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Arithmetic Properties of the Cohomology of Artin Groups

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Abstract. In this paper we compute the cohomology of all Artin groups associated to finite Coxeter groups, with coefficients in the following module R_q : let $R := \mathbb{Q}[q, q^{-1}]$ be the ring of rational Laurent polynomials and let R_q be given by the action defined by mapping each standard generator to the multiplication by $-q$. Case A_n was already considered in a previous paper where the “cohomology table” has nice elementary arithmetic properties. Here also there are similar (more complicated) arithmetic properties for the infinite series, where the methods of proof are similar. For exceptional cases we used a suitable computer program.

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Introduction

Let $R := \mathbb{Q}[q, q^{-1}]$ be the ring of rational Laurent polynomials in one variable q . Let $Br(n)$ be the Artin braid group with n strings and let R_q be the $Br(n)$ -module given by the action over R defined by mapping each standard generator of $Br(n)$ to the multiplication by $-q$. (REMARK: this action coincides also with the determinant of the Burau representation of $Br(n)$). In [DPS] the cohomology of $Br(n)$ with coefficients in R_q was computed. All cohomology modules are cyclotomic fields (or zero) and the table $(H^i(Br(j); R_q))_{i,j}$ has a very interesting arithmetic behaviour.

Here we consider all other Artin' groups, associated to finite irreducible Coxeter groups. We compute the cohomology of these groups with coefficients in the same module R_q , where the action is still given by multiplication by $-q$. The results have a similar flavour to those obtained for the case A_n . (Instead of the cohomology of an Artin group, we prefer to speak of that of its *Coxeter graph*).

The methods of proof are similar to the ones developed in [DPS] (case A_n): we consider a filtration of subcomplexes of the algebraic complex, coming from [S] (see also [DS]), which computes the cohomology of Artin groups.

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The cohomologies of these subcomplexes are related in short exact sequences with the cohomology of Artin groups of type **A** and different ranks. We look at boundary operators in the associated long exact sequences and determine the conditions in order that such boundaries vanish. Then, starting from below, we use induction and results in [DPS] to recover the cohomology of the Artin group.

The exceptional cases are obtained by using a suitable computer program: we present here a table containing all their cohomology groups. This case is included here for completeness: a similar table was already given in [S] (we correct some misprints appearing there).

Let $\Phi_d(q)$ denote the cyclotomic polynomial of primitive d -roots of 1 and let

$$\{d\} := \mathbb{Q}[q, q^{-1}]/\Phi_d(q) = \mathbb{Q}[q]/\Phi_d(q)$$

be the associated cyclotomic field, thought as R -module.

THEOREM (case B_n).

$$H^n(B_n, R_q) = \bigoplus_{r|n} \{2r\}$$

and for $s > 0$

$$H^{n-2s+1}(B_n, R_q) = \bigoplus_{\substack{r \leq \frac{n}{2s}, \\ r|n}} \{2r\}$$

$$H^{n-2s}(B_n, R_q) = \bigoplus_{r \leq \frac{n-1}{2s}, r|n} \{2r\}$$

In case D_n the algebraic complex which computes the cohomology splits as a direct sum of two subcomplexes, which are the invariant and anti-invariant parts with respect to a suitable involution. The invariant subcomplex is isomorphic to the complex for type A_{n-1} . We obtain the following description.

For $n \in \mathbb{N}$, let $S_n := \{k \in \mathbb{N} : k|n \text{ or } k|2(n-1) \text{ but } k \nmid n-1\}$.

THEOREM (case D_n). *With the convention that*

$$\{h\} = \begin{cases} R/(\Phi_h) & \text{if } h \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$

we have that

$$H^n(D_n) = \bigoplus_{2h \in S_n} \{2h\}$$

and, for $s > 0$,

$$H^{n-2s}(D_n) = \bigoplus_{\substack{1 < h \leq \frac{n-2}{2s} \\ 2h \in S_n}} \{2h\} \oplus \left\{ \frac{n-1}{s} \right\}$$

$$H^{n-2s+1}(D_n) = \begin{cases} \bigoplus_{\substack{1 < h \leq \frac{n-2}{2s} \\ 2h \in S_n}} \{2h\} \oplus \left\{ \frac{n}{s} \right\} & \text{if } \frac{n}{s} \text{ is an even integer larger than 2} \\ \bigoplus_{\substack{1 < h \leq \frac{n-2}{2s} \\ 2h \in S_n}} \{2h\} \oplus \left\{ \frac{n}{s} \right\} & \text{otherwise.} \end{cases}$$

The following tables collect our results for type \mathbf{B}_n and \mathbf{D}_n , $n \leq 10$:

	\mathbf{B}_3	\mathbf{B}_4	\mathbf{B}_5	\mathbf{B}_6	\mathbf{B}_7	\mathbf{B}_8	\mathbf{B}_9	\mathbf{B}_{10}
H^1	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}
H^2	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}
H^3	{2}⊕{6}	{2}⊕{4}	{2}	{2}	{2}	{2}	{2}	{2}
H^4		{2}⊕{4}⊕{8}	{2}	{2}⊕{4}	{2}	{2}	{2}	{2}
H^5			{2}⊕{10}	{2}⊕{4}⊕{6}	{2}	{2}⊕{4}	{2}	{2}
H^6				{2}⊕{4}⊕{6}⊕{12}	{2}	{2}⊕{4}	{2}	{2}⊕{4}
H^7					{2}⊕{14}	{2}⊕{4}⊕{8}	{2}⊕{6}	{2}⊕{4}
H^8						{2}⊕{4}⊕{8}⊕{16}	{2}⊕{6}	{2}⊕{4}
H^9							{2}⊕{6}⊕{18}	{2}⊕{4}⊕{10}
H^{10}								{2}⊕{4}⊕{10}⊕{20}

	\mathbf{D}_2	\mathbf{D}_3	\mathbf{D}_4	\mathbf{D}_5	\mathbf{D}_6	\mathbf{D}_7	\mathbf{D}_8	\mathbf{D}_9	\mathbf{D}_{10}
H^1	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}	{2}
H^2	{2}	{3}	{3}	0	0	0	0	0	0
H^3		{4}	{4 ² }	{4}	{3}	{3}	0	0	0
H^4			{2}⊕{4}⊕{6}	{5}	{5}	0	0	{3}	{3}
H^5				{8}	{6 ² }	{4}⊕{6}	{4 ² }	{4}	0
H^6					{2}⊕{6}⊕{10}	{4}⊕{7}	{4}⊕{7}	0	0
H^7						{4}⊕{12}	{4}⊕{8 ² }	{8}	{5}
H^8							{2}⊕{4}⊕{8}⊕{14}	{9}	{6}⊕{9}
H^9								{16}	{6}⊕{10} ²
H^{10}									{2}⊕{6}⊕{10}⊕{18}

The integer cohomology of Artin groups for the series \mathbf{B} and \mathbf{D} was discussed in [G]; that of exceptional cases in [S].

Our results in case of infinite series agree with stability results which were found in [DS1].

The “top” cohomology of Artin groups was obtained in a more general form in [DSS].

1. – Notations and recollections

We recall here the main results of [DPS], generalizing the notations used there.

Let (\mathbf{W}, S) be an irreducible finite Coxeter system of rank m , with associated Coxeter graph $\Gamma := \Gamma_{\mathbf{W}}$. The vertices of Γ correspond to S , which we identify to $\{1, \dots, m\}$ using the standard correspondence of [Bo]. Let $G_{\mathbf{W}}$ be the Artin group of type \mathbf{W} (see [Br]); in case A_m , \mathbf{W} is the symmetric group S_{m+1} and $G_{\mathbf{W}}$ is the braid group $Br(m + 1)$. In general, if the standard presentation for \mathbf{W} is

$$\mathbf{W} = \langle s \in S \mid (s_i s_j)^{m(s_i, s_j)} = 1 \rangle$$

then standard presentation for $G_{\mathbf{W}}$ is

$$G_{\mathbf{W}} = \langle g_s, s \in S \mid g_{s_i} g_{s_j} g_{s_i} g_{s_j} \dots = g_{s_j} g_{s_i} g_{s_j} g_{s_i} \dots \rangle$$

where each member is a word of length $m(s_i, s_j)$ (see [Br], [D]).

Let \mathbf{R} be a commutative ring with 1 and let $q \in \mathbf{R}$ be a unit. Let \mathbf{R}_q be the $G_{\mathbf{W}}$ -module where the action is given by mapping each g_s to the multiplication by $-q$. In [S] an algebraic complex computing the cohomology of $G_{\mathbf{W}}$ with coefficients in \mathbf{R}_q was produced (see also [DS] for generalizations). Here we consider the case $\mathbf{R}_q = R_q$, where R_q is the module given in the introduction.

We think to the cohomology of $G_{\mathbf{W}}$ as that of its Coxeter diagram $\Gamma_{\mathbf{W}}$.

Let $C(\Gamma_{\mathbf{W}})$ be the above complex in case $\Gamma_{\mathbf{W}}$. In dimension k , $C^k = R^{(k)}$ is the free R -module with basis given by all subsets of S of cardinality k .

We indicate such subsets through their characteristic functions, as strings of 0, 1 of length m (which can be multiplied by juxtaposition). The degree $|s|$ of a string s is the number of its 1's.

Given a subset J of S , let $\mathbf{W}_J \subset \mathbf{W}$ be the parabolic subgroup generated by the elements in J . For the corresponding string $s(J)$, let us define

$$s(J)! := \sum_{w \in \mathbf{W}_J} q^{l(w)}$$

where $l(w)$ is the length of a reduced expression of w . Notice that $s(J)! = \prod_{j=1}^m s_j!$, where s_1, \dots, s_m are the strings corresponding to the connected components of the subgraph of $\Gamma_{\mathbf{W}}$ generated by J .

The coboundary $d : C^k \rightarrow C^{k+1}$ is defined by:

$$(1.1) \quad d(s(J)) = \sum_{j \in S \setminus J} (-1)^{\sigma(j, J)} \frac{s(J \cup \{j\})!}{s(J)!} s(J \cup \{j\})$$

The sign $\sigma(j, J)$ is defined as $|\{i \in J : i < j\}|$.

We remark the following computational rules deduced from the above formula:

$$(1.2) \quad d((A0)(0B)) = d(A0)0B + (-1)^{|A|} A0d(0B)$$

$$(1.3) \quad d(A101B) := d(A1)01B + (-1)^{|A|+1} A10d(1B) + (-1)^{|A|+1} \frac{A111B!}{A101B!} A111B$$

In order to make computations, we use the explicit expressions of $s(J)!$ in the different cases. Let us introduce the usual notations

$$[n] := \frac{q^n - 1}{q - 1}, \quad [n]! := \prod_{i=1}^n [i], \quad [n]!! := \prod_{0 \leq i < \frac{n}{2}} [n - 2i]$$

as the q -analogue of the numbers $n, n!, n!!$.

Type A

$$s(J)! = [m + 1]!$$

when the subgraph generated by J is of type A_m (see also [DPS]).

Type B

$$s(J)! = [2m]!!$$

when the subgraph generated by J is of type B_m .

Type D

$$s(J)! = [2(m - 1)]!![m]$$

when the subgraph generated by J is of type D_m .

To end this section, let us recall the main results in [DPS] for case A_m . Hereafter, we denote the complex of case A_m simply by C_m . Let $h \geq 2$ and set:

$$w_h := 01^{h-2}0, \quad dw_h = [h]z_h, \quad z_h := 1^{h-1}0 + (-1)^h 01^{h-1}, \\ b_h := 01^{h-2}, \quad db_h = [h]c_h, \quad c_h := 1^{h-1}.$$

By the rules above one has

$$d(w_h^i) = [h] \sum_{j=0}^{i-1} (-1)^{hj} w_h^j z_h w_h^{i-j-1} = [h]z_h(i), \\ d(w_h^{i-1}b_h) = [h] \left(\sum_{j=0}^{i-2} (-1)^{hj} w_h^j z_h w_h^{i-j-2} b_h + (-1)^{h(i-1)} w_h^{i-1} c_h \right) = [h]v_h(i),$$

where $z_h(i), v_h(i)$ are cocycles (see [DPS, Lemma 1.5]).

The main result in [DPS] is the following

THEOREM (case A_m).

$$(1.4) \quad H^{m-2i+1}C_m = \begin{cases} 0 & \text{if } h := \frac{m}{i} \text{ is not an integer.} \\ \{h\} \text{ generated by } [z_h(i)] & \text{if } h := \frac{m}{i} \text{ is an integer.} \end{cases}$$

$$(1.5) \quad H^{m-2(i-1)}C_m = \begin{cases} 0 & \text{if } h := \frac{m+1}{i} \text{ is not an integer.} \\ \{h\} \text{ generated by } [v_h(i)] & \text{if } h := \frac{m+1}{i} \text{ is an integer.} \end{cases}$$

2. – Case B_m

In this section we compute the cohomology of the complex $J_m := C(B_m)$, using formulas given in previous section. To point out the different behaviour of the last node of B_m , we box the last node of strings of type B .

In order to do induction we shall consider the following subcomplexes and maps.

Take the map

$$\pi_m := \pi : J_m \rightarrow C_{m-1}, \quad \pi(A\boxed{0}) := A; \quad \pi(A\boxed{1}) := 0$$

π is clearly a homomorphism of complexes with kernel the subcomplex L_m^1 having as basis all strings $A\boxed{1}$ ending with 1.

We continue in this way getting maps

$$\pi_m[k] := \pi : L_m^k \rightarrow C_{m-k-1}[k], \quad \pi(A01^{k-1}\boxed{1}) := A; \quad \pi(A1^k\boxed{1}) := 0$$

which induce isomorphisms of complexes:

$$L_m^k / L_m^{k+1} = C_{m-k-1}[k], \quad k = 0, \dots, m - 2$$

up to L_m^{m-1} which has as basis $b_{m+1} = 01^{m-2}\boxed{1}, 1^{m-1}\boxed{1} = c_{m+1} = v_{m+1}(1)$ with differential $db_{m+1} = \frac{[2m]!}{[2(m-1)]!}c_{m+1} = [2m]c_{m+1}$.

The starting point of the computation is of course the computation of the cohomology of L_m^{m-1} which is clearly 0 except for

$$(2.1) \quad H^m(L_m^{m-1}) = \mathbb{Q}[q, q^{-1}][v_{m+1}(1)] = R/([2m]) = \bigoplus_{h|2m} \{h\}$$

We need to do induction and thus it is again necessary to describe also the cohomology of the complexes L_m^k .

3. – Preparation for the Main Theorem

We use an argument similar to that used for type A_n in [DPS]. Consider the exact sequence of complexes

$$0 \rightarrow L_m^{k+1} \rightarrow L_m^k \rightarrow C_{m-k-1}[k] \rightarrow 0$$

and look at a typical piece of the corresponding long exact sequence:

$$(3.1) \rightarrow H^{t-1}C_{m-k-1}[k] \xrightarrow{\delta} H^tL_m^{k+1} \rightarrow H^tL_m^k \xrightarrow{\pi} H^tC_{m-k-1}[k] \rightarrow H^{t+1}L_m^{k+1} \rightarrow$$

We know the terms

$$H^{t-1}C_{m-k-1}[k] = H^{t-k-1}C_{m-k-1}, \quad H^tC_{m-k-1}[k] = H^{t-k}C_{m-k-1}$$

which are either 0 or irreducibles of type $\{h\}$, generated by the appropriate cohomology classes. Therefore we can analyze the change in the module structure from $H^tL_m^{k+1}$ to $H^tL_m^k$.

The main point is the computation of the connecting homomorphisms, that we separate in two cases:

- Case A) $H^{m-2s}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2s+1}L_m^{k+1}$.
- Case B) $H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2(s-1)}L_m^{k+1}$.

3.2 CASE A). We may assume that $H^{m-2s}C_{m-k-1}[k] \neq 0$ then, by induction,

$$H^{m-2s}C_{m-k-1}[k] = H^{m-2s-k}C_{m-k-1} = H^{(m-k-1)-2s+1}C_{m-k-1} = \{h\}$$

if and only if

$$sh = m - (k + 1), \quad h \text{ is an integer}$$

The module $\{h\}$ is irreducible and generated by the class of $z_h(s)$.

LEMMA 3.3. *If $h := \frac{m-(k+1)}{s}$ is an integer then, $\{h\} = H^{m-2s}C_{m-k-1}[k]$ generated by the class of $z_h(s)$ and:*

- a) *If $h|2(k + 1)$ (or $h|2m$) the map δ is equal to 0.*
- b) *If $h \nmid 2(k + 1)$, (or $h \nmid 2m$), the map δ is different from 0.*

PROOF. In order to compute the transform of $z_h(s)$ under the connecting homomorphism we lift $z_h(s)$ to $z_h(s)01^{k-1}\boxed{1} \in L_m^k$ and compute $d(z_h(s)01^{k-1}\boxed{1}) \in L_m^{k+1}$.

We perform the computation as follows: first, compare

$$[h]z_h(s)01^{k-1}\boxed{1} = d(w_h^s)01^{k-1}\boxed{1}$$

with $d(w_h^s)01^{k-1}\boxed{1}$. By formula (1.1) we have

$$d(w_h^s)01^{k-1}\boxed{1} = d(w_h^s)01^{k-1}\boxed{1} + (-1)^{hs}[2(k + 1)]w_h^s1^k\boxed{1}.$$

a) If $h|2(k + 1)$ then

$$d(z_h(s)01^{k-1}\boxed{1}) = (-1)^{hs+1} \frac{[2(k + 1)]}{[h]} d(w_h^s 1^k \boxed{1})$$

is a coboundary in L_m^{k+1} .

b) Assume now that $h \nmid 2(k + 1)$. We have

$$\begin{aligned} & d(w_h^s 1^k \boxed{1}) \\ &= \left\{ [h] \left(\sum_{j=0}^{s-2} (-1)^{hj} w_h^j z_h w_h^{s-j-1} 1^k \boxed{1} + (-1)^{h(s-1)} w_h^{s-1} 1^{h-1} 01^k \boxed{1} \right) \right. \\ & \quad \left. + (-1)^{hs} \frac{[2(h+k)]!!}{[h-1]![2(k+1)]!!} w_h^{s-1} 01^{h+k-1} \boxed{1} \right\} \\ &= \left\{ [h] z_h(s) 1^k \boxed{1} + (-1)^{hs} \left(\frac{[2(h+k)]!!}{[h-1]![2(k+1)]!!} - [h] \right) w_h^{s-1} 01^{h+k-1} \boxed{1} \right\}. \end{aligned}$$

Since h does not divide $2(k + 1)$ we have that Φ_h divides $\frac{[2(h+k)]!!}{[h-1]![2(k+1)]!!} = \frac{[2(k+h)] \dots [2(k+2)]}{[h-1]!}$, hence we can define the cocycle

$$z_{h,k+1}(s) := \Phi_h^{-1} d(w_h^s 1^k \boxed{1}) \in L_m^{k+1}$$

and

$$\frac{[h]}{\Phi_h} d(z_h(s)01^{k-1}\boxed{1}) = (-1)^{hs+1} [2(k + 1)] z_{h,k+1}(s)$$

We claim that $[2(k + 1)] z_{h,k+1}(s)$ is a non zero element in $H^{m-2s+1}(L_m^{k+1})$.

The cohomology class of $z_{h,k+1}(s)$ is non zero since it projects to $\frac{[h]}{\Phi_h}$ times the cohomology class of $v_h(s)$ in $H^{m-2s+1} C_{m-(k+1)-1}[k]$, which is non 0 and generates a module isomorphic to $\{h\}$.

Therefore the claim follows since $h \nmid 2(k + 1)$ and the elements $[2(k + 1)]$, $[h]/\Phi_h$ are not zero in $\{h\}$. □

3.4 CASE B). We assume that $H^{m-2(s-1)-1} C_{m-k-1}[k] \neq 0$ then:

$$H^{m-2(s-1)-1} C_{m-k-1}[k] = H^{(m-k-1)-2(s-1)} C_{m-k-1} = \{h\}$$

if and only if $sh = m - k = (m - k - 1) + 1$, h is an integer.

The module $\{h\}$ is irreducible and generated by the class of $v_h(s)$.

LEMMA 3.5. If $h := \frac{m-k}{s}$ is an integer so that $H^{m-2(s-1)-1} C_{m-k-1}[k] = \{h\}$:

- a) If h is even or it is odd and does not divide k , the map δ is equal to 0.
- b) If h is odd and divides k (or divides m) the map δ is different from 0.

PROOF. As we have done in Lemma 3.3, in order to compute the image of the cohomology class of $v_h(s)$ under the connecting homomorphism:

$$H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2(s-1)}L_m^{k+1}$$

we lift $v_h(s)$ to $v_h(s)01^{k-1}\boxed{1} \in L_m^k$ and compute $d(v_h(s)01^{k-1}\boxed{1}) \in L_m^{k+1}$.

We have

$$d(w_h^s 1^{k-1}\boxed{1}) = \left\{ [h]v_h(s)01^{k-1}\boxed{1} + (-1)^{hs} \frac{[2(h+k-1)]!!}{[h-1]![2k]!!} w_h^{s-1} 01^{h+k-2}\boxed{1} \right\}$$

a) If h does not divide $2k$ then Φ_h divides $\frac{[2(h+k-1)]!!}{[h-1]![2k]!!}$ and so $\frac{[h]}{\Phi_h} v_h(s)$ lifts in L_m^k to the cocycle $\Phi_h^{-1}d(w_h^s 1^{k-1}\boxed{1})$ and thus $\delta = 0$ ($\frac{[h]}{\Phi_h}$ is different from 0 in the irreducible $\{h\}$).

b) If h divides $2k$ thus we can write $v_h(s)01^{k-1}\boxed{1}$ as

$$[h]^{-1} \left\{ d(w_h^s 1^{k-1}\boxed{1}) - (-1)^{hs} \frac{[2(h+k-1)]!!}{[h-1]![2k]!!} w_h^{s-1} 01^{h+k-2}\boxed{1} \right\}.$$

When we apply the differential to this lift we get up to a sign

$$\begin{aligned} & [h]^{-1} \frac{[2(h+k-1)]!!}{[h-1]![2k]!!} d(w_h^{s-1} 01^{h+k-2}\boxed{1}) \\ &= \frac{[2(h+k-1)]!!}{[h-1]![2k]!!} \left\{ z_h(s-1)01^{h+k-2}\boxed{1} \pm \frac{[2(h+k)]}{[h]} w_h^{s-1} 1^{h+k-1}\boxed{1} \right\} \end{aligned}$$

We set

$$v_{h,k+1}(s) := z_h(s-1)01^{h+k-2}\boxed{1} \pm \frac{[2(h+k)]}{[h]} w_h^{s-1} 1^{h+k-1}\boxed{1}$$

Observe that this element lies in L^{h+k-1} and its image in $C_{m-(h+k-1)-1} = C_{m-(h+k)}$ is exactly $z_h(s-1)$. The element $z_h(s-1)$ gives a non 0 cohomology class in the module $H^{(s-1)(h-2)+1}C_{m-(h+k)-1}$ generating a submodule isomorphic to $\{h\}$. Hence the cohomology class of $v_{h,k+1}(s)$ in $H^{m-2s-1}L^{h+k-1}$ is non zero and the module it generates has a factor isomorphic to $\{h\}$.

In order to show that $z_h(s-1)$ gives a non 0 cohomology class in $H^{m-2(s-1)}L^{k+1}$ we prove that the contribution of the factor $\{h\}$ to $H^{m-2(s-1)}L_m^{h+k-1}$ does not disappear when passing to $H^{m-2(s-1)}L_m^{k+1}$.

For this let us remark that whenever we pass from a level $r+1$ of the filtration to r we have the exact sequence:

$$H^{m-2s+1}C_{m-r-1}[r] \xrightarrow{\delta} H^{m-2(s-1)}L_m^{r+1} \rightarrow H^{m-2(s-1)}L_m^r \xrightarrow{\pi} H^{m-2(s-1)}C_{m-r-1}[r]$$

so an irreducible factor in $H^{m-2s}L_m^{r+1}$ of type $\{h\}$ can disappear in $H^{m-2s}L_m^r$ only if $H^{m-2s-1}C_{m-r-1}[r] = \{h\}$ and $\delta \neq 0$. In particular only when $h := \frac{m-r}{s}$, which is not our case. Thus in the inclusion $H^{m-2(s-1)}L_m^{h+k-1} \xrightarrow{\pi} H^{m-2(s-1)}L_m^{k+1}$ the factor $\{h\}$ is not cancelled.

In fact we have shown that, in $H^{m-2(s-1)}L_m^{k+1}$, the element $[z_h(s-1)]$ generates a submodule with a factor of type $\{h\}$.

To finish we verify that if h is odd then $\frac{[2(h+k-1)]!}{[h-1]![2k]!} \neq 0$ in $\{h\}$.

In fact

$$\frac{[2(h+k-1)]!}{[h-1]![2k]!} = \frac{[2(k+h-1)][2(k+h-2)] \dots [2(k+1)]}{[h-1]!}$$

and since h divides $2k$ it must divide k and its next multiple is $k+h$ and so the claim follows. Assume now $h = 2r$ and r divides k , the factor $2(k+r)$ gives 0 in $\{h\}$ finishing the proof of the Lemma. \square

Summarizing:

Case A) $H^{m-2s}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2s+1}L_m^{k+1}$.

If $h := \frac{m-(k+1)}{s}$ is an integer, then $\{h\} = H^{m-2s}C_{m-k-1}[k]$ and

- If $h|2m$ the map δ is equal to 0.
- If $h \nmid 2m$, the map δ is different from 0.

Case B) $H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2(s-1)}L_m^{k+1}$.

If $h := \frac{m-k}{s}$ is an integer, then $H^{m-2(s-1)-1}C_{m-k-1}[k] = \{h\}$ and

- If h is even or it is odd and does not divide m , the map δ is equal to 0.
- If h is odd and divides m the map δ is different from 0.

Consider

$$\begin{aligned} H^{m-2s}C_{m-k-1}[k] &\xrightarrow{\delta} H^{m-2s+1}L_m^{k+1} \rightarrow \\ H^{m-2s+1}L_m^k &\xrightarrow{\pi} H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta'} H^{m-2(s-1)}L_m^{k+1} \end{aligned}$$

and $A(s, k)$ be the class of $H^{m-2s+1}L_m^k$ in the Grothendieck group; then:

If $h := \frac{m-(k+1)}{s}$ is an integer and does not divide $2m$ then passing from $A(s, k+1)$ to $A(s, k)$ we have that $\{h\}$ is deleted.

If $h := \frac{m-k}{s}$ is an integer and h is even or it is odd and does not divide m , passing from $A(s, k+1)$ to $A(s, k)$ we have that $\{h\}$ is added.

Conclusion:

i) h odd and h does not divide m : if $m - hs \geq k$ then $\{h\}$ has been added but if $m - hs \geq k + 1$ then $\{h\}$ has been also deleted so there remain only the odd h which do not divide m and with $m - hs = k$ or $h := \frac{m-k}{s}$ if the latter is an odd integer and $h \nmid m$.

ii) h even: then $m - sh \geq k$ implies that $\{h\}$ is added but if h does not divide $2m$ then if $m - gh \geq k + 1$ it has also been deleted.

So add all even $g \leq \frac{m-k}{s}$ and $g|2m$ and all even g with $g \nmid 2m$ and $\frac{m-(k+1)}{s} < g \leq \frac{m-k}{s}$ thus $g = \frac{m-k}{s}$ if even and does not divide $2m$. It follows:

$$H^{m-2s+1}L_m^k = \begin{cases} \bigoplus_{r \leq \frac{m-k}{2s}, r|m} \{2r\} & \text{unless } h := \frac{m-k}{s} \text{ is an integer and } h \nmid m \\ \bigoplus_{r \leq \frac{m-k}{2s}, r|m} \{2r\} \oplus \{h\} & \text{if } h := \frac{m-k}{s} \text{ is an integer and } h \nmid m \end{cases}$$

Consider now

$$\begin{aligned} H^{m-2(s-1)-1}C_{m-k-1}[k] &\xrightarrow{\delta} H^{m-2(s-1)}L_m^{k+1} \rightarrow \\ H^{m-2(s-1)}L_m^k &\xrightarrow{\pi} H^{m-2(s-1)}C_{m-k-1}[k] \xrightarrow{\delta'} H^{m-2(s-1)+1}L_m^{k+1} \end{aligned}$$

Let $B(s, k)$ be the class of $H^{m-2(s-1)}L_m^k$ in the Grothendieck group; then:

If $h := \frac{m-k}{s}$ is an integer, h is odd and divides m , we have that $\{h\}$ is deleted passing from $B(s, k+1)$ to $B(s, k)$.

If $g := \frac{m-(k+1)}{s-1}$ is an integer if $g|2m$, from $B(s, k+1)$ to $B(s, k)$ we have that $\{g\}$ is added.

Conclusion:

$h|m$ odd: if $m - h(s-1) \geq k+1$ then $\{h\}$ has been added, but if $m - hs \geq k$ then $\{h\}$ has been also deleted; so it remains only the odd $h|m$ with $m - hs = k$ or $h := \frac{m-k}{s}$ if this is an odd integer.

The odd divisors of m which appear in $B(s, k)$ are subject to

$$\frac{m-k}{s} < h \leq \frac{m-(k+1)}{s-1}.$$

The even divisors of $2m$ which appear in $B(s, k)$ are subject to

$$m - (s-1)g \geq k + 1.$$

Therefore:

$$H^{m-2(s-1)}L_m^k = \bigoplus_{r \leq \frac{m-(k+1)}{2(s-1)}, r|m} \{2r\} \quad \bigoplus_{\frac{m-k}{s} < 2r+1 \leq \frac{m-(k+1)}{s-1}, 2r+1|m} \{2r+1\}$$

For $s = 1$ nothing is ever added but only the odd divisors h of m with $m - hs \geq k$ have been deleted.

4. – The Theorem

From previous discussion we get the main theorem in case B_m .

THEOREM.

$$(A) \quad H^{m-2s+1} L_m^k = \begin{cases} \bigoplus_{r \leq \frac{m-k}{2s}, r|m} \{2r\} & \text{unless } h := \frac{m-k}{s} \text{ is an integer and } h \nmid m \\ \bigoplus_{r \leq \frac{m-k}{2s}, r|m} \{2r\} \oplus \{h\} & \text{if } h := \frac{m-k}{s} \text{ is an integer and } h \nmid m \end{cases}$$

$$(B) \quad H^{m-2(s-1)} L_m^k = \bigoplus_{r \leq \frac{m-(k+1)}{2(s-1)}, r|m} \{2r\} \quad \bigoplus_{\frac{m-k}{s} < 2r+1 \leq \frac{m-(k+1)}{s-1}, 2r+1|m} \{2r+1\}$$

For $k = 0$, we have:

$$\begin{aligned} H^{m-2s+1}(B_m) &= \bigoplus_{r \leq \frac{m}{2s}, r|m} \{2r\} \\ H^{m-2(s-1)}(B_m) &= \bigoplus_{r \leq \frac{m-1}{2(s-1)}, r|m} \{2r\} \\ H^m(B_m) &= \bigoplus_{r|m} \{2r\} \end{aligned}$$

5. – Case D_m

We now consider the complex $C(D_m)$ of Section 1. We define an involution of this complex as follows:

$$\begin{aligned} \sigma(A00) &= A00 & \sigma(A11) &= -A11 \\ \sigma(A01) &= A10 & \sigma(A10) &= A01 \end{aligned}$$

It is easy to see that this involution is compatible with the differential, so that

$$C(D_m) = K_m \oplus P_m$$

where K_m is the sub-complex of the σ -invariant elements while P_m is that of the σ -antiinvariant elements.

A basis for K_m is clearly given by all elements of the form $\{1/2(A10 + A01), A00\}$ where A is any string supported in $\{1, \dots, m-2\}$. We have

PROPOSITION. *The unique linear map $p : K_m \rightarrow C_{m-1}$ such that*

$$p(A00) = A0, \quad \text{and} \quad p\left(\frac{1}{2}(A10 + A01)\right) = A1.$$

is an isomorphism of complexes. In particular, for all $s > 0$,

$$H^{m-2s} K_m \sim H^{m-2s} C_{m-1} = \begin{cases} \left\{ \frac{m-1}{s} \right\} & \text{if } s \text{ divides } m-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^{m-2s+1} K_m \sim H^{m-2s+1} C_{m-1} = \begin{cases} \left\{ \frac{m}{s} \right\} & \text{if } s \text{ divides } m \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The proposition follows immediately from the definitions and from Theorem 1.1. □

Since we have that $H^*C(D_m) = H^*K_m \oplus H^*P_m$ it remains to compute the cohomology of P_m . A basis for P_m is given by the elements $\{1/2(A10 - A01), A11\}$ where A is any string supported in $\{1, \dots, m-2\}$.

We now filter P_m by setting $G_m^1 = P_m$, and, if $i = 2, \dots, m-1$, G_m^i equal to the span of the elements $A1^i$, with A any string supported in $1, \dots, m-i$. Let us now recall that in [DPS] we have denoted by F_{m-1}^1 the kernel of the map $\pi : C_{m-1} \rightarrow C_{m-2}$ defined by $\pi(A0) = A$ and $\pi(A1) = 0$ for each A supported in $\{1, \dots, m-2\}$.

We now consider the map $\gamma : G_m^1 \rightarrow F_{m-1}^1$ by $\gamma(A11) = 0$, $\gamma(1/2(A10 - A01)) = A1$. It is easy to see that γ is a map of complexes whose kernel is G_m^2 so that we get an exact sequence

$$0 \rightarrow G_m^2 \rightarrow G_m^1 \xrightarrow{\gamma} F_{m-1}^1 \rightarrow 0$$

If $k \geq 2$, we have a map $\gamma = \gamma_k : G_m^k \rightarrow C_{m-k-1}[k]$ given by $\gamma(A01^k) = A$, $\gamma(A1^{k+1}) = 0$, whose kernel is G_m^{k+1} so that we get an exact sequence

$$0 \rightarrow G_m^{k+1} \rightarrow G_m^k \xrightarrow{\gamma} C_{m-k-1}[k] \rightarrow 0$$

Finally, G_m^{m-1} is the complex

$$\delta : R \rightarrow R$$

translated by $m-1$, where δ is the multiplication by $\frac{[2(m-1)][m]}{[m-1]}$.

This last fact gives the starting point of our computations i.e.

$$H^t G_m^{m-1} = \begin{cases} \frac{R}{\left(\frac{[2(m-1)][m]}{[m-1]}\right)} & \text{if } t = m \\ 0 & \text{otherwise.} \end{cases}$$

6. – Preparation for the main theorem

Our aim in this section is to compute the cohomology of $P_m = G_m^1$. To do this we shall compute the cohomology of each of the complexes G_m^k proceeding by reverse induction on k . We already know the cohomology of G_m^{m-1} , so that we have the starting point of our induction.

We start by considering the case $k > 1$. We have an exact sequence

$$0 \rightarrow G_m^{k+1} \rightarrow G_m^k \xrightarrow{\gamma} C_{m-k-1}[k] \rightarrow 0$$

From this sequence we get a long exact sequence, of which we need to compute the connecting homomorphism, to be able to deduce the cohomology of G_m^k .

We split this in two cases:

$$\text{Case A) } \delta : H^{m-2s}(G_m^{k+1}/G_m^k) \rightarrow H^{m-2s+1}(G_m^{k+1})$$

$$\text{Case B) } \delta : H^{m-2(s-1)-1}(G_m^{k+1}/G_m^k) \rightarrow H^{m-2(s-1)}(G_m^{k+1})$$

Given a positive integer n , we let S_n be the set of integers which are either divisors of n or of $2(n-1)$ but not of $n-1$. In other terms S_n is the set of integers h such that the cyclotomic polynomial Φ_h divides $\frac{[2(n-1)][n]}{[n-1]}$.

In order to compute the cohomology of the complexes G_m^k , we need to compute some coboundaries. Recall that for $1 < k < m-1$, we have an exact sequence

$$0 \rightarrow G_m^{k+1} \rightarrow G_m^k \rightarrow C_{m-k-1}[k] \rightarrow 0$$

Also we have that $H^{m-2s}C_{m-k-1}[k]$ is non zero if and only if s is a proper divisor of $m-k-1$. If this is the case, setting $h = \frac{m-k-1}{s}$, we have $H^{m-2s}C_{m-k-1}[k] = \{h\}$.

LEMMA 6.1. *Let $k > 1$. Consider*

$$d : H^{m-2s}C_{m-k-1}[k] \rightarrow H^{m-2s+1}G_m^{k+1}.$$

Then $d \neq 0$ if and only if the following conditions hold

- 1) s is a proper divisor of $m-k-1$ so that $H^{m-2s}C_{m-k-1}[k] = \{h\}$ with $h = \frac{m-k-1}{s}$.
- 2) h does not lie in S_m .

PROOF. We have already recalled that $H^{m-2s}C_{m-k-1}[k] \neq 0$ if and only if condition 1) is satisfied. If this is the case $H^{m-2s}C_{m-k-1}[k]$ is isomorphic to $\{h\}$ and generated by the class of $z_h(s)$.

So, assuming condition 1), lift $z_h(s)$ to $z_h(s)01^k \in G_m^k$. We have $[h]z_h(s)01^k = d(w_h^s)01^k$. Now

$$d(w_h^s)01^k = d(w_h^s)01^k + (-1)^{hs} \frac{[2k][k+1]}{[k]} w_h^s 1^{k+1}$$

from which

$$[h]d(z_h(s)01^k) = (-1)^{hs+1} \frac{[2k][k+1]}{[k]} d(w_h^s 1^{k+1}).$$

If $h \in S_m$, then since h divides $m - k - 1$, $h \in S_{k+1}$ i.e. $\Phi_h | \frac{[2k][k+1]}{[k]}$ so that the element $\Phi_h^{-1}[h]d(z_h(s)01^k)$ is a coboundary in G_m^{k+1} and $d = 0$.

Assume now $h \notin S_m$. Again $h \notin S_{k+1}$. The equality

$$[h]d(z_h(s)01^k) = (-1)^{hs+1} \frac{[2k][k+1]}{[k]} d(w_h^s 1^{k+1})$$

implies that it is enough to show that the class of the element $[h]^{-1}d(w_h^s 1^{k+1})$ is non zero in $H^{m-2s+1}G_m^{k+1}$. Working modulo G_m^{k+2} , we see that it suffices to see that the class of the image of $[h]^{-1}d(w_h^s 1^{k+1})$ in $G_m^{k+1}/G_m^{k+2} = C_{m-(k+1)-1}[k+1]$ is non zero. A simple direct computation shows that this class equals the class of $v_h(s)$ which we know to be non zero, proving our claim. \square

We now pass to case B) and analyze the coboundary

$$d : H^{m-2(s-1)-1}C_{m-k-1}[k] \rightarrow H^{m-2(s-1)}G_m^{k+1}.$$

Set Q_m equal to the set of divisors of m which are either odd or equal to 2. We have

LEMMA 6.2. *Let $k > 1$. Consider*

$$d : H^{m-2(s-1)-1}C_{m-k-1}[k] \rightarrow H^{m-2(s-1)}G_m^{k+1}.$$

Then $d \neq 0$ if and only if the following conditions hold

- 1) s is a proper divisor of $m - k$ so that $H^{m-2(s-1)-1}C_{m-k-1}[k] = \{h\}$ with $h = \frac{m-k}{s}$.
- 2) $h \in Q_m$.

PROOF. One has that $H^{m-2(s-1)-1}C_{m-k-1}[k]$ is non zero if and only if our condition 1) is satisfied. Furthermore if this is the case, $H^{m-2(s-1)-1}C_{m-k-1}[k] = \{h\}$ and it is generated by the class of $v_h(s)$. So from now on assume that condition 1) is satisfied.

Lift $v_h(s)$ to $v_h(s)01^k \in G_m^k$ then $w_h^s 1^k = w_h^{s-1}b_h 01^k$ and

$$\begin{aligned} d(w_h^s 1^k) &= d(w_h^{s-1}b_h 01^k) = d(w_h^{s-1}b_h)01^k + (-1)^{hs} \gamma w_h^{s-1} 01^{k+h-1} \\ &= [h]v_h(s)01^k + (-1)^{hs} \gamma w_h^{s-1} 01^{k+h-1} \end{aligned}$$

where

$$\gamma = \binom{k+h-1}{k} \frac{[2(k+h-2)]!!}{[k+h-2]}.$$

It is easy to see that $\Phi_h | \gamma$ if and only if $h \notin Q_m$. If this is the case, then

$$\Phi_h^{-1}[h]d(v_h(s)01^k) = (-1)^{hs+1} \Phi_h^{-1} \gamma d(w_h^{s-1} 01^{k+h-1})$$

so that, as in Lemma 6.1, $d = 0$.

So assume $h \in \mathcal{Q}_m$, so that Φ_h does not divide γ .

$$\begin{aligned} [h]d(v_h(s)01^k) &= (-1)^{hs+1} \gamma d(w_h^{s-1}01^{k+h-1}) \\ &= (-1)^{hs+1} \gamma \left\{ [h]z_h(s-1)01^{k+h-1} + (-1)^{h(s-1)} \frac{[2(k+h-1)][k+h]}{[k-h-1]} w_h^{s-1} 1^{k+h} \right\}. \end{aligned}$$

Remark now that one has that Φ_h divides $\gamma' = \gamma \frac{[2(k+h-1)][k+h]}{[k+h-1]}$, so that

$$\Phi_h^{-1}[h]d(v_h(s)01^k) = (-1)^{hs+1} \gamma \frac{[h]}{\Phi_h} z_h(s-1)01^{k+h-1} + (-1)^h \frac{\gamma'}{\Phi_h} w_h^{s-1} 1^{k+h}.$$

This element lies G_m^{k+h-1} and its class in $H^{(s-1)(h-2)+1} C_{m-(k+h-1)-1} = \{h\}$ equals, up to sign, $\Phi_h^{-1}[h]\gamma$ times the class of the generator $z_h(s-1)$. ince by Φ_h does not divide γ , we deduce that the cohomology class of $\Phi_h^{-1}[h]d(v_h(s)01^k)$ does not vanish in $H^{m-2(s-1)} G_m^{k+h-1}$ and generates a module isomorphic to $\{h\}$. Let us see that this submodule does not disappear passing from $H^{m-2(s-1)} G_m^{k+h-1}$ to $H^{m-2(s-1)} G_m^{k+1}$. Consider the sequence

$$H^{m-2s+1} C_{m-r-1}[r] \rightarrow H^{m-2(s-1)} G_m^{r+1} \rightarrow H^{m-2(s-1)} G_m^r$$

with $k+h-1 \geq r \geq k+1$.

Since we know that

$$H^{m-2s+1} C_{m-r-1}[r]$$

is either 0 or, if s divides $m-r$, equals to $\{\frac{m-r}{s}\}$, we deduce that our submodule in $H^{m-2(s-1)} G_m^{k+h-1}$ isomorphic to $\{h\}$ cannot be canceled when passing to $H^{m-2(s-1)} G_m^{k+1}$. This implies that $d \neq 0$, proving our claim. \square

Before we proceed let us recall that by our definitions

$$(6.3) \quad H^t G_m^{m-1} = \begin{cases} \frac{R}{\left(\frac{[2(m-1)]}{[m-1]}[m]\right)} & \text{if } t = m \\ 0 & \text{otherwise.} \end{cases}$$

Notice also that

$$\frac{R}{\left(\frac{[2(m-1)]}{[m-1]}[m]\right)} = \begin{cases} \bigoplus_{h \in S_m} \{h\} & \text{if } m \text{ is odd} \\ \bigoplus_{h \in S_m} \{h\} \oplus \{2\} & \text{if } m \text{ is even.} \end{cases}$$

We can use now the two lemmas above to compute the groups $H^* G_m^{m-k}$ for $1 \leq k \leq m-2$. We have

PROPOSITION 6.4. *Let $1 \leq s \leq \frac{m-1}{2} + 1$ and $1 \leq k \leq m - 2$. Then*

$$H^{m-2(s-1)-1}G_m^{m-k} = \begin{cases} \bigoplus_{\substack{1 < h \leq \frac{k-1}{2s} \\ h \in S_m}} \{2h\} \oplus \left\{ \frac{k}{s} \right\} & \text{if } s \text{ divides } k \text{ and } \frac{k}{s} \notin Q_m \\ \bigoplus_{\substack{1 < h \leq \frac{k-1}{2s} \\ h \in S_m}} \{2h\} & \text{otherwise.} \end{cases}$$

PROOF. By 6.3 we have our result for $k = 1$. Set

$$C_{s,k} = H^{m-2(s-1)-1}G_m^{m-k} / \text{Im} \left(d : H^{m-2s}C_k[m - (k+1)] \rightarrow H^{m-2(s-1)-1}G_m^{m-k} \right),$$

$$K_{s,k} = \text{Ker} \left(d : H^{m-2(s-1)-1}C_k[m - (k+1)] \rightarrow H^{m-2(s-1)}G_m^{m-k} \right).$$

We have an exact sequence

$$0 \rightarrow C_{s,k} \rightarrow H^{m-2(s-1)-1}G_m^{m-(k+1)} \rightarrow K_{s,k} \rightarrow 0$$

Our inductive hypothesis and Lemma 6.2 easily imply that

$$C_{s,k} = \bigoplus_{\substack{1 < h \leq \frac{k}{2s} \\ h \in S_m}} \{2h\}.$$

hand Lemma 6.3 implies that

$$K_{s,k} = 0$$

unless s is a proper divisor of $k+1$ and $\frac{k+1}{s}$ does not lie in Q_m . Assume this is the case. The module $\left\{ \frac{k}{s} \right\}$ is irreducible and does not appear among the irreducible factors of $C_{s,k}$. It follows that the sequence

$$0 \rightarrow C_{s,k} \rightarrow H^{m-2(s-1)-1}G_m^{m-(k+1)} \rightarrow K_{s,k} \rightarrow 0$$

splits in all cases and this together with the above considerations clearly implies our claim.

For $H^{m-2s}G_m^{m-k}$ we have

PROPOSITION 6.5. *Assume $2 \leq k \leq m - 2$, then*

$$H^m G_m^{m-k} = \bigoplus_{\substack{1 < h \leq \frac{k}{2} \\ 2h \in S_m}} \{2h\} \oplus \bigoplus_{\substack{h > k \\ h \in S_m}} \{h\}$$

Let $1 \leq s \leq \frac{m}{2}$. Then

$$H^{m-2s}G_m^{m-k} = \bigoplus_{\substack{1 < h \leq \frac{k}{2(s+1)} \\ 2h \in m}} \{2h\} \oplus \bigoplus_{\substack{\frac{k}{s+1} < h < \frac{k}{s} \\ h \in S_m}} \{h\}$$

PROOF. One obtains the proof of this proposition using the computation of the groups $H^1G_m^{m-1}$ and Lemmas 6.2 and 6.3, in a fashion similar to that of Proposition 6.4. We leave the details to the reader. \square

It remains to compute $H^i G_m^1$, by using the exact sequence

$$(6.6) \quad 0 \rightarrow G_m^2 \rightarrow G_m^1 \rightarrow F_{m-1}^1 \rightarrow 0.$$

Let us recall that we have $H^m F_{m-1}^1 = 0$ and, for $s > 0$

$$H^{m-2s} F_{m-1}^1 = \{h\} \text{ if } h = \frac{m-2}{s} \text{ is an integer, } h > 2$$

$$H^{m-2s+1} F_{m-1}^1 = \{h\} \text{ if } h = \frac{m}{s} \text{ is an integer, } h > 2$$

(the other cohomology groups vanish).

In the first case, $\{h\}$ is generated by the cohomology class of

$$d(z_h(s)0) = (-1)^{hs+1} [2]z_h(s)1 - ([h+1] - [2])w_h^{s-1}01^h,$$

while in the second one it is generated by $v_h(s)$.

On the other hand the cohomology of G_m^2 is given by

$$H^m G_m^2 = \begin{cases} \bigoplus_{2h \in S_m} \{2h\} \oplus \{m\} & \text{if } m \text{ is odd} \\ \bigoplus_{2h \in S_m} \{2h\} & \text{otherwise;} \end{cases}$$

$$H^{m-2s+1} G_m^2 = \begin{cases} \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} \oplus \left\{ \frac{m-2}{s} \right\} & \text{if } \frac{m-2}{s} \text{ is an integer larger than } 2 \\ \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} & \text{otherwise} \end{cases}$$

$$H^{m-2s} G_m^2 = \begin{cases} \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} \oplus \left\{ \frac{m}{s+1} \right\} & \text{if } \frac{m}{s+1} \text{ is an odd integer} \\ \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} & \text{otherwise.} \end{cases}$$

The first statement is clear. As for the second, in order to deduce it from Proposition 6.4 we have to see that there is no integer $h > 1$ with $\frac{m-3}{2s} < h \leq \frac{m-2}{2s}$ and $2h \in S_m$. Indeed, if $2h = \frac{m}{t}$, it is clear that $t > s$, so that we have $\frac{m-3}{2s} < h \leq \frac{m}{2(s+1)}$. This implies $m - 3(s+1) < 0$, hence $h < 3/2$, a contradiction. The case in which $2h$ is a divisor of $2(m-1)$ (but not of $m-1$) is completely analogous and in the same way one deduces our third statement from Proposition 6.5. We leave the details to the reader.

LEMMA 6.7. *The coboundary*

$$d : H^{m-2s} F_{m-1}^1 \rightarrow H^{m-2s+1} G_m^2$$

is always injective.

PROOF. It is clear from the above that we can assume that $h = \frac{m-2}{s}$ is an integer and that $h > 2$, so that $H^{m-2s} F_{m-1}^1 = \{h\}$ is generated by the class of

$$(-1)^{hs+1} [2]z_h(s)1 - ([h+1] - [2])w_h^{s-1}01^h.$$

We lift this class in G_m^1 to

$$\begin{aligned} & \frac{1}{2}((-1)^{hs+1} [2]z_h(s)(10-01) - ([h+1] - [2])w_h^{s-1}01^{h-1}(10-01)) \\ &= \frac{1}{2}((-1)^{hs+1} [2]z_h(s-1)w_h(10-01) + (-1)^{h+1} [2]w_h^{s-1}1^{h-1}0(10-01) \\ & \quad - [2]w_h^{s-1}01^{h-1}(10-01) - ([h+1] - [2])w_h^{s-1}01^{h-1}(10-01)) \\ &= \frac{1}{2}((-1)^{hs+1} [2]z_h(s-1)w_h(10-01) + (-1)^{h+1} [2]w_h^{s-1}1^{h-1}0(10-01) \\ & \quad - [h+1]w_h^{s-1}01^{h-1}(10-01)) \end{aligned}$$

Since the boundary of this chain lies in G_m^2 , in order to compute it it suffices to determine the components which end by 11. One obtains for the boundary the expression:

$$\begin{aligned} & - [2]^2 z_h(s-1)w_h 11 + (-1)^{h(s-1)+1} [2]^2 w_h^{s-1} 1^{h-1} 011 \\ & \quad + (-1)^{hs+1} \frac{[2h]!!}{[r]!} [h+1] w_h^{s-1} 01^{h-1} 11 \\ &= - [2]^2 v_h(s) 011 + (-1)^{hs+1} \frac{[2h]!!}{[r]!} [h+1] w_h^{s-1} 01^{h-1} 11 \end{aligned}$$

The class of this element maps under $H^{m-2s+1} G_m^2 \rightarrow H^{m-2s+1} C_{m-3}[2]$ on the class of $-[2]^2 v_h(s)$ which is a generator of $H^{m-2s+1} C_{m-3}[2] = \{h\}$. So $d \neq 0$. Since the R module $H^{m-2s} F_{m-1}^1 = \{h\}$ is irreducible we deduce that d is injective. □

Let us consider now the boundary

$$d : H^{m-2s+1} F_{m-1}^1 \rightarrow H^{m-2s+2} G_m^2$$

LEMMA 6.8. *The coboundary*

$$d : H^{m-2s+1} F_{m-1}^1 \rightarrow H^{m-2s+2} G_m^2$$

does not vanish iff $h = \frac{m}{s}$ is an odd integer.

PROOF. We can clearly assume that $h = \frac{m}{s}$ is an integer with $h > 2$ so that $H^{m-2s+1}F_{m-1}^1 = \{h\}$ is generated by the class of $v_h(s)$. This class lifts in G_m^1 , to

$$\frac{1}{2}(z_h(s-1)01^{h-3}(10-01) + (-1)^{h(s-1)}w_h^{s-1}1^{h-2}(10-01))$$

if $s > 1$, while, if $s = 1$, it lifts to $1^{m-2}(10-01)$. Computing the boundary one obtains:

$$(6.9) \quad \frac{[2(h-2)]!!}{[h-2]!} \left\{ (-1)^{hs} z_h(s-1)01^{h-1} + (-1)^{h-1} \frac{[2(h-1)]}{[h-1]} w_h^{s-1} 1^h \right\}$$

if $s > 1$ and

$$(-1)^{m-1} \frac{[2(m-1)]!!}{[m-1]!} 1^m$$

if $s = 1$.

It is easy to see that Φ_h divides $\frac{[2(h-2)]!!}{[h-2]!}$ iff h is even.

Assume this is the case, 6.9 is divisible by Φ_h so its class in $H^{m-2s+2}G_m^2$ is zero.

Assume now that h is odd. Then 6.9 gives a non zero class in $H^{m-2s+2}G_m^{h-1}$ since its image in $H^{m-2s+2}C_{m-r}[r-1]$ is a non zero multiple of the class $z_h(s-1)$. We show that when passing from $H^{m-2s+2}G_m^r$ to $H^{m-2s+2}G_m^{r-1}$, $2 < r \leq h-1$, this class remains different from 0.

Indeed, consider the exact sequence

$$0 \rightarrow H^{m-2s+2}G_m^r \rightarrow H^{m-2s+2}G_m^{r-1} \rightarrow H^{m-2s+2}C_{m-r}[r-1] \rightarrow 0$$

We know that

$$H^{m-2s+2}C_{m-r}[r-1] = \begin{cases} \left\{ \frac{m-r}{s-1} \right\} & \text{if } s-1 \text{ divides } m-r \\ 0 & \text{otherwise.} \end{cases}$$

Since in the given interval $h \neq \frac{m-r}{s-1}$, the factor $\{h\}$ appearing in $H^{m-2s+2}G_m^{r-1}$ by Proposition 6.5 must lie in the image of $H^{m-2s+2}G_m^r$, so we have done. \square

Using this lemma and our previous computation of $H^*G_m^2$ it is immediate to see

PROPOSITION 6.10. *The cohomology groups of the complex G_m^1 are*

$$H^m G_m^1 = \bigoplus_{2h \in S_m} \{2h\}$$

and, for $s > 0$,

$$H^{m-2s} G_m^1 = \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\}$$

$$H^{m-2s+1} G_m^1 = \begin{cases} \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} \oplus \left\{ \frac{m}{s} \right\} & \text{if } \frac{m}{s} \text{ is an even integer larger than 2} \\ \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} & \text{otherwise.} \end{cases}$$

7. – The Theorem

Adding the cohomology of A_{m-1} , we get the main theorem:

THEOREM 7.1. *With the convention that*

$$\{h\} = \begin{cases} R/(\Phi_h) & \text{if } h \text{ is an integer} \\ 0 & \text{otherwise,} \end{cases}$$

we have that

$$H^m(D_m) = \bigoplus_{2h \in S_m} \{2h\}$$

and, for $s > 0$,

$$H^{m-2s}(D_m) = \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} \oplus \left\{ \frac{m-1}{s} \right\}$$

$$H^{m-2s+1}(D_m) = \begin{cases} \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} \oplus \left\{ \frac{m}{s} \right\}^{\oplus 2} & \text{if } \frac{m}{s} \text{ is an even integer larger than 2} \\ \bigoplus_{\substack{1 < h \leq \frac{m-2}{2s} \\ 2h \in S_m}} \{2h\} \oplus \left\{ \frac{m}{s} \right\} & \text{otherwise.} \end{cases}$$

REMARK. Notice that $H^i(D_m)$, $i < m$, can have multiple components, in contrast with cases A_m and B_m

8. – The exceptional cases

Here we give the table of cohomology for all exceptional cases. As said in the introduction, this was obtained by using a suitable computer program: we correct here some misprints in the analog table considered in [S].

	$I_2(m)$	H_3	H_4	F_4	E_6	E_7	E_8
H^0	0	0	0	0	0	0	0
H^1	{2}	{2}	{2}	{2}	{2}	{2}	{2}
H^2	{m}	0	0	{2}	0	0	0
H^3		m_{H_3}	0	{2}⊕{3}⊕{6}	0	0	0
H^4			m_{H_4}	m_{F_4}	0	0	0
H^5					{6}⊕{8}	{6}	{2}
H^6					m_{E_6}	{6}	0
H^7						m_{E_7}	{8}⊕{12}
H^8							m_{E_8}

where

$$m_{H_3} = \{2\} \oplus \{6\} \oplus \{10\}$$

$$m_{H_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\} \oplus \{30\}$$

$$m_{F_4} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{6\} \oplus \{8\} \oplus \{12\}$$

$$m_{E_6} = \{3\} \oplus \{6\} \oplus \{9\} \oplus \{12\}$$

$$m_{E_7} = \{2\} \oplus \{6\} \oplus \{14\} \oplus \{18\}$$

$$m_{E_8} = \{2\} \oplus \{3\} \oplus \{4\} \oplus \{5\} \oplus \{6\} \oplus \{8\} \oplus \{10\} \oplus \{12\} \oplus \{15\} \oplus \{20\} \oplus \{24\} \oplus \{30\}.$$

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