

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 28,
n° 3 (1999), p. 405-412

http://www.numdam.org/item?id=ASNSP_1999_4_28_3_405_0

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On Hulls of Meromorphy and a Class of Stein Manifolds

MIHNEA COLȚOIU

To the memory of my friend G. Lupacchiolu

Abstract. If X is a Stein manifold and K a compact subset one may define the meromorphy hull of K in two different ways: with respect to principal hypersurfaces or to arbitrary hypersurfaces. It is shown that the two definitions agree for every compact subset K of X if and only if the following topological condition on X is satisfied: $\text{Hom}(H_2(X, \mathbb{Z}); \mathbb{Z}) = 0$. It is also shown that this condition is equivalent to: for every hypersurface $h \subset X$ and every relatively compact open subset $D \subset\subset X$ there exists $f \in \mathcal{O}(D)$ such that $h \cap D = \{x \in D \mid f(x) = 0\}$.

Finally, several examples are provided, which show that the topological condition $\text{Hom}(H_2(X, \mathbb{Z}); \mathbb{Z}) = 0$ is sharp.

Mathematics Subject Classification (1991): 32A20, 32E10.

0. – Introduction

Let X be a Stein manifold. Then it is known (see e.g. [10], p. 181) that the second Cousin problem on X can be solved for an arbitrary divisor if and only if $H^2(X; \mathbb{Z}) = 0$. On the other hand one may consider the strong Poincaré problem on X : given a meromorphic function F on X find holomorphic functions f, g on X such that the germs f_z, g_z are relative prime at any point and $F = f/g$. As in the Cousin second problem, the strong Poincaré problem can be solved for every meromorphic function F on X if and only if $H^2(X; \mathbb{Z}) = 0$ (see [13], p. 250). Therefore there is a strong connection between global properties of meromorphic functions on X and the purely topological invariant $H^2(X; \mathbb{Z})$. A weaker condition than $H^2(X, \mathbb{Z}) = 0$ is $H^2(X, \mathbb{Z})$ is of torsion. One may easily see (Proposition 2) that this is equivalent to: every hypersurface $h \subset X$ (closed analytic subset of codimension 1) can be defined globally by one equation i.e. there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{f = 0\}$.

In this paper we study a class of Stein manifolds X which satisfy a weaker condition than $H^2(X, \mathbb{Z})$ is of torsion, namely we consider the topological

condition $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$. This condition is related to the study of hulls of meromorphy of compact subsets $K \subset X$.

When $K \subset X$ is a compact subset one may define in a natural way (see [8], [9]) the following two hulls:

$$\begin{aligned} {}_h\widehat{K} &= \{x \in X \mid \text{every hypersurface passing through } x \text{ intersects } K\} \\ {}_H\widehat{K} &= \{x \in X \mid \text{every principal hypersurface passing through } x \text{ intersects } K\} \end{aligned}$$

Obviously ${}_h\widehat{K} \subset {}_H\widehat{K}$ and it is known [8] that they are compact subsets of X . When $X = \mathbb{C}^n$ then ${}_n\widehat{K} = {}_H\widehat{K}$ and it is called the rational convex hull of K (see [16]).

We show (Theorem 1) that on a Stein manifold X the condition

$$(\alpha) \quad {}_h\widehat{K} = {}_H\widehat{K} \text{ for every compact subset } K \subset X$$

is equivalent to the topological condition

$$(\beta) \quad \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0.$$

It is also equivalent to each of the following conditions:

γ) For every hypersurface $h \subset X$ and every relatively compact open subset $D \subset\subset X$ there exists $f \in \mathcal{O}(D)$ such that $h \cap D = \{x \in D \mid f(x) = 0\}$ (in other words the hypersurfaces on X can be defined, set-theoretically, by one equation on compact subsets).

δ) For every $\xi \in H^2(X; \mathbb{Z})$ and every relatively compact open subset $D \subset\subset X$ there exists a positive integer m , depending on ξ and D , such that $m\xi|_D = 0$ ($H^2(X; \mathbb{Z})$ is of torsion on compact subsets).

Finally we give examples showing that the statement of Theorem 1 is sharp: namely, there exists Stein manifolds X satisfying the topological condition $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ but $H^2(X; \mathbb{Z})$ is not of torsion. In particular, on these manifolds there exists hypersurfaces which cannot be defined globally by one equation but they can be defined on each compact subset of X by one equation.

ACKNOWLEDGEMENT. I wish to thank G. Lupaccioli who raised me in autumn 1995, when I was visiting the University of Rome "La Sapienza", the question whether there exist Stein manifolds X and compact subsets $K \subset X$ such that ${}_h\widehat{K} \neq {}_H\widehat{K}$. This was the starting point in writing this paper. I want also to thank my friend and colleague G. Chiriacescu for helpful discussions on the algebraic results needed in the given examples, in particular for the references [1] and [12].

1. – Proof of the results

Let X be a Stein manifold of dimension n . A closed analytic subset $h \subset X$ of pure dimension $(n - 1)$ is called a hypersurface. h is called a principal hypersurface iff there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{f = 0\}$.

For a compact subset $K \subset X$ we consider the following two hulls:

$$\begin{aligned}
 {}_h\widehat{K} &= \{x \in X \mid \text{every hypersurface passing through } x \text{ intersects } K\} \\
 {}_H\widehat{K} &= \{x \in X \mid \text{every principal hypersurface passing through } x \text{ intersects } K\}
 \end{aligned}$$

They are compact subsets of X [8] (p. 50 and p. 52).

Let us recall also the following result [5]

LEMMA 1. *Let X be a Stein manifold, $Y \subset X$ a closed complex submanifold and denote by $N = N_{Y|X}$ the normal bundle of Y in X . Then there exists an open neighborhood U of the null section of N biholomorphic to an open neighborhood U_1 of Y in X by $\varphi : U \xrightarrow{\sim} U_1$ such that the image of the null section by φ is Y .*

Now we can prove:

PROPOSITION 1. *Let X be a Stein manifold and assume that ${}_h\widehat{K} = {}_H\widehat{K}$ for every compact subset $K \subset X$. Then the Kronecker product $H^2(X; \mathbb{Z}) \times H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$ is the null map.*

PROOF. Since X is Stein it follows that $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z})$. On the other hand every line bundle L on X has a section s whose zero set is smooth (see [9], p. 883) and this section defines a positive divisor which corresponds to L . Therefore it is enough to show that the Kronecker product $\langle c(h), \bar{\alpha} \rangle = 0$, where h is a smooth and connected hypersurface, $c(h) \in H^2(X; \mathbb{Z})$ denotes its Chern class and $\bar{\alpha} \in H_2(X; \mathbb{Z})$. In fact $\langle c(h), \bar{\alpha} \rangle$ is the intersection number $\langle h, \bar{\alpha} \rangle$ of h and $\bar{\alpha}$ and can be defined choosing a smooth 2-cycle α (representing $\bar{\alpha}$) intersecting h transversally (see [7], p. 61).

By reductio ad absurdum assume that $z = \langle h, \bar{\alpha} \rangle \neq 0$. Let $N = N_{h|X}$ be the normal bundle of h in X . By Lemma 1 there exists an open neighborhood U of the null section of N biholomorphic to an open neighborhood U_1 of h in X by $\varphi : U \xrightarrow{\sim} U_1$, such that the image of the null section by φ is h . We choose a hermitian metric on N such that $\{w \in N \mid \|w\| \leq 1\} \subset U$ and define $V = \varphi(\{w \in N \mid \|w\| \leq 1\})$. Then V is a closed neighborhood of h and by Thom's isomorphism $H_2(X, X \setminus \overset{\circ}{V}; \mathbb{Z}) \cong H_0(h; \mathbb{Z}) \cong \mathbb{Z}$. In fact, if $x_0 \in h$ is any point and $B_{x_0} = B(x_0, 1) \subset V$ denotes the corresponding closed ball with center x_0 and contained in the fiber then B_{x_0} can be considered as a 2-simplex s of X with boundary contained in $X \setminus \overset{\circ}{V}$, so it defines an element $\bar{s} \in H_2(X, X \setminus \overset{\circ}{V}; \mathbb{Z}) \cong \mathbb{Z}$ and \bar{s} is a generator. In what follows we fix some point $x_0 \in h$.

Consider the exact sequence:

$$H_2(X \setminus \overset{\circ}{V}; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z}) \xrightarrow{i} H_2(X, X \setminus \overset{\circ}{V}; \mathbb{Z}) \cong \mathbb{Z}$$

and let us remark that $i(\bar{\alpha}) \neq 0$ (here i denotes the natural homomorphism at homology induced by the map $(X, \emptyset) \rightarrow (X, X \setminus \overset{\circ}{V})$ which is the identity on X).

Otherwise $\alpha = u + v$ with $u =$ boundary in $X, v =$ cycle in $X \setminus \overset{\circ}{V}$ and it would follow that $z = \langle h, \bar{\alpha} \rangle = 0$ since v does not meet h . Therefore $i(\bar{\alpha}) = \lambda \bar{s}$ with $\lambda \in \mathbb{Z} \setminus \{0\}$. We see that

$$(*) \quad \alpha - \lambda s = \alpha_1 + b$$

with $\alpha_1 =$ chain in $X \setminus \overset{\circ}{V}$ and $b =$ boundary in X (and in fact $\lambda = z$).

Let us define the compact set $K = \text{supp}(\alpha_1) \cup \partial B_{x_0}$. By our assumption ${}_h \widehat{K} = {}_H \widehat{K}$. But $x_0 \in h$ and $h \cap K = \emptyset$ (empty set), so there exists a principal hypersurface $H = \{f = 0\}, f \in \mathcal{O}(X)$ with $x_0 \in H$ and $H \cap K = \emptyset$. We have $\langle H, \alpha - \lambda s \rangle = \langle H, \alpha_1 + b \rangle$ where \langle, \rangle denotes the intersection number. On the other hand $\langle H, \alpha \rangle = 0$ since H is principal, $\langle H, b \rangle = 0$ since b is a boundary, $\langle H, \alpha_1 \rangle = 0$ since $H \cap \text{supp}(\alpha_1) = \emptyset$. But $\langle H, s \rangle \neq 0$ because $H \cap \partial B_{x_0} = \emptyset, H(x_0) = 0$ and on B_{x_0} we have a complex structure (see [7] p. 63, [9] Lemme 5.3). We get $\lambda = 0$ which is a contradiction. Therefore $\langle h, \bar{\alpha} \rangle = 0$ and the proof of Proposition 1 is complete.

THEOREM 1. *Let X be a Stein manifold. Then the following conditions are equivalent:*

- 1) ${}_h \widehat{K} = {}_H \widehat{K}$ for every compact subset K of X
- 2) $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$
- 3) For every $\xi \in H^2(X; \mathbb{Z})$ and every relatively compact open subset $D \subset\subset X$ there exists a positive integer $m = m(D, \xi)$ such that $m\xi|_D = 0$.
- 4) For every hypersurface $h \subset X$ and every relatively compact open subset $D \subset\subset X$ there exists a holomorphic function $f \in \mathcal{O}(D)$ such that one has set-theoretically $h \cap D = \{f = 0\}$.

PROOF. 1) \implies 2)

It is known ([6], p. 132) that the natural morphism (induced by the Kronecker product) $H^2(X; \mathbb{Z}) \rightarrow \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z})$ is surjective. Therefore every $u \in \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z})$ is of the form $u(\bar{\alpha}) = \langle \xi, \bar{\alpha} \rangle$ for some $\xi \in H^2(X; \mathbb{Z})$. By Proposition 1 it follows that $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$.

2) \implies 3)

Let $\xi \in H^2(X; \mathbb{Z}), D \subset\subset X$ a relatively compact open subset, and we have to find a positive integer m such that $m\xi|_D = 0$. We may assume that the boundary of D is smooth, therefore the homology groups of D are finitely generated. We define $\xi_1 = \xi|_D \in H^2(D; \mathbb{Z})$. By our assumption $\langle \xi_1, \bar{\alpha} \rangle = 0$ for every $\bar{\alpha} \in H_2(D; \mathbb{Z})$. Since the homology groups of D are finitely generated it follows from ([6], p. 136) that ξ_1 is a torsion element of $H^2(D; \mathbb{Z})$.

3) \implies 4)

Let $h \subset X$ be a hypersurface and $D \subset\subset X$ a relatively compact open subset. We may assume that D is Stein, therefore $H^2(D; \mathbb{Z}) \cong H^1(D, \mathcal{O}^*) \cdot h$ defines a line bundle $L \in H^1(X, \mathcal{O}^*)$ which has a canonical section $s \in \Gamma(X, L)$ with $h = \{s = 0\}$. Define $\xi = c(L) \in H^2(X; \mathbb{Z})$ the Chern class of L . By our assumption there exists a positive integer m with $m\xi = 0$ on D . Therefore $L^m|_D$ is trivial. Also $s^m \in \Gamma(X, L^m)$ and $f = s^m|_D$ is a holomorphic function on D with $h \cap D = \{f = 0\}$.

4) \implies 1)

Let $K \subset X$ be a compact subset and $x_0 \in X$ such that there exists a hypersurface $h \subset X$ with $x_0 \in h$ and $h \cap K = \emptyset$. We have to find a principal hypersurface $H \subset X$ with $x_0 \in H$ and $H \cap K = \emptyset$. Let $D \subset\subset X$ be a Runge domain with $K \cup \{x_0\} \subset D$. By our assumption there is a holomorphic function $H_1 \in \mathcal{O}(D)$ with $\{H_1 = 0\} = h \cap D$ (set-theoretically).

Let $\varepsilon_0 = \inf\{|H_1(x)| \mid x \in K\} > 0$. Since $D \subset X$ is Runge we can approximate H_1 on $K \cup \{x_0\} \subset D$ by $\tilde{H}_1 \in \mathcal{O}(X)$ such that $|\tilde{H}_1(x) - H_1(x)| \leq \varepsilon_0/4$ if $x \in K$ and $|\tilde{H}_1(x_0)| = |\tilde{H}_1(x_0) - H_1(x_0)| \leq \varepsilon_0/4$. Define $H(x) = \tilde{H}_1(x) - \tilde{H}_1(x_0)$. Then obviously $H(x_0) = 0$ and $H(x) \neq 0$ if $x \in K$.

Thus our theorem is completely proved.

COROLLARY 1. *Let X be a Stein manifold such that $H_1(X; \mathbb{Z}), H_2(X; \mathbb{Z})$ are finitely generated (e.g. X is affine algebraic).*

Then the following conditions are equivalent:

- 1) ${}_h\hat{K} = {}_H\hat{K}$ for every compact subset $K \subset X$.
- 2) $H^2(X; \mathbb{Z})$ is of torsion
- 3) For every hypersurface $h \subset X$ there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{f = 0\}$.

PROOF. Since $H_1(X; \mathbb{Z}), H_2(X; \mathbb{Z})$ are finitely generated it follows from ([6], p. 136) that we have a (non-canonical) isomorphism:

$$(*) \quad H^2(X; \mathbb{Z}) \cong \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) \oplus T_1$$

where T_1 denotes the torsion part of $H_1(X; \mathbb{Z})$.

Now the corollary follows immediately from (*) and Theorem 1.

PROPOSITION 2. *Let X be a connected Stein manifold. Then the following two conditions are equivalent:*

- 1) $H^2(X; \mathbb{Z})$ is of torsion
- 2) For every hypersurface $h \subset X$ there exists $f \in \mathcal{O}(X)$ such that one has set-theoretically $h = \{f = 0\}$.

We first show that 1) \implies 2).

Let $h \subset X$ be a hypersurface and let $L \in H^1(X, \mathcal{O}^*)$ be the corresponding line bundle, therefore there is a canonical section $s \in \Gamma(X, L)$ with $h = \{s = 0\}$. Since X is Stein $H^2(X; \mathbb{Z}) \cong H^1(X, \mathcal{O}^*)$, hence there is a positive integer m

such that L^m is trivial. s^m is a section in $\Gamma(X, L^m)$ and if we set $f = s^m$ then f is a holomorphic function on X such that $h = \{f = 0\}$.

We prove now that 2) \implies 1).

We recall the following result (see [3]): If L is a line bundle over a connected Stein manifold X then there is a section $s \in \Gamma(X, L)$ such that $\{s = 0\}$ is irreducible (in fact the set of sections $s \in \Gamma(X, L)$ with $\{s = 0\}$ irreducible is dense in $\Gamma(X, L)$).

Let now $\xi \in H^2(X; \mathbb{Z}) \cong H^1(X, \mathcal{O}^*)$ and let $L \in H^1(X, \mathcal{O}^*)$ be the corresponding line bundle. We choose $s \in \Gamma(X, L)$ such that $h = \{s = 0\}$ is irreducible. If we consider (h) as a divisor there is a positive integer n such that $L = n(h)$ (n is the order of s along h , which is well defined because h is irreducible). On the hand there exists $f \in \mathcal{O}(X)$ with $h = \{f = 0\}$ (set-theoretically). If m is the order of f along h then $m(h) = 0$. Therefore L^m is the trivial line bundle and consequently $m\xi = 0$. So we have showed that $H^2(X; \mathbb{Z})$ is of torsion, and the proof of Proposition 2 is complete.

REMARK 1. There is a surjective homomorphism group (see [6], p. 132)

$$H^2(X; \mathbb{Z}) \rightarrow \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z})$$

from which it follows that:

$$H^2(X; \mathbb{Z}) \text{ is of torsion} \implies \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0.$$

We shall give examples of Stein manifolds X such that $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ but $H^2(X; \mathbb{Z})$ contains nontorsion elements. Of course, for such Stein manifolds X , every $\xi \in H^2(X; \mathbb{Z})$ is of torsion on compact subsets, i.e. for every $D \subset\subset X$ there is a positive integer $m = m(D, \xi)$ with $m\xi = 0$ on D . But it is possible that $m \rightarrow \infty$ as it will be shown by our next examples.

EXAMPLE 1. In [11] it is given an example of a Stein domain $X \subset \mathbb{C}^2$ with $H_1(X; \mathbb{Z}) = \mathbb{Q}$ (rational numbers) and $H_2(X; \mathbb{Z}) = 0$. Let us study $H^2X; \mathbb{Z}$. There is an exact sequence ([4], p. 153):

$$0 \rightarrow \text{Ext}(H_1(X; \mathbb{Z}); \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z}) \rightarrow \text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) \rightarrow 0$$

Therefore we get $H^2(X; \mathbb{Z}) = \text{Ext}(\mathbb{Q}; \mathbb{Z})$. Clearly $\text{Ext}(\mathbb{Q}; \mathbb{Z})$ is a \mathbb{Q} vector space, so every $\xi \in \text{Ext}(\mathbb{Q}; \mathbb{Z}) \setminus \{0\}$ is a nontorsion element. We shall prove that $\dim_{\mathbb{Q}} \text{Ext}(\mathbb{Q}; \mathbb{Z}) = \infty$.

If p is a prime we denote by P the additive group of those rational numbers whose denominators are powers of p and by $\mathbb{Z}(p^\infty)$ the quotient \mathbb{Z}/P . There is a group isomorphism (see [12], p. 6): $\mathbb{Q}/\mathbb{Z} \cong \bigoplus \mathbb{Z}(p^\infty)$. It follows:

$$\text{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\mathbb{Q}; \bigoplus \mathbb{Z}(p^\infty)) = \bigoplus \text{Hom}(\mathbb{Q}; \mathbb{Z}(p^\infty)).$$

Now for every prime p $\text{Hom}(\mathbb{Q}; \mathbb{Z}(p^\infty)) \neq 0$ since we have a surjective homomorphism $\mathbb{Q} \rightarrow \mathbb{Z}(p^\infty)$ obtained from the composition of two surjective homomorphisms $\mathbb{Q} \rightarrow \bigoplus \mathbb{Z}(p^\infty) \xrightarrow{\text{pr}} \mathbb{Z}(p^\infty)$. It follows that $\dim_{\mathbb{Q}} \text{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) = \infty$.

From the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ applying $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}; \cdot)$ we get the exact sequence of \mathbb{Q} vector spaces:

$$0 \rightarrow \text{Hom}(\mathbb{Q}; \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Q}; \mathbb{Q}) \rightarrow \text{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Ext}(\mathbb{Q}; \mathbb{Z}) \rightarrow \text{Ext}(\mathbb{Q}; \mathbb{Q}) = 0$$

Since $\text{Hom}(\mathbb{Q}; \mathbb{Z}) = 0$, $\text{Hom}(\mathbb{Q}; \mathbb{Q})$ has dimension 1 as a \mathbb{Q} vector space and $\dim_{\mathbb{Q}} \text{Hom}(\mathbb{Q}; \mathbb{Q}/\mathbb{Z}) = \infty$ it follows that $\dim_{\mathbb{Q}} \text{Ext}(\mathbb{Q}; \mathbb{Z}) = \infty$.

EXAMPLE 2. For each integer $m \geq 2$ consider the map $\varphi : \bar{D} \rightarrow \mathbb{R}^4(\bar{D})$ is the closed unit disc, i.e. $\bar{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ given by $\varphi(z) = (z^m, (1 - |z|)z)$ where \mathbb{R}^4 is identified with \mathbb{C}^2 in the usual way. Then $\varphi|_D$ is injective and $\varphi|_{\partial D}$ has degree m .

Since $\partial D = S^1$, if we set $K_m = \varphi(\bar{D})$, then K_m is obtained from S^1 by adding a two cell by a map of degree m (see [6], p. 83). It follows from ([6], p. 89) that $H_1(K_m; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$ and $H_2(K_m; \mathbb{Z}) = 0$.

Consider in \mathbb{R}^4 an infinite real line d , and on d we add the compacts K_m such that $K_m \cap d = a$ point P_m , $K_m \cap K_n = \emptyset$ if $m \neq n$ and $\{K_m\}$ is locally finite. Thus we get a locally finite cellular complex $M \subset \mathbb{R}^4$. One may easily see that $H_1(M; \mathbb{Z}) = \bigoplus_{m \geq 2} \mathbb{Z}/m\mathbb{Z}$ and $H_2(M; \mathbb{Z}) = 0$.

From the exact sequence:

$$0 \rightarrow \text{Ext}(H_1(M; \mathbb{Z}); \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z}) \rightarrow \text{Hom}(H_2(M; \mathbb{Z}); \mathbb{Z}) \rightarrow 0$$

we get $H^2(M; \mathbb{Z}) = \text{Ext}(\bigoplus_{m \geq 2} \mathbb{Z}/m\mathbb{Z}; \mathbb{Z})$. From ([1], § 5, Prop. 7, p. 89) $\text{Ext}(\bigoplus G_m; \mathbb{Z}) \cong \prod \text{Ext}(G_m; \mathbb{Z})$ and by ([4], p. 148) $\text{Ext}(\mathbb{Z}/m\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$. We deduce that $H^2(M; \mathbb{Z}) = \prod_{m \geq 2} \mathbb{Z}/m\mathbb{Z}$. We take now an open neighborhood U of M in \mathbb{R}^4 such that M is a deformation retract of U . Considering the inclusion $\mathbb{R}^4 \subset \mathbb{C}^4$ given by $y_1 = \dots = y_4 = 0$ where $z_k = x_k + iy_k$ are the complex coordinates on \mathbb{C}^4 , there exists by [14] a Stein domain $X \subset \mathbb{C}^4$ such that $X \cap \mathbb{R}^4 = U$ and U is a deformation retract of X . We have $H^2(X; \mathbb{Z}) = \prod_{m \geq 2} \mathbb{Z}/m\mathbb{Z}$ and $H_2(X; \mathbb{Z}) = 0$. The element $(\hat{1}, \hat{1}, \dots, \hat{1}, \dots)$ (taking $\hat{1}$ on all factors of the infinite product) is a nontorsion element of $H^2(X; \mathbb{Z})$.

EXAMPLE 3. In examples 1) and 2) we have $H_2(X; \mathbb{Z}) = 0$. But it is possible to find X with $H_2(X; \mathbb{Z}) \neq 0$, $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ and $H^2(X; \mathbb{Z})$ has nontorsion elements. To see this we replace in example 1) X by $X_1 = X \times \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Then by Künneth formula $H_2(X_1; \mathbb{Z}) = \mathbb{Q}$ and $H_1(X_1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Q}$. It follows that $\text{Hom}(H_2(X_1; \mathbb{Z}); \mathbb{Z}) = 0$ and $H^2(X_1; \mathbb{Z}) = \text{Ext}(H_1(X_1; \mathbb{Z}); \mathbb{Z}) = \text{Ext}(\mathbb{Z} \oplus \mathbb{Q}; \mathbb{Z}) = \text{Ext}(\mathbb{Q}; \mathbb{Z}) \neq 0$.

Similarly we may replace X in example 2) by its product with $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

EXAMPLE 4. By [15] it is possible to construct, for every countable torsion free abelian group G , a compact connected subset $K \subset \mathbb{R}^3$ (in fact a curve)

such that $H_1(\mathbb{R}^3 \setminus K; \mathbb{Z}) \cong G$ and $H_2(\mathbb{R}^3 \setminus K; \mathbb{Z}) = 0$. Taking G such that $\text{Ext}(G; \mathbb{Z})$ contains nontorsion elements and $X \subset \mathbb{C}^3$ a Stein open subset such that $X \cap \mathbb{R}^3 = \mathbb{R}^3 \setminus K$ and $\mathbb{R}^3 \setminus K$ is a deformation retract of X , one gets as above examples of Stein manifolds X with $\text{Hom}(H_2(X; \mathbb{Z}); \mathbb{Z}) = 0$ but $H^2(X; \mathbb{Z})$ has nontorsion elements.

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