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Harnack Inequalities for Schrödinger Operators

WOLFHARD HANSEN

Abstract. Let μ be a signed Radon measure on a domain X in $\mathbb{R}^d$, $d \geq 1$, with Green function G_X, assume that μ is potentially bounded, i.e., that the potential $G_X^{1_B|\mu|}$ is bounded for every ball B in X, and define $d_{\mu^\pm}(x) = \limsup_{y\to x} G_X^{1_B\mu^\pm}(y) - G_X^{1_B\mu^\pm}(x)$, $x \in B$, $\overline{B} \subset X$ (μ is a local Kato measure if and only if $d_{\mu^+} = d_{\mu^-} = 0$). The question, if positive μ-harmonic functions, i.e., positive finely continuous solutions of $\Delta h - h\mu = 0$, satisfy Harnack inequalities, is completely solved: If U is a domain in X admitting a positive μ-harmonic function which is locally bounded and not identically zero, then Harnack inequalities hold for positive μ-harmonic functions on U and every μ-harmonic function on U is locally bounded. In particular, Harnack inequalities always hold as long as $d_{\mu^-} \leq \gamma < 1$ (they may already fail for $d_{\mu^-} \leq 1$, but it is it possible that they hold non-trivially in spite of big values of d_{μ^-}). The results are presented in a general setting covering uniformly elliptic operators and sums of squares of smooth vector fields.

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1. – Introduction

Since many years it is well known that for several classes of linear partial differential operators $\mathcal{L}$ of second order positive (weak) solutions u of Schrödinger equations

$$\mathcal{L}u - u\mu = 0$$

on a domain U satisfy Harnack inequalities

$$\sup_{x\in A} u(x) \leq c \inf_{x\in A} u(x), \qquad A \text{ compact in } U,$$

if μ is a (local) Kato function or – more generally – a signed (local) Kato measure. Over the years, quite a few papers have been written using various techniques and considering different settings (e.g., [AS82], [BHH85], [CFG86],

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[BHH87], [Her85], [HK90], [Sim90], [Han93], [CGL93], [KS93], [Kur94], [Kuw96], [LM97], [Moh98], see Sections 2 and 5 for some comments).

It has also been shown – but this seems to be known primarily among specialists for abstract potential theory (see [FdLP88], [dLP90]) – that Harnack inequalities always hold if $\mathcal{L}$ yields (locally) a Brelot space $(X, \mathcal{H})$ with Green function G_X (true for the operators in the papers quoted before), if $\mu \geq 0$ (more restrictive), and if μ (function or measure) is only *potentially bounded* with respect to G_X and ${}^*G_X$, i.e., if the potentials $G_X^{1_A\mu} = \int_A G_X(\cdot, y)\mu(dy)$ and ${}^*G_X^{1_A\mu} = \int G_X(y, \cdot)\mu(dy)$ are bounded for every compact subset A of X (more general, since μ local Kato is equivalent to having $G_X^{1_A\mu}$ bounded *and* continuous, see the discussion in Section 2). It was accepted that in the classical case and closely related ones Harnack inequalities hold as well if μ^+ is potentially bounded and μ^- is a Kato measure (though there does not seem to exist a correct proof for it in the literature; we give a short proof in Section 5) and that e.g. the method used in [BHH87] would even allow for very small discontinuities of the potentials $G_X^{1_A\mu^-}$. But this is all one could prove in the classical case until now.

So the interesting question arises: What happens if μ^+ and μ^- are potentially bounded, but the oscillation

$$d_{\mu^-}(x) := \limsup_{y \to x} G_X^{1_A\mu^-}(y) - G_X^{1_A\mu^-}(x), \qquad x \in \mathring{A}, \ A \text{ compact},$$

is not assumed to be small? More specifically:

- Do we always have Harnack inequalities as long as $d_{\mu^-} \leq \gamma < 1$?
- May Harnack inequalities already fail for $d_{\mu^-} \leq 1$?
- Is it possible that Harnack inequalities hold in a non-trivial way even if d_{μ^-} admits very big values?
- Can the domains where Harnack inequalities hold for $\Delta - \mu$ be characterized?
- Are Harnack inequalities always valid as long as at least the differences $G_X^{1_A\mu^+} - G_X^{1_A\mu^-}$ are continuous?

The purpose of this paper is to give a positive answer to these questions. The potential theoretic method we shall use will require only *one* property in addition to having a Brelot space with Green function G_X, namely the following *local triangle* property:

(LT) *There exists a covering of X by open sets U such that for some constant $C > 0$ (which may depend on U) and all $x, y, z \in U$*

$$\min(G_X(x, z), G_X(z, y)) \leq C\, G_X(x, y).$$

Clearly, any function $\tilde{G}_X$ which at the diagonal is locally equivalent to a function G_X satisfying (LT) has property (LT) as well. So the local triangle property holds for all Green functions associated with uniformly elliptic operators of type

$$\mathcal{L}=\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\Big(\sum_{i=1}^{d}a_{ij}\frac{\partial}{\partial x_i}\Big)\quad\text{or}\quad\mathcal{L}=\sum_{i,j=1}^{d}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\tag{1.1}$$

where the matrix $a(x)=(a_{ij}(x))$ is symmetric, bounded, measurable, positive definite uniformly in x (see [Her68], [CFG86]) and – in the second, the non-divergence type – Hölder continuous ([Her62], [BM68], [HS82], [HS84]) (of course, both types can be complemented by terms of lower order).

Moreover, it is almost trivial that (LT) is satisfied if X can be covered by open sets U such that

$$G_X\approx\varphi\circ\rho\qquad\text{on } U\times U$$

where ρ is a quasi-metric on U and φ is a positive decreasing numerical function with

$$\varphi(t/2)\le C\varphi(t)\qquad(t\in\mathbb{R}^+)$$

(see Proposition 9.2). In particular, (LT) holds for sub-Laplacians on stratified Lie algebras with homogeneous dimension $Q\ge 3$.

And – last but not least – the local triangle property is satisfied if X can be covered by open sets U such that

$$G_X(x,y)\approx\frac{\rho(x,y)^2}{|B(x,\rho(x,y))|}\qquad\text{for } x,y\in U$$

(see Proposition 9.3) where ρ is any quasi-metric, $B(x,r)=\{y:\rho(x,y)<r\}$, and $B\mapsto|B|$ is any measure which is finite on these balls and has the doubling property $|B(x,2r)|\le C|B(x,r)|$ and weak quadratic increase, i.e., $r^{-2}|B(x,r)|\le Cs^{-2}|B(x,s)|$ if $r<s$. In particular, (LT) holds if $X\subset\mathbb{R}^d$, $d\ge 3$, and $(X,\mathcal{H})$ is given by an operator

$$\mathcal{L}=\sum_{j=1}^{r}X_j^2\tag{1.2}$$

where $X_1,\dots,X_r$ are smooth vector fields on $\mathbb{R}^d$ satisfying Hörmander's condition

$$\operatorname{rank}\operatorname{Lie}[X_1,\dots,X_r]=d\qquad\text{at every } x\in\mathbb{R}^d\tag{1.3}$$

for hypoellipticity of the sum.

Answering the questions, which we raised for the classical case, in the general framework of a Brelot space $(X,\mathcal{H})$ with Green function having property (LT) we may thus obtain results proven for Kato functions in [CFG86] and

[CGL93] by PDE methods and get, as in the classical case, further results for the operators considered there.

Moreover, let us note that the proofs for Harnack inequalities given in [Her87], [dLP90], [Zah96], [Zah] for perturbation of general Brelot spaces by signed Kato measures contain serious gaps or are simply wrong (for the first two papers see [dLP99], in [Zah96] the suggested proof of Theorem 17 fails and [Zah] is based on it). So even for Kato measures our result on Harnack inequalities is new in the general setting.

The reader who is mainly interested in what can be done for Schrödinger operators $\Delta - \mu$ on open subsets of $\mathbb{R}^d$ may simply skip some of the more general parts of the paper, in particular Sections 6 and 7, and imagine to be in the classical situation all the time.

2. – Main results in the classical case

Before introducing μ-harmonic functions in the general setting of a Brelot space let us briefly discuss the main results obtained in the classical case:

Let X be a domain in $\mathbb{R}^d$, $d \geq 1$, with Green function G_X ($\Delta G_X(\cdot, y) = -\delta_y$ for every $y \in X$) and let $\mathcal{M}_{pb}(X)$ denote the set of all signed Radon measures on X which are *potentially bounded*, i.e., such that the potential $G_X^{1_B|\mu|} = \int_B G_X(\cdot, z)\,|\mu|(dz)$ is bounded for every open ball B with $\overline{B} \subset X$. For example, Lebesgue measure λ on X is contained in $\mathcal{M}_{pb}(X)$ and $f\mu \in \mathcal{M}_{pb}(X)$ for any $\mu \in \mathcal{M}_{pb}(X)$ and bounded density f.

We recall that a function V on X is a (local) *Kato function* if

$$\int_{B(x,r)} G_X(x, y)\,|V|(y)\lambda(dy) \to 0 \tag{2.1}$$

locally uniformly in x if r tends to zero (where $B(x,r) = \{y \in X : |y-x| < r\}$). (In [CFG86] these functions are said to belong to the *Stummel class*.) Replacing the measure $|V|\lambda$ in (2.1) by $|\mu|$, μ being a signed measure on X, we may define the more general class $\mathcal{M}_{\mathrm{Kato}}(X)$ of (local) *Kato measures* on X. For example, $(d-1)$-dimensional Hausdorff measure on the intersection of any hyperplane and X is a Kato measure (singular with respect to Lebesgue measure). It is easily seen that $\mathcal{M}_{\mathrm{Kato}}(X)$ is the set of all signed Radon measures on X such that the potentials $G_X^{1_B|\mu|}$, B a ball with $\overline{B} \subset X$, are bounded *and continuous* (see [BHH87]). This shows that

$$\mathcal{M}_{\mathrm{Kato}}(X) \subset \mathcal{M}_{pb}(X).$$

On the other hand, every locally bounded potential is a countable sum of continuous potentials and hence every $\mu \in \mathcal{M}_{pb}^+(X)$ is the limit of an increasing sequence in $\mathcal{M}^+_{\text{Kato}}(X)$. In particular, as Kato measures do not charge polar sets, we have $|\mu|(P) = 0$ for every $\mu \in \mathcal{M}_{pb}(X)$ and for every polar set P.

We recall that two real functions u and u' on an open set U are equal *quasi-everywhere*, $u = u'$ q.e., if $\{u \neq u'\}$ is a polar set and that a function u is called *quasi-continuous*, if for every $\varepsilon > 0$ there exists an open set V of capacity less that ε such that the restriction of u on $U \setminus V$ is continuous. For example, every element in $H^1_{\text{loc}}(U)$ admits a quasi-continuous version (see [DL54]). Furthermore, every finely continuous function is quasi-continuous (the fine topology is the coarsest topology on X such that all positive superharmonic functions on X are continuous). To deal with measures μ which might not be absolutely continuous with respect to λ it is useful to know that two quasi-continuous functions which are equal λ-almost everywhere are equal even quasi-everywhere (see e.g. [BH91]).

DEFINITION 2.1. Given $\mu \in \mathcal{M}_{pb}(X)$ and an open subset U of X, a function $h \in \mathcal{L}^1_{\text{loc}}(U, \lambda + |\mu|)$ is called *μ-harmonic* (*quasi-μ-harmonic* resp.) on U if

$$\Delta h - h\mu = 0 \qquad \text{(in the distributional sense)}$$

and h is finely continuous (finely continuous outside a polar set).

It is easily seen that a locally bounded h is μ-harmonic on U if and only if $h + G_B^{h\mu}$ is harmonic on B for every ball B with $\overline{B} \subset U$ (see Lemma 3.2 where the general case and the equivalence to the definition in [FdLP88] is discussed as well).

If $\mu \in \mathcal{M}_{\text{Kato}}(X)$, then every μ-harmonic function is continuous ([Han93]) and it has been shown in [BHH85] (published in [BHH87]) that defining

$${}^{\mu}\mathcal{H}(U) = \{h \in \mathcal{C}(U) : h \ \mu\text{-harmonic on } U\}, \qquad {}^{\mu}\mathcal{H} = \{{}^{\mu}\mathcal{H}(U) : U \text{ open in } X\}$$

we obtain a Brelot space $(X, {}^{\mu}\mathcal{H})$. In particular, Harnack inequalities hold for μ-harmonic functions, i.e., for every domain U in X and for every compact subset A of U, there exists a constant $c > 0$ such that

$$\sup h(A) \leq c \inf h(A)$$

for every μ-harmonic function $h \geq 0$ on U. At about the same time PDE methods have been used to obtain Harnack inequalities for $\Delta - V$, V being a Kato function ([CFG86]). Both papers [CFG86] and [BHH85]/[BHH87] yield Harnack inequalities for more general uniformly elliptic operators.

An interesting larger subclass of $\mathcal{M}_{pb}(X)$ is the set $\mathcal{M}_{pbc}(X)$ of all $\mu \in \mathcal{M}_{pb}(X)$ such that each function $G_X^{1_B\mu} = G_X^{1_B\mu^+} - G_X^{1_B\mu^-}$ is continuous (while perhaps $G_X^{1_B|\mu|} = G_X^{1_B\mu^+} + G_X^{1_B\mu^-}$ is not). Keuntje showed that, given $\mu \in \mathcal{M}_{pb}(X)$, continuous μ-harmonic functions lead to a harmonic space $(X, {}^{\mu}\mathcal{H})$ if and only if $\mu \in \mathcal{M}_{pbc}(X)$ ([Keu90]). The question, however, if for $\mu \in \mathcal{M}_{pbc}(X)$ the resulting harmonic space is even a Brelot space remained open.

That the method used in [BHH87] to obtain Harnack inequalities for μ-harmonic functions, $\mu \in \mathcal{M}_{\mathrm{Kato}}(X)$, immediately leads to a corresponding result for those $\mu \in \mathcal{M}_{pb}(X)$ having potentials $G_B^{|\mu|}$ which are sufficiently small for small balls B (G_B Green function for B) went unnoticed until Nakai published such a result ([Nak96]). In [dLP90], it is claimed (if we specialize the statement to the classical case) that Harnack inequalities hold for positive locally bounded μ-harmonic functions on a ball B if $G_B^{\mu^-} \le \gamma < 1$. However, as we already noted, the proof for this assertion is not correct (see [dLP99]).

Our first main result is the following (cf. Theorems 11.4, 11.6, and 11.8):

THEOREM 2.2. *Let $\mu \in \mathcal{M}_{pb}(X)$ and let U be a domain in X admitting a locally bounded positive μ-harmonic function on U which is not identically zero* (*or a domain which can be covered by such domains*). *Then the following is true:*

1. *For every quasi-μ-harmonic function h on U there exists a μ-harmonic function $\tilde{h}$ on U such that $\tilde{h} = h$ q.e.*
2. *Every μ-harmonic function on U is locally bounded.*
3. *Harnack inequalities hold for positive μ-harmonic functions on U.*
4. *If $u \in \mathcal{L}^1_{\mathrm{loc}}(U, \lambda + |\mu|)$ is quasi-continuous and $\Delta u - u\mu = 0$, then there exists a* (*unique*) *μ-harmonic function h on U such that $h = u$ q.e.*

4′. *If $\mu = V\lambda$ and $u \in \mathcal{L}^1_{\mathrm{loc}}(U, \lambda)$ such that $Vu \in \mathcal{L}^1_{\mathrm{loc}}(U, \lambda)$ and $\Delta u - Vu = 0$, then there exists a unique μ-harmonic function h on U such that $h = u$ λ-a.e.*

REMARK 2.3. 1) Some regularity is needed if μ is not absolutely continuous with respect to λ: If $\mu \neq 0$ is supported by a (necessarily non-polar) Borel set A having Lebesgue measure zero, then $u := 1_{U \setminus A}$ satisfies $\Delta u = 0 = u\mu$. However, the only finely continuous function h which is λ-a.e. equal to u is the constant 1 which is not μ-harmonic since $\Delta 1 = 0 \neq \mu$. And of course u does not satisfy Harnack inequalities.

2) If $u \in H^1_{\mathrm{loc}}(U)$ is quasi-continuous, then $u \in \mathcal{L}^1_{\mathrm{loc}}(U, |\mu|)$ (see e.g. [Her68], p. 353). So (4) in Theorem 2.2 states in particular that every quasi-continuous solution $u \in H^1_{\mathrm{loc}}(U)$ to $\Delta u - u\mu = 0$ admits a unique quasi-continuous version which is even finely continuous (continuous if μ is a Kato measure).

Let us stress the fact that of course Harnack inequalities hold trivially on every domain U not admitting any positive μ-harmonic function except the constant 0. On the other hand, there is no chance for Harnack inequalities if there exists a single positive μ-harmonic function on the domain U which is not locally bounded (and then there cannot exist any locally bounded positive μ-harmonic function on U). We shall be able to decide when this happens, and examples will illustrate various possible cases.

For every $\mu \in \mathcal{M}^+_{pb}(X)$, $x \in X$, B a ball with closure in X, define

$$d_\mu(x) := \limsup_{y \to x} G_X^{1_B\mu}(y) - G_X^{1_B\mu}(x) = \limsup_{y \to x} G_B^\mu(y) - G_B^\mu(x)$$

and note that the oscillation $d_\mu(x)$ does not depend on the choice of B. In fact

$$d_\mu(x) = \inf\{\|G_B^\mu\|_\infty : B \text{ ball}, x \in B, \overline{B} \subset X\}$$

where $\|\cdot\|_\infty$ denotes supremum norm (for details see Section 13). Of course,

$$\mathcal{M}_{\mathrm{Kato}}(X) = \{\mu \in \mathcal{M}_{pb}(X) : d_{\mu^+} = d_{\mu^-} = 0\}$$

and $d_{\mu^+} = d_{\mu^-}$ for every $\mu \in \mathcal{M}_{pbc}(X)$.

As a consequence of Theorem 2.2 we obtain the following:

COROLLARY 2.4. *For every $\mu \in \mathcal{M}_{pb}(X)$, statements* (1), (2), (3), (4), *and* (4′) *of Theorem 2.2 hold for every domain U where d_{μ^-} is strictly less than* 1.

In particular, the assertion made in [dLP90] is true at least in the classical case.

If $\mu \in \mathcal{M}_{pbc}(X)$, then Harnack inequalities for positive μ-harmonic functions will hold on any domain in X, even if $d_{\mu^\pm}$ is big (cf. Theorem 12.3):

THEOREM 2.5. *For every $\mu \in \mathcal{M}_{pb}(X)$, the following statements are equivalent:*

1. $\mu \in \mathcal{M}_{pbc}(X)$.
2. $(X, {}^\mu\mathcal{H})$ *is a harmonic space.*
3. $(X, {}^\mu\mathcal{H})$ *is a Brelot space.*

The following result implies that there are many measures $\mu \in \mathcal{M}_{pbc}(X) \setminus \mathcal{M}_{\mathrm{Kato}}(X)$ (cf. Corollary 12.6):

THEOREM 2.6. *For every $\mu_1 \in \mathcal{M}^+_{pb}(X)$ there exists $\mu_2 \in \mathcal{M}^+_{pb}(X)$ (having a density with respect to λ) such that $\mu := \mu_1 - \mu_2 \in \mathcal{M}_{pbc}(X)$, $\mu^+ = \mu_1$, $\mu^- = \mu_2$.*

Given any $\mu \in \mathcal{M}_{pb}(X)$, Baire's theorem and Corollary 2.4 imply that there is a *dense* open subset Y of X such that Harnack inequalities for positive μ-harmonic functions will hold on every domain contained in Y. And this is equally true if μ is only assumed to be *potentially finite*, i.e., such that $G_X^{1_B|\mu|} < \infty$ for every ball B with $\overline{B} \subset X$. But that is all we can be sure of in general (cf. Theorems 13.4 and 13.7, Corollary 13.8, and Proposition 14.4):

THEOREM 2.7. *Let μ be a signed Radon measure on X which is potentially finite. Then there exists a dense open subset Y of X such that, for every domain U contained in Y, properties* (1), (2), (3), (4), *and* (4′) *of Theorem 2.2 hold.*

Conversely, given any dense open subset Y of X and any real $\varepsilon > 0$, there exists a measure $\mu \leq 0$ on X (with a density V with respect to Lebesgue measure) having the following properties:

1. *The union of all domains U where Harnack inequalities hold for positive μ-harmonic functions is the set Y.*
2. $G_X^{|\mu|} \leq 1 + \varepsilon$, *$G_X^{|\mu|}$ is continuous on Y (i.e., $d_{|\mu|} = 0$ on Y) and $d_{|\mu|} = 1$ on $X \setminus Y$.*

In particular, for every $0 < \delta \leq 1$, Y is the union of all balls B such that $\|G_B^{|\mu|}\|_\infty < \delta$.

Taking any measure in $\mathcal{M}_{pbc}(X)\setminus\mathcal{M}_{\text{Kato}}(X)$ an application of Theorem 2.5 to multiples of this measure already tells us the following: There are measures $\mu\in\mathcal{M}_{pb}(X)$ such that Harnack inequalities hold for positive μ-harmonic functions on every domain in X in spite of having arbitrarily big oscillations of d_{μ^-}. In contrast with the second part of Theorem 2.7 this may happen even if $\mu^+=0$ (cf. Proposition 14.5):

THEOREM 2.8. *Given any dense open subset Y of X and strictly positive real numbers α and ε, there exists a measure $\mu\le 0$ on X (absolutely continuous with respect to Lebesgue measure) having the following properties:*

1. *The statements (1), (2), (3), (4), and (4′) of Theorem 2.2 hold for every domain U contained in Y.*
2. *$G_X^{|\mu|}\le\alpha+\varepsilon$, $G_X^{|\mu|}$ is continuous on Y and $d_{\mu^-}=\alpha$ on $X\setminus Y$.*

3. – Solutions of Schrödinger equations and μ-harmonic functions

Generalizing the setup discussed in the previous section we shall assume in the following that $(X,\mathcal{H})$ is a connected $\mathcal{P}$-harmonic Brelot space (X locally connected and locally compact with countable base). In particular, $\mathcal{H}$ is a sheaf of vector spaces $\mathcal{H}(U)$ of continuous real functions on U, U open in X, which are called *harmonic* functions. Moreover, the sheaf $\mathcal{H}$ is non-degenerate, there exists a base of $(\mathcal{H})$-regular sets and a strong convergence axiom holds which amounts to having Harnack inequalities for positive harmonic functions on domains in X. For the formal definition we refer the reader to [Her62],[CC72].

Various classes of linear partial differential operators of second order on open subsets X of $\mathbb{R}^d$ lead to Brelot spaces:

a) If

$$\mathcal{L}=\sum_{i,j=1}^{d}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i\frac{\partial}{\partial x_i}+c$$

such that the functions a_{ij},b_i,c are Hölder continuous with exponent $\alpha>0$ and the quadratic forms $\xi\mapsto\sum a_{ij}(x)\xi_i\xi_j$, $x\in X$, are positive definite, then

$$\mathcal{H}(U):=\{u\in\mathcal{C}^{2+\alpha}(U):\mathcal{L}u=0\}$$

yields a Brelot space ([Her62],[BM68]). See [Kro88] for the case where the coefficients are only assumed to be continuous.

b) If

$$\mathcal{L}=\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{d}a_{ij}\frac{\partial}{\partial x_i}+d_i\right)+\sum_{i=1}^{d}b_i\frac{\partial}{\partial x_i}+c$$

such that the functions a_{ij} are measurable, bounded and the matrix $(a_{ij}(x))$ is uniformly elliptic, then (under mild restrictions on the functions b_i,d_i,c, see

[Her68]) we obtain a Brelot space defining a harmonic function u on open subset U of X to be a continuous (version of a) weak solution of $\mathcal{L}u = 0$, i.e., such that $u \in H^1_{\text{loc}}(U)$ and

$$\int \left[\sum_j \left(\sum_i a_{ij} \frac{\partial u}{\partial x_i} + d_j u \right) \frac{\partial \varphi}{\partial x_j} + \left(\sum_i b_i \frac{\partial u}{\partial x_i} + cu \right) \varphi \right] d\lambda = 0 \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(U).$$

c) If $\mathcal{L} = \sum_{j=1}^r X_j^2 + Y$ with smooth vector fields $X_1, \ldots, X_r, Y$ such that (1.3) holds, then we get a Brelot space by

$$\mathcal{H}(U) := \{u \in \mathcal{C}^2(U) : \mathcal{L}u = 0\}, \qquad U \text{ open in } X$$

(see [Bon70], [Bon69], [Her72], [BH86]).

In the examples given above we have Green functions which may (at least locally) be equivalent to the classical Green function (cases (a) and (b)) or rather different (in the degenerate case (c)).

In our abstract situation we shall assume that there is a Green function G_X for $(X, \mathcal{H})$, i.e., that we have a function $G_X : X \times X \to [0, \infty]$ such that the following holds (for an abstract definition of potentials see below):

(i) For every $y \in X$, $G_X(\cdot, y)$ is a potential on X, harmonic on $X \setminus \{y\}$,
(ii) for every $x \in X$, $G_X(x, \cdot)$ is continuous on $X \setminus \{x\}$,
(iii) for every continuous real potential p on X there exists a measure $\nu \geq 0$ on X such that $p = G_X^\nu := \int G_X(\cdot, y)\nu(dy)$.

This implies that the axiom of proportionality holds (i.e., given $y \in X$, any two potentials which are harmonic on $X \setminus \{y\}$ are proportional) (see [Bou79]). Conversely, if all points of X are polar, then the axiom of proportionality implies the existence of a Green function for $(X, \mathcal{H})$ (see [Her62]).

We note that, in particular, G is lower semi-continuous on $X \times X$ and locally bounded off the diagonal.

To get a better understanding of the main ideas let us first suppose that G_X is *symmetric* (as in the classical case), i.e., that

$$G_X(x, y) = G_X(y, x)$$

for all $x, y \in X$ (and hence (ii) is a consequence of (i)). However, we shall mention explicitly when we are using the symmetry of G_X, and in the last section we shall discuss which "adjoint" properties of G_X or μ will allow us to get our results without the hypothesis of symmetry.

Finally, we shall assume for simplicity that the constant function 1 is superharmonic on X (the general case can be reduced to this one dividing by a strictly positive continuous real potential). The local triangle property which we shall need to treat negative perturbations in an effective way will be introduced later (see Section 9).

Let us fix a base $\mathcal{V}$ of relatively compact regular domains for the topology of X such that $X \notin \mathcal{V}$ and, for every $V \in \mathcal{V}$ and for every neighborhood U of $\overline{V}$, there exists a set $W \in \mathcal{V}$ with $\overline{V} \subset W \subset U$. (We could get along without regularity of the sets in $\mathcal{V}$ and the additional properties, but our choice of $\mathcal{V}$ simplifies the following considerations.) In the classical case we could, for example, take the set of all balls B with $\overline{B} \subset X$.

For every $V \in \mathcal{V}$, we have the *harmonic kernel* H_V solving the Dirichlet problem for V (in the classical case and for a ball V, H_V is the Poisson integral operator). For every open subset U of X, define

$$\mathcal{V}(U) = \{V \in \mathcal{V} : \overline{V} \subset U\}.$$

Given an open subset U of X, let $\mathcal{S}^+(U)$ denote the set of all positive superharmonic functions on U, i.e., $\mathcal{S}^+(U)$ is the set of all l.s.c. numerical functions $s \geq 0$ on U such that, for every $V \in \mathcal{V}(U)$, $H_V s$ is harmonic on V and $H_V s \leq s$. By definition, a function $p \in \mathcal{S}^+(U)$ is a *potential* if the constant 0 is the only positive harmonic minorant of p.

For every $V \in \mathcal{V}$, we obtain a Green function G_V on V defining

$$G_V(\cdot, y) = G_X(\cdot, y) - H_V G_X(\cdot, y) \qquad (y \in V). \tag{3.1}$$

The symmetry of G_X implies the symmetry of the functions G_V, $V \in \mathcal{V}$ (see [Her62]).

Let $\mathcal{M}_{pb}(X)$ denote the set of all signed Radon measures μ on X such that, for every compact A in X, the functions $G_X^{1_A \mu^\pm}$ are bounded and countable sums of continuous real potentials on X. One of the many equivalent properties characterizing axiom D, the *axiom of domination*, states that every locally bounded potential is a countable sum of continuous potentials (see [CC72], p. 228). Thus, if $(X, \mathcal{H})$ satisfies axiom D, the set $\mathcal{M}_{pb}(X)$ is the set of all signed Radon measures μ on X such that the functions $G_X^{1_A \mu^\pm}$ are bounded for every compact A in X.

A signed Radon measure μ on X is a (local) Kato measure if the potentials $G_X^{1_A \mu^\pm}$, A compact in X, are continuous and bounded. Let $\mathcal{M}_{\text{Kato}}(X)$ denote the set of all Kato measures on X. So every $\mu \in \mathcal{M}_{pb}^+(X)$ is an increasing limit of measures $\mu_n \in \mathcal{M}_{\text{Kato}}^+(X)$, $n \in \mathbb{N}$. Moreover, measures $\mu \in \mathcal{M}_{pb}(X)$ do not charge semi-polar sets. If $(X, \mathcal{H})$ satisfies axiom D, then every semi-polar set is polar (see [CC72]). Let us recall that a subset P of X is *polar* if there exists a function $s \in \mathcal{S}^+(X)$ such that $P \subset \{s = \infty\}$. In the classical case, polar sets are the sets of (classical) capacity zero, and they are Lebesgue null sets.

Finally, it is easily seen that a signed Radon measure μ on X is in $\mathcal{M}_{pb}(X)$ ($\mathcal{M}_{\text{Kato}}(X)$ resp.) if and only if there is a covering of X by Borel sets A_n such that $A_n \subset \overset{\circ}{A}_{n+1}$ and the functions $G_X^{1_{A_n} \mu^\pm}$ are bounded (continuous resp.) for every $n \in \mathbb{N}$.

In the following we shall always assume that $\mu \in \mathcal{M}_{pb}(X)$ unless explicitly stated otherwise. For every $V \in \mathcal{V}$, we define a kernel K_V^μ by

$$\boxed{K_V^\mu f := G_V^{f\mu} = \int G_V(\cdot, z) f(z)\, d\mu(z) \qquad (f \in \mathcal{B}_b(V))}$$

and obtain a bounded operator on the space $\mathcal{B}_b(V)$ of all bounded Borel functions on V (given the supremum norm $\|\cdot\|_\infty$). Moreover, (3.1) implies that

$$K_W^\mu = K_V^\mu - H_W K_V^\mu \tag{3.2}$$

for all $V, W \in \mathcal{V}$ such that $W \subset V$.

The following two lemmas on solutions of Schrödinger equations $\Delta u - u\mu = 0$ will serve now to motivate the definition of μ-harmonic functions in our general situation. They will be useful later on to apply our results to the classical case and – using corresponding results – to uniformly elliptic operators and sums of squares of vector fields.

Lemma 3.1. *Let U be an open set in $\mathbb{R}^d$ and suppose that μ is absolutely continuous with respect to λ, $\mu = V\lambda$. Then, for every $u \in \mathcal{L}^1_{\mathrm{loc}}(U, \lambda)$ such that $Vu \in \mathcal{L}^1_{\mathrm{loc}}(U, \lambda)$ and*

$$\Delta u - Vu = 0,$$

there exists a quasi-μ-harmonic function h with $h = u$ λ-a.e.

Proof. Given $W \in \mathcal{V}(U)$, the function $G_W^{|uV|\lambda}$ is a potential and

$$\Delta(u + G_W^{uV\lambda}) = \Delta u - uV = 0,$$

hence there exists a (unique) harmonic function g on W such that

$$u + G_W^{uV\lambda} = g \quad \lambda\text{-a.e.}$$

The set $P := \{G_W^{|uV|\lambda} = \infty\}$ is polar and defining

$$\tilde{u} = \begin{cases} g - G_W^{uV\lambda} & \text{on } W \setminus P, \\ 0 & \text{on } P, \end{cases}$$

we obtain a real function $\tilde{u}$ on W such that $\tilde{u}$ is finely continuous on $W \setminus P$ and equal to u λ-a.e. on W.

Now let (W_n) be a sequence in $\mathcal{V}(U)$ covering U and, for every $n \in \mathbb{N}$, choose a polar subset P_n of W_n and a real function $\tilde{u}_n$ on W_n such that $\tilde{u}_n$ is finely continuous on $W_n \setminus P_n$ and

$$\tilde{u}_n = u \quad \lambda\text{-a.e. on } W_n.$$

Then $P = \bigcup_{n=1}^{\infty} P_n$ is polar. For all $m, n \in \mathbb{N}$, we have $\tilde{u}_m = u = \tilde{u}_n$ λ-a.e. on $W_m \cap W_n$ and the functions $\tilde{u}_m$ and $\tilde{u}_n$ are finely continuous on $(W_m \cap W_n) \setminus P$, hence

$$\tilde{u}_m = \tilde{u}_n \quad \text{on } (W_m \cap W_n) \setminus P_n$$

(a λ-null set has no finely interior points). So we may define a real function h on U by

$$h = \begin{cases} \tilde{u}_n & \text{on } W_n \setminus P,\ n \in \mathbb{N}, \\ 0 & \text{on } P, \end{cases}$$

and then h is finely continuous on $U \setminus P$, $h = u$ λ-a.e. on U. In particular, $\Delta h - Vh = 0$. Thus h is quasi-μ-harmonic. □

LEMMA 3.2. *Let h be a measurable real function on an open set U in X. Then the following properties are equivalent in the classical case:*

1. *$h \in \mathcal{L}^1_{\text{loc}}(U, \lambda + |\mu|)$, h is quasi-continuous on U, and $\Delta h - h\mu = 0$.*[(1)]
2. *h is quasi-μ-harmonic on U.*
3. *For every $W \in \mathcal{V}(U)$, there exists a harmonic function g on W and a polar set P such that $G_W^{|h\mu|} < \infty$ on $W \setminus P$ and*

$$h + G_W^{h\mu} = g \quad \text{on } W \setminus P. \tag{3.3}$$

PROOF. (1) $\Longrightarrow$ (2): Given $W \in \mathcal{V}(U)$, there exists a (unique) harmonic function g on W such that $h + G_W^{h\mu} = g$ λ-a.e., i.e.,

$$h + G_W^{(h\mu)^+} = g + G_W^{(h\mu)^-} \quad \lambda\text{-a.e. on } W.$$

Since the terms in this equation are quasi-continuous, we obtain that

$$h + G_W^{(h\mu)^+} = g + G_W^{(h\mu)^-} \quad \text{q.e. on } W,$$

so there exists a polar set P in W such that $\{G_W^{|h\mu|} = \infty\} \subset P$ and $h = g - G_W^{h\mu}$ on $W \setminus P$. Therefore h is finely continuous on $W \setminus P$. A trivial covering argument shows that h is finely continuous on U outside a polar set. Thus h is quasi-μ-harmonic.

(2) $\Longrightarrow$ (1): Any function u on U which is finely continuous on $U \setminus P$, P polar, is quasi-continuous: Indeed, assuming without loss of generality that $u \geq 0$ we may add a function $s \in \mathcal{S}^+(U)$ such that $s = \infty$ on P. Then $u + s$ is finely continuous, hence quasi-continuous. Given $\varepsilon > 0$, we may choose an open set V with capacity less that ε such that $\{s = \infty\} \subset V$ and the restrictions of $u+s$ and s on $U \setminus V$ are continuous. Thus the difference $u|_{U \setminus V}$ is continuous.

(1) Added in proof: In fact, it is known that any real function is quasi-continuous if and only if it is finely continuous outside a polar set (see [FOT94], Theorem 4.6.1). So a proof for the equivalence of (1) and (2) could have been omitted.

(2) $\Longrightarrow$ (3): Let P be a polar set such that h is finely continuous on $U \setminus P$ and fix $W \in \mathcal{V}(U)$. By assumption, $|h\mu|$ is a Radon measure on U, hence $G_W^{|h\mu|}$ is a potential on W and $P' := \{G_W^{|h\mu|} = \infty\}$ is polar. Moreover, $\Delta(h + G_W^{h\mu}) = \Delta h - h\mu = 0$ on W, so there exists a (unique) harmonic function g on W such that

$$h + G_W^{h\mu} = g \quad \lambda\text{–a.e. on } W.$$

Since h, $G_W^{h\mu}$, and g are finely continuous and real on $W \setminus (P \cup P')$, we conclude that

$$h + G_W^{h\mu} = g \qquad \text{on } W \setminus (P \cup P').$$

(3) $\Longrightarrow$ (2): Fix $W \in \mathcal{V}(U)$, a harmonic function g on W and a polar set P such that $G_W^{|h\mu|} < \infty$ on $W \setminus P$ and $h = g - G_W^{h\mu}$ on $W \setminus P$. Then h is of course finely continuous on $W \setminus P$ and $h \in \mathcal{L}^1_{\mathrm{loc}}(W, \lambda)$. Moreover $G_W^{|h\mu|}$ is a potential, hence $h \in \mathcal{L}^1_{\mathrm{loc}}(W, |\mu|)$ and

$$\Delta h - h\mu = \Delta(h + G_W^{h\mu}) = \Delta g = 0 \qquad \text{on } W.$$

A trivial covering argument finishes the proof.

□

REMARK 3.3. 1) If $X \subset \mathbb{R}^d$ and $(X, \mathcal{H}, G_X)$ is given by a sum $\mathcal{L} = \sum_{j=1}^r X_j^2 + Y$ with smooth vector fields $X_1, \dots, X_r, Y$ satisfying (1.3),

$$\mathcal{H}(U) = \{h \in \mathcal{C}^2(U) : \mathcal{L}h = 0\}, \qquad \mathcal{L}G_X(\cdot, y) = -\delta_y,$$

then Lemma 3.1 and Lemma 3.2 will hold as well if we replace Δ by $\mathcal{L}$ (and use Definition 2.1 with $\mathcal{L}$ instead of Δ).

2) Similarly for uniformly elliptic operators in non-divergence form.

3) If $X \subset \mathbb{R}^d$ and $(X, \mathcal{H}, G_X)$ is given by a uniformly elliptic operator $\mathcal{L}$ in divergence form, then the statements of Lemma 3.1 and Lemma 3.2 are true if modified in an obvious way: We have to consider weak solutions u of $\mathcal{L}u - u\mu = 0$, i.e., functions $u \in H^1_{\mathrm{loc}}(U)$ such that

$$\int \left[\sum_j \left(\sum_i a_{ij} \frac{\partial u}{\partial x_i} + d_j u \right) \frac{\partial \varphi}{\partial x_j} + \left(\sum_i b_i \frac{\partial u}{\partial x_i} + cu \right) \varphi \right] d\lambda + \int \varphi u \, d\mu = 0$$

for any $\varphi \in C_0^\infty(U)$ (observe that (3.3) implies that $u \in H^1_{\mathrm{loc}}(U)$, since $g \in H^1_{\mathrm{loc}}(W)$ by definition of $\mathcal{H}(W)$ and $G_W^{u\mu} \in H^1_{\mathrm{loc}}(W)$ (see [Her68]).

Let us now return to the general situation. In view of the Lemma 3.2 and Remarks 3.3 the following definition is justified.

DEFINITION 3.4. Given $\mu \in \mathcal{M}_{pb}(X)$, a measurable real function h on an open subset U of X is called *μ-harmonic* (*quasi-μ-harmonic* resp.) if h is finely continuous (finely continuous outside a polar set) and if, for every $V \in \mathcal{V}(U)$, there exists a harmonic function g on V and a polar subset P of V such that $G_V^{|h\mu|} < \infty$ on $V \setminus P$ and

$$h + K_V^{\mu} h = g \quad \text{on } V \setminus P. \tag{3.4}$$

It is easily seen that fine continuity outside a polar set is already a consequence of (3.4) and that the definition of (quasi-)μ-harmonic functions does not depend on the choice of $\mathcal{V}$. Since we shall not make any use of it, the verification is left to the reader.

If two measurable real functions coincide quasi-everywhere, then of course one is quasi-μ-harmonic if the other is quasi-μ harmonic. Since polar sets have no finely interior points, for every u there exists at most one finely continuous function $\tilde{u}$ such that $\tilde{u} = u$ q.e. In particular, for every quasi-μ-harmonic function h on an open set U, there exists at most one finely continuous real function $\tilde{h}$ on U such that $\tilde{h} = h$ q.e., and then this function $\tilde{h}$ is μ-harmonic.

The following simple lemma will allow us to show the existence of μ-harmonic modifications (see Theorem 11.4).

LEMMA 3.5. *Let h be a quasi-μ-harmonic function on an open set U in X such that, for some polar set P and for every compact subset A of U,*

$$\sup |h|(A \setminus P) < \infty.$$

Then there exists a (unique and locally bounded) μ-harmonic function $\tilde{h}$ on U such that $\tilde{h} = h$ q.e.

PROOF. Fix $V \in V(U)$. By our preceding considerations and a trivial covering argument, it suffices to show that there exists a finely continuous function $\tilde{h}$ on V such that $\tilde{h} = h$ q.e. on V. By definition, there exists a harmonic function g on V and a polar subset P' of V such that $G_V^{|h\mu|} < \infty$ on $V \setminus P'$ and

$$h + G_V^{h\mu} = g \quad \text{on } V \setminus P'.$$

Since $|\mu|(P) = 0$, we obtain that

$$\|G_V^{|h\mu|}\|_\infty \leq \sup |h|(\overline{V} \setminus P) \|G_V^{|\mu|}\|_\infty < \infty.$$

Thus

$$\tilde{h} := g - G_V^{h\mu}$$

is a finely continuous real function on V and $\tilde{h} = h$ q.e. on V. □

We close this section by the following observation which will be extremely useful:

LEMMA 3.6. *Let h be a quasi-μ-harmonic function on an open set U in X, $V \in \mathcal{V}(U)$, and g a harmonic function on V such that $G_V^{|h\mu|} < \infty$ on $V \setminus P$ and $h + K_V^\mu h = g$ on $V \setminus P$. Then $g = H_V h$ and $H_V|h|$ is a harmonic function on V.*

If $h \geq 0$, then $g \geq 0$, and if h is a positive μ-harmonic function on U, then

$$(I + K_V^{\mu^+})h = g + K_V^{\mu^-} h \in \mathcal{S}^+(V). \tag{3.5}$$

PROOF. Choose $V' \in \mathcal{V}(U)$, a polar subset P', and a harmonic function g' on V' such that $\overline{V} \subset V'$, $G_{V'}^{|h\mu|} < \infty$ on $V' \setminus P'$, and

$$h + K_{V'}^\mu h = g' \quad \text{on } V' \setminus P'. \tag{3.6}$$

Then $G_V^{|h\mu|} + H_V G_{V'}^{|h\mu|} = G_{V'}^{|h\mu|} < \infty$ on $V' \setminus P'$ and $|h| \leq |g'| + G_{V'}^{|h\mu|}$ on $V' \setminus P'$, hence $H_V|h| \leq H_V(|g'| + G_{V'}^{|h\mu|}) \in \mathcal{H}(V)$. This implies that the functions $H_V|h|$ and $H_V h$ are harmonic on V and (harmonic measures do not charge polar sets!)

$$H_V h + H_V K_{V'}^\mu h = H_V g' = g' \quad \text{on } V. \tag{3.7}$$

Combining (3.6) and (3.7) we obtain that

$$h + K_V^\mu h = H_V h \quad \text{on } V \setminus P'.$$

So $g = H_V h$ on $V \setminus (P \cup P')$ and hence on V.

Finally, if $h \geq 0$ then $g = H_V h \geq 0$, $g + K_V^{\mu^-} h \in \mathcal{S}^+(V)$, and $h + K_V^{\mu^+} h = g + K_V^{\mu^-} h$ on $V \setminus P$. And if h is μ-harmonic, this equality holds everywhere on V by fine continuity. □

4. – The inverse of $I + K_V^{\mu^+}$

It is not hard to see that, for every $V \in \mathcal{V}$, the operator $I + K_V^{\mu^+}$ on $\mathcal{B}_b(V)$ is invertible (see e.g. [BHH87], [HM90]). For the proof of the crucial Lemma 11.3 we shall need the following result. It generalizes the well known fact that $(I + K_V^{\mu^+})^{-1} s \geq 0$ for every $s \in \mathcal{S}_b^+(V)$.

LEMMA 4.1. *Let $V \in \mathcal{V}$, $s \in \mathcal{S}^+(V)$, $g : V \to [0, \infty]$, let P be a polar subset of V and f a Borel measurable real function on V such that $\{K_V^{\mu^+}|f| = \infty\} \subset P$ and $f + K_V^{\mu^+} f + g = s$ on $V \setminus P$. Then $f + g \geq 0$ on $V \setminus P$.*

PROOF. Fix positive Kato measures ν_n on X with $\nu_n \uparrow \mu^+$, let (A_n) be an increasing sequence of compact subsets of $\{f^+ > 0\}$ such that $\mu^+(\{f^+ > 0\} \setminus \bigcup_{n=1}^{\infty} A_n) = 0$ and define

$$f_n := 1_{A_n} \min(f^+, n) \qquad (n \in \mathbb{N}).$$

Then $(K_V^{\nu_n} f_n)$ is a sequence of continuous real potentials on V which is increasing to $K_V^{\mu^+} f^+$. For every $n \in \mathbb{N}$, $A_n \subset \{f^+ > 0\} \subset \{f^- = 0\}$ and hence

$$s + K_V^{\mu^+} f^- = f^+ + K_V^{\mu^+} f^+ + g \geq K_V^{\nu_n} f_n \qquad \text{on } A_n \setminus P.$$

Since the continuous potential $K_V^{\nu_n} f_n$ is harmonic on $V \setminus A_n$, by minimum principle

$$s + K_V^{\mu^+} f^- \geq K_V^{\nu_n} f_n \qquad \text{on } V \setminus A_n$$

for each $n \in \mathbb{N}$, and hence

$$s + K_V^{\mu^+} f^- \geq K_V^{\mu^+} f^+ \qquad \text{on } V \setminus P$$

(even on V by fine continuity). Thus finally

$$f + g = s + K_V^{\mu^+} f^- - K_V^{\mu^+} f^+ \geq 0 \qquad \text{on } V \setminus P. \qquad \square$$

The following simple consequence will be useful in the next section.

PROPOSITION 4.2. *Let h be a quasi-μ-harmonic function on an open set U in X. Then, for every $V \in \mathcal{V}(U)$,*

$$|h| \leq H_V|h| + K_V^{\mu^-}|h| \quad \textit{q.e. on } V.$$

PROOF. Of course, the statement is an immediate consequence of Lemma 3.6 if $h \geq 0$. In the general case we proceed as follows: Let P be a polar subset of X such that $G_V^{|h\mu|} < \infty$ on $V \setminus P$ and

$$h + K_V^{\mu} h = H_V h \qquad \text{on } V \setminus P.$$

Define $f = 1_{V\setminus P} h$. Then $f + K_V^{\mu^+} f = f + K_V^{\mu^+} h = H_V h + K_V^{\mu^-} h$ on $V \setminus P$, hence

$$f + K_V^{\mu^+} f + (H_V|h| + K_V^{\mu^-}|h|) = (H_V + K_V^{\mu^-})(|h| + h) \qquad \text{on } V \setminus P.$$

Since obviously $(H_V + K_V^{\mu^-})(|h| + h) \in \mathcal{S}^+(V)$, we conclude by Lemma 4.1 that $f + (H_V|h| + K_V^{\mu^-}|h|) \geq 0$ on $V \setminus P$, i.e.,

$$-h \leq H_V|h| + K_V^{\mu^-}|h| \qquad \text{on } V \setminus P.$$

The proof is finished replacing h by $-h$. $\square$

Finally, let us recall some known properties of the inverse of $I + K_V^{\mu^+}$: Fix $s \in \mathcal{S}_b^+(V)$, $V \in \mathcal{V}$. Then the positivity of $f = (I + K_V^{\mu^+})^{-1}s$ implies that

$$s \geq (I + K_V^{\mu^+})^{-1}s. \tag{4.1}$$

More generally, for all $\nu, \nu' \in \mathcal{M}_{pb}^+(X)$,

$$(I + K_V^{\nu})^{-1}s \geq (I + K_V^{\nu+\nu'})^{-1}s,$$

since

$$(I + K_V^{\nu})^{-1} - (I + K_V^{\nu+\nu'})^{-1} = (I + K_V^{\nu})^{-1}K_V^{\nu'}(I + K_V^{\nu+\nu'})^{-1}. \tag{4.2}$$

Logarithmic convexity of $\alpha \mapsto (I + \alpha K_V^{\mu^+})^{-1}s$ (presumably considered first by F. Hirsch as being useful) yields a lower estimate:

$$(I + K_V^{\mu^+})^{-1}s \geq s \exp\left(-\frac{K_V^{\mu^+}s}{s}\right) \tag{4.3}$$

(cf. [FdLP88], [HM90], [Keu90]). In particular,

$$(I + K_V^{\mu^+})^{-1}1 \geq \exp(-\|G_V^{\mu^+}\|_\infty) \tag{4.4}$$

and any inequality $K_V^{\mu^+}s \leq cs$ on some subset A of V will imply that $(I + K_V^{\mu^+})^{-1}s \geq e^{-c}s$ on A.

5. – Classical case: μ^+ potentially bounded, μ^- Kato measure

To illustrate the power of the potential theoretic approach and to introduce the method which will be refined in the subsequent sections let us see how quickly we can get the desired results in the classical case (and for uniformly elliptic operators with Hölder continuous coefficients) if at least μ^- is a Kato measure (or almost a Kato measure).

There exists a constant $c_d > 0$ (depending on the dimension d only) such that, for every ball B in $\mathbb{R}^d$ and for all $x, y, z \in B$,

$$G_B(x,z)G_B(z,y) \leq c_d G_B(x,y)(G_{B'}(x,z) + G_{B'}(z,y)) \tag{5.1}$$

if B' denotes the concentric ball having double radius. This implies that, for every ball B with $\overline{B} \subset X$ and for every $s \in \mathcal{S}^+(B)$,

$$K_B^{\mu^+}s \leq 2c_d\|G_{B'}^{1_B\mu^+}\|_\infty s \tag{5.2}$$

(compare with the proof of Proposition 10.3 or see e.g. [BHH87]) and hence

$$(I + K_B^{\mu^+})^{-1}s \geq \exp(-2c_d\|G_{B'}^{1_B\mu^+}\|_\infty)\, s. \tag{5.3}$$

As we already noted, the following results are more or less known, but their proof is presumably shorter than any other one that can be composed from elements existing in the literature.

THEOREM 5.1. *Let $\mu \in \mathcal{M}_{pb}(X)$ such that μ^- is a Kato measure (or – more generally – that X can be covered by balls B such that $\|G_B^{\mu^-}\|_\infty < 1/(2c_d)$). Then the following holds for every domain U in X:*

1. *For every solution $u \in \mathcal{L}^1_{\text{loc}}(U, \lambda + |\mu|)$ of $\Delta u - u\mu = 0$ (quasi-continuous unless μ is absolutely continuous with respect to λ) there exists a (unique) finely continuous version, i.e., a μ-harmonic function h on U such that $h = u$ $(\lambda + |\mu|)$-a.e.*
2. *Every μ-harmonic function on U is locally bounded.*
3. *Harnack inequalities hold for positive μ-harmonic functions on U.*

PROOF. Fix a ball B such that $\overline{B}$ is contained in the domain U and $\gamma := 2c_d\|G_{B'}^{1_B\mu^-}\|_\infty < 1$ so that by (5.2)

$$K_B^{\mu^-} s \le \gamma s \qquad \text{for every } s \in \mathcal{S}^+(B). \tag{5.4}$$

Let A be a compact subset of B and $c > 0$ such that, for every positive harmonic function $g \ge 0$ on B,

$$\sup g(A) \le c \inf g(A).$$

Now let h be any quasi-μ-harmonic function on U. By Lemma 3.6 and Proposition 4.2, the function $g := H_B|h|$ is harmonic on B, $s := g + K_B^{\mu^-}|h| \in \mathcal{S}^+(B)$ and

$$|h| \le g + K_B^{\mu^-}|h| \quad \text{q.e. on } B$$

(if h is positive, then Lemma 3.2 is sufficient). By induction, for every $n \in \mathbb{N}$,

$$|h| \le \sum_{m=0}^{n-1} (K_B^{\mu^-})^m g + (K_B^{\mu^-})^n |h| \quad \text{q.e. on } B. \tag{5.5}$$

Since $g, s \in \mathcal{S}^+(B)$, we know by (5.4) that, for all $m, n \in \mathbb{N}$,

$$(K_B^{\mu^-})^m g \le \gamma^m g, \qquad (K_B^{\mu^-})^n |h| \le (K_B^{\mu^-})^n s \le \gamma^n s.$$

Therefore (5.5) implies that

$$|h| \le \frac{1}{1-\gamma} g \quad \text{q.e. on } B. \tag{5.6}$$

So (1) and (2) follow from Lemma 3.5 and Lemma 3.2 by a trivial covering argument.

Suppose finally that h is a positive μ-harmonic function on U. By fine continuity, (5.6) implies that

$$h \le \frac{1}{1-\gamma} g \qquad \text{on } B$$

and we obtain that h is locally bounded on U.

Therefore h is in fact bounded on the closure of the ball B we are considering, and the functions s and g are bounded as well. Defining $\beta := 2c_d\|G_{B'}^{1_B\mu^+}\|_\infty$ we get by (3.5) and (5.3) that

$$h = (I + K_B^{\mu^+})^{-1}s \geq e^{-\beta}s \geq e^{-\beta}g.$$

Thus

$$\sup h(A) \leq \frac{1}{1-\gamma}\sup g(A) \leq \frac{c}{1-\gamma}\inf g(A) \leq \frac{c\,e^{\beta}}{1-\gamma}\inf h(A).$$

Again a straightforward covering argument finishes the proof. □

REMARK 5.2. 1) Note that we have an explicit control of the Harnack constants for $\Delta - \mu$ in terms of $\|G_{B'}^{1_B\mu^\pm}\|_\infty$ and the Harnack constants for Δ.

2) Obviously the same proof works for any other Brelot space where we have a Green function G_X and a base of regular sets V such that the magic inequality (5.1) (sometimes called (3G)-inequality, see [CFZ88]) holds for some open neigborhood V' of V. In particular, by results in [HS82], it can be used for uniformly elliptic operators $\mathcal{L} = \sum_{i,j} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i} + c$ with Hölder continuous coefficients.

3) If μ^+ is a Kato measure as well, then we do not need our preceding considerations on the inverse of $I + K_B^{\mu^+}$. Taking B such that in addition $\delta := 2c_d\|G_{B'}^{1_B\mu^+}\|_\infty < 1$, we obtain that $\|K_B^{\mu^+}\| = \|G_B^{\mu^+}\|_\infty < 1$ and $K_B^{\mu^+}s \leq \delta s$, hence

$$h = (I + K_B^{\mu^+})^{-1}s = \sum_{n=0}^{\infty}(-K_B^{\mu^+})^n s = \sum_{n=0}^{\infty}(K_B^{\mu^+})^{2n}(s - K_B^{\mu^+}s) \geq s - K_B^{\mu^+}s \geq (1-\delta)s.$$

4) For Kato measures a similar Neumann series approach is the heart of the analytic proof for Harnack inequalities for locally bounded μ-harmonic functions given in [BHH85] ([BHH87] resp.) (we used $(I + K_B^{\mu})^{-1} = \sum_{n=0}^{\infty}(-K_B^{\mu})^n$ for $2c_d\|G_X^{|\mu|}\|_\infty < 1$). It is also applied in [Han93] to show that μ-harmonic functions are locally bounded (and hence continuous). In both papers it has been noted that the results hold as well for uniformly elliptic operators with Hölder continuous coefficients (p. 134, p. 381 resp.). Harnack inequalities for such operators have recently been studied in [Moh98].

Moreover, knowing that $(X, {}^{\mu}\mathcal{H})$ is a Brelot space it is trivial that, for every compact subset A of U, there exists a continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$|h(x) - h(y)| \leq g(\|x - y\|)h(x)$$

for all positive continuous solutions of $\Delta h - h\mu = 0$ and all $x, y \in A$ (see [BHH87], p. 132). Subsolution estimates are as easily obtained (see [BHH87], p. 133). For a different approach to such estimates see [AS82], [Sim90].

5) The case where (the Kato measure) μ has a density V with respect to Lebesgue measure had been solved before in [AS82] using probabilistic arguments and was treated by PDE methods in [CFG86] and [Sim90] ([CFG86] and [BHH85]/[BHH87] have been written independently in the same year).

6) For comments on [Her87], [dLP90], [Zah96], [Zah] see at the end of the introduction.

6. – μ^+-superharmonic functions

In our general setting we shall have an inequality of type (5.2) at least on compact subsets of our sets $V \in \mathcal{V}$, but only for functions s which are potentials on a larger set V' and with compact superharmonic support. The following considerations will allow us to accomplish the necessary reduction to this case (see the proof of Proposition 10.1).

Let us note first that, for every $V \in \mathcal{V}$, there exists a unique kernel $H_V^{\mu^+}$ such that

$$(I + K_V^{\mu^+})H_V^{\mu^+} = H_V.$$

This follows immediately from the fact that $H_V\varphi \in \mathcal{H}_b^+(V) \subset \mathcal{S}_b^+(V)$ for every $\varphi \in \mathcal{B}_b^+(V)$ and that H_V and $K_V^{\mu^+}$ are kernels. Of course, for every $\varphi \in \mathcal{B}_b(V)$, $H_V^{\mu^+}\varphi$ is μ^+-harmonic on V.

For the remainder of this section let us fix $V \in \mathcal{V}$ and let us assume that $\mu \in \mathcal{M}_{pb}^+(X)$ in order to avoid the continuous appearance of the superscript '+'. A finely continuous numerical function $u \geq 0$ on V will be called *μ-superharmonic* if $u \not\equiv \infty$ and if, for every $W \in \mathcal{V}(V)$,

$$H_W^\mu u \leq u.$$

Of course, the set ${}^\mu\mathcal{S}^+(V)$ of all μ-superharmonic positive functions on V is a min-stable convex cone containing $\mathcal{S}^+(V)$. Our interest in ${}^\mu\mathcal{S}^+(V)$ stems from (3.5) and the following characterization:

LEMMA 6.1. *Let u be a finely continuous positive real function on V. If $u + K_V^\mu u$ is superharmonic, then u is μ-superharmonic. Conversely, if u is bounded and μ-superharmonic, then $u + K_V^\mu u$ is superharmonic.*

PROOF. Define $s := u + K_V^\mu u$ and fix $W \in \mathcal{V}(V)$. If $s \in \mathcal{S}^+(V)$ or u is bounded, then $H_W s$ is a positive harmonic function on W, $0 \leq H_W u \leq H_W s$, and

$$s - H_W s = u + K_V^\mu u - H_W(u + K_V^\mu u) = (I + K_W^\mu)(u - H_W^\mu u).$$

If $u \in {}^{\mu}\mathcal{S}_b^+(V)$, then $u - H_W^{\mu} u \geq 0$, hence $H_W s \leq s$. Since s is finely continuous, this implies that s is superharmonic.

Suppose now that $s \in \mathcal{S}^+(V)$. Then $P := \{s = \infty\}$ is polar and $s - H_W s \in \mathcal{S}^+(W)$. Taking $f := 1_{W \setminus P}(u - H_W^{\mu} u)$ we obtain by Lemma 4.1 that $H_W^{\mu} u \leq u$ on $W \setminus P$. This implies that $H_W^{\mu} u \leq u$ on W, since $H_W^{\mu} u$ is finely l.s.c. □

PROPOSITION 6.2. *Let $\psi \geq 0$ be a finely l.s.c. bounded function on V. Then*

$$^{\mu}R_{\psi} := \inf\{u \in {}^{\mu}\mathcal{S}^+(V) : u \geq \psi\} \in {}^{\mu}\mathcal{S}_b^+(V)$$

and ${}^{\mu}R_{\psi}$ is μ-harmonic on $V \setminus \operatorname{supp}(\psi)$. Moreover, ${}^{\mu_n}R_{\psi} \downarrow {}^{\mu}R_{\psi}$ for every sequence (μ_n) of positive Kato measures increasing to μ. If ψ has compact support in V, then ${}^{\mu}R_{\psi} + K_V^{\mu}({}^{\mu}R_{\psi})$ is a potential on V.

PROOF. Fix positive Kato measures μ_n on X with $\mu_n \uparrow \mu$. Each $(X, {}^{\mu_n}\mathcal{H})$ is a harmonic space with $1 \in {}^{\mu_n}\mathcal{S}_b^+(X)$. Hence we know that the functions

$$u_n := {}^{\mu_n}R_{\psi} \qquad (n \in \mathbb{N})$$

are μ_n-superharmonic on V, μ_n-harmonic on $V \setminus \operatorname{supp}(\psi)$, and $u_n \leq \|\psi\|_{\infty}$.

Since ${}^{\mu_n}\mathcal{S}^+(V) \subset {}^{\mu_{n+1}}\mathcal{S}^+(V) \subset {}^{\mu}\mathcal{S}^+(V)$ for every $n \in \mathbb{N}$, we obtain that the sequence (u_n) is decreasing and

$$\inf u_n \geq {}^{\mu}R_{\psi}.$$

Clearly, ${}^{\mu}R_{\psi} := \inf\{u \in {}^{\mu}\mathcal{S}_b^+(V) : u \geq \psi\}$, since $\min(v, \|\psi\|_{\infty}) \in {}^{\mu}\mathcal{S}_b^+(V)$ for every $v \in {}^{\mu}\mathcal{S}^+(V)$. And if $v \in {}^{\mu}\mathcal{S}_b^+(V)$ such that $v \geq \psi$, then

$$v_n := v + (I + K_V^{\mu_n})^{-1} K_V^{\mu - \mu_n} v$$

satisfies $v_n + K_V^{\mu_n} v_n = v + K_V^{\mu} v \in \mathcal{S}^+(V)$, hence $v_n \in {}^{\mu_n}\mathcal{S}_b^+(V)$, $v_n \geq u_n$ for every $n \in \mathbb{N}$, and therefore

$$v = \lim_{n \to \infty} v_n \geq \inf u_n$$

(note that $(I + K_V^{\mu_n})^{-1} K_V^{\mu - \mu_n} v \leq K_V^{\mu - \mu_n} v$ by (4.1)). Thus

$$u := \inf u_n = {}^{\mu}R_{\psi}. \tag{6.1}$$

Obviously,

$$s_n := u_n + K_V^{\mu} u_n = (u_n + K_V^{\mu_n} u_n) + K_V^{\mu - \mu_n} u_n \tag{6.2}$$

is superharmonic for every $n \in \mathbb{N}$ and

$$s := u + K_V^{\mu} u = \inf s_n. \tag{6.3}$$

Therefore $\widehat{s}^f$ is superharmonic (where $\widehat{\varphi}^f(x) := \text{f-}\liminf_{y\to x}\varphi(y)$) and the set $\{\widehat{s}^f < s\}$ is semi-polar. Since $K_V^\mu u$ is finely continuous, (6.3) implies that

$$\widehat{s}^f = \widehat{u}^f + K_V^\mu u.$$

So the set $\{\widehat{u}^f < u\} = \{\widehat{s}^f < s\}$ is semi-polar, hence $K_V^\mu \widehat{u}^f = K_V^\mu u$,

$$\widehat{u}^f + K_V^\mu \widehat{u}^f = \widehat{s}^f \in \mathcal{S}^+(V).$$

Since $K_V^\mu \widehat{u}^f$ and $\widehat{s}^f$ are finely continuous bounded functions, the function $\widehat{u}^f$ is finely continuous as well, hence $\widehat{u}^f \in {}^\mu\mathcal{S}_b^+(V)$ by Lemma 6.1. But of course $\widehat{u}^f \geq \psi$, since $u \geq \psi$ and ψ is finely l.s.c. Thus $\widehat{u}^f \geq {}^\mu R_\psi = u \geq \widehat{u}^f$, and hence

$${}^\mu R_\psi = \widehat{u}^f \in {}^\mu\mathcal{S}_b^+(V).$$

Let us now convince ourselves that u is μ-harmonic on $V \setminus \operatorname{supp}(\psi)$. So fix a set $W \in \mathcal{V}(V \setminus \operatorname{supp}(\psi))$. Since the functions u_n are μ_n-harmonic on $V \setminus \operatorname{supp}(\psi)$, the functions $u_n + K_W^{\mu_n} u_n$ are harmonic on W. Of course, $0 \leq K_V^{\mu-\mu_n} u_n \leq K_V^{\mu-\mu_n} u_1$ and $\lim_{n\to\infty} K_V^{\mu-\mu_n} u_1 = 0$. Using (6.2) and (6.3) we thus conclude that

$$u + K_V^\mu u = \lim_{n\to\infty}(u_n + K_V^{\mu_n} u_n) = \lim_{n\to\infty}(u_n + K_W^{\mu_n} u_n + H_W K_V^{\mu_n})$$

is harmonic on W. Suppose finally that $\operatorname{supp}(\psi)$ is compact in V. Then there exists a bounded potential q on V such that $q \geq \psi$, hence $u \leq q$ and s being majorized by the potential $q + K_V^\mu q$ on V is a potential itself. □

7. – Uniform Harnack inequalities for positive perturbations

To illustrate the use of Proposition 6.2 let us show how it can be applied to obtain Harnack inequalities if the measure $\mu \in \mathcal{M}_{pb}(X)$ is positive. The control of Harnack constants which our approach yields and which is important in some applications seems to be new. We recall, however, that for measures $\mu \in \mathcal{M}_{pb}^+(X)$ the mere validity of Harnack inequalities for positive μ-harmonic functions is already known (see [FdLP88] for the classical situation, [dLP90] for our general case). The idea of using balayage with respect to μ^+-superharmonic functions to avoid an assumption of type (5.1) is borrowed from [dLP90].

LEMMA 7.1. *Let $U, V \in \mathcal{V}$ such that $\overline{V} \subset U$, let A be a compact subset of V and let A' be a compact neighborhood of $\overline{V}$ in U. Then there exists $c_{A,V,A',U} > 0$ such that, for every $y \in A' \setminus V$ and for every $\mu \in \mathcal{M}_{pb}^+(X)$,*

$$K_U^\mu G_U(\cdot, y) \leq c_{A,V,A',U} \|G_U^\mu\|_\infty G_U(\cdot, y) \quad \textit{on } A.$$

PROOF. We choose an open set W such that $A \subset W$ and $\overline{W} \subset V$. Then

$$c_0 := \inf\{G_U(x,y) : x \in A, y \in A' \setminus V\} > 0,$$
$$c_1 := \sup\{G_U(x,y) : x \in A, y \in U \setminus W\} < \infty,$$
$$c_2 := \sup\{G_U(x,y) : x \in \overline{W}, y \in U \setminus V\} < \infty.$$

Hence

$$c_{A,V,A',U} := \frac{c_1 + c_2}{c_0}$$

is a strictly positive real number. Now fix $x \in A$, $y \in A' \setminus V$, and $\mu \in \mathcal{M}^+_{pb}(X)$. Then, using the symmetry of G_U,

$$\begin{aligned} K^\mu_U(G_U(\cdot,y))(x) &= \int_{U\setminus W} G_U(x,z)G_U(z,y)\,\mu(dz) + \int_W G_U(x,z)G_U(z,y)\,\mu(dz) \\ &\le c_1 \int_{U\setminus W} G_U(z,y)\,\mu(dz) + c_2 \int_W G_U(x,z)\,\mu(dz) \\ &\le (c_1+c_2)\|G^\mu_U\|_\infty \le c_{A,V,A',U}\|G^\mu_U\|_\infty G_U(x,y). \end{aligned}$$

□

Combining Proposition 6.2 with Lemma 7.1 we obtain the following:

PROPOSITION 7.2. *Let $U, V \in \mathcal{V}$ such that $\overline{V} \subset U$, let A be a compact subset of V and let A' be a compact neighborhood of $\overline{V}$ in U. Then, for every $\mu \in \mathcal{M}^+_{pb}(X)$ and for every locally bounded μ-harmonic function $h \ge 0$ on a neighborhood of $\overline{U}$, there exists a potential p on U such that p is harmonic outside $A' \setminus V$ and*

$$p \ge h \ge \exp(-c_{A,V,A',U}\|G^\mu_U\|_\infty)\, p \quad \text{on } A.$$

PROOF. Let W be an open neighborhood of $\overline{V}$ in A', fix $\mu \in \mathcal{M}^+_{pb}(X)$ and a locally bounded μ-harmonic function $h \ge 0$ on a neighborhood of $\overline{U}$, and define a finely l.s.c. function ψ on U by

$$\psi := 1_W h.$$

From Proposition 6.2 we know that

$$p := {}^\mu R_\psi + K^\mu_U({}^\mu R_\psi)$$

is a potential on U which is harmonic on $U \setminus \overline{W}$. Moreover, ${}^\mu R_\psi \le h$ on U, ${}^\mu R_\psi = h$ on W, hence $p \ge h$ on A and on the set W the potential p differs from the function $h + K^\mu_U h \in \mathcal{H}(U)$ by the potential $K^\mu_U(h - {}^\mu R_\psi)$ which is harmonic on W. So p itself is harmonic on W.

Of course, there are continuous real potentials p_n on U, $n \in \mathbb{N}$, such that $p_n \uparrow p$ and each p_n is harmonic outside $A' \setminus V$, i.e., $p_n = G^{\rho_n}_U$, $\operatorname{supp}(\rho_n) \subset$

$A' \setminus V$. We conclude from Lemma 7.1 that $K_U^\mu p_n \le c_{A,V,A',U} \|G_U^\mu\|_\infty p_n$ on A for every $n \in \mathbb{N}$ and hence

$$K_U^\mu p \le c_{A,V,A',U} \|G_U^\mu\|_\infty p \quad \text{on } A.$$

By (4.3), we finally obtain that, for every $x \in A$,

$$h(x) = {}^\mu R_\psi(x) \ge p(x) \exp\left(-\frac{K_U^\mu p(x)}{p(x)}\right) \ge \exp(-c_{A,V,A',U} \|G_U^\mu\|_\infty)\ p(x). \ \square$$

COROLLARY 7.3. *Let A, V, A', U be as in Proposition 7.2 and let $c > 0$ such that* $\sup g(A) \le c \inf g(A)$ *for every harmonic function $g \ge 0$ on V. Then*

$$\sup h(A) \le c\ \exp(c_{A,V,A',U} \|G_U^\mu\|_\infty)\ \inf h(A)$$

for every $\mu \in \mathcal{M}_{pb}^+(X)$ and every μ-harmonic function $h \ge 0$ on a neighborhood of $\overline{U}$.

PROOF. Choosing p as in Proposition 7.2 we have

$$\sup h(A) \le \sup p(A) \le c \inf p(A) \le c\ \exp(c_{A,V,A',U} \|G_U^\mu\|_\infty)\ \inf h(A). \quad \square$$

As usual a covering argument now yields Harnack inequalities for positive μ-harmonic functions on every domain U in X provided $\mu \in \mathcal{M}_{pb}(X)$ is positive.

8. – μ-bounded sets and existence of positive μ-harmonic functions

Given $\mu \in \mathcal{M}_{pb}(X)$ and $U \in \mathcal{V}$, let

$$\boxed{L_U^\mu := (I + K_U^{\mu^+})^{-1} K_U^{\mu^-}.}$$

Then L_U^μ is a positive bounded operator on $\mathcal{B}_b(U)$. It is easily verified that L_U^μ defines a kernel on U and that, denoting this kernel by L_U^μ as well, we have $(I + K_U^{\mu^+}) L_U^\mu = K_U^{\mu^-}$. Finally, we introduce a kernel S_U^μ on U by

$$\boxed{S_U^\mu := \sum_{n=0}^{\infty} (L_U^\mu)^n.} \tag{8.1}$$

Let us note that the functions $K_U^\mu f$ and $L_U^\mu f$ are finely continuous for every $f \in \mathcal{B}_b(U)$. Hence $S_U^\mu f$ is finely l.s.c. for every Borel function $f \ge 0$ on U.

We say that U is *μ-bounded* if the function $S_U^\mu 1$ is bounded. Obviously, U is μ-bounded if $\mu \geq 0$ or if $\|G_U^{\mu^-}\|_\infty < 1$ or, even more generally, if $\|(L_U^\mu)^n 1\|_\infty < 1$ for some $n \in \mathbb{N}$. If U is μ-bounded, then it is easily verified that $I + K_U^\mu$ is invertible and

$$(8.2) \qquad (I + K_U^\mu)^{-1} = \sum_{n=0}^{\infty} (L_U^\mu)^n (I + K_U^{\mu^+})^{-1} = S_U^\mu (I + K_U^{\mu^+})^{-1}.$$

Equality (8.2) reflects the fact that perturbation by μ can be achieved perturbing by μ^+ first and then by μ^-. For details and various characterizations of μ-bounded sets the reader might look at [BHH87] and [HM90]. For us it is important that there is a close connection between μ-boundedness of U and the existence of positive μ-harmonic functions on U (see also Corollary 10.5, Proposition 13.5 and Proposition 13.6). Let us note first that, for every $U \in \mathcal{V}$, there exists a bounded harmonic function $g_0 \geq 1$ on U (it suffices to choose $V \in \mathcal{V}$ containing $\overline{U}$, to define $g = H_V 1$, and to take $g_0 = (\inf g(\overline{U}))^{-1} g|_U$).

PROPOSITION 8.1. *A set $U \in \mathcal{V}$ is μ-bounded if and only if there exists a bounded μ-harmonic function $h \geq 1$ on U, and then*

$$(8.3) \qquad S_U^\mu 1 \leq \|g_0\|_\infty \exp(\|G_U^{\mu^+}\|_\infty \|g_0\|_\infty) \frac{h}{\inf h(U)}$$

for any bounded harmonic function $g_0 \geq 1$ on U.

PROOF. Fix a bounded harmonic function $g_0 \geq 1$ on U and let $f := (I + K_U^{\mu^+})^{-1} g_0$. Then $f \geq \exp(-\|G_U^{\mu^+}\|_\infty \|g_0\|_\infty)$ by (4.4).

Suppose first that U is μ-bounded. Then $h_0 := (I + K_U^\mu)^{-1} g_0$ is a bounded μ-harmonic function on U, $h_0 \geq f$ by (8.2), and we may take $h = h_0 / \inf h_0(U)$.

Now suppose conversely that we have a bounded μ-harmonic function $h \geq 1$ on U and define $h_0 := (\|g_0\|_\infty / \inf h(U))\, h$. Then the function

$$g := h_0 + K_U^\mu h_0 - g_0 = h_0 + K_U^{\mu^+} h_0 - K_U^{\mu^-} h_0 - g_0 = (I + K_U^{\mu^+})(h_0 - L_U^\mu h_0 - f)$$

is harmonic on U and $g \geq -K_U^{\mu^-} h_0$, hence $g \geq 0$. Therefore $h_0 - L_U^\mu h_0 - f \geq 0$ by Lemma 4.1, and we obtain by induction that, for every $m \in \mathbb{N}$,

$$\sum_{n=0}^{m-1} (L_U^\mu)^n f + (L_U^\mu)^m h_0 \leq h_0,$$

hence $S_U^\mu f = \sum_{n=0}^\infty (L_U^\mu)^n f \leq h_0$. This implies (8.3). In particular, U is μ-bounded. □

PROPOSITION 8.2. *Let U be a domain in X and let $h \geq 0$ be a locally bounded μ-harmonic function on U which is not identically zero. Then* $\inf h(A) > 0$ *for every compact subset A of U. In particular, every $V \in \mathcal{V}(U)$ is μ-bounded.*

PROOF. Let $x \in \{h > 0\}$ and take $V \in \mathcal{V}(U)$ containing x. By Lemma 3.6 and (4.3), $s := h + K_V^{\mu^+} h \in \mathcal{S}_b^+(V)$ and

$$h = (I + K_V^{\mu^+})^{-1} s \geq s \exp\left(-\frac{K_V^{\mu^+} s}{s}\right).$$

Since $s(x) \geq h(x) > 0$, we know that $s > 0$ on V. Moreover, $K_V^{\mu^+} s$ is bounded. Since s is l.s.c., we therefore obtain that $\inf h(A) > 0$ for every compact subset A of V. So the set $\{h > 0\}$ is open and has no boundary point in U. Thus $h > 0$ on U, and it follows now immediately that $\inf h(A) > 0$ for every compact subset A of U (cover A by finitely many sets in $\mathcal{V}(U)$). The proof is finished by an application of Proposition 8.1. □

The second statement is also a consequence of the following inequalities needed in Section 11.

PROPOSITION 8.3. *Let $U, V \in \mathcal{V}$, $V \subset U$. Then $L_V^\mu \leq L_U^\mu$ and $S_V^\mu \leq S_U^\mu$. In particular, V is μ-bounded if U is μ-bounded.*

PROOF. This follows easily from general properties of iterated perturbations (see [BHH87]), but for the convenience of the reader we give a direct proof: Using (3.2) we have

$$\begin{aligned} K_V^{\mu^-} = K_U^{\mu^-} - H_V K_U^{\mu^-} &= L_U^\mu + K_U^{\mu^+} L_U^\mu - H_V L_U^\mu - H_V K_U^{\mu^+} L_U^\mu \\ &= L_U^\mu + K_V^{\mu^+} L_U^\mu - H_V L_U^\mu, \end{aligned}$$

hence

$$L_V^\mu = (I + K_V^{\mu^+})^{-1} K_V^{\mu^-} = L_U^\mu - (I + K_V^{\mu^+})^{-1} H_V L_U^\mu$$

where $(I + K_V^{\mu^+})^{-1} H_V L_U^\mu$ is a kernel. □

For our proof of Harnack inequalities on μ-bounded sets the following observation is essential (see Proposition 11.2).

PROPOSITION 8.4. *If $V \in \mathcal{V}$ is μ-bounded, then V is $(\mu - \varepsilon\mu^-)$-bounded for some $\varepsilon > 0$.*

Since

$$L_V^{\mu - \varepsilon\mu^-} = (I + K_V^{\mu^+})^{-1} K_V^{(1+\varepsilon)\mu^-} = (1+\varepsilon) L_V^\mu,$$

Proposition 8.4 is an immediate consequence of the next lemma which will be needed in a discrete situation in Section 14 (see Proposition 14.5).

LEMMA 8.5. *Let K, L be bounded kernels on a measurable space $(E, \mathcal{E})$. If $\sum_{n=0}^{\infty} L^n 1$ is bounded, then $\sum_{n=0}^{\infty}(L+\varepsilon K)^n 1$ is bounded for some $\varepsilon > 0$.*

PROOF. Let M be any bounded kernel on $(E, \mathcal{E})$. Then M acts as positive operator on the Banach space $\mathcal{E}_b$ of all bounded $\mathcal{E}$-measurable functions on E equipped with the norm $\|\cdot\|_\infty$ of uniform convergence. If $\sum_{n=0}^{\infty} M^n 1$ is bounded, then obviously $\sum_{n=0}^{\infty} M^n$ is the inverse of $I-M$ on $\mathcal{E}_b$ and this inverse $(I-M)^{-1}$ is a positive operator. Conversely, if $(I-M)^{-1}$ exists and is positive, then $f := (I-M)^{-1}1 \in \mathcal{E}_b^+$, $f = 1 + Mf$, hence $\sum_{n=0}^{\infty} M^n 1 \le f$, and $\sum_{n=0}^{\infty} M^n 1$ is bounded.

Assume now that $\sum_{n=0}^{\infty} L^n 1$ is bounded, define

$$\alpha = \|K1\|_\infty, \qquad \beta = \|\sum_{n=0}^{\infty} L^n 1\|_\infty,$$

and take $\varepsilon > 0$ such that

$$\varepsilon\,\alpha\,\beta < 1.$$

Then $\|K\| = \alpha$, $\|(I-L)^{-1}\| = \beta$, and $\sum_{n=0}^{\infty}(\varepsilon(I-L)^{-1}K)^n 1 \le \sum_{n=0}^{\infty}(\varepsilon\alpha\beta)^n = (1-\varepsilon\alpha\beta)^{-1}$. Of course, $I-(L+\varepsilon K)$ is invertible and

$$(I-(L+\varepsilon K))^{-1} = (I-\varepsilon(I-L)^{-1}K)^{-1}(I-L)^{-1}.$$

By our preceding considerations $(I-L)^{-1}$ and $(I-\varepsilon(I-L)^{-1}K)^{-1}$ are positive operators, hence $(I-(L+\varepsilon K))^{-1}$ is a positive operator,

$$(I-(L+\varepsilon K))^{-1} = \sum_{n=0}^{\infty}(L+\varepsilon K)^n.$$

In particular, $\sum_{n=0}^{\infty}(L+\varepsilon K)^n 1$ is bounded. We even get more precisely that

$$\begin{aligned}\sum_{n=0}^{\infty}(L+\varepsilon K)^n 1 &= \|(I-(L+\varepsilon K))^{-1}\| \\ &\le \|(I-\varepsilon(I-L)^{-1}K)^{-1}\|\ \|(I-L)^{-1}\| \le \frac{\beta}{1-\varepsilon\alpha\beta}.\end{aligned}$$ □

REMARK 8.6. Assume that $V \in \mathcal{V}$ is μ-bounded. Then Lemma 8.5 implies that, given any $\nu \in \mathcal{M}_{pb}^+(X)$, there always exists $\varepsilon > 0$ such that V is $(\mu-\varepsilon\nu)$-bounded, and it is easily seen that V is $(\mu+\nu)$-bounded.

9. – Local triangle property of the Green function

In the following we shall need the local triangle property we already mentioned in the introduction:

(LT) *There exists a covering of X by open sets U such that for some constant $C > 0$ (which may depend on U) and all $x, y, z \in U$*

$$\boxed{\min(G_X(x,z), G_X(z,y)) \le C\, G_X(x,y)\ .} \tag{9.1}$$

REMARK 9.1. If $W \in \mathcal{V}$ is a neighborhood of $\overline{U}$, then (9.1) holds (with some constant $C > 0$) if and only if there exists a constant $C' > 0$ such that, for all $x, y, z \in U$,

$$\min(G_W(x,z), G_W(z,y)) \le C' G_W(x,y).$$

It suffices to note that for some $c > 0$

$$G_W \le G_X \le c\, G_W \qquad \text{on } U \times U.$$

(The first inequality is trivial. To prove the second inequality let $b_0 := \inf\{G_W(x,y) : x, y \in U\}$ and $b_1 := \sup\{G_X(z,y) : y \in U, z \in \partial W\}$. Fix $x, y \in U$. Then $H_W G_X(\cdot, y) \le b_1$ and hence $G_W(x,y) = G_X(x,y) - (H_W G_X(\cdot,y))(x) \ge G_X(x,y)/2$ if $G_X(x,y) \ge 2b_1$. If, however, $G_X(x,y) < 2b_1$, then $G_W(x,y) \ge b_0 > b_0 G_X(x,y)/(2b_1)$.)

CLASSICAL CASE: If $X = \mathbb{R}^d$, $d \ge 3$, then $G_{\mathbb{R}^d}(x,y) = \kappa_d \|x-y\|^{2-d}$ and (9.1) holds for $U = X$, $C = 2^{d-2}$. Indeed, given $x, y, z \in \mathbb{R}^d$, we have $\|x - z\| \ge \|x-y\|/2$ or $\|z-y\| \ge \|x-y\|/2$. Using Remark 9.1 we get (LT) for any other domain X in $\mathbb{R}^d$, $d \ge 3$.

If $D := \{x \in \mathbb{R}^2 : \|x - x_0\| < r\}$, $x_0 \in \mathbb{R}^2, r > 0$, then

$$G_D(x,y) = \frac{1}{2\pi}\ln(1+\varphi(x,y)) \quad \text{with} \quad \varphi(x,y) = \frac{(r^2 - \|x-x_0\|^2)(r^2-\|y-x_0\|^2)}{r^2\|x-y\|^2}.$$

Let $U := \{x \in \mathbb{R}^2 : \|x - x_0\| < r/2\}$. Then, for all $x, y \in U$,

$$\frac{r^2}{\|x-y\|^2} \ge \varphi(x,y) \ge \frac{1}{2}\frac{r^2}{\|x-y\|^2} \ge \frac{1}{2}.$$

If $x, y, z \in U, x \ne y$ and, say, $\|x - z\| \ge \|x-y\|/2$, then $\varphi(x,z) \le 8\varphi(x,y)$, hence

$$\frac{G_D(x,z)}{G_D(x,y)} \le \frac{\ln(1+8\varphi(x,y))}{\ln(1+\varphi(x,y))} \le 1 + \frac{\ln 8}{\ln(1+\varphi(x,y))} \le 1 + \frac{\ln 8}{\ln\frac{3}{2}}$$

(if $x = y$, then $G_D(x,y) = \infty$ and the desired inequality holds trivially). Using Remark 9.1 we get (LT) for any domain X in $\mathbb{R}^2$ such that X^c is non-polar.

Finally, if X is a an open interval or an open half-line, then G_X is continuous and real on $X \times X$ and (LT) holds trivially.

More generally, we have the following result:

PROPOSITION 9.2. *The local triangle property holds if X is covered by open sets U such that there exists a quasi-metric $\rho < a \le \infty$ on U, a decreasing function $\varphi : [0, a[\to [0, \infty]$, and constants $C_1, C_2 \ge 1$ such that*

$$C_1^{-1} \varphi \circ \rho \le G_X \le C_1 \varphi \circ \rho \quad \text{on } U \times U \tag{9.2}$$

and

$$\varphi(t/2) \le C_2 \varphi(t) \quad \text{for all } 0 < t < a. \tag{9.3}$$

In particular, the local triangle property holds for Green functions associated with sub-Laplacians on stratified Lie algebras with homogeneous dimension $Q \ge 3$ (with $\varphi(t) = t^{2-Q}$).

PROOF. By definition of a quasi-metric, there exists $C_3 > 0$ such that, for all $x, y, z \in U$,

$$\rho(x, y) \le C_3(\rho(x, z) + \rho(z, y)).$$

Of course, we may assume that $C_3 = 2^{k-1}$ for some $k \in \mathbb{N}$ ($k = 1$ if ρ is a metric). Fix $x, y, z \in U$. Then

$$\max(\rho(x, z), \rho(z, y)) \ge \frac{1}{2C_3} \rho(x, y) = 2^{-k} \rho(x, y),$$

without loss of generality $\rho(x, z) \ge 2^{-k} \rho(x, y)$. Then

$$G_X(x, z) \le C_1 \varphi(\rho(x, z)) \le C_1 C_2^k \varphi(\rho(x, y)) \le C_1^2 C_2^k G_X(x, y).$$

For stratified Lie algebras the reader might consult [FS82] and [HH87]). □

PROPOSITION 9.3. *The local triangle property holds if $G_X = \infty$ on the diagonal and if X can be covered by open sets U such that for all $x, y \in U$, $x \ne y$,*

$$C^{-1} \frac{\rho(x, y)^2}{|B(x, \rho(x, y))|} \le G_X(x, y) \le C \frac{\rho(x, y)^2}{|B(x, \rho(x, y))|}$$

where ρ is any metric on X, $B(x, r) = \{y \in X : \rho(x, y) < r\}$ and $B \to |B|$ is any Borel measure on X having the doubling property and weak quadratic increase, i.e., for all $x \in U$ and $0 < r < s \le \sup\{\rho(y, z) : y, z \in U\}$,

$$0 < |B(x, 2r)| \le C_1 |B(x, r)| < \infty, \tag{D}$$

$$r^{-2} |B(x, r)| \le C_2 s^{-2} |B(x, s)|. \tag{WQI}$$

In particular, every Green function G_X associated with an operator $\mathcal{L} = \sum_{j=1}^r X_j^2$ on $X \subset \mathbb{R}^d$, $d \ge 3$, where $X_1, \ldots, X_r$ are smooth vector fields satisfying Hörmander's condition for hypoellipticity has the local triangle property.

PROOF. Fix x, y, z in a set U where (D) and (WQI) hold. It suffices to consider the case $x \neq y$. Then

$$0 < \alpha := \rho(x, y) \le 2\max(\rho(x, z), \rho(z, y)).$$

If $\alpha \le 2\rho(x, z)$, we take $(\tilde{x}, \tilde{y}) = (x, z)$. If $\alpha > 2\rho(x, z)$, then $\alpha \le 2\rho(z, y)$ and taking $(\tilde{x}, \tilde{y}) = (z, y)$ we obtain that $B(x, \alpha) \subset B(\tilde{x}, 2\alpha)$. Defining $\tilde{\alpha} = \rho(\tilde{x}, \tilde{y})$ we have in both cases

$$\alpha \le 2\tilde{\alpha} \quad \text{and} \quad B(x, \alpha) \subset B(\tilde{x}, 2\alpha),$$

hence

$$|B(x, \alpha)| \le |B(\tilde{x}, 2\alpha)| \le C_1|B(\tilde{x}, \alpha)| \le C_1|B(\tilde{x}, 2\tilde{\alpha})| \le C_1^2|B(\tilde{x}, \tilde{\alpha})|.$$

If $\alpha \le \tilde{\alpha}$, then

$$\frac{1}{G_X(x, y)} \le C\frac{|B(x, \alpha)|}{\alpha^2} \le CC_1\frac{|B(\tilde{x}, \alpha)|}{\alpha^2} \le CC_1C_2\frac{|B(\tilde{x}, \tilde{\alpha})|}{\tilde{\alpha}^2} \le C^2C_1C_2\frac{1}{G_X(\tilde{x}, \tilde{y})}.$$

If $\alpha \ge \tilde{\alpha}$, then

$$\frac{1}{G_X(x, y)} \le C\frac{|B(x, \alpha)|}{\alpha^2} \le C\frac{|B(x, \alpha)|}{\tilde{\alpha}^2} \le CC_1^2\frac{|B(\tilde{x}, \tilde{\alpha})|}{\tilde{\alpha}^2} \le C^2C_1^2\frac{1}{G_X(\tilde{x}, \tilde{y})}.$$

The situation where $(X, \mathcal{H})$ is given by $\mathcal{L} = \sum_{j=1}^r X_j^2$ and G_X is the fundamental solution of $-\mathcal{L}$, i.e., $\mathcal{L}G_X(\cdot, y) = -\delta_y$, is a special case (see [SC84], [NSW85], and [CGL93], p. 702): Locally

$$G_X(x, y) \approx \frac{\rho(x, y)^2}{\lambda(B(x, \rho(x, y)))}$$

where ρ is the $(X_1, \dots, X_r)$-control distance, i.e., $\rho(x, y)$ is the infimum of all $T > 0$ such that x and y can be joined by a sub-unitary curve $\gamma : [0, T] \to \mathbb{R}^d$ (i.e., such that $(\dot{\gamma}(t) \cdot \xi)^2 \le \sum_{j=1}^r (X_j(\gamma(t)) \cdot \xi)^2$). Not only the doubling property (see [CGL93]), but also the weak quadratic increase of $\lambda(B(x, r))$ is an immediate consequence of

$$\lambda(B(x, r)) \approx \sum_I |\lambda_I(x)| r^{d(I)}$$

where I varies in a finite set, λ_I is continuous, and $d(I) \ge d \ge 3$ for every I. □

REMARK 9.4. 1) It is easily seen that – as in Proposition 9.2 – it suffices to know that ρ is a quasi-metric.

2) Let us note that the Green function for stratified Lie algebras is symmetric and that for the sum of squares of vector fields we know at least that, given the set U, there exists a constant $C > 0$ such that, for all $x, y \in U$,

$$G_X(y, x) \leq C G_X(x, y)$$

and hence every μ in $\mathcal{M}_{pb}(X)$ ($\mathcal{M}_{\text{Kato}}(X)$ resp.) is also potentially bounded (a Kato measure resp.) with respect to the adjoint Green function ${}^*G_X : (x, y) \mapsto G_X(y, x)$. Moreover, in both cases the axiom of domination holds (see [HH87] and [Hue88]).

It will be crucial for our method to control $\min_{1\leq j\leq n} G_X(z_{j-1}, z_j)$ for any $n \in \mathbb{N}$ and points $z_0, z_1, \dots, z_n$ in a small set U by the value of $G_X(z_0, z_n)$. Having (LT) this is easily achieved:

LEMMA 9.5. *Let U be an open subset of X and $C \geq 1$ such that, for all $x, y, z \in U$,*

$$\min(G_X(x, z), G_X(z, y)) \leq C\, G_X(x, y).$$

Then, for every $n \in \mathbb{N}$ and for all choices of points $z_0, z_1, \dots, z_n \in U$,

$$\min_{1\leq j\leq n} G_X(z_{j-1}, z_j) \leq C n^{\log_2 C} G_X(z_0, z_n). \tag{9.4}$$

PROOF. We first claim that for $k = 0, 1, 2, \dots$

$$\min_{1\leq j\leq n} G_X(z_{j-1}, z_j) \leq C^k G_X(z_0, z_n) \quad \text{if} \quad z_0, \dots, z_n \in U,\ 1 \leq n \leq 2^k. \tag{9.5}$$

This is trivial if $k = 0$. Suppose now that $k \in \mathbb{N}$ is such that (9.5) holds for $k - 1$. Fix $n \in \mathbb{N}$ such that $1 \leq n \leq 2^k$ and take $z_0, \dots, z_n \in U$. To prove (9.5) it suffices to consider the case $n > 2^{k-1} =: m$. Then by assumption

$$\min(G_X(z_0, z_m), G_X(z_m, z_n)) \leq C G_X(z_0, z_n)$$

and by induction hypothesis

$$\min_{1\leq j\leq m} G_X(z_{j-1}, z_j) \leq C^{k-1} G_X(z_0, z_m), \quad \min_{m+1\leq j\leq n} G_X(z_{j-1}, z_j) \leq C^{k-1} G_X(z_m, z_n).$$

Therefore (9.5) holds. To finish the proof it suffices to note that $2^{k-1} \leq n \leq 2^k$ implies that $C^k = C C^{k-1} \leq C C^{\log_2 n} = C n^{\log_2 C}$. □

Recall that in the classical situation on $X = \mathbb{R}^d$, $d \geq 3$, the constant C is equal to 2^{d-2} and hence

$$C n^{\log_2 C} = (2n)^{d-2}.$$

The following is what we really need to deal with negative perturbations (it is very important to admit that z_n is an arbitrary point in V'):

LEMMA 9.6. *Suppose that G_X has the local triangle property. Then for every $V' \in \mathcal{V}$ and $x \in V'$ there exists a neighborhood V of x in V' having the following property:*

(LT′) *There exist $a \geq 1$ and $Q \in \mathbb{N}$ such that*

$$\min_{1\le j\le n} G_{V'}(z_{j-1}, z_j) \le a\, n^Q\, G_{V'}(z_0, z_n) \tag{9.5}$$

for every $n \in \mathbb{N}$ and all choices of points $z_0, z_1, \dots, z_{n-1} \in V$, $z_n \in V'$.

PROOF. Given $x \in V' \in \mathcal{V}$ choose a neighborhood U of x in V' such that $\overline{U} \subset V'$ and (9.4) holds for every $n \in \mathbb{N}$ and all points $z_0, z_1, \dots, z_n \in U$. Moreover, fix a neighborhood V of x with $\overline{V} \subset U$ and a constant $c > 0$ such that $G_X \le c G_{V'}$ on $U \times U$ (see Remark 9.1). There exists $c' \ge 1$ such that

$$\sup g(\overline{V}) \le c' \inf g(\overline{V})$$

for every harmonic function $g \ge 0$ on U. Now fix $z_0, z_1, \dots, z_{n-1} \in V$ and $z_n \in V'$. If $z_n \in V' \setminus U$, then $G_{V'}(\cdot, z_n)$ is harmonic on U and hence

$$G_{V'}(z_{n-1}, z_n) \le c' G_{V'}(z_0, z_n).$$

If $z_n \in U$, then $z_0, z_1, \dots, z_n \in U$ and

$$\begin{aligned}\min_{1\le j\le n} G_{V'}(z_{j-1}, z_j) &\le \min_{1\le j\le n} G_X(z_{j-1}, z_j) \le C\, n^{\log_2 C} G_X(z_0, z_n)\\ &\le cC\, n^{\log_2 C} G_{V'}(z_0, z_n).\end{aligned}$$ □

From now we shall always assume that the Green function G_X has the local triangle property.

DEFINITION 9.7. We shall say that a pair $(V, V') \in \mathcal{V} \times \mathcal{V}$ is *admissible* (with constants a and Q) if $\overline{V} \subset V'$ and (9.5) holds.

Then we know by Lemma 9.6 that for every $x \in X$ and for every neighborhood W of x there exists an admissible pair (V, V') such that $x \in V$ and $V' \subset W$.

10. – First consequences

In the classical case we know the following: If $\mu \in \mathcal{M}_{pb}(X)$, V' is a ball, and $h \ge 0$ is a locally bounded μ-harmonic function on a neighborhood of $\overline{V'}$, then the positive superharmonic function $s = h + K_{V'}^{\mu^+} h$ on V' satisfies

$$s \ge h \ge \exp(-2c_d \| G_{V''}^{1_{V'}\mu^+} \|_\infty)\, s, \tag{10.1}$$

V'' being the concentric ball having twice the radius of V' (see (5.3)).

In our general setting the following result will serve as a substitute for (10.1).

PROPOSITION 10.1. *Let (V, V') be admissible and let A be a compact subset of V. Then there exists $c > 0$ such that, for every $\mu \in \mathcal{M}^+_{pb}(X)$ and for every bounded μ-superharmonic function $t \geq 0$ on V', there exists $u \in \mathcal{B}^+_b(V')$ such that $u \leq t$ on V', $u = t$ in a neighborhood of $\overline{V}$, $s := u + K^{\mu}_{V'}u \in \mathcal{S}^+_b(V')$, and*

$$s \geq t \geq \exp(-c\|G^{\mu}_{V'}\|_\infty)\, s \quad \text{on } A. \tag{10.2}$$

For the proof we shall need Proposition 6.2 and Proposition 10.3. Note that the following lemma differs from Lemma 7.1 by admitting any $y \in A'$.

LEMMA 10.2. *Let A, V, V' be as in Proposition 10.1 and let A' be a compact neighborhood of $\overline{V}$ in V'. Then there exists $c > 0$ such that, for every $\mu \in \mathcal{M}^+_{pb}(X)$ and for every $y \in A'$,*

$$K^{\mu}_{V'}G_{V'}(\cdot, y) \leq c\|G^{\mu}_{V'}\|_\infty G_{V'}(\cdot, y) \quad \text{on } A.$$

PROOF. By Lemma 9.6 there exists $a \geq 1$ such that, for all $x, z \in V$ and $y \in V'$,

$$\min(G_{V'}(x, z), G_{V'}(z, y)) \leq aG_{V'}(x, y)$$

and hence

$$G_{V'}(x, z)G_{V'}(z, y) \leq aG_{V'}(x, y)(G_{V'}(x, z) + G_{V'}(z, y)). \tag{10.3}$$

Moreover

$$c_1 := \sup_{x\in A, z\in V'\setminus V} G_{V'}(x, z) < \infty \quad \text{and} \quad c_2 := \inf_{x\in A, y\in A'} G_{V'}(x, y) > 0.$$

Now fix $\mu \in \mathcal{M}^+_{pb}(X)$, $x \in A$ and $y \in A'$. Integrating (10.3) with respect to μ on V we obtain that

$$\begin{aligned}\big(K^{1_V\mu}_{V'}G_{V'}(\cdot, y)\big)(x) &= \int_V G_{V'}(x, z)G_{V'}(z, y)\, \mu(dz)\\ &\leq aG_{V'}(x,y)\big(G^{1_V\mu}_{V'}(x)+G^{1_V\mu}_{V'}(y)\big) \leq 2a\|G^{1_V\mu}_{V'}\|_\infty G_{V'}(x,y).\end{aligned}$$

Furthermore,

$$\begin{aligned}\big(K^{1_{V'\setminus V}\mu}_{V'}G_{V'}(\cdot, y)\big)(x) &= \int_{V'\setminus V} G_{V'}(x, z)G_{V'}(z, y)\, \mu(dz)\\ &\leq c_1 G^{1_{V'\setminus V}\mu}_{V'}(y) \leq \frac{c_1}{c_2}\|G^{1_{V'\setminus V}\mu}_{V'}\|_\infty G_{V'}(x, y).\end{aligned}$$

Note that we used the symmetry of G in both estimates. Adding these inequalities and taking $c = 2a + c_1/c_2$ we conclude that

$$\big(K^{\mu}_{V'}G_{V'}(\cdot, y)\big)(x) \leq c\|G^{\mu}_{V'}\|_\infty G_{V'}(x, y). \qquad \square$$

PROPOSITION 10.3. *Let A, V, A', V' be as in Lemma 10.2. Then there exists $c > 0$ such that, for every $\mu \in \mathcal{M}^+_{pb}(X)$ and for every potential p on V' with superharmonic support contained in the interior of A',*

$$K^\mu_{V'} p \le c \|G^\mu_{V'}\|_\infty \, p \quad \text{on } A.$$

PROOF. There exist measures $\rho_n \ge 0$ supported by A' such that $G^{\rho_n}_{V'} \uparrow p$. Choosing $c > 0$ according to Lemma 10.2 we obtain by Fubini that, for every $x \in A$ and $n \in \mathbb{N}$,

$$\begin{aligned} K^\mu_{V'} G^{\rho_n}_{V'}(x) &= \int (K^\mu_{V'} G_{V'}(\cdot, y))(x) \, \rho_n(dy) \\ &\le c\|G^\mu_{V'}\|_\infty \int G_{V'}(x, y) \, \rho_n(dy) = c\|G^\mu_{V'}\|_\infty G^{\rho_n}_{V'}(x). \end{aligned}$$

The proof is finished letting n tend to infinity. □

PROOF OF PROPOSITION 10.1. Fix $\mu \in \mathcal{M}^+_{pb}(X)$ and $t \in {}^\mu\mathcal{S}^+_b(V')$. We choose a continuous function φ on V' such that $0 \le \varphi \le 1$ on V', $\varphi = 1$ on a neighborhood of $\overline{V}$, and the support C of φ is contained in the interior of A'. Let

$$u := {}^\mu R_{\varphi t}.$$

Of course, $u \le t$ on V' and hence $u = t$ on $\{\varphi = 1\}$. By Proposition 6.2, $s := u + K^\mu_{V'} u$ is a potential on V', harmonic on $V' \setminus C$. Using (4.3) and Proposition 10.3 we thus obtain that

$$t = u \ge \exp\left(-\frac{K^\mu_{V'} s}{s}\right) s \ge \exp\left(-c\|G^\mu_{V'}\|_\infty\right) s \quad \text{on } A. \tag{10.4}$$

COROLLARY 10.4. *Let (V, V') be admissible and let A be a compact subset of V. Then there exists $c > 0$ such that, for every $\mu \in \mathcal{M}^+_{pb}(X)$ and for every μ-superharmonic function $t \ge 0$ on V',*

$$K^\mu_V t \le \exp(c\|G^\mu_{V'}\|_\infty) \, t \quad \text{on } A. \tag{10.5}$$

PROOF. If t is bounded, then (10.5) follows immediately from Proposition 10.1, since $K^\mu_V t = K^\mu_V u \le K^\mu_{V'} u \le s$. The general case is obtained considering $t_n := \min(t, n)$ and letting n tend to ∞. □

COROLLARY 10.5. *Let U be a domain in X, $\mu \in \mathcal{M}^+_{pb}(X)$, and $t \ge 0$ a μ-superharmonic function on U which is not identically zero. Then* $\inf t(A) > 0$ *for every compact subset A of U.*

PROOF (see also [dLP90]). Since we may replace t by $\inf(t, 1)$, we can of course assume that t is bounded. Let $(V, V') \in \mathcal{V} \times \mathcal{V}$ be admissible such that $\overline{V'} \subset U$ and $t(x) > 0$ for some $x \in V$. Fix a compact subset A of V and take $s \in \mathcal{S}^+_b(V')$ as in Proposition 10.1. Then $s > 0$ on V', since $s(x) > t(x) > 0$, and hence

$$\inf t(A) \ge \exp(-c\|G^\mu_{V'}\|_\infty) \inf s(A) > 0.$$

As in the proof of Proposition 8.2 we now conclude that $t > 0$ on U, $\inf t(A) > 0$ for every compact subset A of U. □

11. – Local boundedness and Harnack inequalities

Suppose for a moment that $\mu^- \in \mathcal{M}_{\text{Kato}}(X)$. Fix $x \in X$ and let (V_0, V') be admissible such that $x \in V_0$. Then there exists a constant $a > 0$ such that

$$\min(G_{V'}(z_0, z_1), G_{V'}(z_1, z_2)) \le a\, G_{V'}(z_0, z_2)$$

for all $z_0, z_1 \in V$, $z_2 \in V'$. Choose $V \in \mathcal{V}$ such that $x \in V \subset V_0$ and $\gamma := 2a\|G_{V'}^{1_V\mu^-}\|_\infty < 1$. Define a kernel M by

$$M := L_{V'}^{\mu^+ - 1_V\mu^-} = (I + K_{V'}^{\mu^+})^{-1} K_{V'}^{1_V\mu^-}.$$

Then we obtain that, for every $y \in V'$,

$$MG_{V'}(\cdot,y) \le K_{V'}^{1_V\mu^-} G_{V'}(\cdot,y) = \int_V G_{V'}(\cdot,z)G_{V'}(z,y)\mu^-(dz) \le \gamma G_{V'}(\cdot,y) \quad \text{on } V.$$

Integrating with respect to positive measures on V' and taking increasing limits we conclude that $Ms \le \gamma s$ on V and hence

$$\sum_{n=0}^{\infty} M^n s \le \frac{1}{1-\gamma} s \quad \text{on } V$$

for every $s \in \mathcal{S}^+(V')$. In our general case we have to work harder to get a similar estimate.

Suppose again that we only have $\mu \in \mathcal{M}_{pb}(X)$ and consider an admissible pair (V, V'). Let M be as above, define

$$M' := L_{V'}^{\mu} = (I + K_{V'}^{\mu^+})^{-1} K_{V'}^{\mu^-}$$

and note that obviously

$$M \le M'.$$

Assuming that V' is μ-bounded, i.e., that $\sum_{n=0}^{\infty}(M')^n 1$ is bounded we want to find a real constant $c \ge 0$ such that $\sum_{n=0}^{\infty} M^n s \le cs$ on V for all $s \in \mathcal{S}^+(V')$. To that end we need perturbed Green functions:

For every $\nu \in \mathcal{M}_{\text{Kato}}^+(X)$ there exists a Green function ${}^\nu G_{V'}$ for V' with respect to $(X, {}^\nu\mathcal{H})$ and ${}^\nu G_{V'}$ is related to $G_{V'}$ by

$${}^\nu G_{V'}(\cdot, y) + K_{V'}^{\nu}({}^\nu G_{V'}(\cdot, y)) = G_{V'}(\cdot, y)$$

for every $y \in V'$. In particular, ${}^\nu G_{V'} \le G_{V'}$ and

$$(I + K_{V'}^{\nu})\left(\int_V {}^\nu G_{V'}(\cdot, y) f(y)\, \mu^-(dy)\right) = \int_V G_{V'}(\cdot, y) f(y)\, \mu^-(dy) = K_{V'}^{1_V\mu^-} f$$

for every $f \in \mathcal{B}_b(V')$, i.e.,

$$(I + K_{V'}^{\nu})^{-1} K_{V'}^{1_V \mu^-} f = \int_V {}^{\nu}G_{V'}(\cdot, y) f(y)\, \mu^-(dy). \tag{11.1}$$

Moreover, it is known that ${}^{\nu}G_{V'}$ is symmetric, i.e., that ${}^{\nu}G_{V'}(x, y) = {}^{\nu}G_{V'}(y, x)$ for all $x, y \in V'$ (see [Bou79] or Lemma 15.1).

If (ν_n) is a sequence of positive Kato measures increasing to μ^+, then, by (4.2),

$$(I + K_{V'}^{\nu_n})^{-1} K_{V'}^{1_V \mu^-} \uparrow (I + K_{V'}^{\mu^+})^{-1} K_{V'}^{1_V \mu^-} = M.$$

Choose $a \geq 1$ and $Q \in \mathbb{N}$ such that

$$\min_{1 \leq j \leq n} G_{V'}(z_{j-1}, z_j) \leq a n^Q G_{V'}(z_0, z_n) \tag{11.2}$$

for every $n \in \mathbb{N}$ and all choices of points $z_0, z_1, \ldots, z_{n-1} \in V$, $z_n \in V'$.

LEMMA 11.1. *For every $n \in \mathbb{N}$ and for all $x \in V$, $y \in V'$,*

$$\big(M^n(G_{V'}(\cdot, y))\big)(x)$$
$$\leq a(n+1)^Q\, G_{V'}(x, y) \left[M^n 1(x) + \sum_{j=0}^{n-1} (M^j 1(x) \cdot K_{V'}^{1_V \mu^-} M^{n-1-j} 1(y)) \right].$$

PROOF. By our preceding considerations we may assume that μ^+ is a Kato measure. Fix $x \in V$, $y \in V'$, and define $s = G_{V'}(\cdot, y)$. Taking $z_0 = x$ and $z_{n+1} = y$ we have by (11.1) that

$$M^n s(x) = \int_V \cdots \int_V {}^{\mu^+}G_{V'}(z_0, z_1) \cdots {}^{\mu^+}G_{V'}(z_{n-1}, z_n) G_{V'}(z_n, y) d\mu^-(z_1) \ldots d\mu^-(z_n).$$

Since ${}^{\mu^+}G_{V'} \leq G_{V'}$, we conclude from (11.2) that

$$M^n s(x) \leq a(n+1)^Q\, s(x) \sum_{j=0}^{n} I_j$$

where

$$I_n = \int_V \cdots \int_V \prod_{i=1}^{n} {}^{\mu^+}G_{V'}(z_{i-1}, z_i)\, d\mu^-(z_1) \ldots d\mu^-(z_n) = M^n 1(x)$$

and, for every $0 \leq j < n$, using the symmetry of $G_{V'}$ and ${}^{\mu^+}G_{V'}$

$$I_j = \int_V \cdots \int_V \prod_{i=1}^{j} {}^{\mu^+}G_{V'}(z_{i-1}, z_i)\, d\mu^-(z_1) \ldots d\mu^-(z_j)$$
$$\cdot \int_V \cdots \int_V \prod_{i=j+1}^{n-1} {}^{\mu^+}G_{V'}(z_i, z_{i+1})\, G_{V'}(z_n, y)\, d\mu^-(z_{j+1}) \ldots d\mu^-(z_n)$$
$$= M^j 1(x) \cdot K_{V'}^{1_V \mu^-} M^{n-1-j} 1(y).$$

□

Let us define

$$\boxed{c^{\mu}_{V,V'} := a(1 + \|G^{1_V\mu^-}_{V'}\|_\infty)\left(\|\sum_{n=0}^{\infty}(n+2)^Q M^n 1\|_\infty\right)^2.}$$

If V' is μ-bounded, then $\sum_{n=1}^{\infty} M^n 1 \le \sum_{n=1}^{\infty}(M')^n 1 \le \|S^{\mu}_{V'}1\|_\infty < \infty$, hence we know by Lemma 8.5 that

$$\|\sum_{n=1}^{\infty}(1+\varepsilon)^n M^n 1\|_\infty < \infty$$

for some $\varepsilon > 0$. Thus

$$c^{\mu}_{V,V'} < \infty$$

if V' is μ-bounded. So the following estimate will be useful:

PROPOSITION 11.2. *For every $s \in \mathcal{S}^+(V')$,*

$$\sum_{n=0}^{\infty} M^n s \le a\, c^{\mu}_{V,V'}\, s \quad \text{on } V. \tag{11.3}$$

PROOF. If (11.3) is true for every function $s = G_{V'}(\cdot, y)$, $y \in V'$, then (11.3) holds for every $s = G^{\rho}_{V'}$, $\rho \ge 0$ a finite Radon measure on V', and hence for every $s \in \mathcal{S}^+(V')$ taking increasing limits.

So fix $y \in V'$ and let $s = G_{V'}(\cdot, y)$, $x \in V$. Define $\gamma = \|\sum_{n=0}^{\infty}(n+2)^Q M^n 1\|_\infty$ and let $K = K^{1_V\mu^-}_{V'}$. Then, by Lemma 11.1,

$$M^n s(x) \le a(n+1)^Q s(x)\left[M^n 1(x) + \sum_{j=0}^{n-1}(M^j 1(x)\cdot K M^{n-1-j}1(y))\right]$$

for every $n \in \mathbb{N}$, and therefore (using the inequality $n+1 \le (j+1)(n-j+1)$ for $0 \le j \le n-1$)

$$\begin{aligned}\sum_{n=0}^{\infty} M^n s(x) &\le a s(x)\left[\sum_{n=0}^{\infty}(n+1)^Q M^n 1(x) + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(j+1)^Q(i+2)^Q M^j 1(x)\cdot K M^i 1(y)\right]\\ &\le a\, s(x)\left[\gamma + \left(\sum_{j=0}^{\infty}(j+1)^Q M^j\right)1(x)\cdot K\left(\sum_{i=0}^{\infty}(i+2)^Q M^i\right)1(y)\right]\\ &\le a\, s(x)\left(\gamma + \gamma^2\|G^{1_V\mu^-}_{V'}\|_\infty\right)\end{aligned}$$

finishing the proof. ☐

The following key lemma will be strong enough to yield local boundedness of μ-harmonic functions, Harnack inequalities for positive μ-harmonic functions, and existence of finely continuous modifications.

LEMMA 11.3. *Let h be a quasi-μ-harmonic function in V' and $s \in \mathcal{S}^+(V')$ such that $|h| \le s$ q.e. on ∂V. Then*

$$|h| \le c^{\mu}_{V,V'} H_V s \quad \text{q.e. on } V.$$

If h is μ-harmonic on V', then $|h| \le c^{\mu}_{V,V'} H_V s$ on V.

PROOF. Let g be a harmonic function on V and P a polar subset of V' such that $|h| \le s$ on $V' \setminus P$, $G_V^{|h\mu|} < \infty$ on $V' \setminus P$, and

$$h + K_V^{\mu} h = g \qquad \text{on } V \setminus P.$$

By Lemma 3.6, $g = H_V h$. Define

$$L := L_V^{\mu} = (I + K_V^{\mu^+})^{-1} K_V^{\mu^-}.$$

We first claim that

$$h \le H_V s + Lh \qquad \text{on } V \setminus P. \tag{11.4}$$

Indeed, we have $(I + K_V^{\mu^+}) L h^{\pm} = K_V^{\mu^-} h^{\pm} = G_V^{h^{\pm}\mu^-} < \infty$ on $V \setminus P$. Defining $f = 1_{V\setminus P}(Lh - h)$ we hence know that $K_V^{\mu^+}|f| < \infty$ on $V \setminus P$. Moreover,

$$f + K_V^{\mu^+} f + H_V s = K_V^{\mu^-} h - h - K_V^{\mu^+} h + H_V s = H_V s - g \qquad \text{on } V \setminus P$$

where $H_V s$ and $H_V s - g = H_V(s - h)$ are positive harmonic functions on V, since by assumption $s - h \ge 0$ q.e. on ∂V. So (11.4) follows from Lemma 4.1.

Since $\mu^-(P) = 0$, a trivial induction now leads to

$$h \le \sum_{n=0}^{k-1} L^n H_V s + L^k h \qquad \text{on } V \setminus P \tag{11.5}$$

for every $k \in \mathbb{N}$. From Proposition 8.3 we know that

$$L \le M = L_{V'}^{\mu^+ - 1_V \mu^-}.$$

By Proposition 11.2,

$$\sum_{n=0}^{\infty} M^n s \le c^{\mu}_{V,V'} s \quad \text{on } V \quad , \quad \sum_{n=0}^{\infty} M^n H_V s \le c^{\mu}_{V,V'} H_V s \quad \text{on } V.$$

In particular,

$$\inf_k L^k |h| \le \inf_k M^k s = 0 \qquad \text{on } V \cap \{s < \infty\},$$

so we conclude from (11.5) that

$$h \le c^{\mu}_{V,V'} H_V s \qquad \text{on } V \setminus (P \cup \{s = \infty\}),$$

hence $h \le c^{\mu}_{V,V'} H_V s$ q.e. on V. If h is μ-harmonic on V', then this inequality holds everywhere on V by fine continuity. The proof is finished replacing h by $-h$. □

THEOREM 11.4. *Let U be a domain in X admitting a locally bounded positive μ-harmonic function which is not identically zero. Then the following holds:*

1. *For every quasi-μ-harmonic function on U there exists a (unique) μ-harmonic function $\tilde{h}$ such that $\tilde{h} = h$ q.e.*
2. *Every μ-harmonic function on U is locally bounded.*

PROOF. Let h be a quasi-μ-harmonic function on U. Fix $x \in U$ and an admissible pair (V, V') such that $x \in V, \overline{V'} \subset U$. Then V' is μ-bounded by Proposition 8.1. By Lemma 3.6, there exists a function $s \in \mathcal{S}^+(V')$ such that $|h| \leq s$ q.e. on V'. Applying Lemma 11.3 we obtain that $|h| \leq c^{\mu}_{V,V'} H_V s$ q.e. on V. The function $H_V s$ is continuous and real on V. If h is even μ-harmonic on U, then $|h| \leq c^{\mu}_{V,V'} H_V s$ by fine continuity showing that h is locally bounded on V, hence locally bounded on U.

In the general case, a trivial covering argument (and the fact that countable unions of polar sets are polar) yields a polar set $\tilde{P}$ in U such that

$$\sup |h|(A \setminus \tilde{P}) < \infty$$

for every compact set A in U. Thus an application of Lemma 3.5 proves (1). □

PROPOSITION 11.5. *Let U be a domain in X admitting a locally bounded positive μ-harmonic function which is not identically zero. Let (V, V') be admissible such that $\overline{V'} \subset U$, and let A be a compact subset of V and $c_0 > 0$ such that*

$$\sup g(A) \leq c_0 \inf g(A)$$

for every harmonic function $g \geq 0$ on V. Then there exists $c > 0$ (the constant from Lemma 10.2) such that, for every μ-harmonic function $h \geq 0$ on U,

$$\sup h(A) \leq c_0 \exp(c\|G^{\mu^+}_{V'}\|_\infty)\, c^{\mu}_{V,V'} \inf h(A). \tag{11.6}$$

PROOF. Fix a μ-harmonic function $h \geq 0$ on U. We know already from Theorem 11.4 that h is locally bounded. Recall that h is μ^+-superharmonic. So by Proposition 10.1 there exists $u \in \mathcal{B}^+_b(V')$ such that

$$u = h \text{ on } \overline{V}, \quad s := u + K^{\mu^+}_{V'} u \in \mathcal{S}^+_b(V'), \quad s \leq \exp(c\|G^{\mu^+}_{V'}\|_\infty)\, h \text{ on } A.$$

Define

$$g := H_V s\,!$$

Of course,

$$g \leq \exp(c\|G^{\mu^+}_{V'}\|_\infty)\, h \quad \text{on } A, \tag{11.7}$$

since $H_V s \leq s$. Moreover, $0 \leq h = u \leq s$ on $\overline{V}$. So Lemma 11.3 implies that

$$h \leq c^{\mu}_{V,V'}\, g \quad \text{on } V. \tag{11.8}$$

Since $\sup g(A) \leq c_0 \inf g(A)$, we finally obtain (11.6) combining (11.7) and (11.8). □

THEOREM 11.6. *Let U be a domain in X admitting a locally bounded positive μ-harmonic function which is not identically zero. Then Harnack inequalities hold for positive μ-harmonic functions on U.*

PROOF. Given $x \in U$, we may choose an admissible pair (V, V') such that $x \in V$ and $\overline{V'} \subset U$. By Proposition 11.5 we know that, for every compact subset A of V and for every μ-harmonic function $h \geq 0$ on U,

$$\sup h(A) \leq c' \inf h(A)$$

where $c' = c_U \exp(c \|G_{V'}^{\mu^+}\|_\infty) c_{V,V'}^{\mu} < \infty$. A covering argument finishes the proof. □

Since every $V \in \mathcal{V}$ such that $\|G_V^{\mu^-}\|_\infty < 1$ is μ-bounded, we immediately have the following (showing in particular that the main assertion of [dLP90] is valid if (LT) holds):

COROLLARY 11.7. *Let Y be the union of all $V \in \mathcal{V}$ such that $\|G_V^{\mu^-}\|_\infty < 1$ ($Y = X$ if $\mu^- \in \mathcal{M}_{\mathrm{Kato}}(X)$!) and let U be a domain in Y.*

Then for every quasi-μ-harmonic function h on U there exists a μ-harmonic function $\tilde{h}$ on U such that $\tilde{h} = h$ q.e. Every μ-harmonic function on U is locally bounded and Harnack inequalities hold for positive μ-harmonic functions on U.

If $X \subset \mathbb{R}^d$ and if $(X, \mathcal{H})$ is associated with an operator $\mathcal{L}$ of type (1.1) or (1.2) we obtain the following consequences relating solutions of $\mathcal{L}u - u\mu = 0$ (in the appropriate sense) directly to the preceding results:

THEOREM 11.8. *Let U be a domain in X admitting a locally bounded positive μ-harmonic function which is not identically zero. Then the following holds:*

1. *If $\mu = V\lambda$ and u is a solution of $\mathcal{L}u - Vu = 0$, then there exists a (unique) μ-harmonic function h on U such that $h = u$ λ-a.e.*
2. *If u is a quasi-continuous solution of $\Delta u - u\mu = 0$, then there exists a (unique) μ-harmonic function h on U such that $h = u$ q.e.*

PROOF. Lemma 3.1, Lemma 3.2, and Theorem 11.4. □

12. – Brelot spaces of μ-harmonic functions

As in the introduction let $\mathcal{M}_{pbc}(X)$ be the set of all $\mu \in \mathcal{M}_{pb}(X)$ such that, for every V in $\mathcal{V}$, $G_X^{1_V\mu} = G_X^{1_V\mu^+} - G_X^{1_V\mu^-}$ is continuous (while $G_X^{1_V\mu^+}$ and $G_X^{1_V\mu^-}$ may have discontinuities). Of course, every Kato measure $\mu \in \mathcal{M}_{pb}(X)$ is contained in $\mathcal{M}_{pbc}(X)$ and

$$\mathcal{M}_{pbc}(X) = \{\mu \in \mathcal{M}_{pb}(X) : G_V^{\mu} \text{ continuous for every } V \in \mathcal{V}\}.$$

At the end of this section we shall see that every $\nu \in \mathcal{M}_{pb}^+(X)$ is the positive part of many measures in $\mathcal{M}_{pbc}(X)$ provided that axiom D is satisfied.

It is a remarkable fact that, for every $\mu \in \mathcal{M}_{pbc}(X)$, sufficiently small open sets are μ-bounded. This is a consequence of the following basic lemmas contained in [Keu90]:

LEMMA 12.1. *Suppose that $\mu \in \mathcal{M}_{pbc}(X)$. Then, for every $x \in X$ and for every $\varepsilon > 0$, there exists a neighborhood V of x such that $\|G_U^\mu\|_\infty \leq \varepsilon$ for every $U \in \mathcal{V}$ contained in V.*

LEMMA 12.2. *If $\mu \in \mathcal{M}_{pb}(X)$ and $U \in \mathcal{V}$ such that $\|G_U^\mu\|_\infty < \frac{1}{2}\exp(-\|G_U^{\mu^+}\|_\infty)$, then U is μ-bounded.*

For the convenience of the reader (and since [Keu90] is still waiting for publication) we repeat the proofs:

PROOF OF LEMMA 12.1. There is a set $W \in \mathcal{V}$ containing x such that $G_W^{|\mu|}(x)(1 - H_W 1) < \varepsilon/3$ on W and a compact neighborhood V of x in W such that $|G_W^\mu - G_W^\mu(x)| < \varepsilon/3$ on V. Let $U \in \mathcal{V}$ such that $U \subset V$. Then $0 \leq 1 - H_U 1 \leq 1 - H_W 1$ and $|H_U G_W^\mu - G_W^\mu(x) H_U 1| < \varepsilon/3$ on U, hence

$$|G_U^\mu| = |G_W^\mu - H_U G_W^\mu| < \varepsilon. \qquad \square$$

PROOF OF LEMMA 12.2. Let $K^\pm = K_U^{\mu^\pm}$, $\alpha = \|G_U^{\mu^+}\|_\infty = \|K^+ 1\|_\infty$. Using the trivial identity

$$(I + K^+)^{-1} K^+ = I - (I + K^+)^{-1}$$

and looking at (4.3) and (4.4), we see that the positive operator $(I + K^+)^{-1} K^+$ on $\mathcal{B}_b(U)$ has a norm

$$\|(I + K^+)^{-1} K^+\| = \|(I + K^+)^{-1} K^+ 1\|_\infty \leq 1 - e^{-\alpha}$$

and hence $\|(I + K^+)^{-1}\| \leq 2 - e^{-\alpha}$. Thus

$$\begin{aligned} L_U^\mu 1 = (I + K^+)^{-1} K^- 1 &= (I + K^+)^{-1} K^+ 1 - (I + K^+)^{-1} G_U^\mu \\ &\leq 1 - e^{-\alpha} + 2\|G_U^\mu\|_\infty < 1. \end{aligned} \qquad \square$$

As already stated in the introduction we have the following result:

THEOREM 12.3. *For every $\mu \in \mathcal{M}_{pb}(X)$ the following statements are equivalent:*

1. $\mu \in \mathcal{M}_{pbc}(X)$.
2. *$(X, {}^\mu\mathcal{H})$ is a harmonic space.*
3. *$(X, {}^\mu\mathcal{H})$ is a Brelot space.*

In particular, for every $\mu \in \mathcal{M}_{pbc}(X)$, every μ-harmonic function is continuous and Harnack inequalities hold for positive μ-harmonic function on every domain in X.

PROOF. (3) $\Longrightarrow$ (2): Trivial.

(2) $\Longrightarrow$ (1) (see [Keu90]): Fix $U_0 \in \mathcal{V}$ and $x \in U_0$. There exists $U \in \mathcal{V}(U_0)$ and $h \in {}^{\mu}H(U)$ such that $x \in U$ and $h(x) = 1$. In particular, h is continuous, and we may assume that $|h| \le 2$ on U (if necessary replace U by a smaller set). Moreover, given $\varepsilon > 0$, there exists an open neighborhood V of x in U such that $|1-h| < \varepsilon$ on V. Finally, since $K_U^{\mu}((1-h)1_{U\setminus V})$ is harmonic on V, there exists a neighborhood W of x in V such that

$$|K_U^{\mu}((1-h)1_{U\setminus V}) - K_U^{\mu}((1-h)1_{U\setminus V})(x)| < \varepsilon \quad \text{on } W.$$

Then, for every $y \in W$,

$$\begin{aligned}&|K_U^{\mu}(1-h)(y) - K_U^{\mu}(1-h)(x)|\\ &\le \varepsilon + |K_U^{\mu}((1-h)1_V)(y) - K_U^{\mu}((1-h)1_V)(x)| \le \varepsilon(1+2\|G_U^{|\mu|}\|_{\infty}),\end{aligned}$$

where the last inequality holds because of $|1-h| < \varepsilon$ on V. So $K_U^{\mu}(1-h)$ is continuous at x. In addition, $K_U^{\mu}h$ is continuous on U, since $h + K_U^{\mu}h$ is harmonic on U. Thus $K_U^{\mu}1$ is continuous at x, and we conclude that $G_{U_0}^{\mu}$ is continuous at x, since $G_{U_0}^{\mu} - G_U^{\mu}$ is harmonic on U.

(1) $\Longrightarrow$ (3): Only little remains to be done to see that $(X, {}^{\mu}\mathcal{H})$ is a Brelot space if $\mu \in \mathcal{M}_{pbc}(X)$. Indeed, based on Lemma 12.1 and Lemma 12.2, J.-M. Keuntje [Keu90] already showed that *continuous* μ-harmonic functions yield a harmonic space and that, for every μ-bounded $V \in \mathcal{V}$, the corresponding μ-harmonic kernel H_V^{μ} is given by

$$H_V^{\mu} = (I + K_V^{\mu})^{-1} H_V.$$

Having a base of μ-bounded sets, we know by Theorem 11.4 and Theorem 11.6 that μ-harmonic functions are locally bounded and that Harnack inequalities hold for positive μ-harmonic functions on domains in X.

Therefore we only have to convince ourselves that every μ-harmonic function is not only finely continuous, but continuous. So let h be a μ-harmonic function on an open subset U of X and fix $V, W \in \mathcal{V}(U)$ such that $\overline{V} \subset W$. Since h is locally bounded on U, we know that $H_V^{\mu}h$ is continuous on V (it is a basic property of harmonic kernels on a harmonic space to yield continuous functions when applied to bounded Borel functions). On the other hand we conclude from Lemma 3.6 that

$$h = (I + K_V^{\mu})^{-1} H_V h = H_V^{\mu} h.$$

Thus h is continuous on V. □

Based on [Net75] and for the classical case, a simple example of a measure $\mu \in \mathcal{M}_{pbc}(X)$ which is not a Kato measure is discussed in [Keu90] . To finish this section we intend to show that, at least if axiom D is satisfied as in the classical case, any $\nu \in \mathcal{M}_{pb}^{+}(X)$ is the positive part of some (many) $\mu \in \mathcal{M}_{pbc}(X)$. For the construction of measures $\nu \in \mathcal{M}_{pb}^{+}(X)$ with given discontinuities for the potentials G_X^{ν} see [Ha98].

PROPOSITION 12.4. *Suppose that $(X, \mathcal{H})$ satisfies axiom D, i.e., that any bounded potential on X is a countable sum of continuous potentials. Let $\nu \in \mathcal{M}^+_{pb}(X)$ and let σ be a positive Radon measure on X. Then there exists a measure $\nu' \in \mathcal{M}^+_{pb}(X)$ which is singular with respect to σ such that $\nu - \nu' \in \mathcal{M}_{pbc}(X)$.*

PROOF. In the classical case, $\sigma(\{x \in X : x_1 = \alpha\}) = 0$ except for (at most) countably many $\alpha \in \mathbb{R}$, so we may choose a dense subset $\{\alpha_m : m \in \mathbb{N}\}$ of $\mathbb{R}$ such that $\sigma(\{x_1 = \alpha_m\}) = 0$ for every $m \in \mathbb{N}$, and take

$$A_n = \{x \in X : x_1 \in \{\alpha_1, \dots, \alpha_n\}\}, \qquad A = \bigcup_{n=1}^{\infty} A_n.$$

In the general case, there exists an uncountable family of disjoint K_σ-sets which are finely dense in X (see e.g. [Han81]). So there exists a finely dense K_σ-set A such that $\sigma(A) = 0$, and we take a sequence (A_n) of compact sets such that $A_n \uparrow A$.

Let us assume first that the support of ν is contained in a relatively compact open subset U of X. Then G^ν_X is a bounded potential and there exist continuous potentials p_k on X, $k \in \mathbb{N}$, such that

$$G^\nu_X = \sum_{k=1}^{\infty} p_k.$$

Keeping $k \in \mathbb{N}$ fixed, the sequence $(\widehat{R}^{A_n \cap U}_{p_k})_{n \in \mathbb{N}}$ is increasing to p_k, since $A = \bigcup_{n=1}^{\infty} A_n$ is finely dense in X and $R^U_{p_k} = p_k$ (by definition, $R^{A_n \cap U}_{p_k}$ is the infimum of all $s \in \mathcal{S}^+(X)$ such that $s \geq p_k$ on $A_n \cap U$, and $\widehat{\varphi}(x) = \liminf_{y \to x} \varphi(y)$). So there exists a natural number n_k such that $\widehat{R}^{A_{n_k} \cap U}_{p_k} + \varepsilon 2^{-k} > p_k$ and, by axiom D, we may find continuous real potentials q_k, q'_k such that

$$q_k + q'_k = \widehat{R}^{A_{n_k} \cap U}_{p_k} \quad \text{and} \quad q_k + \varepsilon 2^{-k} > p_k.$$

We have

$$q_k = G^{\rho_k}_X$$

where the measure ρ_k is supported by $A_{n_k} \cap \overline{U}$ and hence is singular with respect to σ. It now suffices to take

$$\nu' = \sum_{k=1}^{\infty} \rho_k.$$

Indeed, the measure ν' is obviously singular with respect to σ. Moreover,

$$G^\nu_X - G^{\nu'}_X = \sum_{k=1}^{\infty} (p_k - q_k)$$

where the potentials p_k, q_k are continuous and $0 \leq p_k - q_k \leq \varepsilon 2^{-k}$. Thus $0 \leq G_X^\nu - G_X^{\nu'} \leq \varepsilon$ and $G_X^\nu - G_X^{\nu'}$ is continuous.

To finish the proof for an arbitrary $\nu \in \mathcal{M}_{pb}^+(X)$ choose a sequence (U_n) of relatively compact open sets covering X such that $U_1 = \emptyset$ and $\overline{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $\nu_n := 1_{U_{n+1}\setminus U_n}\nu \in \mathcal{M}_{pb}^+(X)$, so we may choose $\nu_n' \in \mathcal{M}_{pb}^+(X)$, singular with respect to σ and supported by $\overline{U}_{n+1}$, such that $\nu_n - \nu_n' \in \mathcal{M}_{pbc}(X)$. Then $\nu' := \sum_{n=1}^\infty \nu_n'$ has the desired properties, since the sum is locally finite. □

REMARK 12.5. Assume that $(X, \mathcal{H})$ satisfies axiom D, that $\nu \in \mathcal{M}_{pb}^+(X)$, and $\rho \geq 0$ is any Radon measure on X such that $\rho(V) > 0$ for every finely open V with $\nu(V) > 0$. Using [BH86], p. 84, it is then possible to construct $\nu' \in \mathcal{M}_{pb}^+(X)$ such that $\nu - \nu' \in \mathcal{M}_{pbc}(X)$ and ν' is absolutely continuous with respect to ρ.

COROLLARY 12.6. *If $(X, \mathcal{H})$ satisfies axiom D, then $\{\mu^+ : \mu \in \mathcal{M}_{pbc}(X)\} = \mathcal{M}_{pb}^+(X)$.*

PROOF. It suffices to apply Proposition 12.4 to the case $\sigma = \nu \in \mathcal{M}_{pb}^+(X)$. Then certainly $(\nu - \nu')^+ = \nu$. □

13. – Harnack inequalities on dense subsets

In this section we shall return to the general situation of a measure $\mu \in \mathcal{M}_{pb}(X)$ where we might not have a base of μ-bounded sets. We shall see, however, that nevertheless there is a dense open set Y in X such that Harnack inequalities hold for positive μ-harmonic functions on domains in Y. This is true even for potentially finite measures (see Corollary 13.8). And in the following section we shall prove that it is impossible to say much more: Any dense open subset of X can be the maximal open subset Y of X such that Harnack inequalities hold on subdomains of Y.

To deal effectively with discontinuities of potentials we introduce the following: Given any $\mu \in \mathcal{M}_{pb}^+(X)$ and $x \in X$, let $d_\mu(x)$ be the oscillation which potentials $G_X^{1_U\mu}$ have at $x \in U$, i.e., we define

$$d_\mu(x) := \limsup_{y \to x} G_X^{1_U\mu}(y) - G_X^{1_U\mu}(x) \tag{13.1}$$

where U is any set in $\mathcal{V}$ containing x (observe that the right side does not depend on the choice of U). Obviously,

$$d_\mu(x) := \limsup_{y \to x} G_U^\mu(y) - G_U^\mu(x) \tag{13.2}$$

for every $U \in \mathcal{V}$ containing x, since $G_X^{1_U\mu} - G_U^\mu = H_U G_X^{1_U\mu}$ is continuous on U. Moreover, d_μ is an u.s.c. function.

Of course, $\mu \in \mathcal{M}_{pb}(X)$ is a Kato measure if and only $d_{|\mu|} = 0$. And $d_{\mu^+} = d_{\mu^-}$ if $\mu \in \mathcal{M}_{pbc}(X)$. Furthermore, it is easily verified that $d_{\alpha\mu} = \alpha d_\mu$ and $\max(d_\mu, d_\nu) \le d_{\mu+\nu} \le d_\mu + d_\nu$ for all $\mu, \nu \in \mathcal{M}_{pb}^+(X)$ and $\alpha \in \mathbb{R}^+$.

PROPOSITION 13.1. *For every $\mu \in \mathcal{M}_{pb}^+(X)$ and every $x \in X$,*

$$d_\mu(x) = \inf\{\|G_U^\mu\|_\infty : x \in U \in \mathcal{V}\}. \tag{13.3}$$

PROOF. Of course, (13.2) implies that $d_\mu(x) \le \|G_U^\mu\|_\infty$ for every $U \in \mathcal{V}$ containing x. Fix $\varepsilon > 0$. There exists $W \in \mathcal{V}$ containing x such that $G_W^\mu(x)(1 - H_W 1) < \varepsilon$ on W. Choose $U \in \mathcal{V}(W)$ such that $x \in U$ and $G_W^\mu < d_\mu(x) + G_W^\mu(x) + \varepsilon$ on U. Since G_W^μ is l.s.c., we may assume that $G_W^\mu > G_W^\mu(x) - \varepsilon$ on $\overline{U}$. Using $H_W 1 \le H_U 1 \le 1$ we conclude that $H_U G_W^\mu > (G_W^\mu(x) - \varepsilon) H_U 1 > G_W^\mu(x) - 2\varepsilon$ and finally

$$G_U^\mu = G_W^\mu - H_U G_W^\mu < d_\mu(x) + G_W^\mu(x) + \varepsilon - (G_W^\mu(x) - 2\varepsilon) = d_\mu(x) + 3\varepsilon. \quad \square$$

COROLLARY 13.2. *If $\mu \in \mathcal{M}_{pb}(X)$ and $x \in X$ such that $d_{\mu^-}(x) < 1$, then every sufficiently small open neighborhood of x is μ-bounded.*

The following lemma shows that the closed set $\{d_{\mu^-} \ge 1\}$ has no interior points:

LEMMA 13.3. *Let $\psi \ge 0$ be a l.s.c. real function on X and*

$$\varphi(x) := \limsup_{y \to x} \psi(y) - \psi(x)$$

for $x \in X$. Then φ is u.s.c. and, for every $\varepsilon > 0$, the set $\{\varphi \ge \varepsilon\}$ is a closed set having no interior points.

PROOF. It is immediately seen that φ is u.s.c. and hence $\{\varphi \ge \varepsilon\}$ is closed for every $\varepsilon > 0$. Fix $x \in X$, $\varepsilon > 0$, and an open neighborhood U of x. There exists a non-empty open subset V of U such that ψ is bounded on V. Indeed, otherwise each $U \cap \{\psi > n\}$, $n \in \mathbb{N}$, would be an open set which is dense in U, hence $U \cap \bigcap_{n=1}^\infty \{\psi > n\}$ would be dense in U. However, the intersection is empty, since we assumed that $\psi < \infty$. Consider now $z \in V$ such that $\psi(z) + \varepsilon > \sup \psi(V)$. Obviously $\sup \psi(V) \ge \limsup_{y \to z} \psi(y)$, hence $\varphi(z) < \varepsilon$. $\square$

Let

$$X_\mu := \bigcup_{V \in \mathcal{V},\ \|S_V^\mu 1\|_\infty < \infty} V, \qquad E_\mu := (X \setminus X_\mu) \cap \bigcup_{V \in \mathcal{V},\ S_V^\mu 1 < \infty} V.$$

Obviously, the set X_μ is open and E_μ is a relatively open subset of the closed set $X \setminus X_\mu$.

By Corollary 13.2, $\{d_{\mu^-} < 1\} \subset X_\mu$, hence by Lemma 13.3, Theorem 11.6, and Theorem 11.8 we obtain the following (see also Corollary 13.8):

THEOREM 13.4. *X_μ is a dense open subset of X. For every domain U in X_μ, μ-harmonic functions on U are locally bounded and Harnack inequalities hold for positive μ-harmonic functions on U.*

Moreover, if $X \subset \mathbb{R}^d$ and $(X, \mathcal{H})$ is associated with an operator $\mathcal{L}$, then statements (1) *and* (2) *of Theorem 11.8 are true for every open subset U of X_μ.*

Later on we shall see that for every dense open subset Y of X there exists a measure $\mu \in \mathcal{M}_{pb}(X)$ such that $Y = X_\mu$ and $d_{\mu^-} = 1_{X\setminus Y}$. Moreover, e.g. the two extreme cases $E_\mu = \emptyset$ and $E_\mu = X \setminus Y$ do occur (see Proposition 14.3 and Proposition 14.4).

Improving Proposition 8.1 and Proposition 8.2 we shall be able to decide if Harnack inequalities hold (formally) also on domains containing points of $X \setminus X_\mu$.

PROPOSITION 13.5. *Let (V, V') be admissible and $S^\mu_{V'}1 < \infty$. Let $g_0 \geq 1$ be a bounded harmonic function on V'. Then $h := S^{\mu^+ - 1_V\mu^-}_{V'}(I + K^{\mu^+}_{V'})^{-1} g_0$ is a μ-harmonic function on V and $\|g_0\|_\infty S^\mu_{V'}1 \geq h \geq \exp(-\|G^{\mu^+}_{V'}\|_\infty \|g_0\|_\infty) S^\mu_V 1$.*

PROOF. Since $K^{\mu^+}_{V'} g_0 \leq \|g_0\|_\infty K^{\mu^+}_{V'} 1 \leq \|G^{\mu^+}_{V'}\|_\infty \|g_0\|_\infty$ and $g_0 \geq 1$, the inequalities follow from (4.1), (4.3), and Proposition 8.3. Defining

$$p_n := [K^{1_V\mu^-}_{V'}(I + K^{\mu^+}_{V'})^{-1}]^n g_0 \qquad (n = 0, 1, 2, \ldots)$$

we have

$$h = \sum_{n=0}^{\infty} (I + K^{\mu^+}_{V'})^{-1} p_n.$$

Every p_n is a potential on V' with superharmonic support contained in $\overline{V}$. Given a compact subset A of V, we know from (4.3) and Proposition 10.3 that there exists $c > 0$ such that, for every $n \in \mathbb{N}$,

$$(I + K^{\mu^+}_{V'})^{-1} p_n \geq p_n \exp\left(-\frac{K^{\mu^+}_{V'} p_n}{p_n}\right) \geq p_n \exp(-c\|G^{\mu^+}_{V'}\|_\infty) \quad \text{on } A$$

and therefore

$$p := \sum_{n=0}^{\infty} p_n \leq \exp(c\|G^{\mu^+}_{V'}\|_\infty)\, h \quad \text{on } A.$$

So $p < \infty$ on V. Of course,

$$h + K^{\mu^+}_{V'} h = p.$$

Therefore $K^{\mu^+}_{V'} h < \infty$ on V. Moreover, taking $\tilde{\mu} := \mu^+ - 1_V\mu^-$ we have $L^{\tilde{\mu}}_{V'} S^{\tilde{\mu}}_{V'} + I = S^{\tilde{\mu}}_{V'}$, hence $L^{\tilde{\mu}}_{V'} h + (I + K^{\mu^+}_{V'})^{-1} g_0 = h$ and

$$K^{1_V\mu^-}_{V'} h + g_0 = h + K^{\mu^+}_{V'} h.$$

This implies that $K_{V'}^{1_V\mu^-}h < \infty$ on V and $h+K_{V'}^{\tilde{\mu}}h = g_0$ on V. Hence $H_V K_{V'}^{\tilde{\mu}}h$ is harmonic on V and

$$h + K_V^{\mu}h = g_0 - H_V K_{V'}^{\tilde{\mu}}h.$$

Thus h is μ-harmonic on V. □

PROPOSITION 13.6. *Let U be a domain in X and let $h \geq 0$ be a μ-harmonic function on U which is not identically zero. Then, for every admissible pair $(V, V') \in \mathcal{V} \times \mathcal{V}$ such that $\overline{V'} \subset U$ and for every $W \in \mathcal{V}(V)$, there exists $\alpha \in \mathbb{R}_+$ such that $S_W^{\mu}1 \leq \alpha h$ on W.*

PROOF. By Corollary 10.5, $\inf h(\overline{V}) > 0$, hence $H_V h$ is a strictly positive harmonic function on V. Since $h + K_V^{\mu}h = H_V h$ on $V \setminus P$, P polar, we obtain by fine continuity that

$$h + K_V^{\mu^+}h = H_V h + K_V^{\mu^-}h \quad \text{on } V.$$

By Corollary 10.4, $K_V^{\mu^+}h < \infty$. So $K_V^{\mu^-}h < \infty$ as well. Since $(I+K_V^{\mu^+})L_V^{\mu}h = K_V^{\mu^-}h$, we obtain that

$$f := h|_V - L_V^{\mu}h$$

is a real function on V satisfying $K_V^{\mu^+}|f| < \infty$. Moreover,

$$f + K_V^{\mu^+}f = h + K_V^{\mu^+}h - K_V^{\mu^-}h = H_V h \quad \text{on } V.$$

Thus $f \geq 0$ by Lemma 4.1. Furthermore , $f = H_V h - K_V^{\mu^+}f$ is finely continuous, so f is a positive μ^+-harmonic function on V. It cannot be identically zero, since $H_V h > 0$ on V. Therefore, fixing $W \in \mathcal{V}(V)$,

$$\gamma := \inf f(W) > 0$$

by Corollary 10.5 (or by Corollary 7.3). Of course, for every $m \in \mathbb{N}$,

$$h = \sum_{n=0}^{m-1}(L_V^{\mu})^n f + (L_V^{\mu})^m h \quad \text{on } V,$$

hence $S_V^{\mu}f \leq h$ and by Proposition 8.3,

$$S_W^{\mu}1 \leq S_V^{\mu}1_W \leq \gamma^{-1}S_V^{\mu}f \leq \gamma^{-1}h.$$ □

THEOREM 13.7. 1) *Let U be a domain such that $U \cap E_{\mu} = \emptyset$. Then Harnack inequalities hold for positive μ-harmonic functions on U.*
2) *For every $x \in E_{\mu}$, there exists $V \in \mathcal{V}$ containing x and a μ-harmonic function $h \geq 0$ on V such that* $\limsup_{y \to x} h(y) = \infty$. *In particular, Harnack inequalities do not hold for positive μ-harmonic functions on subdomains of V containing x.*

PROOF. 1) Let $h \geq 0$ be a μ-harmonic function on U which is not identically zero. Given $x \in U$, we may choose a admissible pair $(V, V') \in \mathcal{V} \times \mathcal{V}$ such that $\overline{V'} \subset U$, $x \in V$, and take $W \in \mathcal{V}(V)$. Then $S_W^\mu 1 < \infty$ by Proposition 13.6, hence $W \setminus X_\mu \subset E_\mu$, $W \subset X_\mu$. Thus $U \subset X_\mu$ and Harnack inequalities hold for positive μ-harmonic functions on U.

2) Fix $x \in E_\mu$. Then $S_{V'}^\mu 1 < \infty$ for some $V' \in \mathcal{V}$ containing x, but of course $\limsup_{y \to x} S_{V'}^\mu 1(y) = \infty$. Otherwise we could find $W \in \mathcal{V}(V')$ with $x \in W$, and $\sup_{y \in W} S_{V'}^\mu(y) < \infty$, and then $S_W^\mu 1 \leq S_{V'}^\mu 1$ would imply that $\|S_W^\mu\|_\infty < \infty$, hence $x \in X_\mu$. We take $V \in \mathcal{V}(V')$ such that $x \in V$ and (V, V') is admissible. Using Proposition 13.5 we obtain a positive μ-harmonic function h on V with $\limsup_{y \to x} h(y) = \infty$. □

If μ is a signed Radon measure on X which is only assumed to be *potentially finite*, i.e., such that $G_X^{1_V|\mu|} < \infty$ for every $V \in \mathcal{V}$ (if axiom D does not hold, we suppose explicitly that every $G_X^{1_V|\mu|}$ is a sum of continuous potentials), then we conclude from Lemma 13.3 that there is a dense open subset $\tilde{X}$ of X such that $\mu|_{\tilde{X}} \in \mathcal{M}_{pb}(\tilde{X})$. So we obtain the following improvement of Theorem 13.4:

COROLLARY 13.8. *Let μ be a signed Radon measure on X which is potentially finite. Then there exists a dense open subset Y of X such that, for every domain U in Y, μ-harmonic functions on U are locally bounded and Harnack inequalities hold for positive μ-harmonic functions on U.*

Moreover, if $X \subset \mathbb{R}^d$ and $(X, \mathcal{H})$ is associated with an operator $\mathcal{L}$, then statements (1) *and* (2) *of Theorem 11.8 are true for every open subset U of Y.*

14. – Construction of general examples

In this section let us suppose that $G_X(y, y) = \infty$ for every $y \in X$.

We shall see to what extent results obtained in previous sections are sharp. In particular, we show that any dense open subset Y may be the maximal open subset of X where Harnack inequalities hold for positive μ-harmonic functions on subdomains. We shall achieve this (see Proposition 14.4) with a measure $\mu = -\nu \leq 0$ such that $G_X^\nu \leq 1+\varepsilon$, G_X^ν is continuous on Y and has oscillation 1 at each point in $X \setminus Y$ (implying that, for every $0 < \delta \leq 1$, Y is the union of all $V \in \mathcal{V}$ with $\|G_V^\nu\|_\infty < \delta$). On the other hand, given any dense open Y in X and any $\alpha > 0$, $\varepsilon > 0$, we may construct $\mu = -\nu \leq 0$ such that the oscillation of G_X^ν at each $x \in X \setminus Y$ is α (implying that $\|G_V^\nu\|_\infty \geq \alpha$ for every $V \in \mathcal{V}$ intersecting $X \setminus Y$) and that nevertheless Harnack inequalities hold for μ-harmonic functions on *any* domain in X (see Proposition 14.5).

Let $\mathcal{P}_\sigma(X)$ denote the convex cone of all real potentials on X which are countable sums of continuous potentials. Moreover, it will be useful to choose a metric for X.

Of course, we know the following: If $\nu \in \mathcal{M}_{pb}^{+}(X)$ and $h \geq 0$ is a $(-\nu)$-harmonic function on X, then h is superharmonic by (3.5). Suppose now, conversely, that p is a strictly positive potential in $\mathcal{P}_\sigma(X)$, say $p = G_X^\rho$, and let

$$\nu := p^{-1}\rho.$$

Then ν is a positive Radon measure on X and obviously, for every $V \in \mathcal{V}$,

$$p + K_V^{-\nu} p = G_X^\rho - G_V^\rho = H_V G_X^\rho$$

is harmonic on V. If p is locally bounded, then $\rho \in \mathcal{M}_{pb}^{+}(X)$ and hence $\nu \in \mathcal{M}_{pb}^{+}(X)$. Applying Theorem 11.4 and Theorem 11.6 we thus obtain the following:

Proposition 14.1. *Let $p = G_X^\rho \in \mathcal{P}_\sigma(X) \setminus \{0\}$. Then $\nu = p^{-1}\rho \in \mathcal{M}_{pb}^{+}(X)$ and p is $(-\nu)$-harmonic on X.*

In particular, $(-\nu)$-harmonic functions are locally bounded and Harnack inequalities hold for positive $(-\nu)$-harmonic functions on every domain in X.

If, however, p is not locally bounded, then we do not know if $p^{-1}\rho \in \mathcal{M}_{pb}^{+}(X)$ (in fact, it may happen that $p^{-1}\rho \notin \mathcal{M}_{pb}^{+}(X)$). So we have to construct special examples (see Proposition 14.4). If we have such an example, then Harnack inequalities will hold for positive $(-\nu)$-harmonic functions on a domain U if and only if p is locally bounded on U.

A simple procedure for the construction of potentials with discontinuities which we may control is the following: Let A be a closed subset of X having no interior points and let S be a countable subset of A^c such that A is the set of limit points of S. Let $a_x \in \mathbb{R}_+$, $x \in S$, and $\varepsilon_x > 0$, $x \in S$, such that $\sum_{x \in S} \varepsilon_x < \infty$. Choose closed balls C_x centered at x and contained in A^c, $x \in S$, which are pairwise disjoint and such that the sum of the radii is finite. For every $x \in S$, there exists $b_x > 0$ such that $b_x G_X(\cdot, x) < \varepsilon_x$ on C_x^c. Then

$$p_x := \inf(a_x + \varepsilon_x, b_x G_X(\cdot, x)) \tag{14.1}$$

is a continuous real potential on X which is harmonic on $X \setminus C_x$, hence

$$p_x = G_X^{\rho_x}$$

where the support $\operatorname{supp}(\rho_x)$ of the measure ρ_x is contained in C_x. Clearly, the set $\{p_x = a_x + \varepsilon_x\}$ is a neighborhood of x containing $\operatorname{supp}(\rho_x)$. Define

$$\rho = \sum_{x \in S} \rho_x \quad , \quad p = G_X^\rho = \sum_{x \in S} p_x .$$

Lemma 14.2. *The function p is contained in $\mathcal{P}_\sigma(X)$, the restrictions $p|_{A^c}$ and $p|_A$ are continuous, and, for every $z \in A$,*

$$d_\rho(z) = \limsup_{x \in S, x \to z} a_x .$$

PROOF. Since $p_x < \varepsilon_x$ on C_x^c, the series $\sum_{x\in S} p_x$ is locally uniformly convergent on A^c and locally uniformly convergent on A. Hence $p|_{A^c}$ and $p|_A$ are continuous functions.

Fix $z \in A$ and $\delta > 0$. There exists a subset S' of S such that $S \setminus S'$ is finite and $\sum_{x\in S'} \varepsilon_x < \delta$. Let

$$p' = \sum_{x\in S'} p_x.$$

Since $p - p' = \sum_{x\in S\setminus S'} p_x$ is a continuous real function, we know that

$$d_\rho(z) = \limsup_{y\to z} p(y) - p(z) = \limsup_{y\to z} \; p'(y) - p'(z).$$

Since $p' \le \sum_{x\in S'} \varepsilon_x < \delta$ on $X \setminus \bigcup_{x\in S'} C_x$ and $p' \le a_x + \delta$ on C_x, $p'(x) \ge a_x$ for every $x \in S'$, we have

$$p'(z) \le \delta, \qquad \limsup_{x\in S', x\to z} a_x \le \limsup_{y\to z} p'(y) \le \limsup_{x\in S', x\to z} a_x + \delta$$

and therefore

$$\limsup_{x\in S', x\to z} a_x - \delta \le d_\rho(z) \le \limsup_{x\in S', x\to z} a_x + \delta.$$

Of course, $\limsup_{x\in S', x\to z} a_x = \limsup_{x\in S, x\to z} a_x$. Since $\delta > 0$ was arbitrary, the proof is finished. □

A first application yields the following:

PROPOSITION 14.3. *Let A be a closed subset of X having no interior points and let $\varepsilon > 0$. Then there exists a potential $p = G_X^\rho \in \mathcal{P}_\sigma(X)$ such that the functions $p|_A$, $p|_{A^c}$ are continuous, $p < 1 + \varepsilon$, $d_\rho = 1_A$, $X_{-\rho} = A^c$, and $E_{-\rho} = \emptyset$.*
In particular, for every domain U in X, $(-\rho)$-harmonic functions on U are locally bounded and Harnack inequalities hold for positive $(-\rho)$-harmonic functions on U.

PROOF. We take ε_x, $x \in S$, such that $\sum_{x\in S} \varepsilon_x < \varepsilon$ and define

$$a_x := 1 \qquad (x \in S).$$

Then by Lemma 14.2 the corresponding potential $p = \sum_{x\in S} p_x = G_X^\rho$ is contained in $\mathcal{P}_\sigma(X)$, the restrictions $p|_A$, $p|_{A^c}$ are continuous, and $d_\rho = 1_A$, $A^c \subset X_{-\rho}$. Since $p_x \le 1 + \varepsilon_x$ on C_x and $p_x < \varepsilon_x$ on $X \setminus C_x$, we obtain that

$$p \le 1 + \sum_{x\in S} \varepsilon_x < 1 + \varepsilon.$$

Now fix $z \in A$ and $V \in \mathcal{V}$ containing z. There exists $x \in V \cap S$ such that $C_x \subset V$. Then $G_X^{\rho_x} = p_x < \varepsilon_x$ on the boundary of V, hence $G_V^{\rho_x} > p_x - \varepsilon_x$. Since $G_X^{\rho_x} = 1 + \varepsilon_x$ on $D_x := \operatorname{supp}(\rho_x)$, we obtain that $G_V^{\rho_x} \ge 1$ on D_x and therefore

$$K_V^\rho 1_{D_x} \ge K_V^{\rho_x} 1_{D_x} = G_V^{\rho_x} \ge 1_{D_x}.$$

This implies that $(K_V^\rho)^n 1_{D_x} \geq 1_{D_x}$ for every $n \in \mathbb{N}$ and

$$S_V^{-\rho}1 = \sum_{n=0}^{\infty}(K_V^\rho)^n 1 \geq \sum_{n=0}^{\infty}(K_V^\rho)^n 1_{D_x} \geq \infty \cdot 1_{D_x}.$$

So $z \notin (X_{-\rho} \cup E_{-\rho})$, $A \cap (X_{-\rho} \cup E_{-\rho}) = \emptyset$. Together with $A^c \subset X_{-\rho}$ this shows that $E_{-\rho} = \emptyset$, $X_{-\rho} = A^c$. The proof is finished by an application of Theorem 13.7. □

In the situation of Proposition 14.3 Harnack inequalities hold for positive $(-\rho)$-harmonic functions even on domains U intersecting A, since there are no such functions except the constant zero. But we may just as well produce the opposite:

PROPOSITION 14.4. *Let $A \neq \emptyset$ be a closed subset of X having no interior points and let $\varepsilon > 0$. Then there exists a potential $p = G_X^\rho \in \mathcal{P}_\sigma(X)$ such that the restrictions $p|_A$, $p|_{A^c}$ are continuous, $d_\rho = \infty \cdot 1_A$, $\nu := p^{-1}\rho \in \mathcal{M}_{pb}^+(X)$, $G_X^\nu < 1 + \varepsilon$, $d_\nu = 1_A$, and p is $(-\nu)$-harmonic.*
In particular, $X_{-\nu} = A^c$, $E_{-\nu} = A$, and Harnack inequalities hold for positive $(-\nu)$-harmonic functions on a domain U in X if and only if $U \cap A = \emptyset$.

PROOF. In our general construction we choose ε_x, $x \in S$, such that $\sum_{x\in S}\varepsilon_x < \varepsilon$ and take

$$a_x := 1 + \frac{1}{\operatorname{dist}(x, A)} \qquad (x \in S).$$

Then, by Lemma 14.2, the corresponding potential $p = \sum_{x\in S} p_x = G_X^\rho$ is contained in $\mathcal{P}_\sigma(X)$, the restrictions $p|_A$, $p|_{A^c}$ are continuous, and $d_\rho = \infty \cdot 1_A$.

Moreover,

$$G_X^\nu = \sum_{x\in S} G_X^{p^{-1}\rho_x} \leq \sum_{x\in S}\frac{p_x}{a_x} \leq 1 + \varepsilon,$$

since $p_x = G_X^{\rho_x}$ satisfies $p \geq p_x \geq a_x$ on $\operatorname{supp}(\rho_x)$, $p_x/a_x \leq p_x \leq \varepsilon_x$ on $X \setminus C_x$, and $p_x/a_x \leq (a_x + \varepsilon_x)/a_x \leq 1 + \varepsilon_x$ on C_x. So $\nu \in \mathcal{M}_{pb}^+(X)$ and p is $(-\nu)$-harmonic.

In particular, Harnack inequalities for positive $(-\nu)$-harmonic functions hold on a domain U in X if and only if $U \cap A = \emptyset$.

Now fix $z \in A$ and $\delta > 0$. There exist $\eta > \eta' > 0$ such that $\sum_{x\in S\cap B(z,\eta)}\varepsilon_x < \delta$, $(1 + \eta\varepsilon)^{-1} > 1 - \delta$, and $C_x \cap B(z, \eta') = \emptyset$ for every $x \in B(z, \eta)^c$. Let $V \in \mathcal{V}(B(z, \eta'))$, $z \in V$. Then

$$G_V^\nu = \sum_{x\in V\cap S} G_V^{p^{-1}\rho_x} \leq \sum_{x\in V\cap S}\frac{p_x}{a_x} < 1 + \delta.$$

Taking $x \in V \cap S$ with $C_x \subset V$ we know that $G_X^{\rho_x}(x) = a_x + \varepsilon_x$ and $G_X^{\rho_x} < \varepsilon_x$ on the boundary of V, hence $G_V^{\rho_x}(x) > (a_x + \varepsilon_x) - \varepsilon_x = a_x$ and

$$\|G_V^\nu\|_\infty \geq G_V^{p^{-1}\rho_x}(x) \geq \frac{1}{a_x + \varepsilon} G_V^{\rho_x}(x) \geq \frac{a_x}{a_x + \varepsilon} > 1 - \delta$$

where the last inequality follows from $1/a_x < \operatorname{dist}(x, A) < \eta$. Using Proposition 13.1 we conclude that $1 - \delta \leq d_\nu(z) \leq 1 + \delta$. Thus $d_\nu(z) = 1$. □

In view of Corollary 13.2 and the preceding examples, it is natural to ask if, for every $\nu \in \mathcal{M}^+_{pb}(X)$, at least $\{d_\nu > 1\} \subset X \setminus X_{-\nu}$. However, this is far from being true as the following result shows:

PROPOSITION 14.5. *Let $A \neq \emptyset$ be a closed subset of X having no interior points and let $\alpha > 0$, $\varepsilon > 0$. Then there exists a potential $p = G_X^\rho \in \mathcal{P}_\sigma(X)$ such that the restrictions $p|_A$, $p|_{A^c}$ are continuous, $p \leq \alpha + \varepsilon$, $d_\rho = \alpha 1_A$, and $X_{-\rho} = X$.*
In particular, Harnack inequalities hold for positive $(-\rho)$-harmonic functions on every domain U in X.

PROOF. We choose S and C_x, $x \in S$, as before. Then we fix a natural number $m > \alpha$ and $\varepsilon_x > 0$, $x \in S$, such that $\sum_{x \in S} \varepsilon_x < \frac{\varepsilon}{2m}$. Consider $x \in S$ and define

$$a_x := \frac{\alpha}{m}.$$

Reducing the radius of C_x if necessary we may assume that there is a harmonic function g_x on a neighborhood of C_x such that

$$a_x + \varepsilon_x < g_x < a_x + 2\varepsilon_x \qquad \text{on } C_x.$$

We choose $b_x > 0$ such that $b_x G_X(\cdot, x) < \varepsilon_x$ on C_x^c and define

$$p_{x1} = \begin{cases} \min(g_x, b_x G_X(\cdot, x)) & \text{on } C_x, \\ b_x G_X(\cdot, x)) & \text{on } C_x^c. \end{cases}$$

Then $\{p_{x1} = g_x\}$ is a neighborhood of x contained in the interior of C_x. We fix a closed ball C_x^2 centered at x such that $p_{x1} = g_x$ on C_x^2 and define p_{x2} in the same way as p_{x1} replacing C_x by C_x^2. Continuing in this manner we obtain potentials

$$p_{xj} = G_X^{\rho_{xj}} \qquad (1 \leq j \leq m)$$

such that the supports $D_{xj} := \operatorname{supp}(\rho_{xj})$, $1 \leq j \leq m$, are pairwise disjoint subsets of C_x and

$$p_{x1}(x), \cdots, p_{xm}(x) \in \left[\frac{\alpha}{m} + \varepsilon_x, \frac{\alpha}{m} + 2\varepsilon_x\right].$$

Defining

$$q_x = \sum_{j=1}^m p_{xj}, \qquad p = \sum_{x \in S} q_x$$

and arguing as in the proof of Proposition 14.5 we obtain that $p \leq \alpha + \varepsilon$, the restrictions $p|_A$, $p|_{A^c}$ are continuous, and $p = G_X^\rho$ with $d_\rho = \alpha 1_A$.

For every $1 \leq j \leq m$ let

$$\rho_j = \sum_{x \in S} \rho_{xj}, \quad p_j = G_X^{\rho_j}, \quad A_j = \bigcup_{x \in S} \operatorname{supp}(\rho_{xj}).$$

Then $A_1, \dots, A_m$ are disjoint sets and for every $y \in A_k$, $1 \le k \le m$,

$$p_j(y) \le \begin{cases} \dfrac{\varepsilon}{m}, & 1 \le k < j, \\ \dfrac{\alpha + \varepsilon}{m}, & j \le k. \end{cases}$$

Define

$$K := K^{\rho}, \qquad s_{jn} := \sup_{y \in A_j} K^n 1(y) \quad (1 \le j \le m, \; n = 0, 1, 2, \dots).$$

We intend to show that

$$\lim_{n \to \infty} s_{jn} = 0 \qquad \text{for every } 1 \le j \le m, \tag{14.2}$$

since then $K^r 1 \le 1/2$ for some $r \in \mathbb{N}$, $\sum_{n=0}^{\infty} K^n 1 = (I + K + \cdots + K^{r-1}) \sum_{n=0}^{\infty} (K^r)^n 1$ is bounded, and hence $X_{-\rho} = X$.

Of course, $s_{10} = \cdots = s_{m0} = 1$ and, for every $n = 0, 1, 2, \dots$ and $y \in A_k$,

$$\begin{aligned} K^{n+1} 1(y) &= \sum_{j=1}^{m} K^{\rho_j} K^n 1(y) \le \sum_{j=1}^{m} s_{jn} K^{\rho_j} 1(y) \\ &= \sum_{j=1}^{m} s_{jn} p_j(y) \le \frac{\alpha}{m} \sum_{j=1}^{k} s_{jn} + \frac{\varepsilon}{m} \sum_{j=1}^{m} s_{jn}, \end{aligned}$$

hence

$$s_{k(n+1)} \le \frac{\alpha}{m} \sum_{j=1}^{k} s_{jn} + \frac{\varepsilon}{m} \sum_{j=1}^{m} s_{jn}. \tag{14.3}$$

Defining t_{kn}, $1 \le k \le m$, $n = 0, 1, 2, \dots$, recursively by

$$t_{k0} = 1, \qquad t_{k(n+1)} = \frac{\alpha}{m} \sum_{j=1}^{k} t_{jn} + \frac{\varepsilon}{m} \sum_{j=1}^{m} t_{jn} \tag{14.4}$$

we conclude from (14.3) that

$$s_{kn} \le t_{kn} \tag{14.5}$$

for all choices of k and n. We introduce two bounded kernels M and N on the finite set $\{1, 2, \dots, m\}$:

$$M(k, \cdot) := \frac{\alpha}{m} \sum_{j=1}^{k} \delta_j, \qquad N(k, \cdot) := \frac{1}{m} \sum_{j=1}^{m} \delta_j$$

(where δ_j denotes Dirac mass at j). Then (14.4) implies that

$$t_{kn} = (M + \varepsilon N)^n 1(k) \tag{14.6}$$

for all $1 \le k \le m$ and $n = 0, 1, 2, \dots$. It is easily verified that $\lim_{n \to \infty} M^n 1 = 0$ and hence $\sum_{n=0}^{\infty} M^n 1$ is bounded. By Lemma 8.5, $\sum_{n=0}^{\infty} (M + \varepsilon N)^n 1$ is bounded provided $\varepsilon > 0$ is small enough. Then we obtain by (14.6) and (14.5) that (14.2) holds. An application of Theorem 13.4 finishes the proof. □

For sake of completeness let us note that our procedure for the construction of potentials yields in fact the following (see [Ha98]):

THEOREM 14.6. *Let $\varphi \geq 0$ be an u.s.c. bounded function on X such that, for every $\delta > 0$, the set $\{\varphi \geq \delta\}$ has no interior points. Then, for every $\varepsilon > 0$, there exists a measure $\mu \in \mathcal{M}^+_{pb}(X)$ such that $d_\mu = \varphi$, $G^\mu_X \leq \|\varphi\| + \varepsilon$.*

REMARK 14.7. Given any measure ν on X whose fine support is X it is possible to choose μ absolutely continuous with respect to ν.

15. – The non-symmetric case

Symmetry of G_X has only been used in Lemma 7.1, in Lemma 10.2, and in Lemma 11.1. Let us now see how these results have to be modified if we do not suppose any more that G_X is symmetric.

The proof of Lemma 7.1 would lead to the inequality

$$K^\mu_U G_U(\cdot, y) \leq c_0^{-1}(c_1\|{}^*G^\mu_U\|_\infty + c_2\|G^\mu_U\|_\infty)G_U(\cdot, y) \qquad \text{on } A$$

where

$${}^*G^\mu_U := \int G_U(z, \cdot)\,\mu(dz).$$

Similarly in the proof of Lemma 10.2. Assuming that μ is potentially bounded with respect to ${}^*G$ we thus obtain all previous results as long as μ^- is a Kato measure (or almost a Kato measure). Note that this condition is satisfied if, for every compact set A in X, there is a constant $C > 0$ such that

$$G(y, x) \leq C\, G(x, y) \qquad \text{for all } x, y \in A.$$

Recall that this holds in particular for the differential operators $\mathcal{L}$ we considered.

Our use of the symmetry in Lemma 11.1 is more serious: We would have to replace $K^{1_V\mu^-}_{V'} M^{n-1-j}1(y)$ by ${}^*K({}^*M)^{n-1-j}1(y)$ where

$$\begin{aligned} {}^*K\varphi(y) &:= \int_V G_{V'}(z, y)\varphi(z)\,\mu^-(dz) = {}^*G^{\varphi 1_V\mu^-}_{V'}(y), \\ {}^*M\varphi(y) &:= \int_V {}^{\mu^+}G_{V'}(z, y)\varphi(z)\,\mu^-(dz). \end{aligned}$$

Therefore, boundedness of $\sum_{n=1}^\infty M^n 1$ has to be complemented by boundedness of the sum $\sum_{n=1}^\infty ({}^*M)^n 1$.

If ${}^*G : (x, y) \mapsto G(y, x)$ is the Green function of an (adjoint) Brelot space, then we may express these supplementary conditions more elegantly by saying e.g. in Theorem 11.4 and Theorem 11.6 that, in addition, $\mu \in \mathcal{M}_{pb}(X, {}^*\mathcal{H})$ and

U has to admit a positive μ-*harmonic function which is not identically zero. This follows immediately from our last lemma which implies that

$$^{\mu^+}(G^*_{V'}) = (^{\mu^+}G_{V'})^* :$$

LEMMA 15.1. *Let* $(E, \mathcal{E}, \nu)$ *be a measure space and* $g, g_1, g_2 : E \times E \to [0, \infty]$ $\mathcal{E} \otimes \mathcal{E}$*-measurable such that*

$$g(x, y) = g_1(x, y) + \int g(x, u) g_1(u, y)\, d\nu(u),$$
$$g(y, x) = g_2(x, y) + \int g(u, x) g_2(u, y)\, d\nu(u)$$

for all $x, y \in E$. *Then* $g_2(y, x) = g_1(x, y)$ *for all* $x, y \in E$ *such that* $g(x, y) < \infty$.

PROOF. Fix $x, y \in E$ such that $g(x, y) < \infty$. Using both identities we obtain that

$$\begin{aligned} g(x, y) &= g_2(y, x) + \int g(u, y) g_2(u, x)\, d\nu(u) \\ &= g_2(y, x) + \int g_1(u, y) g_2(u, x)\, d\nu(u) \\ &\quad + \int \int g(u, v) g_1(v, y) g_2(u, x)\, d\nu(v) d\nu(u) \end{aligned}$$

and

$$\begin{aligned} g(x, y) &= g_1(x, y) + \int g(x, u) g_1(u, y)\, d\nu(u) \\ &= g_1(x, y) + \int g_2(u, x) g_1(u, y)\, d\nu(u) \\ &\quad + \int g(v, u) g_2(v, x) g_1(u, y)\, d\nu(v) d\nu(u). \end{aligned}$$

Thus $g_2(y, x) = g_1(x, y)$. □

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