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Exponential Sums and Additive Problems Involving Square-free Numbers

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Abstract. Let $r_\nu(N)$ denote the number of representations of the integer N as a sum of ν square-free numbers. We obtain unconditional and conditional bounds for the error term in the asymptotic formula for $r_\nu(N)$, when $\nu \geq 3$. The conditional bounds are essentially best possible for $\nu \geq 4$. The unconditional bounds are, for $\nu \geq 3$, essentially best possible with respect to the present knowledge on the distribution of the zeros of the Riemann zeta function. Proofs are based on the circle method. The main ingredients are a new pointwise estimate for the exponential sum $S(\alpha)$ over square-free numbers and a recent bound (see [3]) for the L^2-norm of $S(\alpha)$ restricted to the minor arcs.

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1. – Introduction

Let $r_\nu(N)$ denote the number of representations of the natural number N as the sum of ν square-free numbers. In a series of papers Evelyn and Linfoot [5] established asymptotic formulae for $r_\nu(N)$. When $\nu \geq 2$, they obtained

$$r_\nu(N) = \frac{1}{(\nu-1)!}\left(\frac{6}{\pi^2}\right)^\nu \mathfrak{S}_\nu(N) N^{\nu-1} + O(N^{\nu-1-\theta(\nu)+\varepsilon}), \tag{1}$$

where $\mathfrak{S}_\nu(N)$ is the singular series defined by

$$\mathfrak{S}_\nu(N) = \prod_{p^2 \nmid n}\left(1 - \frac{1}{(1-p^2)^\nu}\right)\prod_{p^2 \mid n}\left(1 - \frac{1}{(1-p^2)^{\nu-1}}\right), \tag{2}$$

and where

$$\theta(2) = \theta(3) = \frac{1}{3}, \quad \theta(\nu) = \frac{1}{2} - \frac{1}{2\nu} \ (\nu \geq 4).$$

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Note that $\theta(\nu)$ is strictly increasing with $\theta(\nu) \to \frac{1}{2}$ as $\nu \to \infty$. When $\nu \geq 3$, these results were obtained by the Hardy-Littlewood circle method, and have been refined by Mirsky [12] to

$$\theta(\nu) = \frac{1}{2} - \frac{1}{4\nu - 2} \quad (\nu \geq 3).$$

When $\nu = 2$, Evelyn and Linfoot used an elementary argument which was much simplified by Estermann [4]. More recently Heath-Brown [9] considered the related problem of counting square-free twins; his method carries over to the problem under consideration and yields the validity of (1) with $\theta(2) = \frac{4}{11}$. In this paper we analyse the error term in (1) more precisely, and link it with the zeros of Dirichlet L-functions. The first theorem improves on the results of Evelyn and Linfoot and of Mirsky for all $\nu \geq 3$.

THEOREM 1. *Let $\nu \geq 3$. Then*

$$r_\nu(N) = \frac{1}{(\nu-1)!}\left(\frac{6}{\pi^2}\right)^\nu \mathfrak{S}_\nu(N) N^{\nu-1} + O(N^{\nu-\frac{3}{2}+\varepsilon}).$$

A further improvement of the exponent in the error term below $\nu - \frac{3}{2}$ would imply a zero-free strip of the Riemann zeta function to the left of $Re(s) = 1$. This is a consequence of the following theorem.

THEOREM 2. *Let ϑ denote the supremum of the real parts of the zeros of the Riemann zeta function. Then, for any $\nu \geq 2$ and any $\varepsilon > 0$,*

$$r_\nu(N) - \frac{1}{(\nu-1)!}\left(\frac{6}{\pi^2}\right)^\nu \mathfrak{S}_\nu(N) N^{\nu-1} = \Omega(N^{\nu-2+\frac{1}{2}\vartheta-\varepsilon}).$$

We have $\vartheta \geq \frac{1}{2}$, with equality should the Riemann hypothesis hold. One may conjecture that the true order of magnitude of the error term in (1) is roughly of size $N^{\nu-\frac{7}{4}}$. If one is prepared to assume the generalised Riemann hypothesis (GRH) for all Dirichlet L-functions, it is possible to confirm this, at least when $\nu \geq 4$.

THEOREM 3. *Suppose that GRH holds. Then, for $\nu \geq 4$,*

$$r_\nu(N) = \frac{1}{(\nu-1)!}\left(\frac{6}{\pi^2}\right)^\nu \mathfrak{S}_\nu(N) N^{\nu-1} + O(N^{\nu-\frac{7}{4}+\varepsilon}).$$

Moreover,

$$r_3(N) = \frac{1}{2}\left(\frac{6}{\pi^2}\right)^3 \mathfrak{S}_3(N) N^2 + O(N^{\frac{37}{28}+\varepsilon}).$$

Thus, when $\nu = 3$, we miss the optimal exponent of N in the error term by $\frac{1}{14}$.

Our proofs of Theorems 1 and 3 utilize the circle method, and therefore follow the path of Evelyn and Linfoot in the first few steps. A new tool for our purposes has become available only very recently. Granville, Vaughan, Wooley and the authors [3] have investigated the mean square of the exponential sum

$$S(\alpha) = \sum_{n \le N} \mu(n)^2 e(\alpha n) \tag{3}$$

restricted to minor arcs. For a precise statement of the result which is relevant to us here, let $1 \le Q \le \frac{1}{2}N$ and denote by $\mathfrak{M}(Q)$ the union of all intervals $|q\alpha - a| \le QN^{-1}$ with $1 \le a \le q \le Q$ and $(a, q) = 1$, and let $\mathfrak{m}(Q) = [QN^{-1}, 1 + QN^{-1}] \setminus \mathfrak{M}(Q)$. Then, Theorem 1.3 of [3] asserts that

$$\int_{\mathfrak{m}(Q)} |S(\alpha)|^2 d\alpha \ll (NQ^{-\frac{1}{2}} + Q^2)N^{\varepsilon}. \tag{4}$$

Note that this is stronger than "square root cancellation" for $N^{2\varepsilon} < Q < N^{\frac{1}{2}-2\varepsilon}$. As was explained in [3], one can deduce from (4), via the circle method, that (1) holds with $\theta(2) = \frac{1}{5}$. This is weaker than the elementary result of Evelyn and Linfoot for $\nu = 2$, but for $\nu \ge 3$ one may hope to improve their work by starting from the orthogonality relation

$$r_\nu(N) = \int_0^1 S(\alpha)^\nu e(-\alpha N) d\alpha, \tag{5}$$

using (4) combined with a reasonably strong pointwise estimate for $S(\alpha)$. Such is provided in the next theorem.

THEOREM 4. *Let $S(\alpha)$ and $\mathfrak{m}(Q)$ be as above. Then, for $1 \le Q \le N^{3/7}$ one has*

$$\sup_{\alpha \in \mathfrak{m}(Q)} |S(\alpha)| \ll N^{1+\varepsilon} Q^{-1}.$$

A similar result occurs *inter alia* in Baker, Brüdern and Harman [2], but is subject to the more stringent condition $Q \le N^{1/3}$.

We shall begin with proving Theorem 4 in Section 2. The argument is based on classical Weyl sum techniques, with an extra idea contained in Lemma 1 below. An application of the circle method yields Theorem 1, with (4) and Theorem 4 as the key ingredients. This will be described in Section 3.

The next two sections are subject to GRH, and are devoted to the proof of Theorem 3. The improved error terms come from enhanced information about the distribution of square-free numbers in arithmetic progressions if GRH is true. At the heart of (4), a version of the Barban-Davenport-Halberstam theorem for square-free numbers due to Warlimont [14] is at work. Subject to GRH one may expect a stronger version of the latter, but as experts in this field will readily recognize, there are familiar but unconventional problems with routine approaches to investigate the distribution of square-free numbers in arithmetic progressions. We are able only to establish the expected bound for a certain variance (see Section 4 below) if the square-free numbers are counted with suitable weights. The weight $e^{-n/N}$ fits well with additive problems. Thus we are forced to formulate the circle method machinery in terms of the abelian transform

$$\tilde{S}(\alpha) = \sum_{n=1}^{\infty} e^{-n/N} \mu(n)^2 e(\alpha n) \tag{6}$$

of the previously defined exponential sum $S(\alpha)$. We can then write $\tilde{S}(\alpha)$ in terms of absolutely convergent Mellin transforms of Dirichlet L-functions (see (22)-(24)), and use arguments inspired by the work of Hardy and Littlewood [7] and Linnik [11] (see also Languasco and Perelli [10]) to establish a conditional improvement of (4). In Section 5, we use this to derive Theorem 4. Finally, in Section 6, we prove Theorem 2 by a simple modification of a classical result of Montgomery and Vaughan [13] concerning the error term in the asymptotic formula for representations by sums of primes.

With little extra effort our results can be extended to sums of k-free numbers; in the interest of brevity we have refrained from doing so.

Some mention should be made of the work of Friedlander and Goldston [6] who considered the analogous problem with primes instead of square-free numbers, subject to GRH. Here, the Ω-result has an $N^{\nu-\frac{3}{2}}$ in place of $N^{\nu-\frac{7}{4}}$, and curiously, they also obtained a matching upper bound for $\nu \geq 4$ summands, but for $\nu = 3$ their result falls somewhat short of the expected one.

Our notational conventions are standard or else explained at the appropriate stage of the argument. Statements involving an ε are true for any $\varepsilon > 0$. Note that this allows us to conclude from $X \ll Z^{\varepsilon}, Y \ll Z^{\varepsilon}$ that one has $XY \ll Z^{\varepsilon}$, for example.

2. – Exponential sums over square-free numbers

We prepare for the proof of Theorem 4 with a simple lemma.

LEMMA 1. *Let $\alpha \in \mathbb{R}$, and let $a \in \mathbb{Z}$, $q \in \mathbb{N}$ be coprime with $|q\alpha - a| \leq q^{-1}$. For $D \geq 1$ and $0 < z \leq \frac{1}{2}$, let*

$$W(D, z) = \#\{D < d \leq 2D : \quad \|\alpha d^2\| \leq z\}.$$

Then

$$W(D, z) \ll zD + D^{1+\varepsilon}(q^{-\frac{1}{2}} + D^{-\frac{1}{2}} + (qz)^{\frac{1}{2}} D^{-1}).$$

PROOF. We may assume that $z \leq \frac{1}{8}$ since the bound is trivial in the opposite case. Let $K = [\frac{1}{2z}]$. The function

$$\Psi_z(t) = \left(\frac{\sin \pi K t}{K \sin \pi t}\right)^2 = \frac{1}{K} \sum_{|k| \leq K} \left(1 - \frac{|k|}{K}\right) e(kt)$$

is non-negative, of period 1 and satisfies $\Psi_z(t) \geq \frac{1}{4}$ for $|t| \leq z$. Hence

$$W(D, z) \leq 4 \sum_{D<d\leq 2D} \Psi_z(\alpha d^2) \leq \frac{4D}{K} + \frac{4}{K} \sum_{0<|k|\leq K} \left(1 - \frac{|k|}{K}\right) T(\alpha k)$$

where

$$T(\beta) = \sum_{D<d\leq 2D} e(\beta d^2).$$

By Lemma 1 of Harman [8],

$$\sum_{0<|k|\leq K} |T(\alpha k)| \ll (KD)^{1+\varepsilon} \left(q^{-\frac{1}{2}} + D^{-\frac{1}{2}} + \left(\frac{q}{KD^2}\right)^{\frac{1}{2}}\right),$$

and the Lemma follows immediately. □

To establish Theorem 4, we rewrite (3) in the form

$$S(\alpha) = \sum_{d\leq\sqrt{N}} \mu(d) \sum_{m\leq Nd^{-2}} e(\alpha d^2 m).$$

We evaluate the inner geometric sum, and then split the outer summation into ranges $D < d \leq 2D$. This routine procedure yields some $D = D(\alpha)$ with $1 \leq D \leq \sqrt{N}$ such that

$$S(\alpha) \ll (\log N) \sum_{D<d\leq D} \min\left(\frac{N}{D^2}, \|\alpha d^2\|^{-1}\right) = (\log N)\Upsilon(\alpha, D),$$

say. The trivial bound $\Upsilon(\alpha, D) \leq ND^{-1}$ is satisfactory when $D > Q$, so we may assume that $D \leq Q$.

Three different arguments are available to estimate $\Upsilon(\alpha, D)$ more precisely. We write $X = NQ^{-1}$. By Dirichlet's theorem, there are coprime integers a, q with $1 \leq q \leq X$ and $|q\alpha - a| \leq X^{-1}$. For $\alpha \in \mathfrak{m}(Q)$ we note that

$$Q < q \leq X. \tag{7}$$

First suppose that $D^2 \leq \frac{1}{8}Q$. Then $4D^2 < \frac{1}{2}q$ by (7), and we can write

$$\|\alpha d^2\| = \left\|\frac{ad^2}{q}\right\| + E$$

where

$$|E| \leq d^2 \left|\alpha - \frac{a}{q}\right| \leq 4D^2 q^{-2} < \tfrac{1}{2}q^{-1}.$$

This shows that $\|\alpha d^2\| \geq \frac{1}{2}\|\frac{ad^2}{q}\|$. We again make use of $4D^2 \leq \frac{1}{2}q$ to infer that

$$\Upsilon(\alpha, D) \ll \sum_{D<d\leq 2D} \left\| \frac{ad^2}{q} \right\|^{-1} \ll \sum_{1\leq u\leq q} \frac{q}{u} \ll X \log X$$

which is more than required.

We may now suppose that $8D^2 > Q$. Now, by Lemma 3.2 of Baker [1],

$$\Upsilon(\alpha, D) \ll \sum_{u\leq 4D^2} \min\left(\frac{N}{D^2}, \|\alpha u\|^{-1}\right) \ll \left(\frac{N}{q} + D^2 + q + \frac{N}{D^2}\right) \log N .$$

If $D^2 \leq NQ^{-1}$ we conclude that $\Upsilon(\alpha, D) \ll N^{1+\varepsilon}Q^{-1}$ which is again acceptable.

It remains to discuss the case where

$$NQ^{-1} \leq D^2 \leq Q^2 . \tag{8}$$

In the notation introduced in Lemma 1, by that Lemma and (7),

$$\begin{aligned}\Upsilon(\alpha, D) &\ll \frac{N}{D^2} W\left(D, \frac{D^2}{N}\right) + \sum_{z=2^j D^2/N} z^{-1} W(D, z) \\ &\ll N^{\varepsilon}\left(D + \frac{N}{Dq^{1/2}} + \frac{N}{D^{3/2}} + \frac{(Nq)^{1/2}}{D}\right) \\ &\ll N^{\frac{1}{2}+\varepsilon} + N^{1+\varepsilon}D^{-1}Q^{-\frac{1}{2}} + N^{1+\varepsilon}D^{-\frac{3}{2}}\end{aligned}$$

By (8), this reduces to $\Upsilon(\alpha, D) \ll N^{1/2+\varepsilon} + N^{1+\varepsilon}D^{-3/2}$. This will be acceptable if $D^{3/2} \geq Q$. We may therefore suppose that $D < Q^{2/3}$. But then, by (8), $NQ^{-1} \leq D^2 < Q^{4/3}$ which is impossible for $Q \leq N^{3/7}$, and the proof of Theorem 4 is complete.

3. – The addition of square-free numbers

This section is devoted to the proof of Theorem 1. We begin with the case $\nu = 3$. The argument departs from the integral (5). Let $\mathfrak{m} = \mathfrak{m}(N^{1/3})$ and $\mathfrak{M} = \mathfrak{M}(N^{1/3})$. Then, by Theorem 4 and (4),

$$\int_{\mathfrak{m}} |S(\alpha)|^3 d\alpha \ll N^{3/2+\varepsilon} ,$$

and (5) yields

$$r_3(N) = \int_{\mathfrak{M}} S(\alpha)^3 e(-\alpha N)d\alpha + O(N^{3/2+\varepsilon}) . \tag{9}$$

We prepare for the treatment of the major arcs $\mathfrak{M}$ by introducing a suitable approximation to $S(\alpha)$. Let $G(q)$ be the multiplicative function defined on prime powers by

$$G(p^l) = \frac{1}{1-p^2} \quad (l = 1 \text{ or } l = 2); \quad G(p^l) = 0 \quad (l \geq 3) . \tag{10}$$

Furthermore, let

$$I(\beta) = \sum_{n \leq N} e(\beta n) . \tag{11}$$

For $1 \leq a \leq q \leq \frac{1}{2}N^{1/2}$, $(a, q) = 1$ and $|q\alpha - a| \leq \frac{1}{2}N^{-1/2}$, write

$$S^*(\alpha) = \frac{6}{\pi^2} G(q) I\left(\alpha - \frac{a}{q}\right) . \tag{12}$$

This defines $S^*(\alpha)$ on $\mathfrak{M}(\frac{1}{2}N^{1/2})$, and for α in this set, we write

$$\Delta(\alpha) = S(\alpha) - S^*(\alpha) . \tag{13}$$

We recall a special case of Lemma 3.2 of [3] which asserts that for $1 \leq Q \leq \frac{1}{2}\sqrt{N}$ one has

$$\int_{\mathfrak{M}(Q)} |\Delta(\alpha)|^2 d\alpha \ll Q^2 N^{\varepsilon} . \tag{14}$$

It is now straightforward to replace $S(\alpha)$ by $S^*(\alpha)$ in (9). Note that (13) gives

$$\begin{aligned} S(\alpha)^3 &= S(\alpha)(S^*(\alpha) + \Delta(\alpha))^2 \\ &= S^*(\alpha)^3 + 3S^*(\alpha)^2\Delta(\alpha) + 2S^*(\alpha)\Delta(\alpha)^2 + S(\alpha)\Delta(\alpha)^2 . \end{aligned}$$

We multiply by $e(-\alpha N)$ and integrate over $\mathfrak{M}$ to infer that

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-\alpha N) d\alpha = \int_{\mathfrak{M}} S^*(\alpha)^3 e(-\alpha N) d\alpha + O(E_1 + E_2 + E_3)$$

where

$$\begin{aligned} E_1 &= \int_{\mathfrak{M}} |S^*(\alpha)^2 \Delta(\alpha)| d\alpha , \\ E_2 &= \int_{\mathfrak{M}} |S^*(\alpha)\Delta(\alpha)^2| d\alpha , \\ E_3 &= \int_{\mathfrak{M}} |S(\alpha)\Delta(\alpha)^2| d\alpha . \end{aligned}$$

Bounds for E_2 and E_3 are readily deduced from (14) and Theorem 4. We write $\mathfrak{N}(Q) = \mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)$. Then, for $1 \le Q \le N^{1/3}$, we deduce from Theorem 4 that

$$\sup_{\alpha\in\mathfrak{N}(Q)} |S(\alpha)| \ll N^{1+\varepsilon}Q^{-1},$$

and (14) now yields

$$\int_{\mathfrak{N}(Q)} |S(\alpha)\Delta(\alpha)^2|d\alpha \ll N^{1+\varepsilon}Q. \tag{15}$$

From (10), we have $G(q) \ll q^{\varepsilon-1}$, and from (11) we see that $I(\beta) \ll N(1 + N|\beta|)^{-1}$. It follows at once that for $1 \le Q \le N^{1/3}$ one has

$$\sup_{\alpha\in\mathfrak{N}(Q)} |S^*(\alpha)| \ll N^{1+\varepsilon}Q^{-1}, \tag{16}$$

and (14) shows that

$$\int_{\mathfrak{N}(Q)} |S^*(\alpha)\Delta(\alpha)^2|d\alpha \ll N^{1+\varepsilon}Q. \tag{17}$$

On summing (15) and (17) over $Q = 2^{-j}N^{1/3}$ with $j \in \mathbb{N}$ we deduce that

$$E_2 + E_3 \ll N^{4/3+\varepsilon}.$$

For the estimation of E_1 we invoke Lemma 4.2 of [3] which shows that

$$\int_{\mathfrak{N}(Q)} |S^*(\alpha)\Delta(\alpha)|d\alpha \ll N^{1/2+\varepsilon}Q^{3/4}.$$

Thus, by (16), we have

$$\int_{\mathfrak{N}(Q)} |S^*(\alpha)^2\Delta(\alpha)|d\alpha \ll N^{3/2+\varepsilon}Q^{-1/4}.$$

Hence, by summing over $Q = 2^{-j}N^{1/3}$ as before, we find that $E_1 \ll N^{3/2+\varepsilon}$. We may now conclude that indeed (9) holds with $S(\alpha)$ replaced by $S^*(\alpha)$, and consequently,

$$r_3(N) = \left(\frac{6}{\pi^2}\right)^3 \sum_{q\le N^{1/3}} c_q(N)G(q)^3 \int_{-q^{-1}N^{-2/3}}^{q^{-1}N^{-2/3}} I(\beta)^3 e(-\beta N)d\beta + O(N^{3/2+\varepsilon})$$

where

$$c_q(N) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{aN}{q}\right).$$

We now observe that by elementary counting,

$$\int_{-1/2}^{1/2} I(\beta)^3 e(-\beta N)d\beta = \tfrac{1}{2}N^2 + O(N),$$

and that for any $X \geq N^{-1}$,

$$\int_X^{1/2} |I(\beta)|^3 d\beta \ll \int_X^{\infty} N^3(1+N|\beta|)^{-3} d\beta \ll X^{-2}.$$

We use the latter with $X = q^{-1}N^{-2/3}$ and then obtain

$$r_3(N) = \frac{1}{2}\left(\frac{6}{\pi^2}\right)^3 N^2 \sum_{q \leq N^{1/3}} c_q(N)G(q)^3 + O(R + N^{3/2+\varepsilon}) \tag{18}$$

where

$$R = N^{4/3} \sum_{q \leq N^{1/3}} q^2 |c_q(N)||G(q)|^3.$$

As we have observed earlier, we have $|G(q)| \ll q^{\varepsilon - 1}$. Moreover, one has the familiar estimate $|c_q(N)| \leq (q, N)$, and therefore

$$R \ll N^{4/3} \sum_{q \leq N^{1/3}} q^{\varepsilon-1}(q, N) \ll N^{4/3+\varepsilon}.$$

By a similar argument, one readily confirms that

$$\sum_{q>Q} |c_q(N)||G(q)|^3 \ll \sum_{q>Q} (q, N) q^{\varepsilon - 3} \ll N^{\varepsilon} Q^{\varepsilon - 2}.$$

Collecting together, (18) finally gives

$$r_3(N) = \frac{1}{2}\left(\frac{6}{\pi^2}\right)^3 N^2 \sum_{q=1}^{\infty} c_q(N)G(q)^3 + O(N^{3/2+\varepsilon}).$$

Since $G(q)$ and $c_q(N)$ are multiplicative functions of q, we may rewrite the infinite sum as an Euler product, and one then finds that

$$\sum_{q=1}^{\infty} c_q(N)G(q)^3 = \mathfrak{S}_3(N).$$

Theorem 1 now follows, at least when $\nu = 3$. For larger values of ν one may proceed as above, estimating the extra generating functions trivially. Details may be omitted.

4. – A conditional mean square estimate

In this section we provide improved versions of (4) and (14) subject to GRH. It will be necessary to work with rapidly convergent Mellin transforms, and therefore we express our results in terms of the abelian transform $\tilde{S}(\alpha)$ defined in (6). Before we can formulate our first lemma, which is still unconditional, we need to introduce some notation. For any Dirichlet character modulo r, let $\tau(\chi) = \sum_{b=1}^{r} \chi(b)e(b/r)$ denote its Gauss sum, and let $L(s,\chi)$ be the Dirichlet L-function. Whenever $r|q$, write

$$D_q(s,\chi) = \prod_{p|q} (1+\chi(p)p^{-s})^{-1}. \tag{19}$$

By comparing Euler products, one readily confirms that for $Re(s) > 1$ one has

$$\sum_{\substack{m=1\\(m,q)=1}}^{\infty} \mu(m)^2 \chi(m) m^{-s} = D_q(s,\chi)\frac{L(s,\chi)}{L(2s,\chi^2)}. \tag{20}$$

Lemma 2. *For any coprime natural numbers a, q and any $\beta \in \mathbb{R}$ one has*

$$\tilde{S}\left(\frac{a}{q}+\beta\right) = \frac{1}{2\pi i}\sum_{d|q}\frac{\mu(d)^2}{\varphi(q/d)}\sum_{\chi \bmod q/d} \overline{\chi}(a)\tau(\chi) \times \int_{2-i\infty}^{2+i\infty} \frac{L(s,\chi)}{L(2s,\chi^2)} D_q(s,\chi) z^{-s}\Gamma(s)\,ds$$

where $z = z(d,\beta) = d(\frac{1}{N} - 2\pi i\beta)$.

Proof. This is a straightforward exercise, so we shall be brief. In (6), we sort terms according to the value of (q,n) and take the resulting formula

$$\tilde{S}\left(\frac{a}{q}+\beta\right) = \sum_{d|q}\sum_{\substack{n=1\\(n,q)=d}}^{\infty} \mu(n)^2 e\left(\frac{an}{q}\right) e(\beta n) e^{-n/N}$$

as the starting point. Now write $q = rd, n = md$ and note that $\mu(md) = 0$ whenever $(m,d) > 1$. This shows that

$$\tilde{S}\left(\frac{a}{q}+\beta\right) = \sum_{d|q}\mu(d)^2 \sum_{\substack{m=1\\(m,r)=(m,d)=1}}^{\infty} \mu(m)^2 e\left(\frac{am}{r}\right) e(\beta dm) e^{-md/N}.$$

The condition $(m,r) = (m,d) = 1$ is equivalent with $(m,q) = 1$. We sort the sum over m according to residue classes $m \equiv b (\bmod r)$ with $1 \le b \le r$ and

$(b, r) = 1$, and then pick up the congruence condition by characters modulo r. This leads to

$$\tilde{S}\left(\frac{a}{q}+\beta\right) = \sum_{d|q} \mu(d)^2 \sum_{\substack{b=1 \\ (b,r)=1}}^{r} e\left(\frac{ab}{r}\right) \sum_{\substack{m=1 \\ (m,q)=1 \\ m\equiv b (\operatorname{mod} r)}}^{\infty} \mu(m)^2 e(\beta dm) e^{-md/N}$$

$$= \sum_{d|q} \frac{\mu(d)^2}{\varphi(r)} \sum_{\chi \operatorname{mod} r} \overline{\chi}(a)\tau(\chi) \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \mu(m)^2 \chi(m) e(\beta dm) e^{-md/N}.$$

But $e(\beta md)e^{-md/N} = e^{-mz}$, and the Mellin transform of e^{-x} is $\Gamma(s)$. Hence, by (20),

$$\sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \mu(m)^2 \chi(m) e(\beta dm) e^{-md/N} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D_q(s, \chi) \frac{L(s, \chi)}{L(2s, \chi^2)} \Gamma(s) z^{-s} ds,$$

and the lemma follows immediately. □

We now wish to shift the line of integration in Lemma 2 to the left. From now on, *we assume that GRH is true*. Let χ_0 denote the principal character $(\operatorname{mod} r)$. Then

$$L(s, \chi_0) = \zeta(s) \prod_{p|r} (1 - p^{-s}),$$

and $L(s, \chi_0)$ has no zeros in $Re(s) > \frac{1}{2}$. Since $\zeta(s)$ is holomorphic on $\mathbb{C}$ except for a simple pole of residue 1 at $s = 1$, it follows that for $\chi = \chi_0$ the integrand in Lemma 2 is holomorphic in $Re(s) > \frac{1}{4}$ except for a simple pole at $s = 1$, with residue

$$z^{-1} D_q(1, \chi_0) L(2, \chi_0^2)^{-1} \prod_{p|r} (1 - p^{-1})$$

$$= (\zeta(2)z)^{-1} \prod_{\substack{p|q \\ p\nmid r}} \left(1 + \frac{1}{p}\right)^{-1} \prod_{p|r} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p|r} \left(1 - \frac{1}{p}\right) = (\zeta(2)z)^{-1} h(q)$$

where

$$h(q) = \prod_{p|q} \left(1 + \frac{1}{p}\right)^{-1}. \tag{21}$$

When $\chi \neq \chi_0$, then $L(s, \chi)$ is entire, and $L(2s, \chi^2)$ again has no zeros in $Re(s) > \frac{1}{4}$. Thus, in this case, the integrand in Lemma 2 is holomorphic for $Re(s) > \frac{1}{4}$.

Fix a real number σ with $\frac{1}{4} < \sigma < \frac{1}{2}$. By Stirling's formula and standard upper bounds for the L-functions, it is readily confirmed that the line of integration in Lemma 2 may be moved to $Re(s) = \sigma$. Since $\tau(\chi_0) = c_r(1) = \mu(r)$ by a well-known evaluation of Ramanujan's sum, this yields

$$\tilde{S}\left(\frac{a}{q} + \beta\right) = \frac{6}{\pi^2} h(q) \sum_{d|q} \frac{\mu(d)^2}{\varphi(q/d)} \mu\left(\frac{q}{d}\right) z^{-1} + \Xi(q, a, \beta), \tag{22}$$

where

$$\Xi(q, a, \beta) = \frac{1}{2\pi i} \sum_{d|q} \frac{\mu(d)^2}{\varphi(q/d)} \sum_{\chi \bmod q/d} \overline{\chi}(a) \tau(\chi) \Xi(q, \chi) \tag{23}$$

with

$$\Xi(q, \chi) = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{L(s, \chi)}{L(2s, \chi^2)} D_q(s, \chi) z^{-s} \Gamma(s) ds . \tag{24}$$

The first term on the right of (22) can be much simplified. Indeed, recalling (21) and (10), one readily verifies that for q a prime power, one has

$$h(q) \sum_{d|q} \frac{\mu(d)^2}{\varphi(q/d)} \frac{\mu(q/d)}{d} = G(q);$$

but then, by multiplicativity, this identity holds for all q. Recalling the definition of $z(d, \beta)$, we may rewrite (22) as

$$\tilde{S}\left(\frac{a}{q} + \beta\right) = \frac{6}{\pi^2} G(q) \left(\frac{1}{N} - 2\pi i \beta\right)^{-1} + \Xi(q, a, \beta), \tag{25}$$

and it should become apparent that $\Xi(q, a, \beta)$ is the proper analogue of the function $\Delta(\alpha)$ in Section 3 and [3]. The following lemma gives a mean square bound for $\Xi(q, a, \beta)$ which improves on (14).

LEMMA 3. *Suppose that* $1 \leq q \leq \sqrt{N}$ *and* $N^{-1} \leq \delta \leq N^{-1/2}$. *Then*

$$\int_{-\delta}^{\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |\Xi(q, a, \beta)|^2 d\beta \ll (q\delta)^{3/2} N^{1+\varepsilon} .$$

PROOF. Since $|\Xi(q, a, -\beta)| = |\Xi(q, a, \beta)|$ it suffices to estimate the integral over $[0, \delta]$. We cover this interval by $[0, 2N^{-1}]$ and $O(\log N)$ intervals of the type $[\eta, 2\eta]$ with $2N^{-1} \leq \eta < \delta$. It follows that for some such η we must have

$$\int_{-\delta}^{\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |\Xi(q, a, \beta)|^2 d\beta \ll K_0 + (\log N) K(\eta) \tag{26}$$

where

$$K(\eta) = \int_{\eta}^{2\eta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} |\Xi(q, a, \beta)|^2 d\beta \tag{27}$$

and K_0 is defined likewise, but with the integration extended over $[0, 2N^{-1}]$.

We now apply Cauchy's inequality to (23). An elementary estimate for the divisor function then shows that

$$|\Xi(q, a, \beta)|^2 \ll q^{\varepsilon} \sum_{d|q} \left(\frac{\mu(d)}{\varphi(r)}\right)^2 \left| \sum_{\chi \bmod r} \overline{\chi}(a)\tau(\chi)\Xi(q, \chi)\right|^2$$

where in accordance with earlier conventions, we wrote $q = rd$. We sum over a, expand the square, use the orthogonality of characters and recall that $\varphi(r)^{-1}|\tau(\chi)|^2 \ll \frac{r}{\varphi(r)} \ll q^{\varepsilon}$ for all χ mod r. This yields

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} |\Xi(q, a, \beta)|^2 \ll q^{\varepsilon} \sum_{d|q} \mu(d)^2 \sum_{\chi \bmod r} |\Xi(q, \chi)|^2 . \tag{28}$$

We estimate K_0 first. In the interest of brevity, write

$$F_q(s, \chi) = d^{-s} \frac{L(s, \chi)}{L(2s, \chi^2)} D_q(s, \chi)\Gamma(s) . \tag{29}$$

When $s = \sigma + it$ with $\frac{1}{4} < \sigma < \frac{1}{2}$ and $t \in \mathbb{R}$, standard bounds for L-series provide the inequalities

$$|L(2s, \chi^2)| \gg (q(|t| + 1))^{-\varepsilon} \quad , \quad |L(s, \chi)| \ll (r(|t| + 1))^{\frac{1}{2}-\sigma+\varepsilon} ,$$

subject to GRH. We also have $|d^{-s}| = d^{-\sigma}$ and $|D_q(s, \chi)| \ll q^{\varepsilon}$ by elementary estimates. Combined with Stirling's formula for $\Gamma(s)$, these inequalities imply that

$$|F_q(s, \chi)| \ll q^{\varepsilon} d^{-\sigma} r^{\frac{1}{2}-\sigma} e^{-\frac{\pi}{2}|t|}(|t| + 1)^{\varepsilon} . \tag{30}$$

Moreover, for $z_0 = z(1, \beta) = \frac{1}{N} - 2\pi i\beta$ we have $|z_0^{-s}| \ll N^{\sigma}$. From (24) we now deduce that

$$|\Xi(q, \chi)| \leq N^{\sigma} \int_{-\infty}^{\infty} |F_q(\sigma + it, \chi)| dt \ll N^{\sigma} q^{\frac{1}{2}-\sigma+\varepsilon} .$$

We may take $\sigma = \frac{1}{4} + \varepsilon$. From (28) it follows that

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} |\Xi(q, a, \beta)|^2 \ll N^{\frac{1}{2}+\varepsilon} q^{\frac{3}{2}} .$$

Integrating over $[0, 2N^{-1}]$ yields $K_0 \ll N^{\varepsilon - \frac{1}{2}} q^{\frac{3}{2}}$ which is acceptable.

More care is required to estimate $K(\eta)$. Although not strictly necessary here, we begin by smoothing the mean square; this process faciliates reference to Languasco and Perelli [10] below. The obvious inequality

$$\int_{\eta}^{2\eta} |\Xi(q, \chi)|^2 d\beta \le \int_1^2 \int_{\frac{1}{2}\theta\eta}^{2\theta\eta} |\Xi(q, \chi)|^2 d\beta \, d\theta$$

is our starting point. Now open the right hand side and recall (24), (29) and the definition of z_0 to infer that

$$\int_{\eta}^{2\eta} |\Xi(q, \chi)|^2 d\beta \le \int_{\sigma - i\infty}^{\sigma + i\infty} \int_{\sigma - i\infty}^{\sigma + i\infty} F_q(s, \chi) \overline{F_q(w, \chi)} \mathcal{I}_\eta(s, w) ds \, dw \tag{31}$$

where

$$\mathcal{I}_\eta(s, w) = \int_1^2 \int_{\frac{1}{2}\theta\eta}^{2\theta\eta} z_0^{-s} \overline{z_0^{-w}} d\beta \, d\theta .$$

An upper bound for $\mathcal{I}_\eta(s, w)$ is required. We write $s = \sigma + it, w = \sigma + it'$ with $\frac{1}{4} < \sigma < \frac{1}{2}$ as before. Note that

$$z_0^{-s} \overline{z_0^{-w}} = |z_0|^{-2\sigma - i(t-t')} \exp((t + t') \arctan 2\pi\beta N) .$$

Since $\arctan \gamma = \frac{\pi}{2} - \arctan \frac{1}{\gamma}$ holds for $\gamma > 0$, we now have

$$z_0^{-s} \overline{z_0^{-w}} = e^{\frac{\pi}{2}(t+t')} f_1(\beta) f_2(\beta)$$

where

$$f_1(\beta) = |z_0|^{-2\sigma - i(t-t')} ; \; f_2(\beta) = \exp\left(-(t + t') \arctan \frac{1}{2\pi\beta N} \right) .$$

If we now write

$$J = J(N, \eta, t, t') = \int_1^2 \int_{\frac{1}{2}\theta\eta}^{2\theta\eta} f_1(\beta) f_2(\beta) d\beta \, d\theta ,$$

then

$$\mathcal{I}_\eta(s, w) = e^{\frac{\pi}{2}(t+t')} J . \tag{32}$$

The integral J has been studied by Languasco and Perelli [10], but only for $\sigma = \frac{1}{2}$ and (in their set-up) $t > 0, t' > 0$. However, an inspection of their argument ([10], pp. 312-314) shows that the estimates remain valid in our more general situation. The inequality (14) of [10] still holds, provided that G_1 now denotes a primitive of a primitive of $f_1(\beta)$. If one then follows the pattern of the estimation of G_1 in [10], one readily confirms that the bound corresponding to (20) of [10] now reads

$$G_1(\beta) \ll \frac{\beta^{2-2\sigma}}{1 + |t - t'|^2}.$$

One then finds that for some suitable $c > 0$, one has

$$J \ll \eta^{1-2\sigma} \left(1 + \left(\frac{|t + t'|}{N\eta}\right)^2\right) (1 + |t - t'|^2)^{-1} \exp\left(-c\left(\frac{t + t'}{N\eta}\right)\right)$$

From this, (32), (31) and (30), we may conclude that

$$\int_{\eta}^{2\eta} |\Xi(q, \chi)|^2 d\beta \ll \eta^{1-2\sigma} q^{1-2\sigma+\varepsilon} U \tag{33}$$

where

$$U = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|tt'| + 1)^{\varepsilon} e^{\frac{\pi}{2}(t+t'-|t|-|t'|)} \frac{1 + \left(\frac{(t + t')}{N\eta}\right)^2}{1 + (t - t')^2} \exp\left(-c\frac{t + t'}{N\eta}\right) dt\, dt'.$$

A straightforward estimation shows that $U \ll N\eta$ (for example, transform the integral via $u = t + t', v = t - t'$). By (33), (27) and (28) it follows that

$$K(\eta) \ll N\eta^{2-2\sigma} q^{2-2\sigma+\varepsilon}.$$

With $\sigma = \frac{1}{4} + \varepsilon$ as before, the Lemma now follows from (26). □

5. – The addition of square-free numbers: conditional results

We return to the main theme of the present paper, and establish the conditional asymptotic formulae in Theorem 3. By orthogonality and (6), we have

$$\int_0^1 \tilde{S}(\alpha)^{\nu} e(-\alpha N) d\alpha = e^{-1} r_{\nu}(N). \tag{34}$$

We shall apply the circle method to (34). Although there is still a dissection into minor and major arcs, there is a certain flavour of a major arc treatment throughout the interval [0, 1].

The intervals $\mathfrak{K}(q,a) = \{\alpha : |q\alpha - a| \le N^{-1/2}\}$ with $1 \le a \le q < \frac{1}{2}\sqrt{N}$ and $(a,q) = 1$ are pairwise disjoint. On the other hand, for any real number α, Dirichlet's theorem yields coprime numbers b, r with $1 \le r \le \sqrt{N}$ and $|r\alpha - b| \le N^{-1/2}$. Therefore, there are numbers $\kappa(q,a), \kappa'(q,a)$ where $\frac{1}{2}\sqrt{N} \le q \le \sqrt{N}$, $1 \le a \le q$, $(a,q) = 1$, with

$$0 \le \kappa'(q,a), \kappa(q,a) \le N^{-1/2},$$

such that the intervals

$$\mathfrak{K}(q,a) = \{\alpha : -\kappa'(q,a) < q\alpha - a < \kappa(q,a)\}$$

together with the $\mathfrak{K}(q,a)$ for $1 \le a \le q \le \frac{1}{2}\sqrt{N}$, $(a,q) = 1$ define a disjoint cover of $[N^{-1/2}, 1 + N^{-1/2}]$. For $\alpha \in \mathfrak{K}(q,a)$ we now write, recalling (25),

$$\tilde{S}^*(\alpha) = \left(\frac{6}{\pi^2}\right) G(q) \left(\frac{1}{N} - 2\pi i \left(\alpha - \frac{a}{q}\right)\right)^{-1} \tag{35}$$

$$\Xi(\alpha) = \tilde{S}(\alpha) - \tilde{S}^*(\alpha) = \Xi\left(q, a, \alpha - \frac{a}{q}\right). \tag{36}$$

Note that we may regard $\tilde{S}^*(\alpha)$ and $\Xi(\alpha)$ as functions of period 1, defined for all real numbers.

Recall the definition of $\mathfrak{M}(Q), \mathfrak{N}(Q)$ and $\mathfrak{m}(Q)$ from Section 3. It is interesting to compare the conclusions of the next Lemma with (14).

LEMMA 4. *Let $1 \le Q \le \frac{1}{2}\sqrt{N}$. Then, subject to GRH,*

$$\int_{\mathfrak{M}(Q)} |\Xi(\alpha)|^2 d\alpha \ll N^{\varepsilon - \frac{1}{2}} Q^{\frac{5}{2}}$$

and

$$\int_0^1 |\Xi(\alpha)|^2 d\alpha \ll N^{\frac{3}{4}+\varepsilon}.$$

PROOF. For the first estimate, use Lemma 3 with $\delta = Q/(qN)$ and sum over $q \le Q$. For the second estimate, proceed likewise with $\delta = q^{-1}N^{-1/2}$ and sum over $q \le \sqrt{N}$. □

LEMMA 5. *Let $\mathfrak{m}(Q)$ be defined as in Theorem 4. Then, for $Q \le N^{3/7}$,*

$$\sup_{\alpha \in \mathfrak{m}(Q)} |\tilde{S}(\alpha)| \ll N^{1+\varepsilon} Q^{-1}.$$

PROOF. This follows from Theorem 4 by partial summation. □

For future reference, we record here a useful corollary. By (35) and (10), we have $|\tilde{S}^*(\alpha)| \ll q^{\varepsilon-1}N(1+N|\alpha-\frac{a}{q}|)^{-1}$ when $\alpha \in \mathfrak{K}(q,a)$. Hence, for $Q \le \frac{1}{2}\sqrt{N}$, we have

$$\sup_{\alpha\in\mathfrak{m}(Q)} |\tilde{S}^*(\alpha)| \ll N^{1+\varepsilon}Q^{-1}.$$

From (36) and Lemma 5, we deduce that for $1 \le Q \le N^{3/7}$, one has

$$\sup_{\alpha\in\mathfrak{m}(Q)} |\Xi(\alpha)| \ll N^{1+\varepsilon}Q^{-1}. \tag{37}$$

We now embark on the main argument. Let $\nu \ge 3$ be a fixed natural number. By (36), we have

$$\tilde{S}(\alpha)^\nu = (\tilde{S}^*(\alpha)+\Xi(\alpha))^\nu = \tilde{S}^*(\alpha)^\nu + O(|\tilde{S}^*(\alpha)^{\nu-1}\Xi(\alpha)| + |\Xi(\alpha)|^\nu).$$

By (34), we conclude that

$$e^{-1}r_\nu(N) = \int_0^1 \tilde{S}^*(\alpha)^\nu e(-\alpha N)d\alpha + O(E_4+E_5). \tag{38}$$

where

$$E_4 = \int_0^1 |\tilde{S}^*(\alpha)|^{\nu-1}|\Xi(\alpha)|d\alpha, \quad E_5 = \int_0^1 |\Xi(\alpha)|^\nu d\alpha.$$

Write $\mathfrak{m} = \mathfrak{m}(N^{3/7})$. From Lemma 4 and (37) we have

$$\int_{\mathfrak{m}} |\Xi(\alpha)|^\nu d\alpha \ll N^{\nu-\frac{5}{4}+\varepsilon}N^{\frac{3}{7}(2-\nu)} \tag{39}$$

When $1 \le Q \le N^{3/7}$, we also deduce from Lemma 4 and (37) that

$$\int_{\mathfrak{N}(Q)} |\Xi(\alpha)|^\nu d\alpha \ll (N^{1+\varepsilon}Q^{-1})^{\nu-2}N^{-\frac{1}{2}}Q^{\frac{5}{2}}, \tag{40}$$

and since $\mathfrak{M} = \mathfrak{M}(N^{3/7})$ is covered by $O(\log N)$ sets $\mathfrak{N}(Q)$ with $Q \le N^{3/7}$, the bounds (39) and (40) yield

$$E_5 \ll N^{\nu-\frac{7}{4}+\frac{1}{14}+\varepsilon} \quad (\nu=3); \qquad E_5 \ll N^{\nu-\frac{7}{4}} \quad (\nu \ge 4). \tag{41}$$

We prepare for the estimation of E_4 with the following upper bound.

LEMMA 6. *Let $k \geq 2$ and $1 \leq Q \leq \frac{1}{2}N^{1/2}$. Then*

$$\int_{\mathfrak{m}(Q)} |\tilde{S}^*(\alpha)|^k d\alpha \ll N^{k-1} Q^{\frac{3}{2}-k+\varepsilon}.$$

If we take this for granted, then we may take $k = 2\nu - 2$, and apply Cauchy-Schwarz's inequality and Lemma 4 to deduce that

$$\int_{\mathfrak{m}(\frac{1}{2}\sqrt{N})} |\tilde{S}^*(\alpha)^{\nu-1}\Xi(\alpha)| d\alpha \leq \left(\int_{\mathfrak{m}(\frac{1}{2}\sqrt{N})} |\tilde{S}^*(\alpha)|^{2\nu-2} d\alpha\right)^{\frac{1}{2}} \left(\int_0^1 |\Xi(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \ll N^{\nu-\frac{7}{4}+\varepsilon}.$$

Likewise, for $1 \leq Q \leq \frac{1}{4}\sqrt{N}$, one finds that

$$\int_{\mathfrak{N}(Q)} |\tilde{S}^*(\alpha)^{\nu-1}\Xi(\alpha)| d\alpha \leq \left(\int_{\mathfrak{m}(Q)} |\tilde{S}^*(\alpha)|^{2\nu-2} d\alpha\right)^{\frac{1}{2}} \left(\int_{\mathfrak{M}(2Q)} |\Xi(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \ll N^{\nu-\frac{7}{4}+\varepsilon} Q^{3-\nu}.$$

The interval $[N^{-1/2}, 1 + N^{-1/2}]$ may be covered by $\mathfrak{m}(\frac{1}{2}\sqrt{N})$ and $O(\log N)$ sets $\mathfrak{N}(Q)$ with $1 \leq Q \leq \frac{1}{4}\sqrt{N}$. Hence, for $\nu \geq 3$, we may conclude that

$$E_4 \ll N^{\nu-\frac{7}{4}+\varepsilon}. \tag{42}$$

Observe that by (41) and (42), we have suitable estimates for the errors in (38).

PROOF OF LEMMA 6. By (35), the integral in question does not exceed

$$\ll \sum_{Q<q\leq\sqrt{N}} q|G(q)|^k \int_{-\infty}^{\infty} \left|\frac{1}{N} - 2\pi i\beta\right|^{-k} d\beta + \sum_{q\leq Q} q|G(q)|^k \int_{-Q/(qN)}^{Q/(qN)} \left|\frac{1}{N} - 2\pi i\beta\right|^{-k} d\beta.$$

The first integral here is $O(N^{k-1})$ whereas the second is $O((Nq/Q)^{k-1})$. It follows that

$$\int_{\mathfrak{m}(Q)} |\tilde{S}^*(\alpha)|^k d\alpha \ll N^{k-1}\left(\sum_{q>Q} q|G(q)|^k + Q^{1-k}\sum_{q\leq Q} q^k|G(q)|^k\right)$$

By (10), we have $G(q) = 0$ unless q is cube-free. For any cube-free q, we may write $q = q_1q_2^2$ with q_1, q_2 square-free, and (10) yields $|G(q)| \ll q^\varepsilon(q_1q_2)^{-2}$. For $k \geq 2$, we infer that

$$\sum_{q\leq Q} q^k|G(q)|^k \ll Q^\varepsilon \sum_{q_1q_2^2\leq Q} q_1^{-k} \ll Q^{\frac{1}{2}+\varepsilon}. \tag{43}$$

Similary, one readily confirms that

$$\sum_{q>Q} q|G(q)|^k \ll Q^{\frac{3}{2}-k+\varepsilon}. \tag{44}$$

The conclusion of Lemma 6 is immediate. □

It remains to evaluate the main term in (38). By Lemma 6 with $Q = \frac{1}{2}\sqrt{N}$, (41) and (42) we have

$$e^{-1}r_\nu(N) = \int_{\mathfrak{M}(\frac{1}{2}\sqrt{N})} \tilde{S}^*(\alpha)^\nu e(-\alpha N)d\alpha + O(N^{\nu-\frac{7}{4}+\gamma(\nu)+\varepsilon}) \tag{45}$$

where $\gamma(3) = \frac{1}{14}, \gamma(\nu) = 0$ $(\nu \geq 4)$. By (35), we may write

$$\begin{aligned} &\int_{\mathfrak{M}(\frac{1}{2}\sqrt{N})} \tilde{S}^*(\alpha)^\nu e(-\alpha N)d\alpha \\ &= \left(\frac{6}{\pi^2}\right)^\nu \sum_{q\leq\frac{1}{2}\sqrt{N}} c_q(N)G(q)^\nu \int_{-(2q\sqrt{N})^{-1}}^{(2q\sqrt{N})^{-1}} \left(\frac{1}{N} - 2\pi i\beta\right)^{-\nu} e(-\beta N)d\beta. \end{aligned} \tag{46}$$

The identity

$$\int_{-\infty}^{\infty} \left(\frac{1}{N} - 2\pi i\beta\right)^{-\nu} e(-\beta N)d\beta = \frac{N^{\nu-1}}{(\nu-1)!e} \tag{47}$$

may be verified by the calculus of residues. With $z = 2\pi i\beta$, we get

$$\int_{-\infty}^{\infty} \left(\frac{1}{N} - 2\pi i\beta\right)^{-\nu} e(-\beta N)d\beta = \frac{N^{\nu-1}}{2\pi i} \int_{-i\infty}^{i\infty} (1-z)^{-\nu} e^{-z}dz. \tag{48}$$

Since $e^{-z} = e^{-1}e^{1-z}$, the power series for e^ζ shows that the function $(1-z)^{-\nu}e^{-z}$ has a pole of order ν, with residue $-((\nu-1)!e)^{-1}$ at $z = 1$, and is holomorphic for all other z. We now integrate this function clockwise over the rectangle with corners $\pm iT, (1\pm i)T$. Then, as $T \to \infty$, the integrals over the horizontal paths as well as over the portion of $Re(z) = T$ tend to 0, by straightforward estimates. The integral over the vertical line from $-iT$ to iT, however, tends to the integral on the right hand side of (48), and (47) follows from the residue theorem.

For $X \geq N^{-1}$ we have

$$\int_X^\infty \left|\frac{1}{N} - 2\pi i\beta\right|^{-\nu} d\beta \ll X^{1-\nu}.$$

Therefore, by (43), (46) and (47),

$$\int_{\mathfrak{M}(\frac{1}{2}\sqrt{N})} \tilde{S}^*(\alpha)^\nu e(-\alpha N)d\alpha$$
$$= \left(\frac{6}{\pi^2}\right)^\nu \frac{N^{\nu-1}}{(\nu-1)!e} \sum_{q\le\frac{1}{2}\sqrt{N}} c_q(N)G(q)^\nu + O(N^{\nu-\frac{7}{4}+\varepsilon}).$$

Now observe that by (10) and multiplicativity,

$$\sum_{q=1}^{\infty} c_q(N)G(q)^\nu = \mathfrak{S}_\nu(N), \tag{49}$$

the sum on the left being absolutely convergent for $\nu \ge 2$ by (44). Another application of (44) yields

$$\int_{\mathfrak{M}(\frac{1}{2}\sqrt{N})} \tilde{S}^*(\alpha)^\nu e(-\alpha N)d\alpha = \left(\frac{6}{\pi^2}\right)^\nu \frac{N^{\nu-1}}{(\nu-1)!e}\mathfrak{S}_\nu(N) + O(N^{\nu-\frac{7}{4}+\varepsilon}),$$

and Theorem 3 follows from (45). Note that we miss the optimal result for $\nu = 3$ only in the estimation of E_5. If it were possible in Theorem 4 to extend the range for Q to $Q \le \sqrt{N}$, then one would, in Theorem 3, obtain the same precision for $\nu = 3$ as is provided for $\nu \ge 4$.

6. – Proof of Theorem 2

Our proof of Theorem 2 closely follows a related result of Montgomery and Vaughan [13], and we are therefore rather sketchy. We consider the power series

$$M(z) = \sum_{n=1}^{\infty} \mu(n)^2 z^n \tag{50}$$

for $z = \varrho$ with $\frac{1}{2} < \varrho < 1$ when $\varrho \to 1$. We write throughout

$$R = \frac{1}{1-\varrho}. \tag{51}$$

This problem is strongly linked with the distribution of the error term

$$E(x) = \sum_{n\le x} \mu(n)^2 - \frac{6x}{\pi^2},$$

since, by partial summation, one has

$$
\begin{aligned}
M(\varrho) - \frac{6}{\pi^2}R &= \sum_{n=1}^{\infty}\left(\mu(n)^2 - \frac{6}{\pi^2}\right)\varrho^n + O(1) \\
&= \frac{1}{R}\sum_{n=1}^{\infty} E(n)\varrho^n + O(1).
\end{aligned} \tag{52}
$$

It is useful now to record here the elementary bound

$$
\sum_{n=1}^{\infty} n^{\xi}\rho^n \ll R^{\xi+1}, \tag{53}
$$

valid whenever $\xi > -1$ (see Lemma 2 of [13]). Elementary number theory shows that $E(x) \ll x^{1/2}$, and Evelyn and Linfoot [5] obtained $E(x) = \Omega(x^{1/4})$ if the Riemann hypothesis holds; if the Riemann hypothesis is false, one has $E(x) = \Omega(x^{\frac{1}{2}\vartheta-\varepsilon})$. From (52) and (53) we may conclude as follows.

LEMMA 7. *In the notation introduced above, we have* $M(\varrho) - \frac{6}{\pi^2}R = O(R^{1/2})$. *Moreover, as* $\varrho \to 1$ *from the left, we have* $M(\varrho) - \frac{6}{\pi^2}R = \Omega(R^{\frac{1}{2}\vartheta-\varepsilon})$. *If the Riemann hypothesis is true, one has* $M(\varrho) - \frac{6}{\pi^2}R = \Omega(R^{1/4})$.

By the previous lemma, we have for any natural number $\nu \geq 2$ that

$$
M(\varrho)^{\nu} - \left(\frac{6}{\pi^2}\right)^{\nu} R^{\nu} = (\nu + o(1))\left(M(\varrho) - \frac{6}{\pi^2}R\right)\left(\frac{6}{\pi^2}\right)^{\nu-1} R^{\nu-1}
$$

which by another appeal to the lemma yields

$$
M(\varrho)^{\nu} - \left(\frac{6}{\pi^2}\right)^{\nu} R^{\nu} = \Omega(R^{\nu-1+\frac{1}{2}\vartheta-\varepsilon}). \tag{54}
$$

Our preparations are now complete. By (53) we see that Theorem 2 will follow provided we are able to establish that

$$
\sum_{n=1}^{\infty}\left(r_{\nu}(n) - \left(\frac{6}{\pi^2}\right)^{\nu}\frac{n^{\nu-1}\mathfrak{S}_{\nu}(n)}{(\nu-1)!}\right)\varrho^n = \Omega(R^{\nu-1+\frac{1}{2}\vartheta-\varepsilon}), \tag{55}
$$

and this we shall now deduce from (54). We begin by observing that whenever $(a, q) = 1$, one has

$$
\sum_{n \leq x} e\left(\frac{an}{q}\right) \ll \|a/q\|^{-1}
$$

whence for $q \geq 2$ we infer that

$$\sum_{n \leq x} c_q(n) \ll \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \|a/q\|^{-1} \ll q \log q \,. \tag{56}$$

When $q = 1$, however, the sum on the left hand side of (56) equals $[x]$. By (49) it follows that

$$\sum_{n \leq x} \mathfrak{S}_\nu(n) = \sum_{q=1}^{\infty} G(q)^\nu \sum_{n \leq x} c_q(n) = x + O\left(\sum_{q=2}^{\infty} |G(q)|^\nu q \log q\right),$$

and for $\nu \geq 2$ the sum in the error term certainly converges, by (10). This yields

$$\sum_{n \leq x} \mathfrak{S}_\nu(n) = x + O(1) \tag{57}$$

for $\nu \geq 2$. Now, since

$$(\nu - 1)! R^\nu = \sum_{n=0}^{\infty} n^{\nu-1} \varrho^n + O(R^{\nu-1})$$

is readily confirmed by elementary considerations, we deduce from (57) by partial summation that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathfrak{S}_\nu(n) n^{\nu-1} \varrho^n - (\nu-1)! R^\nu &= \sum_{n=1}^{\infty} (\mathfrak{S}_\nu(n) - 1) n^{\nu-1} \varrho^n + O(R^{\nu-1}) \\
&= \sum_{n=1}^{\infty} \left(\sum_{m \leq n} \mathfrak{S}_\nu(m) - n\right) (n^{\nu-1}\varrho^n - (n+1)^{\nu-1}\varrho^{n+1}) + O(R^{\nu-1}) \\
&\ll \sum_{n=1}^{\infty} \varrho^n |n^{\nu-1} - (n+1)^{\nu-1}| + R^{-1} \sum_{n=1}^{\infty} (n+1)^{\nu-1} \varrho^n + O(R^{\nu-1}),
\end{aligned}$$

and (53) yields

$$\sum_{n=1}^{\infty} \mathfrak{S}_\nu(n) n^{\nu-1} \varrho^n - (\nu-1)! R^\nu = O(R^{\nu-1}) \,. \tag{58}$$

The required (55) is now immediate from (54) and (58). The proof of Theorem 2 is now complete. If the Riemann hypothesis is true, then the above argument shows that $\vartheta = \frac{1}{2}$, $\varepsilon = 0$ in Theorem 2 are permissible.

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