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Some Properties of Weighted Sobolev Spaces in $\mathbb{R}_+^d$

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Abstract. Duality and complex interpolation are investigated for weighted Sobolev spaces, which then are characterized with the help of fractional powers of the analytic semigroup corresponding to the equation $u_t = (x^1)^2 \Delta u$ in $\mathbb{R}_+^d$ with zero boundary value at $x^1 = 0$.

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1. - Introduction

Weighted Sobolev spaces arise in various issues of the theory of partial differential equations (see [13]). In particular, they are used for studying equations in "bad" domains. One can find corresponding references in [3], [13], and [15]. Quite recently it was realized that these spaces are indispensable in the theory of stochastic partial differential equations (SPDEs) in domains and, moreover, the spaces with full range of the number of "derivatives" are needed in that theory (see, for instance, [10]). It turns out that in one space dimension the corresponding theory reduces to the theory of usual spaces of Bessel potentials just by a logarithmic change of variables (see [9]). In multidimensional case the situation is more delicate. A general unified definition of weighted Sobolev spaces for fractional or negative number of derivatives is given in [7] where some initial properties of these spaces are derived. These properties are sufficient to develop a theory of solvability of SPDEs in domains (see [10]). However if we want to understand certain qualitative properties of solutions, we need a better understanding of the weighted Sobolev spaces. Here we present some results, which are needed, in particular, in investigation of the trace properties for stochastic weighted Sobolev spaces (see [8]). Finally, it is worth noting that weighted Sobolev spaces are also used in [12] where they are applied to porous medium equations (see Remarks 2.4 and 4.3).

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The rest of the article consists of four sections. In Section 2 we recall some properties of weighted Sobolev spaces $H^\gamma_{p,\theta}$ and prove a duality theorem. Section 3 characterizes $H^\gamma_{p,\theta}$ as complex interpolation spaces. In Section 4 we construct an analytic semigroup associated with a degenerate elliptic equation. This semigroup can be used for studying degenerate SPDEs in the spirit of [2]. In the final Section 5 we show that the scale of the spaces $H^\gamma_{p,\theta}$ can be constructed on the basis of fractional powers of the generator of this semigroup.

We use the following standard notation

$$\mathbb{R}^d_+ = \{x = (x^1, x') : x^1 > 0, x' = (x^2, \dots, x^d) \in \mathbb{R}^{d-1}\},$$

$$D_i = \partial/\partial x^i, \quad Du = u_x = (D_1 u, \dots, D_d u).$$

For a multi–index $\alpha = (\alpha_1, \dots, \alpha_d)$ where α_i's are nonnegative integers, we denote

$$D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

By $\mathcal{D}(R^d_+)$ we denote the space of all distributions on $\mathbb{R}^d_+$ that is of all continuous linear functionals on $C_0^\infty(\mathbb{R}^d_+)$, the latter being the space of all infinitely differentiable functions on $\mathbb{R}^d_+$ with compact support belonging to $\mathbb{R}^d_+$.

Any function given on $\mathbb{R}_+ := \mathbb{R}^1_+$ is also considered as a function on $\mathbb{R}^d_+$ independent of x'. We define M^α as an operator of multiplying by $(x^1)^\alpha$, $M = M^1$.

By $H^\gamma_p = H^\gamma_p(\mathbb{R}^d)$ we denote the space of Bessel potentials ($= (1 - \Delta)^{-\gamma/2} L_p$) with norm $\|\cdot\|_{\gamma,p}$. For $\gamma = 0$, we have $H^0_p = L_p$ and we denote $\|\cdot\|_p = \|\cdot\|_{0,p}$.

2. – The definition and basic properties of $H^\gamma_{p,\theta}$

First we recall some definitions and facts from [7].

DEFINITION 2.1. Take and fix a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}_+)$ such that

$$\sum_{n=-\infty}^{\infty} \zeta(e^{x-n}) \geq 1 \quad \forall x \in \mathbb{R}.$$

For $\gamma, \theta \in \mathbb{R}$, and $p \in (1, \infty)$ let $L_{p,\theta} = H^0_{p,\theta}$ and let $H^\gamma_{p,\theta}$ be the set of all distributions u on $\mathbb{R}^d_+$ such that

$$\|u\|^p_{\gamma,p,\theta} := \sum_{n=-\infty}^{\infty} e^{n\theta} \|\zeta A_n u\|^p_{\gamma,p} < \infty,$$

where the operators A_n are defined by $A_n u(x) = u(e^n x)$.

It is easy to check that $L_{p,d} = L_p(\mathbb{R}^d_+)$. Also it turns out that using different functions ζ yields the same spaces with equivalent norms and the set $C_0^\infty(\mathbb{R}^d_+)$ is dense in $H^\gamma_{p,\theta}$. The following less trivial properties of $H^\gamma_{p,\theta}$ will be often used in the future.

LEMMA 2.2 (See [7]). (i) *For any $a > 0$ and $\alpha \in \mathbb{R}$,*

$$\|u(a\,\cdot)\|^p_{\gamma,p,\theta} \le a^{-\theta} N \|u\|^p_{\gamma,p,\theta} \le N \|u(a\,\cdot)\|^p_{\gamma,p,\theta}\,,$$

$$\|M^\alpha u\|_{\gamma,p,\theta} \le N \|u\|_{\gamma,p,\theta+p\alpha} \le N \|M^\alpha u\|_{\gamma,p,\theta}\,,$$

$$\|MDu\|_{\gamma,p,\theta} + \|DMu\|_{\gamma,p,\theta} \le N \|u\|_{\gamma+1,p,\theta}\,.$$

(ii) *There is a sequence $\eta_k \in C_0^\infty(\mathbb{R}^d_+)$ such that, for any $u \in H^\gamma_{p,\theta}$, $\|\eta_k u\|_{\gamma,p,\theta} \le N\|u\|_{\gamma,p,\theta}$ and $\|\eta_k u - u\| \to 0$ as $k \to \infty$.*

(iii) *Let $\mu \le \gamma$, $\infty > q \ge p > 1$, and $\theta, \tau \in \mathbb{R}$ be such that*

$$\gamma - d/p \ge \mu - d/q, \quad \tau/q = \theta/p\,. \tag{2.1}$$

Then for any $u \in H^\gamma_{p,\theta}$ we have

$$u \in H^\mu_{q,\tau}, \quad \|u\|_{\mu,q,\tau} \le N \|u\|_{\gamma,p,\theta}\,. \tag{2.2}$$

(iv) *Assume $\gamma p > d$ and represent $\gamma - d/p$ as $k + \varepsilon$, where k is an integer and $\varepsilon \in (0, 1]$. Let i, j be multi-indices such that $|i| \le k$, $|j| = k$. Then for any $u \in H^\gamma_{p,\theta}$, we have*

$$M^{|i|+\theta/p} D^i u \in C(\mathbb{R}^d_+), \quad M^{k+\varepsilon+\theta/p} D^j u \in \mathcal{C}^\varepsilon_{\text{loc}}(\mathbb{R}^d_+)\,,$$

$$\|M^{|i|+\theta/p} D^i u\|_{C(\mathbb{R}^d_+)} \le N \|u\|_{\gamma,p,\theta}, \quad [M^{k+\varepsilon+\theta/p} D^j u]_{\mathcal{C}^\varepsilon(\mathbb{R}^d_+)} \le N \|u\|_{\gamma,p,\theta}\,,$$

where $\mathcal{C}^\varepsilon$ is the Zygmund space. Furthermore, the above constants N are independent of u, a, λ, and k.

The following extension of Lemma 2.2 (iii) is sometimes useful (cf. Remark 2.4).

LEMMA 2.3. *Let $\mu \le \gamma$, $\infty > q \ge p > 1$, (2.1) be satisfied, and $u \in H^\mu_{q,\tau} \cap H^\gamma_{p,\theta}$.*

(i) *Let either $\tau \ne d - 1$ or $\theta \ne d - 1$. Then*

$$\|u\|_{\mu,q,\tau} \le N \|Mu_x\|_{\gamma-1,p,\theta}\,. \tag{2.3}$$

(ii) *Let either $\tau \ne d - 1 + p$ or $\theta \ne d - 1 + p$. Then*

$$\|u\|_{\mu,q,\tau} \le N \|(Mu)_x\|_{\gamma-1,p,\theta}\,. \tag{2.4}$$

In both assertions N is independent of u.

PROOF. (i) By Corollary 2.12 of [7], if $\tau \neq d-1$, we have

$$\|u\|_{\mu,q,\tau} \leq N\|Mu_x\|_{\mu-1,q,\tau}\,. \tag{2.5}$$

This and the equivalence

$$\|Mu_x\|_{\mu-1,q,\tau} \sim \|u_x\|_{\mu-1,q,\tau+q}$$

together with (2.2) immediately imply (2.3).

If $\theta \neq d-1$, it suffices to estimate the right-hand side of (2.2) by using (2.5) (for γ, p, θ instead of μ, q, τ).

(ii) One gets (2.4) by substituting Mu, $\tau - q$, $\theta - p$ in place of u, τ, θ in (2.3) and using Lemma 2.2 (i). The lemma is proved.

REMARK 2.4. If $1 < p \leq q < \infty$, $s \in \mathbb{R}$, and

$$\sigma := 1 + d/q - d/p \geq 0\,,$$

then, for

$$\mu = 0, \quad \gamma = 1, \quad \tau = d + qs, \quad \theta = d + p(s + \sigma - 1)\,,$$

condition (2.1) is satisfied. In addition, $\tau \neq d-1$ if and only if $s \neq -1/q$. Also for $\tau = d-1$, we have $\theta \neq d-1$ if and only if $(d-1)/p \neq (d-1)/q$. It follows from Lemma 2.3 (and Lemma 2.2 (i)) that, if a) $s \neq -1/q$ or b) $d \geq 2$, $s = -1/q$, and $q > p$, then, for any $u \in C_0^\infty(\mathbb{R}^d_+)$,

$$\|M^s u\|_q = \|M^s u\|_{0,q,d} = \|u\|_{0,q,\tau} \leq N\|Mu_x\|_{0,p,\theta} = N\|M^{s+\sigma}u_x\|_p\,.$$

The inequality between the extreme terms is proved as Theorem 4.2.2 in [12] if $s > -1/q$. We see that, actually, it is true for all s apart from one exceptional value and even this value need not be excluded if $d \geq 2$ and $q > p$.

The following result was noticed by S.V. Lototsky.

THEOREM 2.5. *For $\phi, \psi \in C_0^\infty(\mathbb{R}^d_+)$ define (ϕ, ψ) as the scalar product of ϕ and ψ in $L_2(\mathbb{R}^d_+)$. Then for any $p \in (1, \infty)$, and $\gamma, \theta \in \mathbb{R}$*

$$\|\phi\|_{\gamma,p,\theta} \leq N \sup_{\psi \neq 0} \frac{(\phi, \psi)}{\|\psi\|_{\gamma',p',\theta'}} \leq N\|\phi\|_{\gamma,p,\theta}\,, \tag{2.6}$$

where γ', p', θ' are defined by

$$\gamma' = -\gamma, \quad 1/p + 1/p' = 1, \quad \theta/p + \theta'/p' = d$$

and the constants N are independent of ϕ. Moreover, the relation (ϕ, ψ) can be extended by continuity on all $\phi \in H^\gamma_{p,\theta}$ and $\psi \in H^{\gamma'}_{p',\theta'}$ and then it identifies the dual to $H^\gamma_{p,\theta}$ with $H^{\gamma'}_{p',\theta'}$. In particular, the space $H^\gamma_{p,\theta}$ is reflexive.

PROOF. First we prove (2.6). Choose the function $\zeta(x) = \zeta(x^1)$ so that

$$\zeta \in C_0^\infty(\mathbb{R}_+), \quad \zeta \geq 0, \quad \sum_{n=-\infty}^{\infty} |\zeta(e^{-n}x)|^2 \equiv 1 .$$

Then, by Hölder's inequality,

$$(\phi, \psi) = \sum_{n=-\infty}^{\infty} (\zeta(e^{-n}\cdot)\phi, \zeta(e^{-n}\cdot)\psi) = \sum_{n=-\infty}^{\infty} e^{nd}(\phi(e^n\cdot)\zeta, \psi(e^n\cdot)\zeta)$$

$$\leq \sum_{n=-\infty}^{\infty} e^{nd}\|\zeta A_n\phi\|_{\gamma,p}\|\zeta A_n\psi\|_{\gamma',p'} \leq N\|\phi\|_{\gamma,p,\theta}\|\psi\|_{\gamma',p',\theta'} ,$$

which proves the inequality on the right in (2.6).

To prove the left inequality, fix ϕ and take some numbers $c_n \geq 0$ such that

$$\sum_{n=-\infty}^{\infty} e^{n\theta'}c_n^{p'} = 1 , \tag{2.7}$$

and functions $g_n \in C_0^\infty(\mathbb{R}_+^d)$ such that

$$(\zeta A_n\phi, g_n) = c_n\|\zeta A_n\phi\|_{\gamma,p}, \quad \|g_n\|_{\gamma',p'} \leq 2c_n .$$

Then for $f_n := g_n\zeta$ we have

$$(A_n\phi, f_n) = c_n\|\zeta A_n\phi\|_{\gamma,p}, \quad \|f_n\|_{\gamma',p'} \leq Nc_n ,$$

where N is independent of n, c_n, and ϕ. Now find an integer $m \geq 1$, such that $\zeta A_n\phi \equiv 0$ for $|n| \geq m$, and define

$$\psi = \sum_{k=-m}^{m} A_{-k}f_k .$$

Obviously $\psi \in C_0^\infty(\mathbb{R}_+^d)$. Also notice that $\zeta A_j\zeta \equiv 0$ unless $|j| \leq m_0$, where m_0 can be estimated in terms of ζ alone. We also use that ζ is a pointwise multiplier in $H_{p'}^{\gamma'}$ and that A_j are bounded operators in this space. Then we find

$$\|\psi\|_{\gamma',p',\theta'}^{p'} = \sum_{n=-\infty}^{\infty} e^{n\theta'}\|\zeta A_n\psi\|_{\gamma',p'}^{p'} = \sum_{n=-\infty}^{\infty} e^{n\theta'}\left\|\sum_{\substack{|k|\leq m\\ |n-k|\leq m_0}} \zeta A_{n-k}f_k\right\|_{\gamma',p'}^{p'}$$

$$\leq N\sum_{n=-\infty}^{\infty} e^{n\theta'}\sum_{k:\,|n-k|\leq m_0}\|f_k\|_{\gamma',p'}^{p'} \leq N\sum_{k=-\infty}^{\infty} e^{k\theta'}\|f_k\|_{\gamma',p'}^{p'} \leq N ,$$

where N are independent of c_n and ϕ. Finally,

$$\sum_{n=-\infty}^{\infty} e^{nd} c_n \|\zeta A_n\phi\|_{\gamma,p} = \sum_{|n|\le m} = \sum_{|n|\le m} e^{nd}(A_n\phi, f_n)$$

$$= \sum_{|n|\le m} (\phi, A_{-n} f_n) = (\phi, \psi) \le N \sup_{\psi\neq 0} \frac{(\phi,\psi)}{\|\psi\|_{\gamma',p',\theta'}}.$$

By taking the supremum of the first expression over all c_n satisfying (2.7), we come to the first inequality in (2.6).

After (2.6) has been proved, it is clear that the relation (ϕ, ψ) is indeed extendible by continuity to $\phi \in H^{\gamma}_{p,\theta}$, $\psi \in H^{\gamma'}_{p',\theta'}$ and defines a continuous embedding of $H^{\gamma'}_{p',\theta'}$ into the dual of $H^{\gamma}_{p,\theta}$.

To finish proving the theorem it only remains to prove that any continuous linear functional $l(\phi)$ on $H^{\gamma}_{p,\theta}$ is representable as (ϕ, ψ) with a $\psi \in H^{\gamma'}_{p',\theta'}$. Observe that, for any $\eta \in C_0^\infty(\mathbb{R}^d_+)$, $l(\eta\phi)$ is a bounded linear functional on H^{γ}_p. It follows that $l(\eta\phi) = (\phi, \psi_\eta)$, where $\psi_\eta \in H^{\gamma'}_{p'}$ and that there exists a distribution ψ on $\mathbb{R}^d_+$ such that $l(\phi) = (\phi, \psi)$ for any $\phi \in C_0^\infty(\mathbb{R}^d_+)$ and $\psi\eta \in H^{\gamma'}_{p'}$ for any $\eta \in C_0^\infty(\mathbb{R}^d_+)$. The above proof of the left inequality in (2.6), after interchanging γ, p, θ and γ', p', θ', proves that actually $\psi \in H^{\gamma'}_{p',\theta'}$. The theorem is proved.

Next we point out a multiplicative inequality for the $H^{\gamma}_{p,\theta}$-norms, similar to the one well known for the norms in H^{γ}_p. Let

$$\begin{gathered} \kappa \in [0,1], \quad p_i \in (1,\infty), \quad \gamma_i, \theta_i \in \mathbb{R}, \quad i = 0, 1, \\ \gamma = \kappa\gamma_1 + (1-\kappa)\gamma_0, \quad 1/p = \kappa/p_1 + (1-\kappa)/p_0, \\ \theta/p = \theta_1\kappa/p_1 + \theta_0(1-\kappa)/p_0. \end{gathered} \tag{2.8}$$

It follows from [5] that

$$\|u\|_{\gamma,p} \le N \|u\|^{\kappa}_{\gamma_1,p_1} \|u\|^{1-\kappa}_{\gamma_0,p_0}.$$

Hence

$$e^{n\theta}\|\zeta A_n u\|^p_{\gamma,p} \le (e^{n\theta_1}\|\zeta A_n u\|^{p_1}_{\gamma_1,p_1})^{\kappa p/p_1} (e^{n\theta_0}\|\zeta A_n u\|^{p_0}_{\gamma_0,p_0})^{(1-\kappa)p/p_0}.$$

By applying Hölder's inequality we obtain the following.

THEOREM 2.6. *Assume* (2.8). *Then*

$$\|u\|_{\gamma,p,\theta} \le N \|u\|^{\kappa}_{\gamma_1,p_1,\theta_1} \|u\|^{1-\kappa}_{\gamma_0,p_0,\theta_0}, \tag{2.9}$$

where N is independent of u.

3. – $H^{\gamma}_{p,\theta}$ as complex interpolation spaces

Theorem 2.6 makes naturalthe following result.

THEOREM 3.1. *For* $\kappa \in (0,1)$, $p \in (1,\infty)$, $\gamma_i, \theta_i \in \mathbb{R}$, $i = 0, 1$,

$$\theta = \kappa\theta_1 + (1-\kappa)\theta_0, \quad \gamma = \kappa\gamma_1 + (1-\kappa)\gamma_0,$$

we have $[H^{\gamma_0}_{p,\theta_0}, H^{\gamma_1}_{p,\theta_1}]_\kappa = H^{\gamma}_{p,\theta}$.

To prove this theorem we need a lemma. Fix some constants $\alpha, \beta, a, b \in \mathbb{R}$ and for complex z define

$$\Lambda_z u = \sum_{n=-\infty}^{\infty} e^{n(\alpha z+\beta)} A_{-n}[\zeta(1-\Delta)^{az+b}(\zeta A_n u)].$$

LEMMA 3.2. *We have*

$$\|\Lambda_z u\|_{\gamma,p,\theta} \le N(1+|\lambda|)^q \|u\|_{\nu,p,\tau},$$

where

$$\lambda = \operatorname{Im} z, \quad \mu = \operatorname{Re} z, \quad \nu = 2(a\mu+b)+\gamma, \quad \tau = p(\alpha\mu+\beta)+\theta$$

and N, q *are independent of* z *and* u.

PROOF. By definition

$$\|\Lambda_z u\|^p_{\gamma,p,\theta} = \sum_{m=-\infty}^{\infty} e^{m\theta} \left\| \sum_{n=-\infty}^{\infty} e^{n(\alpha z+\beta)} \zeta A_{m-n}[\zeta(1-\Delta)^{az+b}(\zeta A_n u)] \right\|^p_{\gamma,p}.$$

As in the proof of Theorem 2.5 we use the fact that $\zeta A_{m-n}\zeta \equiv 0$ unless $|m-n| \le m_0$. We also use that $\zeta A_{m-n}\zeta$ are pointwise multipliers in H^{γ}_p and that A_{m-n} is a bounded operator in H^{γ}_p. Then we find that

$$\|\Lambda_z u\|^p_{\gamma,p,\theta} \le N \sum_{|m-n|\le m_0} e^{m\theta+np(\alpha\mu+\beta)} \|(1-\Delta)^{az+b}(\zeta A_n u)\|^p_{\gamma,p}$$

$$\le N \sum_{n=-\infty}^{\infty} e^{n(\theta+p(\alpha\mu+\beta))} \|(1-\Delta)^{ia\lambda}(\zeta A_n u)\|^p_{\gamma+2(a\mu+b),p}.$$

It only remains to notice that, from the theory of Fourier multipliers, it follows that the operator $(1-\Delta)^{ia\lambda}$ is bounded in each H^{γ}_p by a constant times $(1+|\lambda|)^q$. The lemma is proved.

PROOF OF THEOREM 3.1. Since the set $C_0^\infty(\mathbb{R}_+^d)$ is dense in the spaces $H_{p,\theta}^\gamma$, it is also dense in $[H_{p,\theta_0}^{\gamma_0}, H_{p,\theta_1}^{\gamma_1}]_\kappa$ (see Chapter IV of [5]). It follows that we only need to check that, for any $u \in C_0^\infty(\mathbb{R}_+^d)$,

$$\|u\|_{\gamma,p,\theta} \le N\|u\|_{[H_{p,\theta_0}^{\gamma_0}, H_{p,\theta_1}^{\gamma_1}]_\kappa} \le N\|u\|_{\gamma,p,\theta}\,, \tag{3.1}$$

where N is independent of u.

Let $u \in C_0^\infty(\mathbb{R}_+^d) \subset [H_{p,\theta_0}^{\gamma_0}, H_{p,\theta_1}^{\gamma_1}]_\kappa$. Then there is an $H_{p,\theta_0}^{\gamma_0} + H_{p,\theta_1}^{\gamma_1}$-valued bounded continuous function $u(z)$ defined for $0 \le \operatorname{Re} z \le 1$ which is analytic in $0 < \operatorname{Re} z < 1$ and such that, for $\lambda \in \mathbb{R}$,

$$\|u(1+i\lambda)\|_{\gamma_1,p,\theta_1} + \|u(i\lambda)\|_{\gamma_0,p,\theta_0} \le 2\|u\|_{[H_{p,\theta_0}^{\gamma_0}, H_{p,\theta_1}^{\gamma_1}]_\kappa}, \quad u(\kappa) = u\,.$$

Define

$$u_n(z) = e^{z^2+n(\alpha z+\beta)}(1-\Delta)^{az+b}(\zeta^2 A_n u(z))$$

where

$$\alpha = (\theta_1 - \theta_0)/p, \quad \beta = \theta_0/p, \quad a = (\gamma_1 - \gamma_0)/2, \quad b = \gamma_0/2\,.$$

Then, for any $\varphi \in C_0^\infty(\mathbb{R}_+^d)$, the function

$$(u_n(z), \varphi) = e^{z^2+n(\alpha z+\beta)}(\zeta A_n u(z), \zeta(1-\Delta)^{az+b}\varphi)$$

is a scalar continuous function for $0 \le z \le 1$ analytic in $0 < \operatorname{Re} z < 1$. By denoting $\lambda = \operatorname{Im} z$ and assuming without loss of generality that $\gamma_1 \ge \gamma_0$, we get, for $0 \le \operatorname{Re} z \le 1$ and each n,

$$|(u_n(z), \varphi)| \le Ne^{-\lambda^2}\|\zeta A_n u(z)\|_{H_p^{\gamma_0}+H_p^{\gamma_1}}\|(1-\Delta)^{ai\lambda}\varphi\|_{\gamma_1-\gamma_0,p'} \le N\,,$$

where N is independent of λ. Hence the analytic functions $(u_n(z), \varphi)$ are bounded, which by Poisson's formula and owing to

$$\alpha\kappa + \beta = \theta/p, \quad a\kappa + b = \gamma/2\,,$$

implies that

$$\begin{aligned} e^{\kappa^2+n\theta/p}(1-\Delta)^{\gamma/2}(\zeta^2 A_n u) &= u_n(\kappa) \\ &= \int_{\mathbb{R}} u_n(i\lambda)\,\pi_0(d\lambda) + \int_{\mathbb{R}} u_n(1+i\lambda)\,\pi_1(d\lambda)\,, \end{aligned}$$

where π_1 and π_0 are positive measures independent of n satisfying $\pi_1(\mathbb{R}) + \pi_0(\mathbb{R}) = 1$. By Hölder's inequality

$$\begin{aligned}\|u\|^p_{\gamma,p,\theta} &\le N \sum_{n=-\infty}^{\infty} \|u_n(\kappa)\|_p^p \le N \sum_{n=-\infty}^{\infty} \Big(\int_{\mathbb{R}} \|u_n(i\lambda)\|^p_p \pi_0(d\lambda) \\ &\qquad + \int_{\mathbb{R}} \|u_n(1+i\lambda)\|^p_p \pi_1(d\lambda) \Big) \\ &\le N \sup_{\lambda\in\mathbb{R}} \Big(\sum_{n=-\infty}^{\infty} e^{n\theta_0 - \lambda^2} \|(1-\Delta)^{ia\lambda}(\zeta^2 A_n u(i\lambda))\|^p_{\gamma_0,p} \\ &\qquad + \sum_{n=-\infty}^{\infty} e^{n\theta_1-\lambda^2} \|(1-\Delta)^{ia\lambda}(\zeta^2 A_n u(1+i\lambda))\|^p_{\gamma_1,p} \Big) \\ &\le N \sup_{\lambda\in\mathbb{R}} \Big(\sum_{n=-\infty}^{\infty} e^{n\theta_0} \|\zeta A_n u(i\lambda)\|^p_{\gamma_0,p} + \sum_{n=-\infty}^{\infty} e^{n\theta_1} \|\zeta A_n u(1+i\lambda)\|^p_{\gamma_1,p} \Big) \\ &= N \sup_{\lambda\in\mathbb{R}} (\|u(i\lambda)\|^p_{\gamma_0,p,\theta_0} + \|u(1+i\lambda)\|^p_{\gamma_1,p,\theta_1}) .\end{aligned}$$

In this way we get the left inequality in (3.1).

To prove the remaining inequality, take ζ so that

$$\sum_{n=-\infty}^{\infty} \zeta^2(e^{-n}x) \equiv 1 \quad \forall x \in \mathbb{R}_+$$

and define

$$\alpha = (\theta_0 - \theta_1)/p, \quad \beta = \kappa(\theta_1 - \theta_0)/p, \quad a = (\gamma_0 - \gamma_1)/2, \quad b = \kappa(\gamma_1 - \gamma_0)/2 .$$

For $u \in C_0^\infty(\mathbb{R}^d_+)$ define also $u(z) = e^{z^2-\kappa^2}\Lambda_z u$. It follows from Lemma 3.2 that $u(z)$ is an $H^\nu_{q,\tau}$-valued function which is bounded for $|\operatorname{Re} z| \le n$ for any ν, q, τ, n. It is also easy to see that it is an analytic $H^\nu_{q,\tau}$-valued function.

Next, observe that $\alpha\kappa + \beta = a\kappa + b = 0$ and

$$u(\kappa, x) = \sum_{n=-\infty}^{\infty} \zeta^2(e^{-n}x) u(x) = u(x) .$$

Furthermore by Lemma 3.2, if $\operatorname{Re} z = 1$, then

$$\|u(z)\|_{\gamma_1,p,\theta_1} \le N(1+|\lambda|)^q e^{-\lambda^2} \|u\|_{2(a+b)+\gamma_1,p,\theta_1+p(\alpha+\beta)} \le N\|u\|_{\gamma,p,\theta}$$

since $2(a+b)+\gamma_1 = \gamma$ and $\theta_1 + p(\alpha+\beta) = \theta$. Finally, if $\operatorname{Re} z = 0$, then

$$\|u(z)\|_{\gamma_0,p,\theta_0} \le N(1+|\lambda|)^q e^{-\lambda^2} \|u\|_{2b+\gamma_0,p,\theta_0+p\beta} \le N\|u\|_{\gamma,p,\theta} .$$

This proves the theorem.

COROLLARY 3.3. *Let the assumptions of Theorem* 3.1 *be satisfied and let* $q \in (1, \infty)$, $\nu_i, \sigma_i \in \mathbb{R}$. *Define*

$$\nu = \kappa \nu_1 + (1-\kappa)\nu_0, \quad \sigma = \kappa \sigma_1 + (1-\kappa)\sigma_0 .$$

Let S *be an operator defined on* $C_0^\infty(\mathbb{R}^d_+)$ *satisfying*

$$\|Su\|_{\gamma_i, p, \theta_i} \le N_i \|u\|_{\nu_i, q, \sigma_i}$$

for $i = 0, 1$ *with* N_i *independent of* u. *Then*

$$\|Su\|_{\gamma, p, \theta} \le N N_1^\kappa N_0^{1-\kappa} \|u\|_{\nu, q, \sigma} ,$$

where N *is independent of* N_i *and* u.

Indeed, a standard and almost trivial consequence of identifying the interpolation spaces is that $\|Su\|_{\gamma,p,\theta} \le N(N_1 + N_0)\|u\|_{\nu,q,\sigma}$. For the operator $S_c u(x) := (Su)(cx)$, where c is a constant, $c > 0$, we have $\|S_c u\|_{\gamma,p,\theta} \sim c^{-\theta/p}\|Su\|_{\gamma,p,\theta}$ and the corresponding constants N_i become $N N_i c^{-\theta_i/p}$. Therefore,

$$\|Su\|_{\gamma,p,\theta} \le N c^{\theta/p}(N_1 c^{-\theta_1/p} + N_0 c^{-\theta_0/p})\|u\|_{\nu,q,\sigma} .$$

Hence, by taking the inf with respect to $c > 0$, we get our assertion.

4. – An analytic semigroup

For constant b, c define

$$\mathcal{L}_{b,c} = M^2 \Delta + bMD_1 - c .$$

We want to construct and investigate the semigroup with generator $\mathcal{L}_{b,0}$. Notice that, if $d = 1$, this is quite easy to do. Indeed,

$$(D^2 + (b-1)D)(v(e^x)) = (\mathcal{L}_{b,0} v)(e^x) ,$$

so that the properties of the semigroup related to $\mathcal{L}_{b,0}$ can be easily obtained from the well-known properties of the semigroup related to the operator $D^2 + (b-1)D$ with constant coefficients. However for $d \ge 2$ we do not know any easy way to deal with $\mathcal{L}_{b,0}$.

LEMMA 4.1. *For any* b, γ, p, *and* θ, *there exists a constant* $c_0 > 0$ *such that for any complex* z *with* $\operatorname{Re} z \ge c_0$ *and* $u \in H^{\gamma+2}_{p,\theta}$

$$\|u\|_{\gamma+2,p,\theta} + (1 + |z|)\|u\|_{\gamma,p,\theta} \le N\|\mathcal{L}_{b,z} u\|_{\gamma,p,\theta} , \tag{4.1}$$

where the constant N *is independent of* u *and* z.

PROOF. We will use that (see, for instance, [1], more details can be found in [6]), if $a(x)$ is a bounded infinitely differentiable function on $\mathbb{R}^d$ with bounded derivatives and if $a(x) \geq \varepsilon$ with a constant $\varepsilon > 0$, then, for any γ and p, there exists a constant N such that

$$\|v\|_{\gamma+2,p} + (1+|z|)\|v\|_{\gamma,p} \leq N\|zv - a\Delta v\|_{\gamma,p} \tag{4.2}$$

whenever $v \in H_p^{\gamma+2}$ and $\operatorname{Re} z \geq N$.

Having this in mind and observing that $(x^1)^2$ is a "good" function on the support of ζ, write

$$\|u\|^p_{\gamma+2,p,\theta} + |z|^p\|u\|^p_{\gamma,p,\theta} = \sum_{n=-\infty}^{\infty} e^{n\theta}\{\|\zeta A_n u\|^p_{\gamma+2,p} + |z|^p\|\zeta A_n u\|^p_{\gamma,p}\}$$

$$\leq N \sum_{n=-\infty}^{\infty} e^{n\theta}\|(z - M^2\Delta)(\zeta A_n u)\|^p_{\gamma,p} .$$

Here

$$\Delta(\zeta A_n u) = (A_n u)\Delta\zeta + (A_n M^2 \Delta u)M^{-2}\zeta + 2(A_n M D_1 u)M^{-1}D_1\zeta ,$$

so that

$$(z - M^2\Delta)(\zeta A_n u) = \zeta A_n(bMD_1 u - \mathcal{L}_{b,z}u)$$
$$- (A_n u)M^2\Delta\zeta - 2(A_n M D_1 u)MD_1\zeta .$$

In addition, by Lemma 2.2 and and Theorem 2.6

$$\|MD_i u\|_{\gamma,p,\theta} \leq N\|u\|_{\gamma+1,p,\theta} \leq N\|u\|^{1/2}_{\gamma+2,p,\theta}\|u\|^{1/2}_{\gamma,p,\theta} . \tag{4.3}$$

Therefore

$$\|u\|^p_{\gamma+2,p,\theta} + |z|^p\|u\|^p_{\gamma,p,\theta}$$
$$\leq N\|\mathcal{L}_{b,z}u\|^p_{\gamma,p,\theta} + N\|u\|^{p/2}_{\gamma+2,p,\theta}\|u\|^{p/2}_{\gamma,p,\theta} + N\|u\|^p_{\gamma,p,\theta} ,$$

which obviously leads to (4.1) if c_0 is large enough. The theorem is proved.

By Theorem 2.8 of [7] the operator $\mathcal{L}_{b,c} : H^{\gamma+2}_{p,\theta} \to H^{\gamma}_{p,\theta}$ acts onto and has a bounded inverse if c is real and large enough. From [7] we also know that $C_0^\infty(\mathbb{R}^d_+)$ and $H^{\gamma+2}_{p,\theta}$ are dense in $H^{\gamma}_{p,\theta}$. This combined with Theorem 13.2 of [4] and Lemma 4.1 proves the following result.

THEOREM 4.2. *For any b, γ, p, and θ, the operator $\mathcal{L}_{b,0}$ is a generator of an analytic semigroup T_t^b acting in $H^{\gamma}_{p,\theta}$.*

REMARK 4.3. Given $u_0 \in H^{\gamma}_{p,\theta}$, Theorem 4.2 allows one to find a unique $H^{\gamma}_{p,\theta}$-valued function $u(t)$ defined for $t \geq 0$ which has strong $H^{\gamma-2}_{p,\theta}$-derivative in time, equals u_0 at $t = 0$, and satisfies the parabolic equation

$$u_t = (x^1)^2 \Delta u \quad t \geq 0 . \tag{4.4}$$

Unique solvability in spaces $H^{\gamma}_{p,\theta}$ of equations like (4.4) with 1 in place of $(x^1)^2$ are studied in [7] and with x^1 in place of $(x^1)^2$ are studied in [12], where the results are applied to porous medium equations.

It turns out that there is a range of invertibility of $\mathcal{L}_{b,z}$ common to all γ.

THEOREM 4.4. *For any b, p, and θ there exists a constant $c_0 = c_0(b, p, \theta)$ such that, if* $\operatorname{Re} z \geq c_0$, *then $(-\mathcal{L}_{b,z})^{-1}$ is a bounded operator from $H^{\gamma}_{p,\theta}$ onto $H^{\gamma+2}_{p,\theta}$ for any γ.*

PROOF. Since $\mathcal{L}_{b,z}$ is a bounded operator from $H^{\gamma+2}_{p,\theta}$ to $H^{\gamma}_{p,\theta}$ (Lemma 2.2), we can replace "onto" in the assertion with "into". Also, bearing in mind the method of continuity and the fact that, for any given γ, p, θ, there exists c such that $(-\mathcal{L}_{b,c})^{-1}$ is a bounded operator from $H^{\gamma}_{p,\theta}$ into $H^{\gamma+2}_{p,\theta}$ (Theorem 2.8 of [7]), one easily sees that it suffices to prove that, for any b and θ, there exists a constant c_0 such that, if $\operatorname{Re} z \geq c_0$ and $\gamma \in \mathbb{R}$, then

$$\|u\|_{\gamma+2,p,\theta} \leq N \|\mathcal{L}_{b,z} u\|_{\gamma,p,\theta} , \tag{4.5}$$

where N is independent of u.

According to Theorem 2.8 of [7] and Lemma 4.1 one can choose c_0 so that, for $\operatorname{Re} z \geq c_0$, $(-\mathcal{L}_{b,z})^{-1}$ and $(-\mathcal{L}^*_{b,z})^{-1}$ are bounded operators from $H^0_{q,\tau}$ into $H^2_{q,\tau}$ for $q = p, p'$ and $\tau = \theta, \theta'$, where $\mathcal{L}^*_{b,z}$ is the formal adjoint operator for $\mathcal{L}_{b,z}$, $p' = p/(p-1)$, $\theta' = p'd - \theta p'/p$. It turns out that this c_0 suits other values of γ as well and moreover, for $\operatorname{Re} z \geq c_0$ and $\gamma \in \mathbb{R}$, the following assertion holds

(A) both $(-\mathcal{L}_{b,c})^{-1}$ and $(-\mathcal{L}^*_{b,c})^{-1}$ are bounded operators from $H^{\gamma}_{q,\tau}$ into $H^{\gamma+2}_{q,\tau}$, with $(q, \tau) = (p, \theta)$ and $= (p', \theta')$.

To prove this, first notice that it suffices to consider only $\gamma \geq 0$. Indeed, by using Theorem 2.5 one easily sees that if (A) holds, then $(-\mathcal{L}_{b,z})^{-1} = ((-\mathcal{L}^*_{b,z})^{-1})^*$ and $(-\mathcal{L}^*_{b,z})^{-1} = ((-\mathcal{L}_{b,z})^{-1})^*$ are bounded operators from $H^{-\gamma-2}_{q,\tau}$ into $H^{-\gamma}_{q,\tau}$, with $(q, \tau) = (p, \theta)$ and $= (p', \theta')$, and by interpolation (Corollary 3.3) these operators are also bounded as operators from $H^{\nu}_{q,\tau}$ into $H^{\nu+2}_{q,\tau}$ for any $\nu \in [-\gamma - 2, \gamma]$.

From this argument we also see that it suffices to consider the case in which γ is a (large) integer. As we have mentioned in the beginning of the proof, we need to prove (4.5) for $\mathcal{L}_{b,z}$ and $\mathcal{L}^*_{b,z}$ and q, τ in place of p, θ. By our choice of c_0 and by Theorem 2.8 of [7], for $n = 1, 2, \dots$, $\operatorname{Re} z \geq c_0$, and c_1 large enough, we have

$$\|u\|_{2n,p,\theta} \leq N \|\mathcal{L}_{b,c_1}^{n-1} u\|_{2,p,\theta} \leq N \|\mathcal{L}_{b,z} \mathcal{L}_{b,c_1}^{n-1} u\|_{0,p,\theta}$$

$$= N \|\mathcal{L}_{b,c_1}^{n-1} \mathcal{L}_{b,z} u\|_{0,p,\theta} \leq N \|\mathcal{L}_{b,z} u\|_{2(n-1),p,\theta} .$$

Similarly we treat $\mathcal{L}_{b,z}^*$ and other values of q, τ. The theorem is proved.

REMARK 4.5. One can give a probabilistic representation for T_t^b which will be needed in the future. Observe (see, for instance, Lemma 13.3 of [4]) that for $f \in H_{p,\theta}^{\gamma}$ and c large enough we have

$$(4.6) \qquad (-\mathcal{L}_{b,c})^{-1} f = \int_0^\infty e^{-ct} T_t^b f \, dt .$$

On the other hand, let $x_t(x)$ be a unique solution of the stochastic equation

$$(4.7) \qquad dx_t = \sqrt{2} x_t^1 \, dw_t + b x_t^1 e_1 \, dt$$

with initial data $x_0 = x$, where $w_t = (w_t^1, w_t')$ is a d–dimensional Wiener process and $e_1 = (1, 0, \dots, 0)$. Define the matrix-valued random process σ_t by

$$(\sigma_t x)^1 = e^{\xi_t} x^1, \quad (\sigma_t x)' = x' + x^1 \eta_t ,$$

$$\xi_t := w_t^1 \sqrt{2} + (b-1)t, \quad \eta_t := \sqrt{2} \int_0^t e^{\xi_s} \, dw_s' .$$

It is shown in the proof of Theorem 2.8 of [7] that $x_t(x) = \sigma_t x$ and, if $f \in C_0^\infty(\mathbb{R}_+^d)$ and c is large enough, then the function

$$(4.8) \qquad u(x) := E \int_0^\infty e^{-ct} f(x_t(x)) \, dt = E \int_0^\infty e^{-ct} f(\sigma_t x) \, dt$$

belongs to $H_{p,\theta}^{\gamma+2}$ and solves the equation $f = -\mathcal{L}_{b,c} u$. By the way formula (4.8) makes sense for all $x \in \mathbb{R}^d$ and not only for $x^1 > 0$ and as easy to see $x_t(x) = \sigma_t x$ stays in $\mathbb{R}^d \cap \{x^1 \leq 0\}$ if $x^1 \leq 0$ so that $(-\mathcal{L}_{b,c})^{-1} f(x) = 0$ if $x^1 \leq 0$.

By comparing (4.6) and (4.8) and remembering that c is any large number we get that for any $f \in C_0^\infty(\mathbb{R}_+^d)$ and $t \geq 0$ we have

$$(4.9) \qquad T_t^b f(x) = Ef(x_t(x)) = Ef(\sigma_t x) .$$

Next, for $l = 1, 2, \dots$ define

$$\phi(l, z) := (-1)^{l+1} \frac{\sin \pi z}{\pi} (2l-1)! \prod_{k=1-l}^{l-1} (z+k)^{-1} .$$

It turns out that $\phi(l,z)$ is an analytic function for $|\operatorname{Re} z| < l$.

REMARK 4.6. Lemma 4.1 and Theorem 4.4 imply that, for $c \geq c_0(b,p,\theta)$, the operator $-\mathcal{L}_{b,c}$, acting on $H^{\gamma}_{p,\theta}$ with domain $H^{\gamma+2}_{p,\theta}$, is positive in terminology of [4] and, as follows from Section 14 of [4], for any $l = 1,2,\dots$ and $u \in H^{\gamma+2l}_{p,\theta}$, the right-hand side in the formula

$$(4.10)\qquad (-\mathcal{L}_{b,c})^z u := \phi(l,z)\int_0^\infty t^{l-1+z}(t-\mathcal{L}_{b,c})^{-2l}(-\mathcal{L}_{b,c})^l u\,dt$$

converges as the Bochner integral in $H^{\gamma}_{p,\theta}$ for $|\operatorname{Re} z| < l$. Furthermore, the value of the right-hand side in (4.11) is independent of l as long as $u \in H^{\gamma+2l}_{p,\theta}$.

REMARK 4.7. For $1 > z > 0$ there is a different formula for $(-\mathcal{L}_{b,c})^z$ in terms of T^b_t (see, for instance, formula (9.11.5) of [18])

$$(-\mathcal{L}_{b,c})^z = \frac{1}{\Gamma(-z)}\int_0^\infty t^{-z-1}[e^{-ct}T^b_t - 1]\,dt\,,$$

which along with (4.9) implies that, for any $f \in C^\infty_0(\mathbb{R}^d_+)$,

$$(4.11)\qquad (-\mathcal{L}_{b,c})^z f(x) = \frac{1}{\Gamma(-z)}\int_0^\infty t^{-z-1}[e^{-ct}Ef(x_t(x)) - f(x)]\,dt\,.$$

As in Remark 4.5 formula (4.11) makes sense for all $x \in \mathbb{R}^d$ and $(-\mathcal{L}_{b,c})^z f(x) = 0$ if $x^1 \leq 0$.

REMARK 4.8. The operators $(\mathcal{L}_{b,c})^z$ enjoy the same scaling property as $\mathcal{L}_{b,c}$. Indeed, as easy to check, for any constant $\alpha > 0$ we have $\mathcal{L}_{b,c}[u(\alpha\,\cdot)] = [\mathcal{L}_{b,c}u](\alpha\,\cdot)$. It follows from here and from (4.10) that

$$(\mathcal{L}_{b,c})^z[u(\alpha\,\cdot)] = [(\mathcal{L}_{b,c})^z u](\alpha\,\cdot)\,.$$

Furthermore, for any $\alpha \in \mathbb{R}$

$$\begin{aligned}\mathcal{L}_{b,c}M^\alpha u &= M^{2+\alpha}\Delta u + 2\alpha M^{1+\alpha}D_1 u + \alpha(\alpha-1)M^\alpha u\\ &\quad + bM^{1+\alpha}D_1 u + \alpha b M^\alpha u - cM^\alpha u = M^\alpha \mathcal{L}_{b+2\alpha,c-\alpha b-\alpha(\alpha-1)}u\,.\end{aligned}$$

It follows that

$$\begin{aligned}(-\mathcal{L}_{b,c})^z &= M^\alpha(-\mathcal{L}_{b+2\alpha,c-\alpha b-\alpha(\alpha-1)})^z M^{-\alpha}\,,\\ (-\mathcal{L}_{b,c})^z M^\alpha &= M^\alpha(-\mathcal{L}_{b+2\alpha,c-\alpha b-\alpha(\alpha-1)})^z\,.\end{aligned}$$

Few other properties of $(-\mathcal{L}_{b,c})^z$ which we use in Section 5 we collect in the following lemmas.

LEMMA 4.9. *Let* $b, \theta \in \mathbb{R}$, $z \in \mathbb{Z}$, *and* $c \geq c_0(b, p, \theta)$.

(i) *If* $u \in H^{\infty}_{p,\theta} := \bigcap_{\gamma} H^{\gamma}_{p,\theta}$, *then* $(-\mathcal{L}_{b,c})^z u \in H^{\infty}_{p,\theta}$. *In particular* $\psi(-\mathcal{L}_{b,c})^z \phi \in C_0^{\infty}(\mathbb{R}^d_+)$ *for any* $\psi, \phi \in C_0^{\infty}(\mathbb{R}^d_+)$.

(ii) *For any* $z_1, z_2 \in \mathbb{Z}$ *and* $u \in H^{\infty}_{p,\theta}$, *we have* $(-\mathcal{L}_{b,c})^{z_1}((-\mathcal{L}_{b,c})^{z_2} u) = (-\mathcal{L}_{b,c})^{z_1+z_2} u$.

(iii) *If* $c_i \geq c_0(b, p, \theta)$, $z_i \in \mathbb{Z}$, $i = 1, 2$, *then the operators* $(-\mathcal{L}_{b,c_1})^{z_1}$ *and* $(-\mathcal{L}_{b,c_2})^{z_2}$ *commute on* $H^{\infty}_{p,\theta}$.

(iv) *For any generalized function u on* $\mathbb{R}^d_+$ *and any* $\phi \in C_0^{\infty}(\mathbb{R}^d_+)$, *the generalized function* $(-\mathcal{L}_{b,c})^z(\phi u)$ *on* $\mathbb{R}^d_+$ *is well defined, belongs to* $H^{-\infty}_{p,\theta} := \bigcup_{\gamma} H^{\gamma}_{p,\theta}$, *and satisfies*

$$(\psi, (-\mathcal{L}_{b,c})^z(\phi u)) = (\phi(-\mathcal{L}^*_{b,c})^z \psi, u) \quad \forall \psi \in C_0^{\infty}(\mathbb{R}^d_+) \tag{4.12}$$

(*where* $\mathcal{L}^*_{b,c}$ *is the formal adjoint to* $\mathcal{L}_{b,c}$).

PROOF. Assertion (i) follows from Remark 4.6 and Lemma 2.2 (iv). Assertion (ii) is proved in Section 14 of [4]. Assertion (iii) is easily checked if z_i are integers. In the general case it immediately follows from (4.10).

To prove assertion (iv) notice that for any generalized function u and any $\phi \in C_0^{\infty}(\mathbb{R}^d_+)$ there is a $\gamma \in \mathbb{R}$ such that $\phi u \in H^{\gamma}_{p,\theta}$. Therefore, $(-\mathcal{L}_{b,c})^z(\phi u)$ is defined indeed. Formula (4.12) follows from definition (4.10) and the fact that for integers l obviously $((-\mathcal{L}_{b,c})^{\pm l})^* = (-\mathcal{L}^*_{b,c})^{\pm l}$. The lemma is proved.

LEMMA 4.10. *Let* $\gamma, b, \theta \in \mathbb{R}$, *and* $z > 0$. *Then there exists* $c_0 \geq 0$ *such that for any* $c \geq c_0$ *and* $u \in H^{\infty}_{p,\theta}$ *we have*

$$\|u\|_{\gamma,p,\theta} \leq N \|(-\mathcal{L}_{b,c})^z u\|_{\gamma,p,\theta} ,$$

where N is independent of u.

PROOF. One knows that (see, for instance, [18]), owing to $z > 0$, we have

$$(-\mathcal{L}_{b,c})^{-z} u = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-ct} T^b_t u \, dt ,$$

which can be rewritten in terms of the process $x_t(x)$ by using (4.9). Then as in the proof of Theorem 2.7 of [7] we get that, for c large enough and $u \in C_0^{\infty}(\mathbb{R}^d_+)$,

$$\|(-\mathcal{L}_{b,c})^{-z} u\|_{\gamma,p,\theta} \leq N \|u\|_{\gamma,p,\theta} ,$$

where N is independent of u. Passing to the limit, we have this inequality for all $u \in H^{\infty}_{p,\theta}$. Finally, by substituting $(-\mathcal{L}_{b,c})^z u$ in place of u and using Lemma 4.9, we get our assertion. The lemma is proved.

5. – Equivalent norms in $H^{\gamma}_{p,\theta}$

Let an interval $(r_1, r_2) \subset \mathbb{R}_+$ contain the closed support of ζ and let $a(r)$ be an infinitely differentiable strictly positive function on $\mathbb{R}$ such that $a(r) = r$ on (r_1, r_2), $a(r) = 1$ for $r \le r_1/2$, $a(r) = 1$ for $r \ge 2r_2$. Define

$$\hat{\mathcal{L}}_{b,c} = a^2(x)\Delta + ba(x)D_1 - c\,.$$

Similarly to (4.2), for any b, γ, and p, there exists constants $N, c_0 > 0$ such that for any complex λ with $\operatorname{Re}\lambda \ge c_0$ and $v \in H^{\gamma+2}_{p,\theta}$ one has

$$\|v\|_{\gamma+2,p} + (1 + |\lambda|)\|v\|_{\gamma,p} \le N\|\hat{\mathcal{L}}_{b,\lambda}v\|_{\gamma,p}\,.$$

This allows one to define powers of $-\hat{\mathcal{L}}_{b,c}$, the semigroup $\hat{T}_t$ corresponding to $\hat{\mathcal{L}}_{b,0}$, and get a probabilistic representation of $(-\hat{\mathcal{L}}_{b,0})^z$ if $0 < z < 1$. In this situation, if c is large enough, then for any $f \in C_0^\infty(\mathbb{R}^d)$,

$$(-\hat{\mathcal{L}}_{b,c})^z f(x) = \frac{1}{\Gamma(-z)}\int_0^\infty t^{-z-1}[e^{-ct}Ef(\hat{x}_t(x)) - f(x)]\,dt\,, \tag{5.1}$$

where $\hat{x}_t(x)$ is a unique solution of the equation

$$dx_t = \sqrt{2}\,a(x_t^1)\,dw_t + ba(x_t^1)e_1\,dt, \quad x_0 = x\,. \tag{5.2}$$

Furthermore, by Seeley's theorem from [14] the operator $(-\hat{\mathcal{L}}_{b,c})^{-z}$ is a pseudo-differential operator of order $-2z$. By Corollary in Section 6.5 of [16] we have $\|(1-\Delta)^z(-\hat{\mathcal{L}}_{b,c})^{-z}u\|_p \le N\|u\|_p$ for $u \in C_0^\infty(\mathbb{R}^d)$, which implies that, for any $0 < z < 1$ and $u \in H^z_p$,

$$\|u\|_{2z,p} \le N\|(-\hat{\mathcal{L}}_{b,c})^z u\|_p \tag{5.3}$$

with N independent of u provided c is large enough.

Lemma 5.1. *Take a function $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta = 1$ on (r_1, r_2). Then for any $0 < z < 1$ and $u \in C_0^\infty(\mathbb{R}^d_+)$ we have*

$$\|\zeta u\|_{2z,p} \le N\|\zeta(-\mathcal{L}_{b,c})^z u\|_p + N\|\eta u\|_{z,p} \tag{5.4}$$

with N independent of u provided c is large enough.

Proof. The operator $\zeta(-\hat{\mathcal{L}}_{b,c})^z - (-\hat{\mathcal{L}}_{b,c})^z\zeta\,\cdot$ is a pseudo-differential operators of order $2z - 1 < z$ (see [16]). Therefore, (5.3) implies

$$\|\zeta\eta u\|_{2z,p} \le N\|(-\hat{\mathcal{L}}_{b,c})^z(\zeta\eta u)\|_p \le N\|\zeta(-\hat{\mathcal{L}}_{b,c})^z(\eta u)\|_p + N\|\eta u\|_{z,p}\,. \tag{5.5}$$

Next let $\tau(x)$ be the first exit time of $x_t^1(x)$ from (r_1, r_2). Since the coefficients of equations (4.7) and (5.2) coincide on (r_1, r_2), $\tau(x)$ is also the first exit time of $\hat{x}_t^1(x)$ from (r_1, r_2), so that for $x^1 \in (r_1, r_2)$ we have

$$E(\eta u)(\hat{x}_t(x))I_{0<t\le\tau(x)} = Eu(\hat{x}_t(x))I_{0<t\le\tau(x)} = Eu(x_t(x))I_{0<t\le\tau(x)}\,. \tag{5.6}$$

Also observe that for $x^1 \in (r_1, r_2)$, $1 < r < p$, and

$$\Gamma(-z)\hat{I}(x) := E\int_0^\infty t^{-z-1}e^{-ct}(\eta u)(\hat{x}_t(x))I_{\tau(x)<t}\,dt$$

we have

$$\begin{aligned}|\hat{I}(x)|^r &\le N\left(E\int_{\tau(x)}^\infty t^{-(z+1)r/(r-1)}e^{-ct}\,dt\right)^{r-1} E\int_0^\infty e^{-ct}|(\eta u)(\hat{x}_t(x))|^r\,dr\\ &\le N(E(\tau(x))^{-\alpha})^{r-1}(-\hat{\mathcal{L}}_{b,c})^{-1}(|\eta u|^r)(x)\,,\end{aligned}$$

where $\alpha = (z+1)r/(r-1) - 1$. Owing to uniform nondegeneracy of x_t^1 on (r_1, r_2) and the fact that the support of ζ lies strictly inside this interval, on the support of ζ the function $E(\tau(x))^{-\alpha}$ (which only depends on x^1) is bounded by a constant. It follows that

$$\|\zeta\hat{I}\|_p \le N\|\zeta(-\hat{\mathcal{L}}_{b,c})^{-1}(|\eta u|^r)\|_{p/r}^{1/r} \le N\|\,|\eta u|^r\|_{p/r}^{1/r} = N\|\eta u\|_p\,. \tag{5.7}$$

To finish the auxiliary work define $I(x)$ as $\hat{I}(x)$ with $x_t(x)$ in place of $\hat{x}_t(x)$ and notice that similarly to (5.7) (remember that $L_p(\mathbb{R}^d_+) = L_{p,d} = H^0_{p,d}$ so that considering $(-\mathcal{L}_{b,c})^{-1}$ in $L_p(\mathbb{R}^d_+)$ makes sense),

$$\|\zeta I\|_p \le N\|\zeta(-\mathcal{L}_{b,c})^{-1}(|\eta u|^r)\|_{p/r}^{1/r} \le N\|\,|\eta u|^r\|_{p/r}^{1/r} = N\|\eta u\|_p\,. \tag{5.8}$$

Now using (5.6), for $x^1 \in (r_1, r_2)$ write

$$\begin{aligned}(-\hat{\mathcal{L}}_{b,c})^z(\eta u)(x) &= \frac{1}{\Gamma(-z)}\int_0^\infty t^{-z-1}[e^{-ct}Eu(x_t(x))I_{0<t\le\tau(x)} - u(x)]\,dt\\ &+ \frac{1}{\Gamma(-z)}E\int_0^\infty t^{-z-1}e^{-ct}u(\hat{x}_t(x))I_{\tau(x)<t}\,dt = (-\mathcal{L}_{b,c})^z u(x) + \hat{I}(x) - I(x)\,.\end{aligned}$$

Combining (5.7), (5.8), and (5.5), we obviously come to (5.4). The lemma is proved.

THEOREM 5.2. *For any γ, ν, p, θ, b, there exist a constant $c_0 > 0$ such that, for any $c \ge c_0$, the operator $(-\mathcal{L}_{b,c})^\gamma$ is extendible to a bounded operator from $H^{2\gamma+\nu}_{p,\theta}$ into $H^\nu_{p,\theta}$ and, if we keep the same notation for the extension, then for any $u \in H^{\gamma+\nu}_{p,\theta}$ we have*

$$\|u\|_{2\gamma+\nu,p,\theta} \le N\|(-\mathcal{L}_{b,c})^\gamma u\|_{\nu,p,\theta} \le N\|u\|_{2\gamma+\nu,p,\theta}\,, \tag{5.9}$$

where the constants N are independent of u.

PROOF. We only need prove (5.9) for a dense subset in $H^{2\gamma+\nu}_{p,\theta}$, say for $u \in C_0^\infty(\mathbb{R}^d)$. In addition, the formula $(-\mathcal{L}_{b,c})^\nu(-\mathcal{L}_{b,c})^\gamma = (-\mathcal{L}_{b,c})^{\gamma+\nu}$ shows that it suffices to consider $\nu = 0$. Finally, if $\gamma = n + \alpha$ with $n \in \{0, \pm 1, \dots\}$ and $\alpha \in (0, 1)$, we notice that by Theorem 2.8 of [7] the operator $(-\mathcal{L}_{b,c})^{-n}$ (for c large) is a bounded operator with bounded inverse acting from $H^{2\alpha}_{p,\theta}$ onto $H^{2n+2\alpha}_{p,\theta}$ so that

$$\|u\|_{2\gamma,p,\theta} = \|(-\mathcal{L}_{b,c})^{-n}(-\mathcal{L}_{b,c})^n u\|_{2n+2\alpha,p,\theta} \sim \|(-\mathcal{L}_{b,c})^n u\|_{2\alpha,p,\theta}\,,$$

which shows that we may confine ourselves to the case $\gamma = \alpha \in (0, 1)$.

In this case by Lemma 5.1 and by the scaling property of $(-\mathcal{L}_{b,c})^z$ from Remark 4.6

$$\begin{aligned}\|u\|^p_{2\gamma,p,\theta} &= \sum_{n=-\infty}^{\infty} e^{n\theta}\|\zeta A_n u\|^p_{2\gamma,p} \\ &\le N\sum_{n=-\infty}^{\infty} e^{n\theta}\|\zeta A_n(-\mathcal{L}_{b,c})^\gamma u\|^p_p + N\sum_{n=-\infty}^{\infty} e^{n\theta}\|\eta A_n u\|^p_{\gamma,p} \\ &\le N\|(-\mathcal{L}_{b,c})^\gamma u\|^p_{0,p,\theta} + N\|u\|^p_{\gamma,p,\theta}\,.\end{aligned}$$

To finish proving the left inequality in (5.9) it only remains to use interpolation Theorem 2.6 and Lemma 4.10. To prove the remaining one it suffices to take $(-\mathcal{L}_{b,c})^{-\gamma}v$ in place of u in the first one. The theorem is proved.

COROLLARY 5.3. *If c is large enough, the operators M^α and $(-\mathcal{L}_{b,c})^\gamma$ are interchangeable in the sense that*

$$\|(-\mathcal{L}_{b,c})^\gamma M^\alpha u\|_{\nu,p,\theta} \le N\|M^\alpha(-\mathcal{L}_{b,c})^\gamma u\|_{\nu,p,\theta} \le N\|(-\mathcal{L}_{b,c})^\gamma M^\alpha u\|_{\nu,p,\theta}\,.$$

In addition,

$$\|M^\alpha(-\mathcal{L}_{b,c})^\gamma u\|_{\nu,p,\theta} \le N\|M^\alpha u\|_{\gamma+\nu,p,\theta}$$

Indeed, for large c each participating norm is equivalent to

$$\|u\|_{2\gamma+\nu,p,\theta+\alpha p}\,.$$

REFERENCES

[1] S. AGMON, *On the eigenfunctions and on the eigenvalues of general elliptic boundary problems*, Comm. Pure Appl. Math., No. 2, **15** (1961), 119-147.

[2] G. DA PRATO – J. ZABCZYK, "Stochastic equations in infinite dimensions", Cambridge University Press, New York, NY, 1992.

[3] P. GRISVARD, "Elliptic Problems in nonsmooth domains", Monographs and Studies in Mathematics 24, Pittman, Boston-London-Melbourn, 1985.

[4] M. A. KRASNOSELSKII – E. I. PUSTYLNIK – P. E. SOBOLEVSKII – P. P. ZABREJKO, "Integral operators in spaces of summable functions", Nauka, Moscow, 1966 in Russian; English translation: Noordhoff International Publishing, Leyden, 1976.

[5] S. G. KREIN – JU. I. PETUNIN – E. M. SEMENOV, "Interpolation of linear operators", Nauka, Moscow, 1978 in Russian; English translation in Translations of Math. Monographs, Vol. 54, Amer. Math. Soc., Providence, RI, 1982.

[6] N. V. KRYLOV, *Parabolic equations in the plane and semigroups*, Differentsialnye Uravnenija, No. 12, **5** (1969), 2216-2244 in Russian; English translation in Differential Equations **5** (1969), 1661-1667.

[7] N. V. KRYLOV, *Weighted Sobolev spaces and Laplace's equation and the heat equation in a half space*, Comm. Partial Differential Equations, **24** (1999), 1611-1653.

[8] N. V. KRYLOV, *Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces*, Preprint No. 99-091, SFB 343, University of Bielefeld.

[9] N. V. KRYLOV – S. V. LOTOTSKY, *A Sobolev space theory of SPDEs with constant coefficients on a half line*, SIAM J. Math. Anal., **30** (1999) (1998 in electronic form http://www.siam.org/journals/sima/30-2/32690.html), 298-325.

[10] N. V. KRYLOV – S. V. LOTOTSKY, *A Sobolev space theory of SPDEs with constant coefficients in a half space*, to appear in SIAM J. Math Anal.

[11] N. V. KRYLOV – B. L. ROZOVSKII, *On the Cauchy problem for stochastic linear partial differential equations*, Izvestija Acad. Nauk SSSR, ser. math., No. 6, **41** (1977), 1329-1347 in Russian; English translation Math. USSR-Izvestija, No. 6, **11**, (1977), 1267-1284.

[12] H. KOCH, "Non-Euclidean singular integrals and the porous medium equation", Habilitationsschrift, Facultät für Math., Ruprecht-Karls-Universität, Heidelberg, 1998.

[13] J.-L. LIONS – E. MAGENES, "Problèmes aux limites non homogènes et applications, 1", Dunod, Paris, 1968.

[14] R. T. SEELEY, *Complex powers of an elliptic operator*, In: "Proc. of Symposia in Pure Math.", Vol. 10, American Math. Soc, Providence, RI, 1967.

[15] V. A. SOLONNIKOV, *Solvability of the classical initial-boundary-value problems for the heat-conduction equation in a dihedral angle*, Zapiski Nauchnykh Seminarov LOMI, **138** (1984), 146-180 in Russian; English translation in Journal of Soviet Math., No. 5, **32** (1986), 526-546.

[16] E. M. STEIN, "Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals", Princeton Univ. Press, Princeton, NJ, 1993.

[17] H. TRIEBEL, "Theory of function spaces II", Birkhäuser Verlag, Basel-Boston-Berlin, 1992.

[18] K. YOSIDA, "Functional analysis", Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.

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