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Distance Between Components in Optimal Design Problems with Perimeter Penalization

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Abstract. We consider minimal energy configurations of mixtures of two materials in $\Omega \subset \mathbb{R}^2$, where the energy includes a penalization of the length of the interface between the materials. We show that the distance between any two components of either material is positive away from the boundary.

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1. – Introduction

Many problems in materials science, such as optimal design, phase transitions, liquid crystals, etc., involve interfacial energies. A model problem is the optimal design of composites of two or more materials, with an energy penalty on the length of the interface separating the materials. Optimal design without such a penalty has been the subject of extensive study (see especially [8], [7], and the English translations of these and other papers in [2]). More recently, [1] and [6] independently investigated effects due to the presence of interfacial energy in a broad class of problems that includes problems of optimal design of composites of two materials. They considered minimizing functionals of the form

$$(u, A) \mapsto \int_\Omega \sigma_A |\nabla u|^2 dx + P_\Omega(A)$$

where $A \subset \Omega \subset \mathbb{R}^N$ is measurable, σ_A is a positive two-valued function taking the smaller value in A and the larger in A^c, $u \in H^1(\Omega)$, and $P_\Omega(A)$ is the perimeter of A in Ω. In [6], this problem is considered subject to a Dirichlet condition on u and a constraint with respect to where A meets $\partial\Omega$, and in [1], a zero Dirichlet condition is assumed, but there are additional terms in the energy motivated by problems of optimal design. A is interpreted to be the region occupied by one material and A^c is occupied by the other, σ_A represents

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the conductivities of the two materials, u is the temperature, and a term of the form $-2\int_\Omega fudx$ can be included in the energy, where $f \in L^\infty(\Omega)$ represents a source density (see Example 2.5 in [1]).

In this paper, we study the components of optimal sets A when $\Omega \subset \mathbb{R}^2$. First, we consider the problem of minimizing

$$E(u, A) := \int_\Omega \sigma_A |\nabla u|^2 dx - 2\int_\Omega fudx + P_\Omega(A) + C|A| \tag{1.1}$$

over $u \in H_0^1(\Omega)$ and measurable A. We take $\sigma_A = 1$ in A and $\sigma_A = L > 1$ in A^c. Typically in optimal design, one assumes a fixed amount of the material A (i.e., fixed measure of A). Here, we first replace this constraint with the $C|A|$ term (where $|A|$ is the Lebesgue measure of A). This eases some of the analysis, as well as possibly reflecting the fact that in practice A would simply be the more expensive material, resulting in an added cost of $C|A|$. At the end of the paper, we indicate why the $C|A|$ term can be replaced by a constraint on the measure of A. As we explain below, our results are easy when there is no term in $|A|$ or constraint on $|A|$, but instead there is a constraint involving $\partial A \cap \partial\Omega$.

This energy E is in the class considered in [1]. The main results in [1] are the existence of minimizers and a proof that optimal sets A are essentially open. In fact, they prove that up to a set of $\mathcal{H}^1$ measure zero, there is no difference between the measure theoretic boundary of A and its topological boundary (see the proof of Theorem 2.2 in [1]). [6] proves the critical Hölder regularity $u \in C^{0,1/2}(\Omega)$, and the partial regularity of ∂A.

It is our conjecture that stronger regularity holds. In particular, that optimal u are Lipschitz. From this, it would follow from quasiminimal surface theory that A and its complement each occupy a finite number of connected regions with C^1 boundaries (for a relatively quick proof that does not rely on quasiminimal surface theory, see [5]). The idea to show u is Lipschitz is based on the fact that ∇u can only blow up at a singularity in ∂A. Here we show that there cannot be singularities in ∂A due to boundaries of different components intersecting, thereby demonstrating that pairs of components of either material must be a positive distance apart in any $\Omega' \subset\subset \Omega$. Note that this is simple in the case of the problem considered in [6], since the constraint on A there doesn't affect compact perturbations of A. In that case, it is immediate from equation (3.1) below (with $C = 0$) that components cannot touch. We also point out that, in any case, it is immediate from the relative isoperimetric inequality for $\Omega \subset \mathbb{R}^N$ that A^c has a finite number of components.

Our approach is as follows. For a minimizing pair (u, A), we know from the analysis in [6] that

$$\int_{B_r(y)} |\nabla u|^2 dx \le kr \text{ for some } k > 0 \text{ and all } B_r(y) \subset \Omega\,. \tag{1.2}$$

Furthermore, since

$$-\mathrm{div}(\sigma_A \nabla u) = f\,,$$

u satisfies a Caccioppoli inequality. In particular, it is easy to see from Section 2.2 of [4] that for $c > 0$, there exists $c' > 0$ such that

$$\begin{aligned} &\text{if } \int_{B_r(y)} |\nabla u|^2 dx \geq cr, \\ &\text{then } \int_{B_{r/2}(y)} |\nabla u|^2 dx \leq c' \fint_{B_r(y)\setminus B_{r/2}(y)} |u - \bar{u}|^2 dx \,, \end{aligned} \tag{1.3}$$

where $\bar{u}$ is the average of u over $B_r(y)$. We can then study the minimization of the functional

$$\mathcal{A} \mapsto \int_\Omega \sigma_{\mathcal{A}} |\nabla u|^2 dx + P_\Omega(\mathcal{A}) + C|\mathcal{A}|$$

assuming only that u satisfies (1.2) and the above Caccioppoli inequality (1.3) (and knowing that for a minimizer A, we have partial regularity of ∂A, etc.).

We first show, in Lemma 3.1, that if two components of a minimizer A meet at a point in Ω, then in small neighborhoods of this point, the components must become elongated and fit in a relatively narrow rectangle T. In Theorem 3.3, we then consider small disjoint balls whose union contains most of T, and whose radii are large compared to T's width. It follows that there must be high concentrations of $|\nabla u|^2$ in these balls, yet the measure of these components of A are relatively small. Using Lemma 3.2, we show that either a significant proportion of $|\nabla u|^2$ must "spill over" to A^c in these balls, or there must be a quantity of A nearby besides what lies in T. In either case, it reduces E to expand A, as this expansion either weighs more of $|\nabla u|^2$ by the lower value of σ_A in A, or it consumes part of the boundaries of the other nearby components, which outweighs the increase in perimeter caused by the expansion. We then show that our analysis applies to components of A^c. Finally, we indicate why our results hold if the $C|A|$ term is replaced by a constraint on the total measure of A.

We note that proving a version of Lemma 3.1 for dimensions greater than 2 seems to be significantly more complicated (or even impossible), while the proof of Lemma 3.2 is not dimension dependent (except that for $\Omega \subset \mathbb{R}^N$, factors of r would be replaced by factors of r^{N-1}).

2. – Preliminaries and notation

For a measurable set $E \subset \Omega \subset \mathbb{R}^2$, $|E|$ denotes its Lebesgue measure and the perimeter of E in Ω is

$$P_\Omega(E) := \sup \left\{ \int_E \mathrm{div}\phi dx : \phi \in C_0^1(\Omega; \mathbb{R}^2), |\phi| \leq 1 \right\} .$$

We denote by $B_r(x)$ the open ball centered at x with radius r. If $P_\Omega(E) < \infty$, then we define the measure theoretic boundary $\partial_* E$ in Ω to be

$$\{x \in \Omega : \limsup_{r\to 0} r^{-2}|E \cap B_r(x)| > 0 \text{ and } \limsup_{r\to 0} r^{-2}|E^c \cap B_r(x)| > 0\},$$

and there exists a $\mathcal{H}^1$ measurable function $\nu_E : \partial_* E \to S^1$ satisfying

$$\int_E \operatorname{div}\phi dx = \int_{\partial_* E} \phi \cdot \nu_E d\mathcal{H}^1 \tag{2.1}$$

for all $\phi \in C_0^1(\Omega; \mathbb{R}^2)$, where $\mathcal{H}^1$ is the 1 dimensional Hausdorff measure and S^1 is the unit circle. Note that we then have $P_\Omega(E) = \mathcal{H}^1(\partial_* E)$.

It is easy to see from the definition of measure theoretic boundary that if $E = E_1 \cup E_2$, then

$$P_\Omega(E) \le P_\Omega(E_1) + P_\Omega(E_2)\,, \tag{2.2}$$

and this inequality can be strict. However, it follows from the proof of Theorem 2.2 in [1] that an optimal A satisfies $\mathcal{H}^1(\partial_* A) = \mathcal{H}^1(\overline{\partial_* A})$, is equivalent to a set that is the interior of its closure, and taking A to be this open set, we have $\mathcal{H}^1(\partial A \backslash \partial_* A) = 0$. Hence, we can consider the at most countable components A_i of A, and write $A = \cup A_i$. It follows from Lemma 3.3 and the proof of Lemma 3.4 in [5] that

$$P_\Omega(A) = P_\Omega(A_i) + P_\Omega(A \backslash A_i)\,, \tag{2.3}$$

and this equality holds if A_i is replaced with $A_i \cup A_j$, or any union of components. The effect of this is that if we add a set S to A, then from (2.2), $P_\Omega(A \cup S) \le P_\Omega(A_i \cup S) + P_\Omega(A \backslash A_i)$, and so by (2.3) $P_\Omega(A) - P_\Omega(A \cup S) \ge P_\Omega(A_i) - P_\Omega(A_i \cup S)$. Again, the same holds if A_i is replaced with any union of components. This fact will be used repeatedly below.

3. – Distance between components

We first study the local geometry of A near intersections of its components in Ω.

LEMMA 3.1. *Let A_i and A_j be components of A, and suppose $x \in \partial A_i \cap \partial A_j \cap \Omega$. For $0 < r <$ dist $(x, \partial\Omega)$, we can choose components A_i' and A_j' of $A_i \cap B_r(x)$ and $A_j \cap B_r(x)$ respectively, such that $x \in \partial A_i' \cap \partial A_j'$ and there exists a rectangle T_r containing $A_i' \cup A_j'$ with length $2r$ and width $o(r)$.*

Proof. Notice first that for $r > 0$, only a finite number of components of $A_i \cap B_r(x)$ intersect $B_{r/2}(x)$ (since A has finite perimeter), and so x is in the boundary of one (or more) of these components. The same is true for A_j, and so we can choose components A_i', A_j' whose boundaries contain x. For the rest of this proof, we will write A_i for A_i', etc.

Let $z_i \in A_i$, $z_j \in A_j$, and let l be the line segment from z_i to z_j and d be the diameter of $B_r(x)$ parallel to l. Set $\beta :=$ dist (d, l) and let $0 < \varepsilon < \beta$. Set $D := A_i \cup A_j \cup B_\varepsilon(x)$, which is connected. Let $\Gamma \subset D$ be a simple arc containing x and intersecting l only at the endpoints of Γ, so that for a segment $l' \subset l$, $\Gamma \cup l'$ is a Jordan curve. We denote its interior by I, and note that I is connected.

Take a basis vector e_1 to be parallel to l and construct $\phi = \phi_1 e_1 + \phi_2 e_2 \in C_0^1(I; \mathbb{R}^2)$ with $|\phi| \leq 1$, $\phi_2 = 0$, and $|\int_{A_i} \frac{\partial}{\partial x_1} \phi_1 dy|$ arbitrarily close to β. It then follows from (2.1) that $|\int_{I \cap \partial_* A_i} e_1 \cdot \nu_{A_i} d\mathcal{H}^1| \geq \beta$. The same holds for A_j, and a similar argument shows

$$\left| \int_{I \cap \partial_*(A_i \cup A_j)} e_2 \cdot \nu_{A_i \cup A_j} d\mathcal{H}^1 \right| \geq \mathcal{H}^1(l \backslash [A_i \cup A_j]) .$$

We then have

$$\mathcal{H}^1(I \cap [\partial_* A_i \cup \partial_* A_j]) \geq \left(4\beta^2 + \mathcal{H}^1(l \backslash [A_i \cup A_j])^2\right)^{\frac{1}{2}} =: \alpha .$$

Adding $I \cup B_\varepsilon(x)$ to A therefore reduces perimeter by at least $\alpha - \mathcal{H}^1(l \backslash [A_i \cup A_j]) - 2\pi\varepsilon$ and increases E by no more than $C\pi r^2$. Hence, it must be that

$$\alpha - \mathcal{H}^1(l \backslash [A_i \cup A_j]) \leq C\pi r^2 . \tag{3.1}$$

It follows that β is $o(r)$, and the lemma follows. □

The Hölder continuity of u follows from natural modifications to the proof in [6]. Indeed, the limits of the blow-up sequences in Lemma 2.2 of that paper and the corresponding limits for the problem here are the same. Theorem 2, showing $u \in C^{0,\frac{1}{2}}$, applies here with very minor modification. In fact, we won't actually use Hölder continuity, but rather the inequality (1.2) from which the Hölder continuity follows.

We now show that, in balls where there is a concentration of $|\nabla u|^2$, there is a lower bound on either the proportion of A, or on how much of $|\nabla u|^2$ must "spill over" from A to A^c. The strategy for this proof was inspired by Lemma 2.2 in [6].

Lemma 3.2. *Given $c > 0$, there exists $\gamma > 0$ such that if $0 < r <$dist$(y, \partial\Omega)$ and*

$$\int_{B_{r/2}(y)} |\nabla u|^2 dx \geq cr ,$$

then either

$$\text{i)}\quad |B_r(y)\cap A| \geq \gamma |B_r(y)| \quad \textit{or}$$

$$\text{ii)}\quad \int_{B_r(y)\setminus A} |\nabla u|^2 dx \geq \gamma r\,.$$

PROOF. Suppose to the contrary that there exists a sequence $\{x_i, r_i\}$ such that

$$\int_{B_{r_i/2}(x_i)} |\nabla u|^2 dx \geq c r_i\,,$$

$$|B_i \cap A| < \frac{1}{i}|B_i|\,,$$

and

$$\int_{B_i\setminus A} |\nabla u|^2 dx < \frac{1}{i} r_i\,,$$

where B_i denotes $B_{r_i}(x_i)$. Define $v_i \in H^1(B_1(0))$ by

$$v_i(x) := \frac{u(x_i + x r_i) - \bar{u_i}}{||\nabla u||_{L^2(B_i)}}\,,$$

where $\bar{u_i}$ is the average of u on B_i. Also, define $A_i := \frac{A - x_i}{r_i}$ and now denote $B_1(0)$ by B_1. Then

$$\text{a)}\quad |B_1 \cap A_i| < \frac{1}{i}\pi \qquad \text{and}$$

$$\text{b)}\quad \int_{B_1\setminus A_i} |\nabla v_i|^2 dx < \frac{1}{i}\frac{1}{c}\int_{B_{1/2}} |\nabla v_i|^2 dx\,.$$

Notice that v_i are bounded in $H^1(B_1)$ so, for a subsequence, $v_i \rightharpoonup v$ in $H^1(B_1)$ for some v. Note also that a) implies $|B_1 \cap A_i| \to 0$. From b), and since v_i are bounded in $H^1(B_1)$, it follows that $||\nabla v_i||_{L^2(B_1\setminus A_i)} \to 0$. Let $\delta > 0$, and choose $T_\delta \subset B_1$ measurable such that $|B_1\setminus T_\delta| < \delta$ and $\chi_{A_i} \to 0$ uniformly on T_δ. Then for i sufficiently large, $T_\delta \cap A_i = \emptyset$ and we have

$$\int_{T_\delta} |\nabla v|^2 dx \leq \liminf_{i\to\infty} \int_{T_\delta} |\nabla v_i|^2 dx = 0\,.$$

It follows that $\int_{B_1} |\nabla v|^2 dx = 0$, and so $v = \text{constant} = 0$ since the average of v is 0.

We now have that $v_i \to 0$ in $L^2(B_1)$ and $||\nabla v_i||_{L^2(B_1)} = 1$. We show that this contradicts the Caccioppoli inequality from (1.3)

$$\int_{B_{r/2}} |\nabla u|^2 dx \leq c' \fint_{B_r} |u - \bar{u}|^2 dx\,.$$

It is easy to see that v_i inherit a corresponding inequality

$$\int_{B_{1/2}} |\nabla v_i|^2 dx \le c' \fint_{B_1} |v_i|^2 dx\,.$$

Since $v_i \to 0$ in $L^2(B_1)$, both terms in this inequality go to 0. However, we have from (1.2) that $||\nabla u||^2_{L^2(B_i)} \le kr_i$, and so

$$\int_{B_{1/2}} |\nabla v_i|^2 dx = \frac{1}{||\nabla u||^2_{L^2(B_i)}} \int_{B_{r_i/2}(x_i)} |\nabla u|^2 dx \ge \frac{1}{kr_i} cr_i = \frac{c}{k} > 0$$

which is a contradiction. □

We now prove our main result, a theorem showing that E can be reduced near points in Ω where boundaries of components intersect.

THEOREM 3.3. *Let A_i and A_j be components of A. Then* $\operatorname{dist}(A_i, A_j) > 0$ *away from $\partial\Omega$.*

PROOF. We suppose $x_0 \in \partial A_i \cap \partial A_j \cap \Omega$ and show that E can be reduced in $B_r(x_0)$ for sufficiently small r. Using Lemma 3.2 with $c = \frac{1}{L-1}$, we find γ as specified by that lemma and for each $0 < r <$dist$(x_0, \partial\Omega)$, we consider a rectangle T_r guaranteed by Lemma3.1. We also will use the notation A_i for A_i', etc., as we did in the proof of Lemma 3.1. We then essentially cover the longer axis of T_r with disjoint balls $B_{r'}(x_\alpha)$, where $r' = \eta r$ and η is the reciprocal of a positive integer (the number of balls) and will be specified later.

We first estimate $\int_{B_{r'/2}(x_\alpha)} |\nabla u|^2 dx$. Define $\tilde{A} := A \setminus (B_{r'/2}(x_\alpha) \cap (A_i \cup A_j))$, and note that replacing A with $\tilde{A}$ reduces perimeter by $\mathcal{H}^1((\partial A_i \cup \partial A_j) \cap B_{r'/2}(x_\alpha))$ and increases perimeter by $\mathcal{H}^1((A_i \cup A_j) \cap \partial B_{r'/2}(x_\alpha))$ which we know from Lemma 3.1 is $o(r)$. For $r > 0$ small enough, $\frac{2}{r'}\mathcal{H}^1((\partial A_i \cup \partial A_j) \cap B_{r'/2}(x_\alpha))$ is arbitrarily close to 4 (or is greater than 4), while $\frac{2}{r'}\mathcal{H}^1((A_i \cup A_j) \cap \partial B_{r'/2}(x_\alpha))$ is arbitrarily close to zero. Hence, for r small enough, replacing A with $\tilde{A}$ reduces perimeter by at least r'. However, this replacement shifts some Dirichlet energy off of A, resulting in an increase in energy of no more than

$$(L-1) \int_{A \cap B_{r'/2}(x_\alpha)} |\nabla u|^2 dx\,.$$

Since this replacement cannot reduce E, we must have

$$(L-1) \int_{A \cap B_{r'/2}(x_\alpha)} |\nabla u|^2 dx \ge r'$$

and so

$$\int_{B_{r'/2}(x_\alpha)} |\nabla u|^2 dx \ge \frac{r'}{L-1}\,.$$

Now consider the smallest rectangle T' enclosing all the balls $B_{r'}(x_\alpha)$. If we add this set to A, perimeter is increased by no more than $4r'$ and $C|A|$ is increased by no more than $4Crr'$. We now show that this is more than offset by the total reduction in E.

To see the reduction in each ball, consider one of the balls $B_{r'}(x_\alpha)$ and suppose first inequality ii) of Lemma 3.2 holds. Then adding T' to A reduces energy in this ball by at least $(L-1)\gamma r'$. Next, suppose inequality i) holds. Since

$$\frac{1}{r^2}|B_r \cap (A_i \cup A_j)| \to 0 \text{ as } r \to 0\,,$$

it must be that, for r sufficiently small,

$$|B_{r'}(x_\alpha) \cap A \backslash (A_i \cup A_j)| \geq \frac{1}{2}\gamma |B_{r'}(x_\alpha)|\,.$$

Hence, denoting by B one of the two components of $B_{r'}(x_\alpha)\backslash(\bar{A}_i \cup \bar{A}_j)$, it must be that $|B \cap A| \geq \frac{\gamma}{4}|B_{r'}(x_\alpha)|$ and $|B_{r'}(x_\alpha)\backslash(B \cap A)| \geq \frac{\gamma}{4}|B_{r'}(x_\alpha)|$. By the relative isoperimetric inequality (see, e.g., Theorem 2 (ii) in Section 5.6.2 of [3], Theorem 5.4.3 in [9]), it follows that

$$\mathcal{H}^1(\partial(A \cap B) \cap B) \geq \frac{1}{\bar{c}}\sqrt{\frac{\gamma}{4}\pi}\, r'\,,$$

where $\bar{c}$ is the constant from this isoperimetric inequality. Adding T' to A removes this boundary, and so reduces E in this ball by at least $\frac{1}{\bar{c}}\sqrt{\frac{\gamma}{4}\pi}\, r'$. Setting

$$m := \min\left\{\frac{1}{\bar{c}}\sqrt{\frac{\gamma}{4}\pi},\, (L-1)\gamma\right\}\,,$$

as T' contains $\frac{1}{\eta}$ balls, there is a total reduction of at least $\frac{m}{\eta}r'$. Finally, since $\frac{m}{\eta}r' > 4r' + 4Crr'$ for η sufficiently small, E is reduced by adding T' to A. This contradicts (u, A) being a minimizer, and it must be that $\partial A_i \cap \partial A_j \cap \Omega = \emptyset$. □

If O_i, O_j are components of A^c, it is not hard to extend the previous analysis to show that they too cannot meet. Considering a ball of radius r centered at a point in $\partial O_i \cap \partial O_j$, an argument similar to Lemma 3.1 shows that a component of $\bar{A}$ in this ball must be contained in a rectangle T_r as specified by that lemma. Theorem 3.3 then applies just as for intersections of ∂A_i and ∂A_j.

Lastly, we sketch why this analysis holds when the term $C|A|$ is replaced by a constraint on $|A|$. Throughout, the role of $C|A|$ has been to provide the ability to make small additions to A with a cost proportional to area, and this cost has been overcome by other energy reductions. If $|A|$ is constrained, such additions are not possible. However, it is enough to show that, for a minimizer (u, A) of the constrained problem, the small additions to A can be achieved by removing the same amount of A from elsewhere. As long as this can be done with a cost bounded by a constant times the measure of the removed set, our analysis holds for the constrained problem.

From [6], we know that $\mathcal{H}^1$ almost every point in ∂A has a neighborhood in which ∂A is $C^{1,\alpha}$. Consider such a point x, and choose $r > 0$ such that $\partial A \cap B_r(x)$ can be considered the graph of a $C^{1,\alpha}$ function $f : [-r, r] \to \mathbb{R}$, with A the epigraph of f. If f is not concave, then some of A can be removed while reducing perimeter. Since within this ball $\nabla u \in L^\infty$, removing a subset of A from this ball increases the first term of E in proportion to the measure of the subset. Hence, we need only consider the case where any removal of A increases perimeter, i.e., f is concave (and so it has negative mean curvature). If we then keep u fixed, add a $C|A|$ term, and minimize over only variations of f, a simple calculation shows that for C big enough, for the new minimizing function $\tilde{f}$ and corresponding set $\tilde{A}$, the mean curvature of $\tilde{f}$ will be less than that of f, so $\tilde{f} \geq f$. Hence, $\tilde{A}$ will have less area by ΔA, $P_\Omega(\tilde{A})$ will be larger by Δl, and so $C\Delta A \geq \Delta l + (L-1)\int_{A\setminus\tilde{A}} |\nabla u|^2 dy$, so that if A is replaced by $\tilde{A}$ in the constrained problem, the energy cost per area is bounded by C. We note that the idea behind these calculations is similar to that for the existence of Lagrange multipliers for the constrained problem.

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