.1) which satisfies (5.7), (5.8), (5.13), (5.15). For $\delta > 0$ and $\eta > 0$ given, let ψ_δ^+ and ψ_δ^- , and ψ_η^+ and ψ_η^- be functions in $C_c^\infty(\Omega)$ which satisfy (5.17)-(5.24). We then have*

$$(6.17) \quad \begin{cases} \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) \psi_\delta^+ \psi_\eta^+ dx = \omega(\eta, \delta, \varepsilon), \\ \int_{\{-n \leq u_\varepsilon \leq k\}} (k - T_k(u_\varepsilon)) \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus = \omega(\eta, n, \delta, \varepsilon), \end{cases}$$

$$(6.18) \quad \begin{cases} \int_\Omega a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) \psi_\delta^- \psi_\eta^- dx = \omega(\eta, \delta, \varepsilon), \\ \int_{\{-k \leq u_\varepsilon \leq n\}} (k + T_k(u_\varepsilon)) \psi_\delta^- \psi_\eta^- d\lambda_{\varepsilon,0}^\ominus = \omega(\eta, n, \delta, \varepsilon). \end{cases}$$

REMARK 6.4. As in Remark 6.2, some comments are in order. The first result in (6.17) can be seen as giving some properties of $T_k(u_\varepsilon)$ near the set E^+ : it expresses the fact that $a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon)$ is uniformly (in ε) small on a small neighbourhood of E^+ .

On the other hand, since $k - T_k(s) \geq 0$, the second result in (6.17) expresses the fact that u_ε is very large (greater than k for every k) “near” E^+ , where $\lambda_{\varepsilon,0}^\oplus$ tends to concentrate. The same remarks can be made on (6.18).

Finally observe that in the second part of both (6.17) and (6.18) the order in which the parameters converge is first ε , next δ , then n , and finally η . The introduction of the parameter η (and therefore of the “double” test function $\psi_\delta^+ \psi_\eta^+$) is made necessary here by the fact that we need to pass to the limit first in ε , then in δ and then in n , and not first in ε and then in n as we did in Lemma 6.1. Lemma 6.3 is the only place where this double test function is needed. It is used to control the term (B) in the proof (see (6.21) below). Note that this term is zero (and thus there is no need of using the double test function) if μ_ε (and therefore u_ε) is nonnegative.

PROOF OF LEMMA 6.3. Let $k > 0$ be fixed, and let n be an integer with $n > k$. Let $h_n(s) = H_{n,n}(s)$, where $H_{n,n}$ is defined in (4.3). Reasoning as in

the proof of Lemma 6.1 (that is, using again (5.4)), and observing that $h_n(u_\varepsilon)$ is bounded by 1, we get that for n fixed

$$(6.19) \quad h_n(u_\varepsilon) \rightarrow h_n(u) \quad \text{almost everywhere and weakly}^* \text{ in } L^\infty(\Omega),$$

$$(6.20) \quad h_n(u_\varepsilon) \rightarrow h_n(u) \quad \text{weakly in } W^{1,p}(\Omega).$$

We choose $w = (k - T_k(u_\varepsilon)) h_n(u_\varepsilon) \psi_\delta^+ \psi_\eta^+$ as test function in (6.2) (setting $w^{+\infty} = w^{-\infty} = 0$, this can be done since $w = 0$ almost everywhere on $\{|u_\varepsilon| > 2n\}$), and we obtain

$$\begin{aligned} (A) \quad & - \int_{\Omega} a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) h_n(u_\varepsilon) \psi_\delta^+ \psi_\eta^+ dx \\ (B) \quad & + \int_{\Omega} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon h'_n(u_\varepsilon) (k - T_k(u_\varepsilon)) \psi_\delta^+ \psi_\eta^+ dx \\ (C) \quad & + \int_{\Omega} a(x, \nabla u_\varepsilon) \cdot \nabla \psi_\delta^+ h_n(u_\varepsilon) (k - T_k(u_\varepsilon)) \psi_\eta^+ dx \\ (D) \quad & + \int_{\Omega} a(x, \nabla u_\varepsilon) \cdot \nabla \psi_\eta^+ h_n(u_\varepsilon) (k - T_k(u_\varepsilon)) \psi_\delta^+ dx \\ (E) \quad & = \int_{\Omega} f_\varepsilon (k - T_k(u_\varepsilon)) h_n(u_\varepsilon) \psi_\delta^+ \psi_\eta^+ dx \\ (F) \quad & - \langle \operatorname{div}(g_\varepsilon), (k - T_k(u_\varepsilon)) h_n(u_\varepsilon) \psi_\delta^+ \psi_\eta^+ \rangle \\ (G) \quad & + \int_{\Omega} (k - T_k(u_\varepsilon)) h_n(u_\varepsilon) \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus \\ (H) \quad & - \int_{\Omega} (k - T_k(u_\varepsilon)) h_n(u_\varepsilon) \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\ominus. \end{aligned}$$

Since n is larger than k , we have $k - T_k(u_\varepsilon) = 2k$ on $\{-2n < u_\varepsilon \leq -n\}$, and $k - T_k(u_\varepsilon) = 0$ on $\{n \leq u_\varepsilon < 2n\}$, so that we get

$$(B) = \frac{2k}{n} \int_{\{-2n < u_\varepsilon \leq -n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \psi_\delta^+ \psi_\eta^+ dx.$$

Since the integrands are nonnegative, and $\psi_\delta^+ \leq 1$, we have

$$0 \leq (B) \leq \frac{2k}{n} \int_{\{-2n < u_\varepsilon \leq -n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon \psi_\eta^+ dx.$$

Thus, by the first assertion of (6.5), which can be applied since $\phi_\eta^\oplus = \psi_\eta^+$ satisfies the second assertion of (6.3) thanks to (5.22), we conclude that

$$(6.21) \quad 0 \leq (B) \leq \omega_\eta(n, \varepsilon) + \eta = \omega(\eta, n, \varepsilon).$$

Furthermore, since for k fixed, $k - T_k(u_\varepsilon)$ converges to $k - T_k(u)$ in the weak* topology of $L^\infty(\Omega)$, and since (6.19) holds, using the fact that $\text{supp}(h_n) = [-2n, 2n]$ we obtain

$$(6.22) \quad \begin{aligned} \text{(C)} &= \int_{\Omega} a(x, \nabla T_{2n}(u)) \cdot \nabla \psi_\delta^+ h_n(u)(k - T_k(u)) \psi_\eta^+ dx + \omega_{\eta,n,\delta}(\varepsilon) \\ &= \omega_{\eta,n}(\delta, \varepsilon), \end{aligned}$$

where the last statement is due to the fact that ψ_δ^+ converges strongly to zero in $W_0^{1,p}(\Omega)$ (see (5.20)), that $a(x, \nabla T_{2n}(u))$ belongs to $(L^{p'}(\Omega))^N$, while $h_n(u)(k - T_k(u)) \psi_\eta^+$ belongs to $L^\infty(\Omega)$. Similarly, we have

$$(6.23) \quad \begin{aligned} \text{(D)} &= \int_{\Omega} a(x, \nabla T_{2n}(u)) \cdot \nabla \psi_\eta^+ h_n(u)(k - T_k(u)) \psi_\delta^+ dx + \omega_{\eta,n,\delta}(\varepsilon) \\ &= \omega_{\eta,n}(\delta, \varepsilon), \end{aligned}$$

using for the second statement the fact that ψ_δ^+ tends to zero almost everywhere. Furthermore we have, by Proposition 2.8,

$$(6.24) \quad \text{(E)} = \int_{\Omega} f(k - T_k(u)) h_n(u) \psi_\delta^+ \psi_\eta^+ dx + \omega_{\eta,n,\delta}(\varepsilon) = \omega_{\eta,n}(\delta, \varepsilon),$$

since f_ε converges weakly to f in $L^1(\Omega)$, since (6.19) holds, and since $k - T_k(u_\varepsilon)$ converges to $k - T_k(u)$ weakly* in $L^\infty(\Omega)$ and almost everywhere; the second limit is performed using again the fact that ψ_δ^+ converges to zero in the weak* topology of $L^\infty(\Omega)$, while the term $f(k - T_k(u)) h_n(u) \psi_\eta^+$ belongs to $L^1(\Omega)$. Moreover we have

$$\text{(F)} = - \langle \text{div}(g), (k - T_k(u)) h_n(u) \psi_\delta^+ \psi_\eta^+ \rangle + \omega_{\eta,n,\delta}(\varepsilon),$$

since $-\text{div}(g_\varepsilon)$ converges to $-\text{div}(g)$ strongly in $W^{-1,p'}(\Omega)$, while, as ε tends to zero, $(k - T_k(u_\varepsilon)) h_n(u_\varepsilon) \psi_\delta^+ \psi_\eta^+$ converges to $(k - T_k(u)) h_n(u) \psi_\delta^+ \psi_\eta^+$ weakly in $W_0^{1,p}(\Omega)$ (this easily follows from the weak convergence of $T_k(u_\varepsilon)$ to $T_k(u)$ in $W_0^{1,p}(\Omega)$ and from (6.19) and (6.20)). Thus, since we have that $(k - T_k(u)) h_n(u) \psi_\delta^+ \psi_\eta^+$ converges strongly to zero in $W_0^{1,p}(\Omega)$ as δ tends to zero (this is due to (5.20)),

$$(6.25) \quad \text{(F)} = \omega_{\eta,n}(\delta, \varepsilon).$$

Finally, thanks to (5.22),

$$(6.26) \quad |(\text{H})| \leq 2k \int_{\Omega} \psi_\delta^+ d\lambda_{\varepsilon,0}^\ominus \leq 2k \int_{\Omega} \psi_\delta^+ d\lambda_\varepsilon^\ominus = \omega(\delta, \varepsilon).$$

Putting together (6.21)-(6.26), we have

$$(6.27) \quad -(A) + (G) = \omega(\eta, n, \delta, \varepsilon).$$

Since $n > k$ we have

$$-(A) = \int_{\Omega} a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) \psi_\delta^+ \psi_\eta^+ dx,$$

and

$$(G) \geq \int_{\{-n \leq u_\varepsilon \leq k\}} (k - T_k(u_\varepsilon)) \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus,$$

so that (6.27) implies (6.17).

The estimate (6.18) is obtained in the same way, choosing as test function

$$(k + T_k(u_\varepsilon)) h_n(u_\varepsilon) \psi_\delta^- \psi_\eta^- ,$$

and using the corresponding properties of ψ_δ^- , ψ_η^- , $\lambda_\varepsilon^\oplus$, and $\lambda_\varepsilon^\ominus$. □

7. – Far from E

This section will be devoted to the proof of the following result.

LEMMA 7.1. *Let k be a positive real number. Let $f_\varepsilon, g_\varepsilon, \lambda_\varepsilon^\oplus$, and $\lambda_\varepsilon^\ominus$ be sequences which satisfy (3.9)-(3.13), and let u_ε be a sequence of renormalized solutions of (6.1) which satisfies (5.7), (5.8), (5.13), (5.15). For $\delta > 0$ and $\eta > 0$ given, let ψ_δ^+ and ψ_δ^- , and ψ_η^+ and ψ_η^- , be functions in $C_c^\infty(\Omega)$ which satisfy (5.17)-(5.24). Define*

$$(7.1) \quad \Phi_{\delta,\eta} = \psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\eta^- .$$

We then have

$$(7.2) \quad \begin{aligned} & \int_{\Omega} a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) (1 - \Phi_{\delta,\eta}) dx \\ &= \int_{\Omega} a(x, \nabla T_k(u)) \cdot \nabla T_k(u) (1 - \Phi_{\delta,\eta}) dx + \omega(\eta, \delta, \varepsilon). \end{aligned}$$

REMARK 7.2. The meaning of (7.2) is, roughly speaking, that $a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon)$ strongly converges to $a(x, \nabla T_k(u)) \cdot \nabla T_k(u)$ in $L^1(\Omega)$ if we “stay away” from E .

We split the proof into three lemmas. We begin with the following result.

LEMMA 7.3. *Under the hypotheses of Lemma 7.1 we have*

$$(7.3) \quad \int_{\Omega} a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) (1 - \Phi_{\delta,\eta}) \, dx - \int_{\Omega} a(x, \nabla u) \cdot \nabla \Phi_{\delta,\eta} T_k(u) \, dx \\ = \int_{\Omega} f (1 - \Phi_{\delta,\eta}) T_k(u) \, dx - \langle \operatorname{div}(g), (1 - \Phi_{\delta,\eta}) T_k(u) \rangle + \omega(\eta, \delta, \varepsilon).$$

PROOF. We choose $w = (1 - \Phi_{\delta,\eta}) T_k(u_\varepsilon)$ as test function in (6.2) (setting $w^{+\infty} = k(1 - \Phi_{\delta,\eta})$ and $w^{-\infty} = -k(1 - \Phi_{\delta,\eta})$, this can be done since $w = k(1 - \Phi_{\delta,\eta})$ almost everywhere on $\{u_\varepsilon > k\}$ and $w = -k(1 - \Phi_{\delta,\eta})$ almost everywhere on $\{u_\varepsilon < -k\}$), and we obtain

$$(A) \quad \int_{\Omega} a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) (1 - \Phi_{\delta,\eta}) \, dx \\ (B) \quad - \int_{\Omega} a(x, \nabla u_\varepsilon) \cdot \nabla \Phi_{\delta,\eta} T_k(u_\varepsilon) \, dx \\ (C) \quad = \int_{\Omega} f_\varepsilon (1 - \Phi_{\delta,\eta}) T_k(u_\varepsilon) \, dx \\ (D) \quad - \langle \operatorname{div}(g_\varepsilon), (1 - \Phi_{\delta,\eta}) T_k(u_\varepsilon) \rangle \\ (E) \quad + \int_{\Omega} (1 - \Phi_{\delta,\eta}) T_k(u_\varepsilon) \, d\lambda_{\varepsilon,0}^\oplus \\ (F) \quad - \int_{\Omega} (1 - \Phi_{\delta,\eta}) T_k(u_\varepsilon) \, d\lambda_{\varepsilon,0}^\ominus \\ (G) \quad + \int_{\Omega} (1 - \Phi_{\delta,\eta}) k \, d\mu_{\varepsilon,s}^+ \\ (H) \quad - \int_{\Omega} (1 - \Phi_{\delta,\eta}) k \, d\mu_{\varepsilon,s}^-.$$

Since $\Phi_{\delta,\eta}$ belongs to $C_c^\infty(\Omega)$, $a(x, \nabla u_\varepsilon)$ converges to $a(x, \nabla u)$ strongly in $(L^q(\Omega))^N$ for every $q < \frac{N}{N-1}$ by (5.15), and $T_k(u_\varepsilon)$ converges weakly* in $L^\infty(\Omega)$ and almost everywhere to $T_k(u)$, we have

$$(7.4) \quad (B) = - \int_{\Omega} a(x, \nabla u) \cdot \nabla \Phi_{\delta,\eta} T_k(u) \, dx + \omega_{\eta,\delta}(\varepsilon).$$

Since for η and δ fixed, $(1 - \Phi_{\delta,\eta}) T_k(u_\varepsilon)$ converges to $(1 - \Phi_{\delta,\eta}) T_k(u)$ weakly* in $L^\infty(\Omega)$ and almost everywhere in Ω , while f_ε converges weakly to f in $L^1(\Omega)$, we get, by Proposition 2.8,

$$(7.5) \quad (C) = \int_{\Omega} f (1 - \Phi_{\delta,\eta}) T_k(u) \, dx + \omega_{\eta,\delta}(\varepsilon).$$

Since for η and δ fixed, $(1 - \Phi_{\delta,\eta}) T_k(u_\varepsilon)$ converges to $(1 - \Phi_{\delta,\eta}) T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, and since $-\operatorname{div}(g_\varepsilon)$ converges to $-\operatorname{div}(g)$ strongly in $W^{-1,p'}(\Omega)$, we obtain

$$(7.6) \quad (D) = - \langle \operatorname{div}(g), (1 - \Phi_{\delta,\eta}) T_k(u) \rangle + \omega_{\eta,\delta}(\varepsilon).$$

For (E) and (G) we have, using the inequality $\mu_{\varepsilon,s}^+ \leq \lambda_{\varepsilon,s}^\oplus$ (see (3.18)) and (5.23),

$$(7.7) \quad \begin{aligned} |(E) + (G)| &\leq k \int_{\Omega} (1 - \Phi_{\delta,\eta}) d\lambda_{\varepsilon}^\oplus \\ &\leq k \int_{\Omega} (1 - \psi_{\delta}^+ \psi_{\eta}^+) d\lambda_{\varepsilon}^\oplus = \omega(\eta, \delta, \varepsilon). \end{aligned}$$

In the same way, using the inequality $\mu_{\varepsilon,s}^- \leq \lambda_{\varepsilon,s}^\ominus$ (see (3.18)) and (5.24), we get

$$(7.8) \quad (F) + (H) = \omega(\eta, \delta, \varepsilon).$$

Putting together (7.4)-(7.8), we obtain (7.3). \square

LEMMA 7.4. *Under the hypotheses of Lemma 7.1, we have*

$$(7.9) \quad \frac{1}{n} \int_{\{n \leq |u_{\varepsilon}| < 2n\}} a(x, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} (1 - \Phi_{\delta,\eta}) dx = \omega(\eta, \delta, n, \varepsilon).$$

PROOF. We split the left hand side of (7.9) into the sum of four terms:

$$\begin{aligned} (A) \quad & \frac{1}{n} \int_{\{n \leq u_{\varepsilon} < 2n\}} a(x, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} (1 - \psi_{\delta}^+ \psi_{\eta}^+) dx, \\ (B) \quad & - \frac{1}{n} \int_{\{n \leq u_{\varepsilon} < 2n\}} a(x, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} \psi_{\delta}^- \psi_{\eta}^- dx, \\ (C) \quad & \frac{1}{n} \int_{\{-2n < u_{\varepsilon} \leq -n\}} a(x, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} (1 - \psi_{\delta}^- \psi_{\eta}^-) dx, \\ (D) \quad & - \frac{1}{n} \int_{\{-2n < u_{\varepsilon} \leq -n\}} a(x, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} \psi_{\delta}^+ \psi_{\eta}^+ dx. \end{aligned}$$

For every term above we can apply the result of Lemma 6.1. Indeed, if we define $\phi_{\delta+\eta}^\ominus = 1 - \psi_{\delta}^+ \psi_{\eta}^+$, we have, by (5.23),

$$\int_{\Omega} \phi_{\delta+\eta}^\ominus d\mu_s^+ \leq \delta + \eta,$$

and so $\phi_{\delta+\eta}^\ominus$ satisfies (6.3); this implies, by (6.4), that

$$0 \leq (A) \leq \omega_{\eta,\delta}(n, \varepsilon) + \delta + \eta = \omega(\eta, \delta, n, \varepsilon).$$

The same thing clearly holds if we define $\phi_{\delta+\eta}^\oplus = 1 - \psi_{\delta}^- \psi_{\eta}^-$ and use (5.24), so that we get

$$0 \leq (C) \leq \omega_{\eta,\delta}(n, \varepsilon) + \delta + \eta = \omega(\eta, \delta, n, \varepsilon).$$

Moreover, if we set $\phi_{\delta,\eta}^{\ominus} = \psi_{\delta}^{-} \psi_{\eta}^{-}$, we have

$$\int_{\Omega} \phi_{\delta,\eta}^{\ominus} d\mu_s^+ \leq \int_{\Omega} \psi_{\eta}^{-} d\mu_s^+ \leq \eta,$$

by (5.21); this implies, again by Lemma 6.1, that

$$|(\mathbf{B})| \leq \omega_{\eta}(n, \varepsilon) + \eta = \omega(\eta, \delta, n, \varepsilon).$$

The same computation, with $\phi_{\delta,\eta}^{\oplus} = \psi_{\delta}^+ \psi_{\eta}^+$, yields

$$|(\mathbf{D})| \leq \omega(\eta, \delta, n, \varepsilon).$$

Putting together the estimates we have obtained on the four terms, we get (7.9). □

LEMMA 7.5. *Under the hypotheses of Lemma 7.1, we have*

$$\begin{aligned} (7.10) \quad & \int_{\Omega} a(x, \nabla T_k(u)) \cdot \nabla T_k(u) (1 - \Phi_{\delta,\eta}) dx - \int_{\Omega} a(x, \nabla u) \cdot \nabla \Phi_{\delta,\eta} T_k(u) dx \\ & = \int_{\Omega} f (1 - \Phi_{\delta,\eta}) T_k(u) dx - \langle \text{div}(g), (1 - \Phi_{\delta,\eta}) T_k(u) \rangle + \omega(\eta, \delta). \end{aligned}$$

PROOF. Let n be an integer with $n > k$, and define $h_n(s) = H_{n,n}(s)$ as in the proof of Lemma 6.3; then $h_n(u_{\varepsilon})$ satisfies (6.19) and (6.20). Moreover, using the definition of h_n and arguing as in the proof of (6.11), we obtain

$$(7.11) \quad h_n(u) \rightarrow 1 \quad \text{almost everywhere and weakly}^* \text{ in } L^{\infty}(\Omega),$$

$$(7.12) \quad h_n(u) \rightarrow 1 \quad \text{strongly in } W^{1,p}(\Omega).$$

We choose $(1 - \Phi_{\delta,\eta}) T_k(u) h_n(u_{\varepsilon})$ as test function in (6.2), with $w^{+\infty} = w^{-\infty} = 0$, and we obtain

$$\begin{aligned} (\mathbf{A}) \quad & \int_{\Omega} a(x, \nabla u_{\varepsilon}) \cdot \nabla T_k(u) (1 - \Phi_{\delta,\eta}) h_n(u_{\varepsilon}) dx \\ (\mathbf{B}) \quad & - \int_{\Omega} a(x, \nabla u_{\varepsilon}) \cdot \nabla \Phi_{\delta,\eta} T_k(u) h_n(u_{\varepsilon}) dx \\ (\mathbf{C}) \quad & + \int_{\Omega} a(x, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} T_k(u) (1 - \Phi_{\delta,\eta}) h_n'(u_{\varepsilon}) dx \\ (\mathbf{D}) \quad & = \int_{\Omega} f_{\varepsilon} (1 - \Phi_{\delta,\eta}) T_k(u) h_n(u_{\varepsilon}) dx \\ (\mathbf{E}) \quad & - \langle \text{div}(g_{\varepsilon}), (1 - \Phi_{\delta,\eta}) T_k(u) h_n(u_{\varepsilon}) \rangle \\ (\mathbf{F}) \quad & + \int_{\Omega} (1 - \Phi_{\delta,\eta}) T_k(u) h_n(u_{\varepsilon}) d\lambda_{\varepsilon,0}^{\oplus} \\ (\mathbf{G}) \quad & - \int_{\Omega} (1 - \Phi_{\delta,\eta}) T_k(u) h_n(u_{\varepsilon}) d\lambda_{\varepsilon,0}^{\ominus}. \end{aligned}$$

Since $a(x, \nabla u_\varepsilon) h_n(u_\varepsilon) = a(x, \nabla T_{2n}(u_\varepsilon)) h_n(u_\varepsilon)$, and since $a(x, \nabla T_{2n}(u_\varepsilon))$ converges to $a(x, \nabla T_{2n}(u))$ weakly in $(L^{p'}(\Omega))^N$ while $h_n(u_\varepsilon)$ converges almost everywhere to $h_n(u)$ and is bounded by 1, we obtain that the sequence $a(x, \nabla u_\varepsilon) h_n(u_\varepsilon)$ converges to $a(x, \nabla T_{2n}(u)) h_n(u)$ weakly in $(L^{p'}(\Omega))^N$ as ε tends to zero. For $n > k$, we have

$$(7.13) \quad \begin{aligned} \text{(A)} &= \int_{\Omega} a(x, \nabla T_{2n}(u)) \cdot \nabla T_k(u) (1 - \Phi_{\delta, \eta}) h_n(u) dx + \omega_{\eta, \delta, n}(\varepsilon) \\ &= \int_{\Omega} a(x, \nabla T_k(u)) \cdot \nabla T_k(u) (1 - \Phi_{\delta, \eta}) dx + \omega_{\eta, \delta}(n, \varepsilon). \end{aligned}$$

Moreover, we have

$$(7.14) \quad \begin{aligned} \text{(B)} &= - \int_{\Omega} a(x, \nabla u) \cdot \nabla \Phi_{\delta, \eta} T_k(u) h_n(u) dx + \omega_{\eta, \delta, n}(\varepsilon) \\ &= - \int_{\Omega} a(x, \nabla u) \cdot \nabla \Phi_{\delta, \eta} T_k(u) dx + \omega_{\eta, \delta}(n, \varepsilon); \end{aligned}$$

the first statement holds true since $a(x, \nabla u_\varepsilon)$ converges strongly to $a(x, \nabla u)$ in $(L^q(\Omega))^N$, for every $q < \frac{N}{N-1}$, while $\Phi_{\delta, \eta}$ belongs to $C_c^\infty(\Omega)$, and $h_n(u_\varepsilon)$ satisfies (6.19); the second one is due to (7.11). Furthermore, by the result of Lemma 7.4,

$$(7.15) \quad |(\text{C})| \leq \frac{k}{n} \int_{\{n \leq |u_\varepsilon| < 2n\}} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon (1 - \Phi_{\delta, \eta}) dx = \omega(\eta, \delta, n, \varepsilon).$$

As for the right hand side, we have

$$(7.16) \quad \begin{aligned} \text{(D)} &= \int_{\Omega} f (1 - \Phi_{\delta, \eta}) T_k(u) h_n(u) dx + \omega_{\eta, \delta, n}(\varepsilon) \\ &= \int_{\Omega} f (1 - \Phi_{\delta, \eta}) T_k(u) dx + \omega_{\eta, \delta}(n, \varepsilon), \end{aligned}$$

by Proposition 2.8 together with (6.19), and the weak convergence of f_ε to f in $L^1(\Omega)$; the second statement is due to (7.11). Moreover,

$$(7.17) \quad \begin{aligned} \text{(E)} &= - \langle \operatorname{div}(g), (1 - \Phi_{\delta, \eta}) T_k(u) h_n(u) \rangle + \omega_{\eta, \delta, n}(\varepsilon) \\ &= - \langle \operatorname{div}(g), (1 - \Phi_{\delta, \eta}) T_k(u) \rangle + \omega_{\eta, \delta}(n, \varepsilon), \end{aligned}$$

where the first statement holds true since $-\operatorname{div}(g_\varepsilon)$ converges to $-\operatorname{div}(g)$ strongly in $W^{-1, p'}(\Omega)$, while $h_n(u_\varepsilon)$ satisfies (6.20); the second statement is due to the fact that $h_n(u)$ satisfies (7.12). The two remaining terms are estimated as follows using (5.23) and (5.24):

$$(7.18) \quad |(\text{F})| \leq k \int_{\Omega} (1 - \Phi_{\delta, \eta}) d\lambda_{\varepsilon, 0}^\oplus \leq k \int_{\Omega} (1 - \Phi_{\delta, \eta}) d\lambda_\varepsilon^\oplus \leq \omega(\eta, \delta, \varepsilon),$$

and (in analogous way),

$$(7.19) \quad |(\text{G})| \leq \omega(\eta, \delta, \varepsilon).$$

Putting together (7.13)-(7.19), we have proved (7.10), since all integral terms in (7.10) depend neither on ε nor on n . \square

PROOF OF LEMMA 7.1. To obtain (7.2) it is enough to put together (7.3) and (7.10). \square

8. – Proof of the stability result: conclusion

In this section we conclude the proof of the stability theorem. Recall that u_ε is a sequence of renormalized solutions of (6.1), and that in the previous steps (see Section 5) we have already extracted a subsequence, still denoted by u_ε , which satisfies (5.7), (5.8), (5.13), (5.15).

STEP 5: strong convergence of the truncates in $W_0^{1,p}(\Omega)$.

Let η and δ be fixed positive real numbers, and let $\Phi_{\delta,\eta} = \psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\eta^-$ as in Section 7. We have

$$\begin{aligned} \text{(A)} & \int_{\Omega} [a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) - a(x, \nabla T_k(u)) \cdot \nabla T_k(u)] dx \\ \text{(B)} & = \int_{\Omega} [a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) - a(x, \nabla T_k(u)) \cdot \nabla T_k(u)] (1 - \Phi_{\delta,\eta}) dx \\ \text{(C)} & + \int_{\Omega} a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) (\psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\eta^-) dx \\ \text{(D)} & - \int_{\Omega} a(x, \nabla T_k(u)) \cdot \nabla T_k(u) (\psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\eta^-) dx. \end{aligned}$$

Thanks to (7.2) (proved in Lemma 7.1) we have

$$(8.1) \quad \text{(B)} = \omega(\eta, \delta, \varepsilon).$$

Thanks to (6.17) and (6.18) (proved in Lemma 6.3), we have

$$(8.2) \quad \text{(C)} = \omega(\eta, \delta, \varepsilon),$$

while, being $a(x, \nabla T_k(u)) \cdot \nabla T_k(u)$ in $L^1(\Omega)$, and since $\psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\eta^-$ converges to zero in the weak* topology of $L^\infty(\Omega)$, we have

$$(8.3) \quad \text{(D)} = \omega(\eta, \delta).$$

Thus, by (8.1)-(8.3), we have

$$\text{(A)} = \omega(\eta, \delta, \varepsilon),$$

which implies

$$(8.4) \quad \int_{\Omega} a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) dx = \int_{\Omega} a(x, \nabla T_k(u)) \cdot \nabla T_k(u) dx + \omega(\varepsilon).$$

Since $a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon)$ is a sequence of nonnegative functions that converges almost everywhere to $a(x, \nabla T_k(u)) \cdot \nabla T_k(u)$, (8.4) implies that

$$a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) \rightarrow a(x, \nabla T_k(u)) \cdot \nabla T_k(u) \quad \text{strongly in } L^1(\Omega).$$

By (2.6) we have

$$\alpha |\nabla T_k(u_\varepsilon)|^p \leq a(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon),$$

so that, since $\nabla T_k(u_\varepsilon)$ converges almost everywhere to $\nabla T_k(u)$, the Vitali theorem yields that

$$\nabla T_k(u_\varepsilon) \rightarrow \nabla T_k(u) \quad \text{strongly in } (L^p(\Omega))^N,$$

and this concludes the proof of this step.

STEP 6: the limit function is a renormalized solution.

We will prove that u satisfies Definition 2.25; in view of the equivalence between the various definitions of renormalized solution (see Theorem 2.33) this will prove the stability theorem. The fact that u satisfies conditions (a) and (b) of Definition 2.13 has been proved in (5.9) and (5.14). Let us prove (2.27).

Let h be a function in $W^{1,\infty}(\mathbf{R})$ such that h' has compact support in \mathbf{R} , and let φ be a function in $W^{1,r}(\Omega) \cap L^\infty(\Omega)$, with $r > N$, such that $h(u)\varphi$ belongs to $W_0^{1,p}(\Omega)$. Define, as usual, $h(+\infty)$ as the limit of $h(s)$ at $+\infty$ and $h(-\infty)$ as the limit of h at $-\infty$ (note that h is constant for $|s|$ large). Since h is bounded and continuous, by the dominated convergence theorem the sequence $h(u_\varepsilon)$ is bounded in $L^\infty(\Omega)$ and converges to $h(u)$ strongly in $L^p(\Omega)$ and weakly* in $L^\infty(\Omega)$. If M is such that $\text{supp}(h') \subseteq [-M, M]$, we have, almost everywhere in Ω ,

$$|\nabla h(u_\varepsilon)|^p = |\nabla u_\varepsilon h'(u_\varepsilon)|^p \leq c |\nabla T_M(u_\varepsilon)|^p.$$

Since $\nabla T_M(u_\varepsilon)$ is bounded in $(L^p(\Omega))^N$ by (5.8), we conclude that $h(u_\varepsilon)$ converges to $h(u)$ weakly in $W^{1,p}(\Omega)$.

We now choose $h(u_\varepsilon)\varphi$ as test function in the renormalized equation (2.27) satisfied by u_ε with μ_ε as right hand side (decomposed as in (3.16) and (3.17)). We obtain

$$\begin{aligned} \text{(A)} \quad & \int_{\Omega} a(x, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon h'(u_\varepsilon) \varphi \, dx \\ \text{(B)} \quad & + \int_{\Omega} a(x, \nabla u_\varepsilon) \cdot \nabla \varphi h(u_\varepsilon) \, dx \\ \text{(C)} \quad & = \int_{\Omega} f_\varepsilon h(u_\varepsilon) \varphi \, dx \\ \text{(D)} \quad & - \langle \text{div}(g_\varepsilon), h(u_\varepsilon) \varphi \rangle \\ \text{(E)} \quad & + \int_{\Omega} h(u_\varepsilon) \varphi \, d\lambda_{\varepsilon,0}^{\oplus} \\ \text{(F)} \quad & - \int_{\Omega} h(u_\varepsilon) \varphi \, d\lambda_{\varepsilon,0}^{\ominus} \\ \text{(G)} \quad & + h(+\infty) \int_{\Omega} \varphi \, d\mu_{\varepsilon,s}^+ \\ \text{(H)} \quad & - h(-\infty) \int_{\Omega} \varphi \, d\mu_{\varepsilon,s}^-. \end{aligned}$$

As $\text{supp}(h') \subseteq [-M, M]$, we have

$$\begin{aligned}
 (8.5) \quad (A) &= \int_{\Omega} a(x, \nabla T_M(u_\varepsilon)) \cdot \nabla T_M(u_\varepsilon) h'(u_\varepsilon) \varphi \, dx \\
 &= \int_{\Omega} a(x, \nabla T_M(u)) \cdot \nabla T_M(u) h'(u) \varphi \, dx + \omega(\varepsilon) \\
 &= \int_{\Omega} a(x, \nabla u) \cdot \nabla u h'(u) \varphi \, dx + \omega(\varepsilon),
 \end{aligned}$$

since $\nabla T_M(u_\varepsilon)$ converges to $\nabla T_M(u)$ strongly in $(L^p(\Omega))^N$, while $\nabla u_\varepsilon h'(u_\varepsilon)$ converges to $\nabla u h'(u)$ weakly in $(L^p(\Omega))^N$, and φ belongs to $L^\infty(\Omega)$. Furthermore

$$(8.6) \quad (B) = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi h(u) \, dx + \omega(\varepsilon),$$

since $a(x, \nabla u_\varepsilon)$ converges strongly to $a(x, \nabla u)$ in $(L^q(\Omega))^N$, for every $q < \frac{N}{N-1}$ by (5.15), since φ is in $W^{1,r}(\Omega)$, with $r > N$, and since $h(u_\varepsilon)$ is weakly* convergent in $L^\infty(\Omega)$ to $h(u)$.

As for the right hand side, we have

$$(8.7) \quad (C) = \int_{\Omega} f h(u) \varphi \, dx + \omega(\varepsilon),$$

in view of Proposition 2.8, since f_ε converges to f weakly in $L^1(\Omega)$, while $h(u_\varepsilon) \varphi$ converges to $h(u) \varphi$ weakly* in $L^\infty(\Omega)$ and almost everywhere. We then have

$$(8.8) \quad (D) = -\langle \text{div}(g), h(u) \varphi \rangle + \omega(\varepsilon),$$

since $-\text{div}(g_\varepsilon)$ converges strongly to $-\text{div}(g)$ in $W^{-1,p'}(\Omega)$, while $h(u_\varepsilon) \varphi$ converges weakly to $h(u) \varphi$ in $W_0^{1,p}(\Omega)$ as ε tends to zero.

As for the other terms, we have

$$\begin{aligned}
 (8.9) \quad (E) + (G) &= \int_{\Omega} (h(u_\varepsilon) - h(+\infty)) \varphi \, d\lambda_{\varepsilon,0}^\oplus + h(+\infty) \int_{\Omega} \varphi \, d\lambda_\varepsilon^\oplus \\
 &\quad - h(+\infty) \int_{\Omega} \varphi \, dv_\varepsilon,
 \end{aligned}$$

where we have set $v_\varepsilon = \lambda_{\varepsilon,s}^\oplus - \mu_{\varepsilon,s}^+$. Since, by (3.18), and by the nonnegativity of $\lambda_{\varepsilon,0}^\oplus$, $0 \leq v_\varepsilon \leq \lambda_\varepsilon^\oplus$, and since by hypothesis (3.12) $\lambda_\varepsilon^\oplus$ converges to μ_s^+ in the narrow topology of measures, we conclude that, up to a subsequence, the sequence v_ε converges in the narrow topology to a measure ν such that $0 \leq \nu \leq \mu_s^+$. As $\mu_{\varepsilon,s} = \lambda_{\varepsilon,s}^\oplus - \lambda_{\varepsilon,s}^\ominus$ (see (3.17) in Remark 3.5), we also have $v_\varepsilon = \lambda_{\varepsilon,s}^\ominus - \mu_{\varepsilon,s}^-$. Since $\lambda_{\varepsilon,0}^\ominus \geq 0$, we have $v_\varepsilon \leq \lambda_\varepsilon^\ominus$, from which we conclude

that $0 \leq \nu \leq \mu_s^-$. Since μ_s^+ and μ_s^- are mutually singular, we have $\nu = 0$, so that the whole sequence ν_ε converges to 0 in the narrow topology of measures. Therefore

$$(8.10) \quad h(+\infty) \int_{\Omega} \varphi d\nu_\varepsilon = \omega(\varepsilon).$$

On the other hand, since by hypothesis (3.12) $\lambda_\varepsilon^\oplus$ converges to μ_s^+ in the narrow topology of measures, and since φ is continuous and bounded, we have

$$(8.11) \quad h(+\infty) \int_{\Omega} \varphi d\lambda_\varepsilon^\oplus = h(+\infty) \int_{\Omega} \varphi d\mu_s^+ + \omega(\varepsilon).$$

As for the other term, we observe that, if $\text{supp}(h') \subseteq [-M, M]$, then $h(u_\varepsilon) - h(+\infty)$ is zero on the set $\{u_\varepsilon > M\}$, and so

$$(8.12) \quad \begin{aligned} & \left| \int_{\Omega} (h(u_\varepsilon) - h(+\infty)) \varphi d\lambda_{\varepsilon,0}^\oplus \right| \\ & \leq 2 \|h\|_{L^\infty(\mathbf{R})} \|\varphi\|_{L^\infty(\Omega)} \int_{\{u_\varepsilon \leq M\}} d\lambda_{\varepsilon,0}^\oplus \\ & = 2 \|h\|_{L^\infty(\mathbf{R})} \|\varphi\|_{L^\infty(\Omega)} \int_{\{u_\varepsilon \leq M\}} (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_{\varepsilon,0}^\oplus \\ & \quad + 2 \|h\|_{L^\infty(\mathbf{R})} \|\varphi\|_{L^\infty(\Omega)} \int_{\{u_\varepsilon \leq M\}} \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus. \end{aligned}$$

For the first term of the right hand side of (8.12), we have, by (5.23)

$$(8.13) \quad 0 \leq \int_{\{|u_\varepsilon| \leq M\}} (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_{\varepsilon,0}^\oplus \leq \int_{\Omega} (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_\varepsilon^\oplus = \omega(\eta, \delta, \varepsilon).$$

For the second term of the right hand side of (8.12), we note that for $k = M + 1$ one has

$$0 \leq \chi_{(-\infty, M)}(t) \leq k - T_k(t) \quad \forall t \in \mathbf{R}.$$

Therefore we have, for $n > k$,

$$\begin{aligned} 0 & \leq \int_{\{u_\varepsilon \leq M\}} \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus \leq \int_{\Omega} (k - T_k(u_\varepsilon)) \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus \\ & \leq \int_{\{-n \leq u_\varepsilon \leq k\}} (k - T_k(u_\varepsilon)) \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus + 2k \int_{\{u_\varepsilon < -n\}} \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus, \end{aligned}$$

in which we use the second assertion of (6.17) for the first integral, and the second assertion of (6.5) and the fact that

$$\int_{\Omega} \psi_\delta^+ \psi_\eta^+ d\mu_s^- \leq \int_{\Omega} \psi_\eta^+ d\mu_s^- \leq \eta,$$

for the second one, to obtain

$$(8.14) \quad 0 \leq \int_{\{u_\varepsilon \leq M\}} \psi_\delta^+ \psi_\eta^+ d\lambda_{\varepsilon,0}^\oplus \leq \omega(\eta, n, \delta, \varepsilon) + \omega_\eta(n, \varepsilon) + \eta.$$

From (8.10)-(8.14) we get

$$(8.15) \quad (E) + (G) = h(+\infty) \int_\Omega \varphi d\mu_s^+ + \omega(\varepsilon),$$

and in the same way,

$$(8.16) \quad (F) + (H) = -h(-\infty) \int_\Omega \varphi d\mu_s^- + \omega(\varepsilon).$$

Putting together (8.5)-(8.8) and (8.15), (8.16) we obtain

$$\begin{aligned} \int_\Omega a(x, \nabla u) \cdot \nabla(h(u) \varphi) dx &= \int_\Omega f h(u) \varphi dx - \langle \operatorname{div}(g), h(u) \varphi \rangle \\ &\quad + h(+\infty) \int_\Omega \varphi d\mu_s^+ - h(-\infty) \int_\Omega \varphi d\mu_s^-, \end{aligned}$$

that is (2.27), as $\mu_0 = f - \operatorname{div}(g)$. This concludes the proof of Theorem 3.4. \square

REMARK 8.1. It is crucial in the proof of Theorem 3.4 that the measures $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ are nonnegative and converge to two measures which are singular with respect to the p -capacity and which are mutually singular. Taking $\lambda_\varepsilon^\oplus = \mu_\varepsilon^+$ and $\lambda_\varepsilon^\ominus = \mu_\varepsilon^-$, the two examples below show that the truncates of u_ε do not converge strongly in $W_0^{1,p}(\Omega)$ if $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ are nonnegative and converge to two measures which are not mutually singular. Taking $\lambda_\varepsilon^\oplus = \mu_\varepsilon^+ - \mu_\varepsilon^-$, and $\lambda_\varepsilon^\ominus = 0$, these examples also show that the strong convergence of the truncates fails if $\lambda_\varepsilon^\oplus$ (or $\lambda_\varepsilon^\ominus$) is not nonnegative, even if $\lambda_\varepsilon^\oplus$ and $\lambda_\varepsilon^\ominus$ converge to two mutually singular measures (which are here equal to 0).

EXAMPLE 8.2. In [7], Remark 3.2 (see also [8], Section 2) the authors consider a sequence u_ε of solution in $H_0^1(\Omega)$ ($u_\varepsilon = 1 - w^\varepsilon$ in the notation of [8]) of the problem

$$\begin{cases} -\Delta u_\varepsilon = \mu_\varepsilon^+ - \mu_\varepsilon^- & \text{on } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_ε^+ and μ_ε^- are two sequences of nonnegative measures in $H^{-1}(\Omega)$, which both converge to (the same multiple of) the Lebesgue measure on Ω in the narrow topology of measures, as well as in the strong topology of $H^{-1}(\Omega)$ for what concerns μ_ε^- , but only in the weak topology of $H^{-1}(\Omega)$ for what concerns μ_ε^+ . The sequence u_ε satisfies $0 \leq u_\varepsilon \leq 1$, and converges weakly to 0 in $H_0^1(\Omega)$, but not strongly. Thus $T_k(u_\varepsilon) = u_\varepsilon$ for $k \geq 1$ does not converge strongly in $H_0^1(\Omega)$.

One may think that the result fails only because μ_ε approximates a measure which is not concentrated on a set of zero 2-capacity. This is not the case, as the following example shows.

EXAMPLE 8.3. Let $\Omega = B_1(0) = \{x \in \mathbf{R}^N : |x| < 1\}$, with $N > 2$; for ε such that $\varepsilon \sqrt[N]{6} < 1$, consider the sequence u_ε of solutions of

$$\begin{cases} -\Delta u_\varepsilon = \mu_\varepsilon^+ - \mu_\varepsilon^- & \text{on } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\mu_\varepsilon^+ = \frac{1}{\varepsilon^N} (\chi_{B_\varepsilon(0)} + \chi_{B_{\sqrt[N]{6}\varepsilon}(0) \setminus B_{\sqrt[N]{5}\varepsilon}(0)}), \quad \mu_\varepsilon^- = \frac{1}{\varepsilon^N} \chi_{B_{\sqrt[N]{5}\varepsilon}(0) \setminus B_{\sqrt[N]{3}\varepsilon}(0)}.$$

It is easily seen that both μ_ε^+ and μ_ε^- converge in the narrow topology of measures to the same multiple $\frac{2\sigma_{N-1}}{N} \delta_0$ of the Dirac mass at the origin. The solution u_ε is radial and satisfies

$$u_\varepsilon(x) = \frac{1}{\varepsilon^{N-2}} v\left(\frac{|x|}{\varepsilon}\right)$$

where v is a C^1 function defined on \mathbf{R}^+ such that

$$v(r) = -\frac{A}{2N} r^2 + \frac{B}{N(N-2)} \frac{1}{r^{N-2}} + \frac{C}{2(N-2)}$$

for $r > 0$, with

$$\begin{cases} A = 1, & B = 0, & C = 1 + 3^{2/N} - 2 \cdot 5^{2/N} + 6^{2/N} & \text{for } 0 \leq r \leq 1 \\ A = 0, & B = 1, & C = 3^{2/N} - 2 \cdot 5^{2/N} + 6^{2/N} & \text{for } 1 < r \leq \sqrt[N]{3} \\ A = -1, & B = 4, & C = -2 \cdot 5^{2/N} + 6^{2/N} & \text{for } \sqrt[N]{3} < r \leq \sqrt[N]{5} \\ A = 1, & B = -6, & C = 6^{2/N} & \text{for } \sqrt[N]{5} < r \leq \sqrt[N]{6} \\ A = 0, & B = 0, & C = 0 & \text{for } \sqrt[N]{6} < r. \end{cases}$$

Moreover one can easily prove that $v'(r) = 0$ if and only if $r = 0$, $\sqrt[N]{4}$ or $\sqrt[N]{6}$, that the function v is strictly decreasing on $(0, \sqrt[N]{4})$ and strictly increasing on $(\sqrt[N]{4}, \sqrt[N]{6})$, and that $u(0) > 0$. Since $v(\sqrt[N]{6}) = 0$, v has, on $(0, \sqrt[N]{6})$, a unique zero, which we denote by z ; clearly, $v'(z) < 0$.

Since $v \equiv 0$ for $r \geq \sqrt[N]{6}$ we have $u_\varepsilon \equiv 0$ on $B_1(0) \setminus B_{\sqrt[N]{6}\varepsilon}(0)$, so that u_ε converges pointwise to zero except at the origin and hence almost everywhere in Ω . Furthermore, since $|\mu_\varepsilon^+ - \mu_\varepsilon^-| = \mu_\varepsilon^+ + \mu_\varepsilon^-$ is bounded in $L^1(\Omega)$, then u_ε is bounded in $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$ (see [4], or (5.6) above). This implies (by Rellich theorem) that u_ε converges strongly to zero in $L^1(\Omega)$. Let us study the behaviour of $T_k(u_\varepsilon)$.

For fixed k , $\nabla T_k(u_\varepsilon)$ remains bounded in $(L^2(\Omega))^N$ independently of ε ; let us compute the limit of its norm as ε tends to 0. By definition of u_ε and thanks to easy changes of variable, one has

$$(8.17) \quad \begin{aligned} \int_{\Omega} |\nabla T_k(u_\varepsilon)|^2 dx &= \int_{\{|u_\varepsilon| \leq k\}} |\nabla u_\varepsilon|^2 dx \\ &= \frac{\sigma_{N-1}}{\varepsilon^{N-2}} \int_{\{|v| \leq k\varepsilon^{N-2}\}} |v'(r)|^2 r^{N-1} dr. \end{aligned}$$

In view of the form of the graph of the function v , the set $\{r : 0 < r < \sqrt[N]{6}, |v(r)| \leq k\varepsilon^{N-2}\}$ is the union of two disjoint intervals $(\rho_\varepsilon^-, \rho_\varepsilon^+)$ and $(\tau_\varepsilon, \sqrt[N]{6})$, with $v(\rho_\varepsilon^-) = k\varepsilon^{N-2}$, $v(\rho_\varepsilon^+) = v(\tau_\varepsilon) = -k\varepsilon^{N-2}$, and $\rho_\varepsilon^- < z < \rho_\varepsilon^+ < \sqrt[N]{4} < \tau_\varepsilon < \sqrt[N]{6}$. Since by definition both $v(\rho_\varepsilon^-)$ and $v(\rho_\varepsilon^+)$ converge to zero as ε tends to zero, the continuity of v implies that both ρ_ε^- and ρ_ε^+ tend to z . Hence, since v is C^1 , we have

$$v'(z) = \lim_{\varepsilon \rightarrow 0^+} \frac{v(\rho_\varepsilon^+) - v(\rho_\varepsilon^-)}{\rho_\varepsilon^+ - \rho_\varepsilon^-} = \lim_{\varepsilon \rightarrow 0^+} \frac{-2k\varepsilon^{N-2}}{\rho_\varepsilon^+ - \rho_\varepsilon^-}.$$

Thus, recalling that $v'(z) < 0$,

$$\rho_\varepsilon^+ - \rho_\varepsilon^- = -\frac{2k\varepsilon^{N-2}}{|v'(z)|} (1 + \omega(\varepsilon)).$$

Analogous calculations (using $v'(\sqrt[N]{6}) = 0$, and $v''(\sqrt[N]{6}) = -1$) yield

$$\sqrt[N]{6} - \tau_\varepsilon = \sqrt{2k\varepsilon^{\frac{N-2}{2}}} (1 + \omega(\varepsilon)).$$

Therefore it follows from (8.17), from the mean value theorem, and from the continuity of v (recall that $v''(\sqrt[N]{6}) = -1$), that, for k fixed,

$$\begin{aligned} \int_{\Omega} |\nabla T_k(u_\varepsilon)|^2 dx &= \frac{\sigma_{N-1}}{\varepsilon^{N-2}} \int_{\rho_\varepsilon^-}^{\rho_\varepsilon^+} |v'(r)|^2 r^{N-1} dr + \frac{\sigma_{N-1}}{\varepsilon^{N-2}} \int_{\tau_\varepsilon}^{\sqrt[N]{6}} |v'(r)|^2 r^{N-1} dr \\ &= \frac{\sigma_{N-1}}{\varepsilon^{N-2}} \frac{2k\varepsilon^{N-2}}{|v'(z)|} (1 + \omega(\varepsilon)) (|v'(z)|^2 z^{N-1} + \omega(\varepsilon)) \\ &\quad + \frac{\sigma_{N-1}}{\varepsilon^{N-2}} c_N k^{3/2} \varepsilon^{\frac{3(N-2)}{2}} (1 + \omega(\varepsilon)) \\ &= 2k\sigma_{N-1}|v'(z)|z^{N-1} + \omega(\varepsilon), \end{aligned}$$

so that $T_k(u_\varepsilon)$ does not converge strongly to 0 in $H_0^1(\Omega)$. Hence the result of Theorem 3.2 does not hold.

Similarly one can prove that

$$\frac{1}{n} \int_{\{n \leq u_\varepsilon < 2n\}} |\nabla u_\varepsilon|^2 dx = \sigma_{N-1}|v'(z)|z^{N-1} + \omega(n, \varepsilon),$$

which has to be confronted with (2.23), which in the present case would assert that

$$\frac{1}{n} \int_{\{n \leq u_\varepsilon < 2n\}} |\nabla u_\varepsilon|^2 dx = \omega(n, \varepsilon)$$

since the limit measure of $\mu_\varepsilon^+ - \mu_\varepsilon^-$ is zero.

9. – A property of the difference of two solutions

In this section we prove a property of the difference of two renormalized solutions of (1.1) corresponding to two different measures.

THEOREM 9.1. *Suppose that a satisfies (2.6)-(2.8). Let u and \hat{u} be renormalized solutions of (1.1), with data μ and $\hat{\mu}$ respectively. Then, for every $k > 0$ we have*

$$(a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot (\nabla u - \nabla \hat{u}) \chi_{\{|u-\hat{u}|<k\}} \in L^1(\Omega)$$

and

$$(9.1) \quad \int_{\{|u-\hat{u}|<k\}} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot (\nabla u - \nabla \hat{u}) dx \leq k(|\mu| + |\hat{\mu}|)(\Omega).$$

If we assume that $p \geq 2$, and a stronger hypothesis than (2.8) on a , namely

$$(9.2) \quad (a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^p,$$

for almost every x in Ω and for every ξ, ξ' in \mathbf{R}^N , where $\alpha > 0$ is a given constant (an hypothesis satisfied for example by $a(x, \xi) = |\xi|^{p-2} \xi$), then a consequence of (9.1) is the following theorem.

THEOREM 9.2. *Suppose that $p \geq 2$ and that a satisfies (2.6), (2.7) and (9.2). Let u and \hat{u} be renormalized solutions of (1.1), with data μ and $\hat{\mu}$ respectively. Then, for every $k > 0$ we have $T_k(u - \hat{u})$ in $W_0^{1,p}(\Omega)$ and*

$$(9.3) \quad \alpha \int_{\Omega} |\nabla T_k(u - \hat{u})|^p dx \leq k(|\mu|(\Omega) + |\hat{\mu}|(\Omega)).$$

PROOF OF THEOREM 9.2. From (9.1) and (9.2) we obtain

$$(9.4) \quad \alpha \int_{\{|u-\hat{u}|<k\}} |\nabla u - \nabla \hat{u}|^p dx \leq k(|\mu|(\Omega) + |\hat{\mu}|(\Omega)).$$

for every $k > 0$.

In order to prove that $T_k(u - \hat{u})$ belongs to $W_0^{1,p}(\Omega)$ (recall that $\nabla u - \nabla \hat{u}$ may not be $\nabla(u - \hat{u})$, due to the definition of gradient we use), we consider, for every $n > 0$, the function $T_k(T_n(u) - T_n(\hat{u}))$, which belongs to $W_0^{1,p}(\Omega)$ by (a) of Definition (2.13). It is easy to prove that, for $n > k$,

$$\begin{aligned} \int_{\Omega} |\nabla T_k(T_n(u) - T_n(\hat{u}))|^p dx &\leq \int_{\{|u-\hat{u}|<k\}} |\nabla u - \nabla \hat{u}|^p dx \\ &\quad + \int_{\{n-k < |u| < n\}} |\nabla u|^p dx + \int_{\{n-k < |\hat{u}| < n\}} |\nabla \hat{u}|^p dx. \end{aligned}$$

By (9.4) and (5.10) the right hand side of the previous inequality is bounded uniformly with respect to n . Therefore the function $T_k(T_n(u) - T_n(\hat{u}))$ is bounded in $W_0^{1,p}(\Omega)$ uniformly with respect to n . Since this function converges to $T_k(u - \hat{u})$ almost everywhere in Ω as n tends to infinity, we conclude that $T_k(u - \hat{u})$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$. Using Lemma 2.12 with $\lambda = -1$ we have that

$$(\nabla u - \nabla \hat{u})\chi_{\{|u-\hat{u}|<k\}} = \nabla(u - \hat{u})\chi_{\{|u-\hat{u}|<k\}} = \nabla T_k(u - \hat{u}),$$

which allows us to deduce (9.3) from (9.4). □

PROOF OF THEOREM 9.1. We will use the fact that both u and \hat{u} are renormalized solutions in the sense of Definition 2.29; thus we have that, for every $n > 0$, there exists nonnegative measures $\lambda_n^+, \lambda_n^-, \hat{\lambda}_n^+$ and $\hat{\lambda}_n^-$ in $\mathcal{M}_0(\Omega)$ such that

$$(9.5) \quad \int_{\{|u|<n\}} a(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\{|u|<n\}} \varphi \, d\mu_0 + \int_{\Omega} \varphi \, d\lambda_n^+ - \int_{\Omega} \varphi \, d\lambda_n^-,$$

$$(9.6) \quad \int_{\{|\hat{u}|<n\}} a(x, \nabla \hat{u}) \cdot \nabla \varphi \, dx = \int_{\{|\hat{u}|<n\}} \varphi \, d\hat{\mu}_0 + \int_{\Omega} \varphi \, d\hat{\lambda}_n^+ - \int_{\Omega} \varphi \, d\hat{\lambda}_n^-,$$

for every φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We fix $k > 0$ and choose as test function $\varphi = T_k(T_n(u) - T_n(\hat{u}))$ in (9.5), and $\varphi = T_k(T_n(\hat{u}) - T_n(u))$ in (9.6). Since λ_n^+ and λ_n^- converge in the narrow topology to μ_s^+ and μ_s^- respectively, we have

$$\begin{aligned} & \left| \int_{\Omega} T_k(T_n(u) - T_n(\hat{u})) \, d\lambda_n^+ - \int_{\Omega} T_k(T_n(u) - T_n(\hat{u})) \, d\lambda_n^- \right| \\ & \leq k \lambda_n^+(\Omega) + k \lambda_n^-(\Omega) = k |\mu_s|(\Omega) + \omega(n). \end{aligned}$$

Analogously, we have

$$\left| \int_{\Omega} T_k(T_n(\hat{u}) - T_n(u)) \, d\hat{\lambda}_n^+ - \int_{\Omega} T_k(T_n(\hat{u}) - T_n(u)) \, d\hat{\lambda}_n^- \right| \leq k |\hat{\mu}_s|(\Omega) + \omega(n).$$

Furthermore,

$$\left| \int_{\{|u|<n\}} T_k(T_n(u) - T_n(\hat{u})) \, d\mu_0 \right| \leq k |\mu_0|(\Omega),$$

and

$$\left| \int_{\{|\hat{u}|<n\}} T_k(T_n(\hat{u}) - T_n(u)) \, d\hat{\mu}_0 \right| \leq k |\hat{\mu}_0|(\Omega).$$

Summing up (9.5) and (9.6), and using the previous estimates, we thus obtain that, for every $n > 0$,

$$(9.7) \quad \begin{aligned} & \int_{\{|u|<n\}} a(x, \nabla u) \cdot \nabla T_k(T_n(u) - T_n(\hat{u})) \, dx \\ & + \int_{\{|\hat{u}|<n\}} a(x, \nabla \hat{u}) \cdot \nabla T_k(T_n(\hat{u}) - T_n(u)) \, dx \\ & \leq k (|\mu|(\Omega) + |\hat{\mu}|(\Omega)) + \omega(n). \end{aligned}$$

We now write

$$\begin{aligned} & \int_{\{|u|<n\}} a(x, \nabla u) \cdot \nabla T_k(T_n(u) - T_n(\hat{u})) \, dx \\ &= \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} a(x, \nabla u) \cdot \nabla T_k(T_n(u) - T_n(\hat{u})) \, dx \\ &+ \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|\geq n \end{smallmatrix} \right\}} a(x, \nabla u) \cdot \nabla T_k(T_n(u) - T_n(\hat{u})) \, dx . \end{aligned}$$

The first integral of the right hand side is equal to

$$\int_{\left\{ \begin{smallmatrix} |u|<n, |\hat{u}|<n \\ |u-\hat{u}|<k \end{smallmatrix} \right\}} a(x, \nabla u) \cdot (\nabla u - \nabla \hat{u}) \, dx ,$$

while the second one is equal to

$$\int_{\left\{ \begin{smallmatrix} |u|<n \\ \hat{u}\geq n \end{smallmatrix} \right\}} a(x, \nabla u) \cdot \nabla T_k(T_n(u) - n) \, dx + \int_{\left\{ \begin{smallmatrix} |u|<n \\ \hat{u}\leq -n \end{smallmatrix} \right\}} a(x, \nabla u) \cdot \nabla T_k(T_n(u) + n) \, dx ,$$

where both terms are easily seen to be nonnegative using (2.6). We thus have

$$\begin{aligned} (9.8) \quad & \int_{\left\{ \begin{smallmatrix} |u|<n, |\hat{u}|<n \\ |u-\hat{u}|<k \end{smallmatrix} \right\}} a(x, \nabla u) \cdot (\nabla u - \nabla \hat{u}) \, dx \\ & \leq \int_{\{|u|<n\}} a(x, \nabla u) \cdot \nabla T_k(T_n(u) - T_n(\hat{u})) \, dx . \end{aligned}$$

Reasoning in an analogous way, we have

$$\begin{aligned} (9.9) \quad & \int_{\left\{ \begin{smallmatrix} |\hat{u}|<n, |u|<n \\ |u-\hat{u}|<k \end{smallmatrix} \right\}} a(x, \nabla \hat{u}) \cdot (\nabla \hat{u} - \nabla u) \, dx \\ & \leq \int_{\{|\hat{u}|<n\}} a(x, \nabla \hat{u}) \cdot \nabla T_k(T_n(\hat{u}) - T_n(u)) \, dx . \end{aligned}$$

Using (9.8) and (9.9) we thus obtain from (9.7)

$$\int_{\left\{ \begin{smallmatrix} |u|<n, |\hat{u}|<n \\ |u-\hat{u}|<k \end{smallmatrix} \right\}} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot (\nabla u - \nabla \hat{u}) \, dx \leq k(|\mu|(\Omega) + |\hat{\mu}|(\Omega)) + \omega(n) .$$

Since the integrands is nonnegative by (2.8), we can pass to the limit as n tends to infinity using the Fatou lemma to obtain that

$$(a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot (\nabla u - \nabla \hat{u}) \chi_{\{|u-\hat{u}|<k\}} \in L^1(\Omega)$$

and (9.1). □

10. – Some partial uniqueness results

In this section we first prove the uniqueness of the renormalized solution in the linear case. Then we prove, in the nonlinear case, that two renormalized solutions with the same data which are “comparable” (in a sense which will be specified) are actually equal. Let us explicitly observe that in the nonlinear case we do not prove uniqueness *stricto sensu* for general measures μ in $\mathcal{M}_b(\Omega)$ and “noncomparable” solutions. However, uniqueness has been proved in the case of measures in $L^1(\Omega)$, or more generally in $\mathcal{M}_0(\Omega)$ (see [1], [5], and [23], [25], [26]). In the case $p = N$, uniqueness has been recently proved in [16], [12] (see also Remark 10.8, below).

10.1. – Uniqueness in the linear case

THEOREM 10.1. *Let A be a uniformly elliptic matrix with $L^\infty(\Omega)$ coefficients, which is therefore such that $a(x, \xi) = A(x)\xi$ satisfies (2.6)-(2.8) with $p = 2$, and let μ be a measure in $\mathcal{M}_b(\Omega)$. Then the renormalized solution of (1.1) is unique, and coincides with the solution introduced by Stampacchia in [29].*

PROOF. Let u be a renormalized solution of (1.1). By the equivalence theorem (Theorem 2.33), u satisfies Definition 2.21. Choosing in (2.25) $h(s) = H_{n,n}(s)$, where $H_{n,n}(s)$ is defined by (4.3), and $\varphi = v$, where v belongs to $H_0^1(\Omega) \cap C_b^0(\Omega)$, we obtain

$$\begin{aligned} \text{(A)} \quad & \int_{\Omega} A(x) \nabla u \cdot \nabla v H_{n,n}(u) dx \\ \text{(B)} \quad & - \frac{1}{n} \int_{\{n \leq u < 2n\}} A(x) \nabla u \cdot \nabla u v dx \\ \text{(C)} \quad & + \frac{1}{n} \int_{\{-2n < u \leq -n\}} A(x) \nabla u \cdot \nabla u v dx \\ \text{(D)} \quad & = \int_{\Omega} H_{n,n}(u) v d\mu_0. \end{aligned}$$

Using (2.23) and (2.24) we have

$$\text{(B)} = - \int_{\Omega} v d\mu_s^+ + \omega(n), \quad \text{(C)} = \int_{\Omega} v d\mu_s^- + \omega(n),$$

while Lebesgue dominated convergence theorem (observe that $H_{n,n}(u)$ converges to 1 μ_0 -almost everywhere as n tends to infinity in view of Remarks 2.11 and 2.18) yields

$$\text{(D)} = \int_{\Omega} v d\mu_0 + \omega(n).$$

Thus

$$(10.1) \quad \text{(A)} = \text{(D)} - \text{(B)} - \text{(C)} = \int_{\Omega} v d\mu + \omega(n).$$

If $A^*(x)$ is the transpose matrix of $A(x)$, and S_n is the function in $W^{1,\infty}(\mathbf{R})$ defined by

$$S_n(s) = \int_0^s H_{n,n}(t) dt,$$

we have

$$(10.2) \quad (A) = \int_{\Omega} A^*(x) \nabla v \cdot \nabla u H_{n,n}(u) dx = \int_{\Omega} A^*(x) \nabla v \cdot \nabla S_n(u) dx.$$

Let now g be any function in $C_c^\infty(\Omega)$; by the De Giorgi regularity theorem (see, e.g., [29]), the unique solution v of

$$(10.3) \quad \begin{cases} v \in H_0^1(\Omega), \\ \int_{\Omega} A^*(x) \nabla v \cdot \nabla \varphi dx = \int_{\Omega} g \varphi dx, \quad \forall \varphi \in H_0^1(\Omega), \end{cases}$$

belongs to $C_b^0(\Omega)$, and can thus be used as test function in (10.1) and (10.2). On the other hand, since $S_n(u)$ belongs to $H_0^1(\Omega)$, we deduce from (10.2) and (10.3) that

$$(A) = \int_{\Omega} A(x) \nabla u \cdot \nabla v H_{n,n}(u) dx = \int_{\Omega} g S_n(u) dx.$$

Since $S_n(u)$ converges to u strongly in $L^1(\Omega)$ as n tends to infinity, this implies that

$$(A) = \int_{\Omega} g u dx + \omega(n),$$

which together with (10.1) yields

$$\int_{\Omega} g u dx = \int_{\Omega} v d\mu, \quad \text{for every } g \text{ in } C_c^\infty(\Omega),$$

where v is defined by (10.3). Thus u is a solution of the problem in the duality sense, as defined by G. Stampacchia in [29]. Since the Stampacchia solution is unique, so is the renormalized solution of (1.1). \square

REMARK 10.2. Another proof of Theorem 10.1 can be given in the spirit of [10], since any renormalized solution is a reachable solution in the sense of [10], even in the nonlinear case (see Remarks 2.28 and 2.32). Let us sketch this alternative proof. By the equivalence theorem (Theorem 2.33), any renormalized solution u of (1.1) satisfies Definition 2.29. In the linear case, $T_k(u)$ is therefore the unique solution of

$$\begin{cases} T_k(u) \in H_0^1(\Omega), \\ \int_{\Omega} A(x) \nabla T_k(u) \cdot \nabla \varphi dx = \langle \mu_k, \varphi \rangle, \quad \forall \varphi \in H_0^1(\Omega), \end{cases}$$

where $\mu_k = \mu_0 \llcorner \{|u| < k\} + \lambda_k^+ - \lambda_k^-$ belongs to $H^{-1}(\Omega) \cap \mathcal{M}_b(\Omega)$ (see Remark 2.32). Thus $T_k(u)$ is the Stampacchia solution (defined by duality in the sense of [29]) of the problem with datum μ_k . Since μ_k converges to μ in the narrow topology of measures, as k tends to infinity, and since the Stampacchia solution is continuous with respect to the datum for this topology, it follows that u , the limit as k tends to infinity of $T_k(u)$ in $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, is the Stampacchia solution of the linear problem (1.1). This proves the uniqueness of u .

10.2. – Partial uniqueness in the nonlinear case

In order to prove partial uniqueness results in the nonlinear case, we make further hypotheses on the function a , namely the strong monotonicity and the local Lipschitz continuity, or the Hölder continuity, with respect to ξ : we assume that

$$(10.4) \quad \begin{cases} (a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^p, & \text{if } p \geq 2, \\ (a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \alpha \frac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{2-p}}, & \text{if } p < 2, \end{cases}$$

and

$$(10.5) \quad \begin{cases} |a(x, \xi) - a(x, \xi')| \leq \gamma [b(x) + |\xi| + |\xi'|]^{p-2} |\xi - \xi'|, & \text{if } p \geq 2, \\ |a(x, \xi) - a(x, \xi')| \leq \gamma |\xi - \xi'|^{p-1}, & \text{if } p < 2, \end{cases}$$

for almost every x in Ω and for every ξ, ξ' in \mathbf{R}^N , where $\gamma > 0$ is a constant and b is a nonnegative function in $L^p(\Omega)$. These hypotheses are satisfied for example by the function $a(x, \xi) = |\xi|^{p-2}\xi$.

We then have the following theorem.

THEOREM 10.3. *Suppose that a satisfies (2.6), (2.7), (2.8), and (10.5). Let μ be a measure in $\mathcal{M}_b(\Omega)$, and let u and \hat{u} be two renormalized solutions of (1.1), relative to the measure μ . If moreover*

$$(10.6) \quad \frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx + \frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |\hat{u}| < 2n \\ |u| < 2n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx = \omega(n),$$

then $u = \hat{u}$.

PROOF. Define $h_n(s) = H_{n,n}(s)$, where $H_{n,n}$ is given by (4.3), and choose

$$h_n(u) h_n(\hat{u}) T_k(u - \hat{u})$$

as test function in both equations (2.25) written for u and \hat{u} . Subtracting the equations, we obtain

$$\begin{aligned}
 \text{(A)} \quad & \int_{\Omega} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot \nabla T_k(u - \hat{u}) h_n(u) h_n(\hat{u}) dx \\
 \text{(B)} \quad & + \int_{\Omega} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot \nabla u h'_n(u) h_n(\hat{u}) T_k(u - \hat{u}) dx \\
 \text{(C)} \quad & + \int_{\Omega} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot \nabla \hat{u} h'_n(\hat{u}) h_n(u) T_k(u - \hat{u}) dx \\
 & = 0.
 \end{aligned}$$

Since $h_n(u) \hat{h}_n(u)$ is nonnegative and converges to 1 almost everywhere, and since by (2.8) the function $(a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot \nabla T_k(u - \hat{u})$ is nonnegative, we have, by Fatou lemma,

$$\int_{\Omega} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot \nabla T_k(u - \hat{u}) dx \leq \text{(A)} + \omega(n).$$

If we prove that

$$|(\text{B})| + |(\text{C})| = \omega(n),$$

we will then have

$$\int_{\Omega} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot \nabla T_k(u - \hat{u}) dx = 0,$$

which implies the result by (2.8). We will only prove that

$$(10.7) \quad |(\text{B})| = \omega(n),$$

since the proof will hold also for $|(\text{C})|$.

If $p \geq 2$, then by (10.5), and since both h_n and h'_n have compact support, we have

$$\begin{aligned}
 |(\text{B})| & \leq \frac{\gamma k}{n} \int_{\left\{ \begin{array}{l} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{array} \right\}} [b(x) + |\nabla u| + |\nabla \hat{u}|]^{p-2} |\nabla u - \nabla \hat{u}| |\nabla u| dx \\
 & \leq \frac{\gamma k}{n} \int_{\left\{ \begin{array}{l} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{array} \right\}} [b(x) + |\nabla u| + |\nabla \hat{u}|]^{p-1} |\nabla u - \nabla \hat{u}| dx.
 \end{aligned}$$

Applying Hölder inequality with exponents p and p' , we then have

$$\begin{aligned}
 |(\mathbf{B})| &\leq \gamma k \left(\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx \right)^{\frac{1}{p}} \\
 &\quad \times \left(\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} [b(x) + |\nabla u| + |\nabla \hat{u}|]^p dx \right)^{\frac{1}{p'}} \\
 &\leq \gamma k \left(\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx \right)^{\frac{1}{p}} \\
 &\quad \times \left(\frac{c}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} 3^p [|b(x)|^p + |\nabla u|^p + |\nabla \hat{u}|^p] dx \right)^{\frac{1}{p'}}.
 \end{aligned}$$

Using (4.5) with $k = n$, and (2.6), we have

$$\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u|^p dx \leq \frac{1}{n} \int_{\{n \leq |u| < 2n\}} |\nabla u|^p dx \leq c,$$

while, using again (4.5), and (2.6), we get

$$\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla \hat{u}|^p dx \leq \frac{1}{n} \int_{\{|\hat{u}| < 2n\}} |\nabla \hat{u}|^p dx \leq c.$$

Moreover, we also have

$$\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |b(x)|^p dx \leq \frac{1}{n} \int_{\Omega} |b(x)|^p dx = \omega(n) \leq c.$$

Finally, using hypothesis (10.6), we have

$$\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx = \omega(n).$$

Putting together the last estimates we obtain (10.7), which concludes the proof of the theorem in the case $p \geq 2$.

If $p < 2$, using the second assumption of (10.5), we have

$$|(\mathbf{B})| \leq \frac{\gamma k}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^{p-1} |\nabla u| dx,$$

so that, by Hölder inequality,

$$|(\mathbf{B})| \leq \gamma k \left(\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx \right)^{\frac{1}{p'}} \left(\frac{1}{n} \int_{\left\{ \begin{smallmatrix} n \leq |u| < 2n \\ |\hat{u}| < 2n \end{smallmatrix} \right\}} |\nabla u|^p dx \right)^{\frac{1}{p}},$$

and this implies the result as before. □

The following theorems give other sufficient conditions for uniqueness, which are stronger than (10.6).

THEOREM 10.4. *Suppose that a satisfies (2.6), (2.7), (2.8), and (10.5). Let μ be a measure in $\mathcal{M}_b(\Omega)$. Let u and \hat{u} be two renormalized solutions of (1.1), relative to the measure μ . If moreover one of the following conditions holds:*

$$(10.8) \quad \frac{1}{n} \int_{\left\{ \begin{array}{l} |u| < n \\ |\hat{u}| < n \end{array} \right\}} |\nabla u - \nabla \hat{u}|^p dx = \omega(n),$$

$$(10.9) \quad \frac{1}{n} \int_{\{|u - \hat{u}| < n\}} |\nabla u - \nabla \hat{u}|^p dx = \omega(n),$$

then $u = \hat{u}$.

PROOF. It is indeed clear that (10.8) implies (10.6). Since

$$\{|u| < n, |\hat{u}| < n\} \subseteq \{|u - \hat{u}| < 2n\},$$

(10.9) implies (10.8). □

REMARK 10.5. Observe that the assumptions of Theorem 10.4 imply in particular that the integrals which appear in (10.8) or (10.9) are finite, so that $T_n(u - \hat{u})$ belongs to $W_0^{1,p}(\Omega)$ for every $n > 0$ (see the proof of Theorem 9.2), a property that we have obtained in general only in the case $p \geq 2$, and under a strong monotonicity assumption on a .

THEOREM 10.6. *Suppose that a satisfies (2.6), (2.7), (10.4), and (10.5). Let μ be a measure in $\mathcal{M}_b(\Omega)$. Let u and \hat{u} be two renormalized solutions of (1.1), relative to the measure μ . If moreover*

$$(10.10) \quad u - \hat{u} \in L^\infty(\Omega),$$

then $u = \hat{u}$.

PROOF. Let $M = \|u - \hat{u}\|_{L^\infty(\Omega)} + 1$. Assume first that $p \geq 2$. We will prove that (10.9) holds, which implies the uniqueness result. Indeed

$$\{|u - \hat{u}| < n\} = \{|u - \hat{u}| < M\} = \Omega$$

for $n \geq M$, and therefore

$$(10.11) \quad \frac{1}{n} \int_{\{|u - \hat{u}| < n\}} |\nabla u - \nabla \hat{u}|^p dx = \frac{1}{n} \int_{\{|u - \hat{u}| < M\}} |\nabla u - \nabla \hat{u}|^p dx.$$

Since $T_k(u)$, $T_k(\hat{u})$, and $T_k(u - \hat{u})$ belong to $W_0^{1,p}(\Omega)$, by Definition 2.9 and Theorem 9.2 we have by Lemma 2.12 (with $\lambda = -1$),

$$(\nabla u - \nabla \hat{u})\chi_{\{|u - \hat{u}| < M\}} = \nabla(u - \hat{u})\chi_{\{|u - \hat{u}| < M\}} = \nabla T_M(u - \hat{u}).$$

We deduce from (10.11) and from (9.3) that

$$\frac{1}{n} \int_{\{|u-\hat{u}|<n\}} |\nabla u - \nabla \hat{u}|^p dx = \frac{1}{n} \int_{\Omega} |\nabla T_M(u - \hat{u})|^p dx \leq \frac{2M}{n\alpha} |\mu|(\Omega) = \omega(n),$$

i.e. (10.9).

Assume now that $1 < p < 2$; we will prove that (10.8) holds, which implies the uniqueness result. Indeed by Hölder inequality (with $2/p > 1$) we have

$$\begin{aligned} & \frac{1}{n} \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx \\ &= \frac{1}{n} \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} \frac{|\nabla u - \nabla \hat{u}|^p}{(|\nabla u| + |\nabla \hat{u}|)^{(2-p)p/2}} (|\nabla u| + |\nabla \hat{u}|)^{(2-p)p/2} dx \\ &\leq \left(\frac{1}{n}\right)^{p/2} \left(\int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} \frac{|\nabla u - \nabla \hat{u}|^2}{(|\nabla u| + |\nabla \hat{u}|)^{2-p}} dx \right)^{p/2} \\ &\quad \times \left(\frac{1}{n} \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} (|\nabla u| + |\nabla \hat{u}|)^p dx \right)^{(2-p)/2}. \end{aligned}$$

Since $\{|u - \hat{u}| < M\} = \Omega$, we have in view of the strong monotonicity (10.4) and of (9.1),

$$\begin{aligned} & \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} \frac{|\nabla u - \nabla \hat{u}|^2}{(|\nabla u| + |\nabla \hat{u}|)^{2-p}} dx \\ &\leq \frac{1}{\alpha} \int_{\{|u-\hat{u}|<M\}} (a(x, \nabla u) - a(x, \nabla \hat{u})) \cdot (\nabla u - \nabla \hat{u}) dx \\ &\leq \frac{2M}{\alpha} |\mu|(\Omega). \end{aligned}$$

On the other hand, in view of (2.22)

$$\begin{aligned} \frac{1}{n} \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} (|\nabla u| + |\nabla \hat{u}|)^p dx &\leq \frac{2^{p-1}}{n} \int_{\Omega} (|\nabla T_n(u)|^p + |\nabla T_n(\hat{u})|^p) dx \\ &\leq \frac{2^p}{\alpha} |\mu|(\Omega). \end{aligned}$$

From these results we deduce that

$$\frac{1}{n} \int_{\left\{ \begin{smallmatrix} |u|<n \\ |\hat{u}|<n \end{smallmatrix} \right\}} |\nabla u - \nabla \hat{u}|^p dx \leq \left(\frac{1}{n}\right)^{p/2} C = \omega(n),$$

i.e., (10.8). □

REMARK 10.7. In the case where the measure μ belongs to $\mathcal{M}_0(\Omega)$ (and not only to $\mathcal{M}_b(\Omega)$), uniqueness of the renormalized solutions has been proved in [5], [23], [25], [26]. Let us emphasize that this uniqueness result holds for any renormalized solutions, and not only for “comparable” renormalized solutions.

If (10.5) holds, this uniqueness result for $\mu \in \mathcal{M}_0(\Omega)$ is actually a particular case of Theorem 10.4. Indeed when u and \hat{u} are renormalized solutions, we have

$$\frac{1}{n} \int_{\left\{ \begin{array}{l} |u| < 2n \\ |\hat{u}| < 2n \end{array} \right\}} |\nabla u - \nabla \hat{u}|^p dx \leq \frac{2^{p-1}}{n} \int_{\Omega} |\nabla T_n(u)|^p dx + \frac{2^{p-1}}{n} \int_{\Omega} |\nabla T_n(\hat{u})|^p dx .$$

Using (2.6) and (2.19) with the test function $w = T_n(u)$ (which corresponds to $w^{+\infty} = n$ and $w^{-\infty} = -n$) we have

$$\frac{1}{n} \int_{\Omega} |\nabla T_n(u)|^p dx \leq \frac{1}{\alpha n} \int_{\Omega} a(x, \nabla u) \cdot \nabla T_n(u) dx = \frac{1}{\alpha} \int_{\Omega} \frac{T_n(u)}{n} d\mu_0$$

since $\mu_s = 0$ in the present case. By Lebesgue theorem, we have

$$\int_{\Omega} \frac{T_n(u)}{n} d\mu_0 = \omega(n)$$

since $|T_n(u)/n| \leq 1$ and $T_n(u)/n$ tends to 0 μ_0 -almost everywhere. This result and the similar one for \hat{u} , imply (10.8) and therefore $u = \hat{u}$.

REMARK 10.8. In the special case $p = N$, uniqueness (as well as existence) has been recently proved in [16], [13], and [12]. In those papers, the authors prove the existence and uniqueness of a solution u in the sense of distributions, specifying the class E_N to which the solution belongs. In the papers [16] and [13], E_N is the “grand Sobolev space” $W_0^{1,N}(\Omega)$, defined by

$$W_0^{1,N}(\Omega) = \left\{ v \in W_0^{1,q}(\Omega) \ \forall q < N : \sup_{q < N} (N - q) \int_{\Omega} |\nabla v|^q dx < +\infty \right\} .$$

while in the paper [12] E_n is the space

$$E_N = \left\{ v \in W_0^{1,1}(\Omega) : \nabla v \in (L^{N,\infty}(\Omega))^N \right\} ,$$

where $L^{N,\infty}(\Omega)$ is the usual Marcinkiewicz space.

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