

ANNALI DELLA  
SCUOLA NORMALE SUPERIORE DI PISA  
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4<sup>e</sup> série, tome 28, n° 2 (1999), p. 287-322*

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## Multiplier Algebras, Banach Bundles, and One-Parameter Semigroups

WOJCIECH CHOJNACKI

**Abstract.** Several results are proved concerning representations of multiplier algebras that arise as extensions of representations of underlying Banach algebras. These results are then used to rederive Kiszyński's generalisation of the Hille-Yosida theorem and to establish two generalisations of the Trotter-Kato theorem, one of which, involving Banach bundles, is abstract and the other is classical in character.

**Mathematics Subject Classification (1991):** 46H15 (primary), 43A10, 43A20, 47A10, 47D06 (secondary).

### 1. – Introduction

Throughout the paper vector spaces are assumed to be over a fixed field  $\mathbb{F}$  of scalars, which is either the field  $\mathbb{R}$  of real numbers or the field  $\mathbb{C}$  of complex numbers. The results will be valid for each particular choice of the ground field  $\mathbb{F}$ .

Let  $\mathbb{R}_+$  be the set of all non-negative numbers and let  $\mathbb{R}_+^\bullet$  be the set of all positive numbers. For each  $\lambda \in \mathbb{R}$ , denote by  $\epsilon_\lambda$  the function

$$\epsilon_\lambda(t) = e^{\lambda t} \quad (t \in \mathbb{R}_+).$$

Given  $w \in \mathbb{R}_+$ , let  $L_w^1(\mathbb{R}_+)$  be the space of equivalence classes (under equality almost everywhere) of Lebesgue measurable functions  $f$  on  $\mathbb{R}_+$  for which  $|f|\epsilon_w$  is Lebesgue integrable. With addition and scalar multiplication derived from the pointwise addition and scalar multiplication of functions, and with the norm given by

$$\|f\|_{1,w} = \int_{\mathbb{R}_+} |f(t)|e^{wt} dt$$

(where the same symbol  $f$  is used to denote both a function and its equivalence class),  $L_w^1(\mathbb{R}_+)$  is a Banach space. With the convolution

$$(f * g)(t) = \int_0^t f(t-s)g(s) ds \quad (\text{almost all } t \in \mathbb{R}_+)$$

as product, it becomes a Banach algebra; here we adopt the standard convention according to which “almost all” with no further qualification is synonymous with “almost all with respect to Lebesgue measure”.

Let  $\mathcal{A}$  be a Banach algebra. An indexed family  $\{r_\lambda\}_{\lambda \in U}$  of elements of  $\mathcal{A}$ , where  $U$  is a subset of  $\mathbb{F}$ , is called a *pseudo-resolvent* in  $\mathcal{A}$  if the following *Hilbert equation* is satisfied:

$$(1.1) \quad r_\lambda - r_\mu = (\mu - \lambda)r_\lambda r_\mu \quad (\lambda, \mu \in U).$$

Given  $w \in \mathbb{R}_+$  and a pseudo-resolvent  $r = \{r_\lambda\}_{\lambda \in (w, +\infty)}$  in  $\mathcal{A}$ , let

$$c_r = \sup\{(\lambda - w)^n \|r_\lambda^n\| \mid n \in \mathbb{N}, \lambda \in (w, +\infty)\}.$$

It is directly verified that, for any  $w \in \mathbb{R}_+$ ,  $\rho_w = \{\epsilon_{-\lambda}\}_{\lambda \in (w, +\infty)}$  is pseudo-resolvent in  $L_w^1(\mathbb{R}_+)$  such that  $c_{\rho_w} = 1$ . The pseudo-resolvents  $\rho_w$  play a fundamental role in the following result of J. Kiszyński [27] (see also [3], [5]):

**THEOREM 1.1 (Kiszyński).** *Let  $\mathbf{A}$  be a Banach algebra, let  $w \in \mathbb{R}_+$ , let  $r = \{r_\lambda\}_{\lambda \in (w, +\infty)}$  be a pseudo-resolvent in  $\mathbf{A}$ . Then the following conditions are equivalent:*

- (i)  $c_r < +\infty$ ;
- (ii) *there exists a continuous homomorphism  $\phi: L_w^1(\mathbb{R}_+) \rightarrow \mathbf{A}$  such that  $\phi(\epsilon_{-\lambda}) = r_\lambda$  for each  $\lambda \in (w, +\infty)$ .*

*Furthermore, if a continuous homomorphism  $\phi: L_w^1(\mathbb{R}_+) \rightarrow \mathbf{A}$  satisfying  $\phi(\epsilon_{-\lambda}) = r_\lambda$  for each  $\lambda \in (w, +\infty)$  exists, then it is unique and  $\|\phi\| = c_r$ .*

In [27] the above theorem is used to establish a generalisation of the Hille-Yosida theorem on the generation of one-parameter semigroups of operators and a generalisation of the Trotter-Kato theorem on the convergence of sequences of one-parameter semigroups. Both these generalisations operate with pseudo-resolvents rather than with resolvents of closed, densely-defined operators. Pseudo-resolvents give rise, via Theorem 1.1, to representations of  $L_w^1(\mathbb{R}_+)$ , and these in turn engender semigroups. In the process an intimate connection is revealed between semigroup theory and the representation theory of Banach algebras.

The main purpose of the present paper is to investigate further this connection. In [27] a one-parameter semigroup is developed from a continuous representation of  $L_w^1(\mathbb{R}_+)$  by differentiating, in the sense of the strong operator topology, a certain integrated semigroup formed with the use of the representation, this semigroup being restricted to the space where the representation is non-degenerate. Here we take a different approach: first, by invoking a general

principle, the non-degenerate part of a continuous representation of  $L_w^1(\mathbb{R}_+)$  is extended to a continuous representation of the multiplier algebra of  $L_w^1(\mathbb{R}_+)$ ; next, using the fact that the multiplier algebra of  $L_w^1(\mathbb{R}_+)$  is identifiable with a convolution algebra of measures, an appropriate semigroup is obtained as part of a canonical form for the extended representation obtained in the first step. The latter technique of semigroup generation is analogous to a familiar method of evolving a unitary representation of a locally compact group from a non-degenerate Hilbert space  $*$ -representation of the corresponding group algebra (cf. [37, Chap. 10.1]).

Use of representations of multiplier algebras arising as extensions of representations of underlying Banach algebras, advocated in this paper, does not lead to any new result in the case of the generalised Hille-Yosida theorem. The situation is different, however, as far as the generalised Trotter-Kato theorem is concerned. Consideration of extended Banach algebra representations acting on special spaces, namely on spaces of cross-sections of some Banach bundles, permits establishing a fairly general, abstract version of the Trotter-Kato theorem. Applying this result to  $L_w^1(\mathbb{R}_+)$  and certain Banach bundles leads immediately to a generalisation of the Trotter-Kato theorem, not involving Banach bundles explicitly, of which the classical statement of the Trotter-Kato theorem and that due to Kisyński are special cases.

The approach taken here emphasises the presence of a bounded metric approximate identity in the algebra  $L_w^1(\mathbb{R}_+)$ . The abstract results underpinning the use of representations of  $L_w^1(\mathbb{R}_+)$  apply to any Banach algebra possessing a bounded metric approximate identity. One implication of this is that once an analogue of Theorem 1.1 is established in which  $L_w^1(\mathbb{R}_+)$  is replaced by another Banach algebra with a bounded metric approximate identity and  $c_r$  is replaced by another entity characterising pseudo-resolvents, appropriate versions of the Hille-Yosida and Trotter-Kato theorems can readily be obtained. Therefore not only can our treatment serve to clarify the role of the algebra  $L_w^1(\mathbb{R}_+)$ , but it can also play some role in developing new results.

The rest of the paper is organised as follows. Section 2 contains prerequisites concerning multiplier algebras and representations, including a crucial theorem on the extension of representations. Section 3 focuses on  $L_w^1(\mathbb{R}_+)$ , describing a characterisation of the multiplier algebra of  $L_w^1(\mathbb{R}_+)$  along with a canonical form of the representations of this algebra derived from representations of  $L_w^1(\mathbb{R}_+)$ . In Section 4 the material of Sections 2 and 3 is applied to rederive Kisyński's generalisation of the Hille-Yosida theorem. Section 5 discusses Banach bundles and establishes an abstract version of the Trotter-Kato theorem. Finally, in Section 6 the main result of the previous section is used to establish a theorem that simultaneously generalises the classical version and Kisyński's version of the Trotter-Kato theorem.

## 2. – Multiplier algebras and representations

In this section, we review certain notions and results from the representation theory of Banach algebras. After introducing preliminary material concerning multiplier algebras and representations, we present a fundamental result concerning the extension of representations of Banach algebras to representations of the corresponding multiplier algebras.

### 2.1. – Multiplier algebras

Given vector spaces  $E$  and  $F$ , let  $\mathcal{L}(E, F)$  be the space of all linear operators from  $E$  into  $F$ . For topological vector spaces  $E$  and  $F$ , denote by  $\mathcal{L}(E, F)$  the space of all continuous linear operators from  $E$  into  $F$ . Occasionally, given a vector space  $E$  and a complete locally convex topological vector space  $F$ , we shall regard  $\mathcal{L}(E, F)$  as being endowed with the strong operator topology, under which  $\mathcal{L}(E, F)$  is a complete locally convex topological vector space. For normed spaces  $E$  and  $F$ ,  $\mathcal{L}(E, F)$  will always be viewed as a normed space, equipped with the norm

$$\|S\| = \sup_{\|x\| \leq 1} \|Sx\| \quad (S \in \mathcal{L}(E, F)).$$

When  $E$  is a normed space and  $F$  is a Banach space,  $\mathcal{L}(E, F)$  is a Banach space. If  $E = F$ , we abbreviate, as is customary,  $\mathcal{L}(E, E)$  to  $\mathcal{L}(E)$ , and  $\mathcal{L}(E, E)$  to  $\mathcal{L}(E)$  (in the latter case we tacitly assume that  $E$  is a topological vector space).  $\mathcal{L}(E)$  and  $\mathcal{L}(E)$  are unital algebras, the identity operator  $\text{id}_E$  on  $E$  being the common identity of both algebras. When  $E$  is a Banach space,  $\mathcal{L}(E)$  is a Banach algebra.

Let  $\mathbf{A}$  be a Banach algebra. For each  $a \in \mathbf{A}$ , let  $L_a$  and  $R_a$  be the linear maps in  $\mathcal{L}(\mathbf{A})$  defined by

$$L_a b = ab \quad \text{and} \quad R_a b = ba \quad (b \in \mathbf{A}).$$

The map  $\mathbf{A} \ni a \mapsto L_a \in \mathcal{L}(\mathbf{A})$  is called the *left regular representation* of  $\mathbf{A}$  and the map  $\mathbf{A} \ni a \mapsto R_a \in \mathcal{L}(\mathbf{A})$  is called the *right regular representation* of  $\mathbf{A}$ . Denote by  $L_{\mathbf{A}}$  and  $R_{\mathbf{A}}$  the respective images of these maps. An operator  $S \in \mathcal{L}(\mathbf{A})$  is said to be a *left (right) multiplier*, or *left (right) centraliser*, of  $\mathbf{A}$  if, for all  $a, b \in \mathbf{A}$ ,

$$(Sa)b = S(ab) \quad (a(Sb) = S(ab)).$$

Equivalently,  $S \in \mathcal{L}(\mathbf{A})$  is a left (right) multiplier of  $\mathbf{A}$  if, for all  $a \in \mathbf{A}$ ,

$$(2.1) \quad SL_a = L_{Sa} \quad (SR_a = R_{Sa}).$$

Denote by  $\text{Mul}_l(\mathbf{A})$  ( $\text{Mul}_r(\mathbf{A})$ ) the collection of all left (right) multipliers of  $\mathbf{A}$ . It is easily verified that  $\text{Mul}_l(\mathbf{A})$  and  $\text{Mul}_r(\mathbf{A})$  are unital Banach subalgebras of

$\mathcal{L}(\mathbf{A})$ . These are called the *left multiplier algebra* of  $\mathbf{A}$  and the *right multiplier algebra* of  $\mathbf{A}$ , respectively. From (2.1) it is clear that  $L_{\mathbf{A}}$  ( $R_{\mathbf{A}}$ ) is a left ideal of  $\text{Mul}_l(\mathbf{A})$  ( $\text{Mul}_r(\mathbf{A})$ ). If  $\mathbf{A}$  is commutative, then  $\text{Mul}_l(\mathbf{A}) = \text{Mul}_r(\mathbf{A})$ ; accordingly, we can abbreviate  $\text{Mul}_l(\mathbf{A})$  and  $\text{Mul}_r(\mathbf{A})$  to  $\text{Mul}(\mathbf{A})$ , and call the latter the *multiplier algebra* of  $\mathbf{A}$ .

From this point on we shall consider “left-handed” objects only, leaving the reader to make the minor modifications necessary in the alternative case.

A net  $\{e_\alpha\}_{\alpha \in A}$  in  $\mathbf{A}$ , where  $A$  is a directed set, is a *left (two-sided) approximate identity* for  $\mathbf{A}$  if  $\lim_{\alpha \in A} e_\alpha a = a$  ( $\lim_{\alpha \in A} e_\alpha a = \lim_{\alpha \in A} a e_\alpha = a$ ) for each  $a \in \mathbf{A}$ . A left approximate identity is *bounded* if  $\sup_{\alpha \in A} \|e_\alpha\| < +\infty$ , and is *metric* if  $\lim_{\alpha \in A} \|e_\alpha\| = 1$ . Given a bounded left approximate identity  $\{e_\alpha\}_{\alpha \in A}$ , let

$$K_0 = \liminf_{\alpha \in A} \|e_\alpha\| \quad \text{and} \quad K = \sup_{\alpha \in A} \|e_\alpha\|.$$

We have the following elementary result:

**PROPOSITION 2.1.** *Let  $\mathbf{A}$  be a Banach algebra with a bounded left approximate identity  $\{e_\alpha\}_{\alpha \in A}$ . Then, for each  $S \in \text{Mul}_l(\mathbf{A})$ , there is an  $A$ -indexed net  $\{a_\alpha\}_{\alpha \in A}$  in  $\mathbf{A}$ , such that*

$$(2.2) \quad Sb = \lim_{\alpha \in A} L_{a_\alpha} b \quad (b \in \mathbf{A})$$

and

$$(2.3) \quad \liminf_{\alpha \in A} \|a_\alpha\| \leq K_0 \|S\| \quad \text{and} \quad \sup_{\alpha \in A} \|a_\alpha\| \leq K \|S\|.$$

**PROOF.** For each  $\alpha \in A$ , set  $a_\alpha = S e_\alpha$ . We check that the net  $\{a_\alpha\}_{\alpha \in A}$  has the required properties. For each  $b \in \mathbf{A}$ ,

$$\lim_{\alpha \in A} a_\alpha b = \lim_{\alpha \in A} S(e_\alpha b) = S\left(\lim_{\alpha \in A} e_\alpha b\right) = Sb,$$

and so (2.2) is established. Inequalities (2.3) are obvious. □

The following is now immediate.

**PROPOSITION 2.2.** *If  $\mathbf{A}$  is a Banach algebra with a bounded left approximate identity  $\{e_\alpha\}_{\alpha \in A}$ , then  $\text{Mul}_l(\mathbf{A})$  coincides with the closure of  $L_{\mathbf{A}}$  in  $\mathcal{L}(\mathbf{A})$  in the strong operator topology. If in addition  $\mathbf{A}$  is commutative, then so too is  $\text{Mul}(\mathbf{A})$ .*

## 2.2. – Representations of Banach algebras

Let  $\mathbf{A}$  be a Banach algebra and let  $E$  be a Banach space. A homomorphism from  $\mathbf{A}$  into  $\mathcal{L}(E)$  is called a *representation* of  $\mathbf{A}$  on  $E$ . A representation is continuous if it is continuous as a homomorphism of Banach algebras.

Suppose that  $\mathbf{A}$  has a bounded left approximate identity  $\{e_\alpha\}_{\alpha \in A}$ . Let  $\phi$  be a continuous representation of  $\mathbf{A}$  on  $E$ . Define the *regularity space*  $\mathcal{R}_\phi$  of  $\phi$  to be the closed linear span of  $\{\phi(a)x \mid a \in \mathbf{A}, x \in E\}$ . It is readily seen that

$$(2.4) \quad \mathcal{R}_\phi = \left\{ x \in E \mid \lim_{\alpha \in A} \phi(e_\alpha)x = x \right\}.$$

This equality can obviously serve as justification for the qualification “regularity space”. We say that  $\phi$  is *non-degenerate* if  $\mathcal{R}_\phi$  is all of  $E$ . It is evident that  $\mathcal{R}_\phi$  is an invariant subspace for all the  $\phi(a)$  ( $a \in \mathbf{A}$ ). Restricting each  $\phi(a)$  to  $\mathcal{R}_\phi$  defines a representation of  $\mathbf{A}$  on  $\mathcal{R}_\phi$  which we term the *non-degenerate part* of  $\phi$  and denote  $\phi \upharpoonright \mathcal{R}_\phi$ . Observe that the non-degenerate part of  $\phi$  fully determines  $\phi$  itself. Indeed, for each  $a \in \mathbf{A}$  and each  $x \in E$ ,

$$\phi(a)x = \lim_{\alpha \in A} \phi(e_\alpha a)x = \lim_{\alpha \in A} \phi(e_\alpha)\phi(a)x = \lim_{\alpha \in A} [\phi \upharpoonright \mathcal{R}_\phi](e_\alpha) \cdot \phi(a)x.$$

If  $\{e_\alpha\}_{\alpha \in A}$  is a two-sided approximate identity for  $\mathbf{A}$ , the above relation can be rewritten in an even more suggestive form as follows:

$$\phi(a)x = \lim_{\alpha \in A} [\phi \upharpoonright \mathcal{R}_\phi](a) \cdot \phi(e_\alpha)x.$$

Much of the subsequent development will rest on the following generalisation of the so-called factorisation theorem of P. J. Cohen [7], that was found independently by E. Hewitt [20], P. C. Curtis, Jr. and A. Figá-Talamanca [8], and S. L. Gulick, T. S. Liu and A. C. M. van Rooij [19]:

**THEOREM 2.3 (Hewitt *et al.*).** *Let  $\mathbf{A}$  be a Banach algebra possessing a bounded left approximate identity, let  $E$  be a Banach space, and let  $\phi$  be a continuous representation of  $\mathbf{A}$  on  $E$ . Then*

$$\mathcal{R}_\phi = \{\phi(a)x \mid a \in \mathbf{A}, x \in E\}.$$

An extensive literature is devoted to this and related theorems: relevant references include [1], [2], [4, Chap. 1, Sec. 11, Corol. 11], [11], [16, Chap. V, Sec. 9.2], [28], [31, Thm. 5.2.2], [32, Chap. 8], and [33], [34], [35], [38], [39], [40], [41].

### 2.3. – Extension of representations

A direct adaptation of a result of B. E. Johnson [25, Thm. 21] gives the following:

**THEOREM 2.4.** *Let  $\mathbf{A}$  be a Banach algebra possessing a bounded left approximate identity, let  $E$  be a Banach space, and let  $\phi$  be a continuous representation of  $\mathbf{A}$  on  $E$ . Then there is a unique representation  $\tilde{\phi}$  of  $\text{Mul}_l(\mathbf{A})$  on  $\mathcal{R}_\phi$  such that*

$$(2.5) \quad \tilde{\phi}(L_a)x = \phi(a)x \quad (a \in \mathbf{A}, x \in \mathcal{R}_\phi).$$

*The mapping  $\tilde{\phi}: \text{Mul}_l(\mathbf{A}) \rightarrow \mathcal{L}(\mathcal{R}_\phi)$  is continuous under the strong operator topologies on  $\text{Mul}_l(\mathbf{A})$  and  $\mathcal{L}(\mathcal{R}_\phi)$ , the topology on  $\text{Mul}_l(\mathbf{A})$  being the strong operator topology of  $\mathcal{L}(\mathbf{A})$  relativised to  $\text{Mul}_l(\mathbf{A})$ . Furthermore,  $\tilde{\phi}: \text{Mul}_l(\mathbf{A}) \rightarrow \mathcal{L}(\mathcal{R}_\phi)$  is continuous in the norm topologies of  $\text{Mul}_l(\mathbf{A})$  and  $\mathcal{L}(\mathcal{R}_\phi)$ , and*

$$(2.6) \quad \|\phi\| \leq \|\tilde{\phi}\| \leq K_0 \|\phi\|.$$

**PROOF.** Use (2.5) to define a linear mapping  $\tilde{\phi}: L_{\mathbf{A}} \rightarrow \mathcal{L}(\mathcal{R}_\phi)$ . That the definition is correct is seen as follows. Suppose that  $L_a = 0$  for some  $a \in \mathbf{A}$ . If  $x \in \mathcal{R}_\phi$ , then, by Theorem 2.3, there exist  $b \in \mathbf{A}$  and  $y \in E$  such that  $x = \phi(b)y$ . Since  $ab = L_a b = 0$ , we have  $\phi(a)x = \phi(ab)y = 0$ , showing that  $\tilde{\phi}$  is indeed well defined.

For  $x \in \mathcal{R}_\phi$  written as  $x = by$ , where  $b \in \mathbf{A}$  and  $y \in E$ , and for  $a \in \mathbf{A}$ , we have

$$(2.7) \quad \|\tilde{\phi}(L_a)x\| = \|\phi(a)\phi(b)y\| = \|\phi(ab)y\| \leq \|\phi\| \|ab\| \|y\| = \|\phi\| \|y\| \|L_a b\|.$$

This estimate implies immediately the continuity of  $\tilde{\phi}$  under the strong operator topologies on  $L_{\mathbf{A}}$  and  $\mathcal{L}(\mathcal{R}_\phi)$ .

Endow the space  $\mathcal{L}(\mathcal{R}_\phi)$  with the strong operator topology. Since  $\mathcal{R}_\phi$  is complete, so too is  $\mathcal{L}(\mathcal{R}_\phi)$ . Now, in accordance with Proposition 2.2,  $L_{\mathbf{A}}$  is dense in  $\text{Mul}_l(\mathbf{A})$  under the strong operator topology, so we can extend  $\tilde{\phi}$  by continuity to a mapping from  $\text{Mul}_l(\mathbf{A})$  into  $\mathcal{L}(\mathcal{R}_\phi)$ . Denote the corresponding extension again as  $\tilde{\phi}$ . Clearly,  $\tilde{\phi}$  is continuous under the strong operator topologies on  $\text{Mul}_l(\mathbf{A})$  and  $\mathcal{L}(\mathcal{R}_\phi)$ .

We proceed to show that the range of  $\tilde{\phi}$  is a subset of  $\mathcal{L}(\mathcal{R}_\phi)$ . Given  $S \in \text{Mul}_l(\mathbf{A})$ , select a net  $\{a_\alpha\}_{\alpha \in A}$  in  $\mathbf{A}$  for which (2.2) and (2.3) hold. Then, for each  $x \in \mathcal{R}_\phi$ ,

$$(2.8) \quad \tilde{\phi}(S)x = \lim_{\alpha \in A} \tilde{\phi}(L_{a_\alpha})x = \lim_{\alpha \in A} \phi(a_\alpha)x$$

and further

$$\|\tilde{\phi}(S)x\| \leq \|\phi\| \liminf_{\alpha \in A} \|a_\alpha\| \|x\| \leq \|\phi\| K_0 \|S\| \|x\|.$$

This estimate clearly implies that  $\tilde{\phi}(S)$  is a member of  $\mathcal{L}(\mathcal{R}_\phi)$  and

$$\|\tilde{\phi}(S)\| \leq K_0 \|\phi\| \|S\|.$$



Note that the last inequality shows that  $\tilde{\phi}$  is continuous in the norm topologies on  $\text{Mul}_l(\mathbf{A})$  and  $\mathcal{L}(\mathcal{R}_\phi)$ , and that  $\|\tilde{\phi}\| \leq K_0 \|\phi\|$ . In view of (2.5),

$$\|\phi(a)\| = \|\tilde{\phi}(L_a)\| \leq \|\tilde{\phi}\| \|L_a\| \leq \|\tilde{\phi}\| \|a\|,$$

whence  $\|\phi\| \leq \|\tilde{\phi}\|$ . Thus (2.6) is established.

It is apparent that  $\tilde{\phi}$  is linear. Applying (2.8) to  $x$  written as  $x = \phi(b)y$ , where  $b \in \mathbf{A}$  and  $y \in E$ , and taking into account that  $Sb = \lim_{\alpha \in A} a_\alpha b$  and that the mapping  $a \mapsto \phi(a)y$  is continuous, we see that

$$(2.9) \quad \tilde{\phi}(S)x = \phi(Sb)y,$$

or equivalently

$$(2.10) \quad \tilde{\phi}(S)\tilde{\phi}(L_b) = \tilde{\phi}(L_{Sb}).$$

Using the last equality repeatedly, we obtain, for any  $S, T \in \text{Mul}_l(\mathbf{A})$  and any  $b \in \mathbf{A}$ ,

$$\tilde{\phi}(ST)\tilde{\phi}(L_b) = \tilde{\phi}(L_{STb}) = \tilde{\phi}(S)\tilde{\phi}(L_{Tb}) = \tilde{\phi}(S)\tilde{\phi}(T)\tilde{\phi}(L_b).$$

Bearing in mind that

$$(2.11) \quad \{\tilde{\phi}(L_b)x \mid b \in \mathbf{A}, x \in E\} = \{\phi(b)x \mid b \in \mathbf{A}, x \in E\} = \mathcal{R}_\phi,$$

we conclude that  $\tilde{\phi}(ST) = \tilde{\phi}(S)\tilde{\phi}(T)$ . Substituting  $\text{id}_\mathbf{A}$  for  $S$  in (2.10) and again resorting to (2.11) yields  $\tilde{\phi}(\text{id}_\mathbf{A}) = \text{id}_{\mathcal{R}_\phi}$ . Thus  $\tilde{\phi}$  is a homomorphism of unital algebras.

It remains to prove that  $\tilde{\phi}$  is the only representation of  $\text{Mul}_l(\mathbf{A})$  on  $\mathcal{R}_\phi$  satisfying (2.5). Suppose that  $\psi$  is a representation of  $\text{Mul}_l(\mathbf{A})$  on  $\mathcal{R}_\phi$  such that  $\psi(L_a)x = \phi(a)x$  for  $a \in \mathbf{A}$  and  $x \in E$ . Let  $S \in \text{Mul}_l(\mathbf{A})$  and let  $x \in \mathcal{R}_\phi$  be written as  $x = \phi(b)y$ , where  $b \in \mathbf{A}$  and  $y \in E$ . Taking into account (2.1), we find that

$$\psi(S)\psi(L_b) = \psi(SL_b) = \psi(L_{Sb}),$$

whence

$$\psi(S)x = \psi(S)\phi(b)y = \psi(S)\psi(L_b)y = \psi(L_{Sb})y = \phi(Sb)y.$$

Combining this with (2.9), we obtain  $\psi = \tilde{\phi}$ . □

### 3. – $L_w^1(\mathbb{R}_+)$ , its multiplier algebra, and its extended representations

We wish to apply the material from the previous section to algebras of the form  $L_w^1(\mathbb{R}_+)$ . This requires some preparation that we now undertake. Throughout this section  $w$  will be a fixed non-negative number. We begin by indicating a family of bounded metric approximate identities for  $L_w^1(\mathbb{R}_+)$ . Next we address the problem of characterising the multiplier algebra  $\text{Mul}(L_w^1(\mathbb{R}_+))$ . Finally, we determine a canonical form of the representations of  $\text{Mul}(L_w^1(\mathbb{R}_+))$  arising as extensions of the non-degenerate parts of representations of  $L_w^1(\mathbb{R}_+)$ .

#### 3.1. – Some approximate units for $L_w^1(\mathbb{R}_+)$

For each  $\lambda \in (w, +\infty)$ , set

$$(3.1) \quad e_\lambda = \lambda \epsilon_{-\lambda}.$$

PROPOSITION 3.1. *For each  $w' \in (w, +\infty)$ ,  $\{e_\lambda\}_{\lambda \in (w', +\infty)}$  is a bounded metric approximate identity for  $L_w^1(\mathbb{R}_+)$ .*

PROOF. Fix  $w' \in (w, +\infty)$ . Direct verification shows that, for each  $\lambda \in (w, +\infty)$ ,

$$(3.2) \quad \|e_\lambda\|_{1,w} = \frac{\lambda}{\lambda - w}.$$

Hence, for each  $\lambda \in (w', +\infty)$ ,

$$\|e_\lambda\|_{1,w} \leq \frac{w'}{w' - w}$$

and

$$\lim_{\lambda \rightarrow +\infty} \|e_\lambda\|_{1,w} = 1.$$

To complete the proof, it remains to show that

$$(3.3) \quad \lim_{\lambda \rightarrow +\infty} e_\lambda * f = f \quad (f \in L_w^1(\mathbb{R}_+)).$$

In view of (1.1), for each  $\mu \in (w, +\infty)$  and each  $\lambda \in (w, +\infty)$  with  $\lambda \neq \mu$ ,

$$e_\lambda * \epsilon_{-\mu} = \frac{\lambda}{\lambda - \mu} \epsilon_{-\mu} - \frac{1}{\lambda - \mu} e_\lambda.$$

Hence, taking into account (3.2),

$$(3.4) \quad \lim_{\lambda \rightarrow +\infty} e_\lambda * \epsilon_{-\mu} = \epsilon_{-\mu}$$

for all  $\mu \in (w, +\infty)$ , and so (3.3) holds for  $f$  being a finite linear combination of the  $\epsilon_{-\mu}$  ( $\mu \in (w, +\infty)$ ). A standard argument using, say, the Stone-Weierstrass theorem shows that the set  $\{\epsilon_{-\mu} \mid \mu \in (w, +\infty)\}$  is linearly dense in  $L_w^1(\mathbb{R}_+)$ . Combining this with the fact that the net  $\{e_\lambda\}_{\lambda \in (w', +\infty)}$  is bounded, we conclude that (3.3) holds for all  $f \in L_w^1(\mathbb{R}_+)$ .  $\square$

### 3.2. – A characterisation of $\text{Mul}(L_w^1(\mathbb{R}_+))$

Let  $\mathfrak{B}(\mathbb{R}_+)$  be the  $\sigma$ -algebra of Borel subsets of  $\mathbb{R}_+$ . For a non-negative Borel function  $f$  on  $\mathbb{R}_+$  and a non-negative Borel measure on  $\mathbb{R}_+$ , let  $f\mu$  be the Borel measure on  $\mathbb{R}_+$  given by

$$(f\mu)(A) = \int_A f d\mu \quad (A \in \mathfrak{B}(\mathbb{R}_+)).$$

Let  $M_w(\mathbb{R}_+)$  be the collection of all Borel measures  $\mu$  on  $\mathbb{R}_+$  for which  $\epsilon_w|\mu|$  is bounded, where  $|\mu|$  denotes the total variation of  $\mu$ . Given  $t \in \mathbb{R}_+$ , denote by  $\delta_t$  the Dirac measure concentrated at  $t$ . With set-wise linear operations, with the convolution multiplication defined by

$$(\mu * \nu)(A) = \int_{\mathbb{R}_+} \mu(A \ominus t) d\nu(t) \quad (A \in \mathfrak{B}(\mathbb{R}_+)),$$

where  $A \ominus t = \{s \in \mathbb{R}_+ \mid s = a - t \text{ for some } a \in A\}$ , with  $\delta_0$  as an multiplicative identity element, and with the norm given by

$$\|\mu\|_w = \int_{\mathbb{R}_+} e^{wt} d|\mu|(t),$$

$M_w(\mathbb{R}_+)$  is a unital Banach algebra. The linear combinations of Dirac measures on  $\mathbb{R}_+$  form a subalgebra of  $M_w(\mathbb{R}_+)$  that we denote by  $M_w^{\text{fin}}(\mathbb{R}_+)$ .

For each  $f \in L_w^1(\mathbb{R}_+)$ , let  $\nu_f$  be the measure in  $M_w(\mathbb{R}_+)$  defined as

$$\nu_f(A) = \int_A f(t) dt \quad (A \in \mathfrak{B}(\mathbb{R}_+)).$$

The mapping  $f \mapsto \nu_f$  is a Banach algebra isomorphism of  $L_w^1(\mathbb{R}_+)$  onto the algebra formed by all measures in  $M_w(\mathbb{R}_+)$  that are absolutely continuous with respect to Lebesgue measure. We identify  $L_w^1(\mathbb{R}_+)$  with its image via  $f \mapsto \nu_f$ . Under this identification,  $L_w^1(\mathbb{R}_+)$  becomes an ideal of  $M_w(\mathbb{R}_+)$ . If  $f \in L_w^1(\mathbb{R}_+)$  and  $\mu \in M_w(\mathbb{R}_+)$ , then  $f * \mu$  is a member of  $L_w^1(\mathbb{R}_+)$  determined by  $\nu_f * \mu = \nu_{f*\mu}$  and given by

$$(f * \mu)(t) = \int_{[0,t]} f(t-s) d\mu(s) \quad (\text{almost all } t \in \mathbb{R}_+).$$

Furthermore,  $\|f * \mu\|_w \leq \|\mu\|_w \|f\|_{1,w}$ , which implies that, given  $\mu \in M_w(\mathbb{R}_+)$ , setting

$$T_\mu f = f * \mu \quad (f \in L_w^1(\mathbb{R}_+))$$

defines a multiplier of  $L_w^1(\mathbb{R}_+)$  satisfying  $\|T_\mu\| \leq \|\mu\|_w$ . We shall soon see that  $\|T_\mu\| = \|\mu\|_w$ . It is straightforwardly verified that the mapping  $T: M_w(\mathbb{R}_+) \ni \mu \mapsto T_\mu \in \text{Mul}(L_w^1(\mathbb{R}_+))$  is a contractive homomorphism. In fact, more is true:

**THEOREM 3.2.** *The mapping  $T$  is an isometric homomorphism of  $M_w(\mathbb{R}_+)$  onto  $\text{Mul}(L_w^1(\mathbb{R}_+))$ .*

This theorem is a natural adaptation of a result of J. G. Wendel [43] on the form of the multiplier algebras of group algebras (see also [16, Chap. VIII, Sec. 1.25] and [31, Sec. 1.9.13] for Wendel’s theorem and [10], [17], [18], [44] for related results). Before giving the proof of Theorem 3.2, we make a few remarks about notation.

For a vector space  $E$ , let  $E'$  be the algebraic dual of  $E$ . Denote by  $E \times E' \ni (x, x') \mapsto \langle x, x' \rangle \in \mathbb{F}$  the duality relation between  $E$  and  $E'$ . For a topological vector space  $E$ , let  $E^*$  represent the topological dual space of  $E$ .

Let  $C_{b,-w}(\mathbb{R}_+)$  be the space of all continuous functions  $f$  on  $\mathbb{R}_+$  for which  $\epsilon_{-w}f$  is bounded, and let  $C_{0,-w}(\mathbb{R}_+)$  be the space of all continuous functions  $f$  on  $\mathbb{R}_+$  for which  $\epsilon_{-w}f$  vanishes at infinity. Under the norm  $\|f\|_{\infty,-w} = \sup_{s \in \mathbb{R}_+} e^{-ws} |f(s)|$ ,  $C_{b,-w}(\mathbb{R}_+)$  and  $C_{0,-w}(\mathbb{R}_+)$  are Banach spaces. Every function in  $C_{b,-w}(\mathbb{R}_+)$  is integrable with respect to every measure in  $M_w(\mathbb{R}_+)$ , and setting

$$\langle f, \mu \rangle = \int_{\mathbb{R}_+} f d\mu \quad (f \in C_{b,-w}(\mathbb{R}_+), \mu \in M_w(\mathbb{R}_+))$$

defines a duality relation between  $C_{b,-w}(\mathbb{R}_+)$  and  $M_w(\mathbb{R}_+)$ . This duality can be used to obtain the following characterisation of the topological dual of  $C_{0,-w}(\mathbb{R}_+)$ : The mapping that assigns to each measure  $\mu \in M_w(\mathbb{R}_+)$  the linear functional  $C_{0,-w}(\mathbb{R}_+) \ni f \mapsto \langle f, \mu \rangle \in \mathbb{F}$  is a linear isometry from  $C_{0,-w}(\mathbb{R}_+)$  onto  $M_w(\mathbb{R}_+)^*$ .

Let  $L_{-w}^\infty(\mathbb{R}_+)$  be the space of all equivalence classes of essentially bounded measurable functions  $f$  on  $\mathbb{R}_+$  for which  $\epsilon_{-w}f$  is bounded, equipped with the norm  $\|f\|_{\text{ess},\infty,-w} = \text{ess sup}_{s \in \mathbb{R}_+} e^{-ws} |f(s)|$ . Setting

$$\langle f, k \rangle = \int_{\mathbb{R}_+} f(x)k(x) dx \quad (f \in L_w^1(\mathbb{R}_+), k \in L_{-w}^\infty(\mathbb{R}_+))$$

defines a duality relation between  $L_w^1(\mathbb{R}_+)$  and  $L_{-w}^\infty(\mathbb{R}_+)$ . This duality permits determining the topological dual of  $L_w^1(\mathbb{R}_+)$  as follows: The mapping that assigns to each  $k \in L_{-w}^\infty(\mathbb{R}_+)$  the linear functional  $L_w^1(\mathbb{R}_+) \ni f \mapsto \langle f, k \rangle \in \mathbb{F}$  is a linear isometry from  $L_{-w}^\infty(\mathbb{R}_+)$  onto  $L_w^1(\mathbb{R}_+)^*$ .

**PROOF OF THEOREM 3.2.** It suffices to show that each  $S \in \text{Mul}(L_w^1(\mathbb{R}_+))$  can be represented as  $T_\mu$  for some  $\mu \in M_w(\mathbb{R}_+)$  satisfying  $\|\mu\|_w \leq \|S\|$ . Let  $S \in \text{Mul}(L_w^1(\mathbb{R}_+))$ . By Propositions 2.1 and 3.1, there exists a net  $\{s_\alpha\}_{\alpha \in A}$  in  $L_w^1(\mathbb{R}_+)$  such that

$$(3.5) \quad Sf = \lim_{\alpha \in A} s_\alpha * f \quad (f \in L_w^1(\mathbb{R}_+)),$$

where the limit is taken in the  $\|\cdot\|_{1,w}$  norm, and

$$\sup_{\alpha \in A} \|s_\alpha\|_{1,w} \leq \|S\|.$$

By the relative weak\* compactness of bounded sets in  $C_{0,-w}(\mathbb{R}_+)^*$ , there exists a subnet  $\{s_\beta\}_{\beta \in B}$  and a measure  $\mu \in M_w(\mathbb{R}_+)$  with  $\|\mu\|_w \leq \|S\|$  such that  $\lim_{\beta \in B} \langle g, \nu_{s_\beta} \rangle = \langle g, \mu \rangle$  for each  $g \in C_{0,-w}(\mathbb{R}_+)$ . Fix  $f \in L_w^1(\mathbb{R}_+)$  and  $\psi \in C_{0,-w}(\mathbb{R}_+)$  arbitrarily. Define a function  $h$  by

$$h(x) = \int_x^{+\infty} f(y-x)\psi(y) dy \quad (x \in \mathbb{R}_+).$$

A standard argument shows that  $h$  is continuous and belongs to  $L_w^1(\mathbb{R}_+)$ . Moreover,

$$|h(x)| \leq e^{wx} \sup_{y \geq x} e^{-wy} |\psi(y)| \|f\|_{1,w},$$

which shows that  $\epsilon_{-w}h$  vanishes at infinity. Consequently,

$$(3.6) \quad \lim_{\beta \in B} \int_{\mathbb{R}_+} h s_\beta dx = \int_{\mathbb{R}_+} h d\mu.$$

Using Fubini's theorem, one verifies at once that, for each  $\beta \in B$ ,

$$\int_{\mathbb{R}_+} h s_\beta dx = \int_{\mathbb{R}_+} (f * s_\beta) \psi dx$$

and

$$\int_{\mathbb{R}_+} h d\mu = \int_{\mathbb{R}_+} (f * \mu) \psi dx.$$

With these identities, (3.6) can be rewritten as

$$(3.7) \quad \lim_{\beta \in B} \int_{\mathbb{R}_+} (f * s_\beta) \psi dx = \int_{\mathbb{R}_+} (f * \mu) \psi dx.$$

But, in view of (3.5), we have

$$\lim_{\beta \in B} \int_{\mathbb{R}_+} (f * s_\beta) \psi dx = \int_{\mathbb{R}_+} (Sf) \psi dx.$$

Therefore

$$\int_{\mathbb{R}_+} (Sf) \psi dx = \int_{\mathbb{R}_+} (f * \mu) \psi dx.$$

Since  $\psi$  was an arbitrary member of  $C_{0,-w}(\mathbb{R}_+)$ , we see that  $Sf = f * \mu$ , as was to be proved.  $\square$

Multipliers associated with Dirac measures have a particularly simple form. For each  $t \in \mathbb{R}_+$ ,  $T_{\delta_t}$  is the *forward shift* operator by  $t$ , given by

$$(3.8) \quad (T_{\delta_t} f)(s) = 1_{[t, +\infty)}(s) f(s - t) \quad (f \in L_w^1(\mathbb{R}_+), \text{ almost all } s \in \mathbb{R}_+),$$

where, as usual, the symbol  $1_A$  denotes the characteristic function of the set  $A$ . Clearly,  $\|\delta_t\|_w = e^{wt}$  holds for each  $t \in \mathbb{R}_+$ , therefore

$$(3.9) \quad \|T_{\delta_t}\| = e^{wt} \quad (t \in \mathbb{R}_+).$$

For any  $s, t \in \mathbb{R}_+$  we have  $\delta_s * \delta_t = \delta_{s+t}$ , and so  $T_{\delta_s} T_{\delta_t} = T_{\delta_{s+t}}$ . From (3.8) we conclude that the mapping  $t \mapsto T_{\delta_t}$  is continuous under the strong operator topology on  $\text{Mul}(L_w^1(\mathbb{R}_+))$ . Thus  $\{T_{\delta_t}\}_{t \in \mathbb{R}_+}$  is a strongly continuous one-parameter semigroup on  $L_w^1(\mathbb{R}_+)$ .

### 3.3. – A canonical form of the extended representations of $L_w^1(\mathbb{R}_+)$

Representations of  $\text{Mul}(L_w^1(\mathbb{R}_+))$  arising as extensions of the non-degenerate parts of representations of  $L_w^1(\mathbb{R}_+)$  can always be expressed in a certain canonical form. The theorem describing this form is an analogue of a theorem describing the form of non-degenerate Hilbert space \*-representations of group algebras (cf. [37, Thm. 10.1']). The proof given below parallels Johnson's proof of the latter result (see [24, pp. 606-607]).

**THEOREM 3.3.** *Let  $E$  be a Banach space, and let  $\phi$  be a continuous representation of  $L_w^1(\mathbb{R}_+)$  on  $E$ . For each  $t \in \mathbb{R}_+$ , set*

$$(3.10) \quad S_t = \tilde{\phi}(T_{\delta_t}).$$

*Then  $\{S_t\}_{t \in \mathbb{R}_+}$  is a strongly continuous one-parameter group on  $\mathcal{R}_\phi$  such that*

$$(3.11) \quad \|S_t\| \leq \|\tilde{\phi}\| e^{wt} \quad (t \in \mathbb{R}_+),$$

*and, for each  $\mu \in M_w(\mathbb{R}_+)$  and each  $x \in \mathcal{R}_\phi$ ,*

$$(3.12) \quad \tilde{\phi}(T_\mu)x = \int_{\mathbb{R}_+} S_t x \, d\mu(t),$$

*where the integral is to be interpreted as a Bochner integral.*

**PROOF.**  $\{T_{\delta_t}\}_{t \in \mathbb{R}_+}$  is a strongly continuous one-parameter semigroup on  $L_w^1(\mathbb{R}_+)$ , so, by Theorem 2.4,  $\{S_t\}_{t \in \mathbb{R}_+}$  is a strongly continuous one-parameter semigroup on  $\mathcal{R}_\phi$ . The estimate (3.11) follows immediately from (3.9).

For each  $x \in \mathcal{R}_\phi$ , the  $\mathcal{R}_\phi$ -valued function  $t \mapsto S_t x$  is continuous and hence strongly  $|\mu|$ -measurable whatever  $\mu \in M_w(\mathbb{R}_+)$ . Moreover, in view of (3.11), the function  $t \mapsto \|S_t x\|$  is a member of  $C_{b,-w}(\mathbb{R}_+)$  and as such is  $|\mu|$ -integrable.

It follows that the function  $t \mapsto S_t x$  is  $|\mu|$ -integrable in the sense of Bochner. In particular, the integral on the right-hand side of (3.12) is well defined.

Equation (3.12) is evident for measures in  $M_w^{\text{fin}}(\mathbb{R}_+)$ . Let  $\sigma(M_w(\mathbb{R}_+), C_{b,-w}(\mathbb{R}_+))$  be the coarsest locally convex topology on  $M_w(\mathbb{R}_+)$  for which all the mappings  $\mu \mapsto \langle f, \mu \rangle$  ( $f \in C_{b,-w}(\mathbb{R}_+)$ ) are continuous; clearly,  $\sigma(M_w(\mathbb{R}_+), C_{b,-w}(\mathbb{R}_+))$  is generated by the family of seminorms

$$p_{f_1, \dots, f_n}(\mu) = \sum_{i=1}^n |\langle f_i, \mu \rangle| \quad (\mu \in M_w(\mathbb{R}_+)),$$

where  $f_1, \dots, f_n \in C_{b,-w}(\mathbb{R}_+)$  and  $n \in \mathbb{N}$ . To prove (3.12) for arbitrary measures in  $M_w(\mathbb{R}_+)$ , let  $\mu \in M_w(\mathbb{R}_+)$  and suppose for a moment that there is a net  $\{\mu_\alpha\}_{\alpha \in A}$  in  $M_w^{\text{fin}}(\mathbb{R}_+)$  converging to  $\mu$  in the  $\sigma(M_w(\mathbb{R}_+), C_{b,-w}(\mathbb{R}_+))$ -topology and such that  $\{T_{\mu_\alpha}\}_{\alpha \in A}$  converges to  $T_\mu$  in the strong topology of  $\text{Mul}(L_w^1(\mathbb{R}_+))$ . Then, by Theorem 2.4,  $\{\tilde{\phi}(T_{\mu_\alpha})\}_{\alpha \in A}$  converges to  $\tilde{\phi}(T_\mu)$  in the strong topology of  $\mathcal{L}(\mathcal{R}_\phi)$ . Given  $x \in \mathcal{R}_\phi$  and  $x^* \in \mathcal{R}_\phi^*$ , the function  $t \mapsto \langle S_t x, x^* \rangle$  belongs to  $C_{b,-w}(\mathbb{R}_+)$ , and so

$$\begin{aligned} \langle \tilde{\phi}(T_\mu)x, x^* \rangle &= \lim_{\alpha \in A} \langle \tilde{\phi}(T_{\mu_\alpha})x, x^* \rangle = \lim_{\alpha \in A} \int_{\mathbb{R}_+} \langle S_t x, x^* \rangle d\mu_\alpha(t) \\ &= \int_{\mathbb{R}_+} \langle S_t x, x^* \rangle d\mu(t), \end{aligned}$$

which immediately implies (3.12).

To end the proof, we need to establish the existence of the approximating net  $\{\mu_\alpha\}_{\alpha \in A}$ . Equip  $M_w(\mathbb{R}_+)$  with the coarsest locally convex topology  $\tau$  for which all the mappings  $\mu \mapsto T_\mu f$  ( $f \in L_w^1(\mathbb{R}_+)$ ) and  $\mu \mapsto \langle g, \mu \rangle$  ( $g \in C_{b,-w}(\mathbb{R}_+)$ ) are continuous; this topology is, of course, determined by the family of seminorms

$$q_{f_1, \dots, f_n, g_1, \dots, g_n}(\mu) = \sum_{i=1}^n \|T_\mu f_i\|_{1,w} + \sum_{i=1}^n |\langle g_i, \mu \rangle| \quad (\mu \in M_w(\mathbb{R}_+)),$$

where  $f_1, \dots, f_n \in L_w^1(\mathbb{R}_+)$ ,  $g_1, \dots, g_n \in C_{b,-w}(\mathbb{R}_+)$  and  $n \in \mathbb{N}$ . We have to prove that  $M_w^{\text{fin}}(\mathbb{R}_+)$  is dense in  $M_w(\mathbb{R}_+)$  in the  $\tau$ -topology. By the Hahn-Banach theorem, it suffices to show that any linear functional on  $M_w(\mathbb{R}_+)$  which is continuous in the  $\tau$ -topology and vanishes on  $M_w^{\text{fin}}(\mathbb{R}_+)$  is null.

Let  $Z \in M_w(\mathbb{R}_+)'$  be a linear functional continuous in the  $\tau$ -topology, vanishing on  $M_w^{\text{fin}}(\mathbb{R}_+)$ . The  $\tau$ -continuity of  $Z$  means that, for some  $n \in \mathbb{N}$ , there exist  $\{f_i\}_{i=1}^n$  in  $L_w^1(\mathbb{R}_+)$ ,  $\{g_i\}_{i=1}^n$  in  $C_{b,-w}(\mathbb{R}_+)$ , and  $C \in \mathbb{R}_+$  such that, for each  $\nu \in M_w(\mathbb{R}_+)$ ,

$$(3.13) \quad |\langle \nu, Z \rangle| \leq C \left( \sum_{i=1}^n \|T_\nu f_i\|_{1,w} + \sum_{i=1}^n |\langle g_i, \nu \rangle| \right).$$

Let  $X = (L_w^1(\mathbb{R}_+))^n \oplus \mathbb{F}^n$ . Endow  $X$  with the norm

$$\|(h_1, \dots, h_n, x_1, \dots, x_n)\| = \sum_{i=1}^n \|h_i\|_{1,w} + \sum_{i=1}^n |x_i|,$$

where  $\{h_i\}_{i=1}^n$  and  $\{x_i\}_{i=1}^n$  are arbitrary sequences in  $L_w^1(\mathbb{R}_+)$  and  $\mathbb{F}$ , respectively. Let  $X_0$  be the subspace of  $X$  composed of the elements of the form

$$(T_\nu f_1, \dots, T_\nu f_n, \langle g_1, \nu \rangle, \dots, \langle g_n, \nu \rangle) \quad (\nu \in M_w(\mathbb{R}_+)).$$

Define  $F \in X_0'$  by

$$F((T_\nu f_1, \dots, T_\nu f_n, \langle g_1, \nu \rangle, \dots, \langle g_n, \nu \rangle)) = \langle \nu, Z \rangle \quad (\nu \in M_w(\mathbb{R}_+)).$$

It follows from (3.13) that  $F$  is well defined and that it is continuous in the norm topology inherited from  $X$ . By the Hahn-Banach theorem,  $F$  can be extended to a continuous linear functional  $\tilde{F}$  defined on the whole of  $X$ . It is clear that  $\tilde{F}$  takes the form

$$\tilde{F}((h_1, \dots, h_n, x_1, \dots, x_n)) = \sum_{i=1}^n \langle h_i, e_i \rangle + \sum_{i=1}^n a_i x_i$$

for some  $\{e_i\}_{i=1}^n$  in  $L_w^\infty(\mathbb{R}_+)$  and some  $\{a_i\}_{i=1}^n$  in  $\mathbb{F}$ . Hence, for each  $\nu \in M_w(\mathbb{R}_+)$ ,

$$(3.14) \quad \langle \nu, Z \rangle = \sum_{i=1}^n \langle T_\nu f_i, e_i \rangle + \sum_{i=1}^n a_i \langle g_i, \nu \rangle.$$

In particular, taking into account (3.8), we find that, for each  $t \in \mathbb{R}_+$ ,

$$\langle \delta_t, Z \rangle = \sum_{i=1}^n \int_t^{+\infty} f_i(s-t)e_i(s) ds + \sum_{i=1}^n a_i g_i(t).$$

But  $Z$  vanishes on  $M_w^{\text{fin}}(\mathbb{R}_+)$ , so

$$\sum_{i=1}^n \int_t^{+\infty} f_i(s-t)e_i(s) ds + \sum_{i=1}^n a_i g_i(t) = 0$$

for all  $t \in \mathbb{R}_+$ . Fix  $\nu \in M_w(\mathbb{R}_+)$  arbitrarily. Integrating both sides of the last equation with respect to  $\nu$ , we obtain

$$(3.15) \quad \sum_{i=1}^n \int_{\mathbb{R}_+} \left[ \int_t^{+\infty} f_i(s-t)e_i(s) ds \right] d\nu(t) + \sum_{i=1}^n a_i \int_{\mathbb{R}_+} g_i(t) d\nu(t) = 0.$$

By Fubini's theorem, for each  $i \in \{1, \dots, n\}$ ,

$$\begin{aligned} \int_{\mathbb{R}_+} \left[ \int_t^{+\infty} f_i(s-t)e_i(s) ds \right] d\nu(t) &= \int_{\mathbb{R}_+} \left[ \int_{[0,s]} f_i(s-t) d\nu(t) \right] e_i(s) ds \\ &= \langle T_\nu f_i, e_i \rangle. \end{aligned}$$

Combining this relation with (3.14) and (3.15), we see that  $\langle \nu, Z \rangle = 0$ , and further, in view of the arbitrariness of  $\nu$ , that  $Z$  is null, as was to be proved.  $\square$



#### 4. – Kisyński's generalisation of the Hille-Yosida theorem

In this section, we apply the results from the previous sections to rederive Kisyński's generalisation of the Hille-Yosida theorem (cf. [27]).

##### 4.1. – The generalised Hille-Yosida theorem

Let  $E$  be a Banach space, let  $w \in \mathbb{R}_+$ , and let  $R = \{R_\lambda\}_{\lambda \in (w, +\infty)}$  be a pseudo-resolvent in  $\mathcal{L}(E)$  such that  $c_R < +\infty$ . Define the *regularity space* of  $R$  as

$$(4.1) \quad \mathcal{R}_R = \left\{ x \in E \mid \lim_{\lambda \rightarrow +\infty} \lambda R_\lambda x = x \right\}.$$

According to Theorem 1.1, there exists a unique bounded representation  $\phi$  of  $L_w^1(\mathbb{R}_+)$  on  $E$  such that  $R_\lambda = \phi(\epsilon_{-\lambda})$  for each  $\lambda \in (w, +\infty)$ ; it will be termed the *representation associated with  $R$* . Keeping the  $e_\lambda$  as in (3.1), we clearly have  $\lambda R_\lambda = \phi(e_\lambda)$  for each  $\lambda \in (w, +\infty)$ . Since  $\{e_\lambda\}_{\lambda \in (w', +\infty)}$  is a bounded metric approximate identity for  $L_w^1(\mathbb{R}_+)$  if only  $w' > w$ , it follows from (2.4) and (4.1) that

$$(4.2) \quad \mathcal{R}_R = \mathcal{R}_\phi.$$

In particular,  $\mathcal{R}_R$  is invariant for all the  $\phi(f)$  ( $f \in L_w^1(\mathbb{R}_+)$ ), and any element of  $\mathcal{R}_R$  can be represented as  $\phi(f)x$  for some  $f \in L_w^1(\mathbb{R}_+)$  and  $x \in E$ .

We are now ready to state Kisyński's result:

**THEOREM 4.1 (Generalised Hille-Yosida Theorem).** *Let  $E$  be a Banach space, let  $w \in \mathbb{R}_+$ , let  $R = \{R_\lambda\}_{\lambda \in (w, +\infty)}$  be a pseudo-resolvent in  $\mathcal{L}(E)$  with  $c_R < +\infty$ , and let  $\phi$  be the representation of  $L_w^1(\mathbb{R}_+)$  associated with  $R$ . Then there exists a unique one-parameter semigroup  $\{S_t\}_{t \in \mathbb{R}_+}$  on  $\mathcal{R}_R$  such that*

$$(4.3) \quad S_t \phi(f)x = \phi(T_{\delta_t} f)x \quad (t \in \mathbb{R}_+, f \in L_w^1(\mathbb{R}_+), x \in E).$$

The semigroup  $\{S_t\}_{t \in \mathbb{R}_+}$  is strongly continuous and satisfies

$$(4.4) \quad R_\lambda x = \int_{\mathbb{R}_+} e^{-\lambda t} S_t x dt \quad (\lambda \in (w, +\infty), x \in \mathcal{R}_R)$$

and

$$(4.5) \quad \|S_t\| \leq c_R e^{wt} \quad (t \in \mathbb{R}_+).$$

**PROOF.** We first prove the existence statement. Extend  $\phi$  to a corresponding representation  $\tilde{\phi}$  of  $\text{Mul}(L_w^1(\mathbb{R}_+))$  on  $E$ . For each  $t \in \mathbb{R}_+$ , let  $S_t \in \mathcal{L}(\mathcal{R}_\phi)$  be

given as in (3.10). From Theorem 3.3 and (4.2), we see that  $\{S_t\}_{t \in \mathbb{R}_+}$  is a strongly continuous semigroup on  $\mathcal{R}_R$ . By (2.5) and (2.10),

$$S_t \phi(f) = \tilde{\phi}(T_{\delta_t}) \tilde{\phi}(L_f) = \tilde{\phi}(L_{T_{\delta_t} f}) = \phi(T_{\delta_t} f)$$

for each  $t \in \mathbb{R}_+$  and each  $f \in L_w^1(\mathbb{R}_+)$ . Now (2.6) together with the fact  $L_w^1(\mathbb{R}_+)$  has a bounded metric approximate identity yields  $\|\tilde{\phi}\| = \|\phi\|$ . Combining this equality with  $\|\phi\| = c_R$  (which is part of Theorem 1.1) and (3.11), we obtain (4.5).

If  $f \in L_w^1(\mathbb{R}_+)$ , then

$$\tilde{\phi}(T_{v_f}) = \tilde{\phi}(L_f) = \phi(f).$$

In particular, for each  $\lambda \in (w, +\infty)$ ,

$$\tilde{\phi}(T_{v_{\epsilon_{-\lambda}}}) = \phi(\epsilon_{-\lambda}) = R_\lambda.$$

But, by (3.12), for each  $x \in \mathcal{R}_\phi$ ,

$$\tilde{\phi}(T_{v_{\epsilon_{-\lambda}}})x = \int_{\mathbb{R}_+} S_t x \, dv_{\epsilon_{-\lambda}}(t) = \int_{\mathbb{R}_+} e^{-\lambda t} S_t x \, dt.$$

Thus (4.4) is established.

Finally, the uniqueness of  $\{S_t\}_{t \in \mathbb{R}_+}$  follows from (4.3) and the fact that every member of  $\mathcal{R}_R$  can be represented as  $\phi(f)x$  for some  $f \in L_w^1(\mathbb{R}_+)$  and some  $x \in E$ . □

### 4.2. – A link with the classical Hille-Yosida theorem

The main, sufficiency part of the Hille-Yosida theorem, concerns the generation of a one-parameter semigroup of operators given a pseudo-resolvent whose range space is dense in an ambient Banach space. As we shall see now, this part of the Hille-Yosida theorem can easily be deduced from Theorem 4.1.

Let  $E$  be a Banach space, let  $w \in \mathbb{R}_+$ , and let  $R = \{R_\lambda\}_{\lambda \in (w, +\infty)}$  be a pseudo-resolvent in  $\mathcal{L}(E)$  such that  $c_R < +\infty$ . It immediately follows from the Hilbert equation (1.1) that all the  $R_\lambda$  ( $\lambda \in (w, +\infty)$ ) have a common null space  $\text{Ker } R$  and a common range  $\text{Im } R$ . Another consequence of (1.1) is that  $\text{Im } R \subset \mathcal{R}_R$ . Since  $\mathcal{R}_R$  is closed in  $E$  (being identified with  $\mathcal{R}_\phi$ ), the closure of  $\text{Im } R$  in  $E$ ,  $\overline{\text{Im } R}$ , is contained in  $\mathcal{R}_R$ . On the other hand, (4.1) implies that  $\mathcal{R}_R \subset \overline{\text{Im } R}$ . Therefore

$$(4.6) \quad \mathcal{R}_R = \overline{\text{Im } R}.$$

A quick glance at (4.1) reveals that

$$(4.7) \quad \text{Ker } R \cap \mathcal{R}_R = \{0\}.$$

We note also that if  $\text{Ker } R$  is null, then  $R$  is a resolvent of a closed operator whose domain coincides with  $\text{Im } R$ .

To derive the sufficiency part of the classical Hille-Yosida theorem, suppose that  $\overline{\text{Im } R} = E$ . Then, by (4.6),  $\mathcal{R}_R = E$ , and further, by (4.7),  $\text{Ker } R$  is zero. Consequently,  $R$  is a resolvent of a closed operator  $A$  whose domain coincides with  $\text{Im } R$ . Resorting to Theorem 4.1, we conclude that there exists a strongly continuous semigroup  $\{S_t\}_{t \in \mathbb{R}_+}$  on  $E$  satisfying (4.4) for all  $x \in E$  and (4.5). Note that (4.4) immediately implies that  $A$  is the (infinitesimal) generator of the semigroup. Now the existence of a strongly continuous semigroup  $\{S_t\}_{t \in \mathbb{R}_+}$  satisfying (4.4) and (4.5), and linked to  $R$  via  $A$ , is precisely what the sufficiency part of the standard Hille-Yosida theorem asserts.

### 4.3. – Additional comments

Theorem 4.1 affirms the existence of a strongly continuous one-parameter semigroup of operators on a closed subspace of a Banach space  $E$ , given a pseudo-resolvent in  $\mathcal{L}(E)$ . This assertion is standard fare (cf. [9, Chap. XIII, Sec. 1, Subsec. 4, Thm., p. 311] and [36, pp. 44 and 53]). It can easily be derived from the classical Hille-Yosida theorem. A novel supplement to the assertion, due to Kisyński, is the expression for the engendered semigroup in terms of a representation of an appropriate space  $L_w^1(\mathbb{R}_+)$  and the semigroup of forward shifts in  $L_w^1(\mathbb{R}_+)$ . In view of this essential addition, Theorem 4.1 has been termed by Kisyński the *algebraic version* of the Hille-Yosida theorem. One consequence of this strengthened form of the Hille-Yosida theorem is a version of Trotter-Kato theorem given below. Other consequences include: (i) a theorem concerning the generation of a one-parameter semigroup, acting on the bidual  $E^{**}$  of a Banach space  $E$ , such that the semigroup trajectories passing through elements of the  $*$ -weak sequential closure of  $E$  in  $E^{**}$  are  $*$ -weakly Borel measurable [26]; (ii) a result on the Favard classes of semigroups associated with pseudo-resolvents [6].

## 5. – Banach bundles and an abstract Trotter-Kato theorem

The classical Trotter-Kato theorem operates with a sequence of possibly distinct Banach spaces converging, in a certain sense, to a Banach space which may be different from all the spaces forming the sequence. Converging sequences of Banach spaces are special instances of so-called continuous fields of Banach spaces. The key concept for studying such fields is that of a Banach bundle. Objects of fundamental significance for representation theory are various spaces of cross-sections of Banach bundles (cf. [13], [16]).

Here we first discuss Banach bundles, various spaces of cross-sections of Banach bundles, and Banach algebra representations on both Banach bundles and spaces of cross-sections. Following this, we establish a bundle-theoretic

version of the Trotter-Kato theorem. When applied to suitably chosen Banach bundles, this version will yield a generalisation of the classical Trotter-Kato theorem.

### 5.1. – Basic definitions

Let  $B$  be a fixed topological Hausdorff space. A *bundle over  $B$*  is a triple  $\xi = (E, B, \pi)$ , in which  $E$  is a Hausdorff topological space and  $\pi: E \rightarrow B$  is a continuous open surjection.  $E$  is said to be the *bundle* or *total space* of  $\xi$ ,  $B$  the *base space* of  $\xi$ , and  $\pi$  the *bundle projection*. For each  $b \in B$ ,  $\pi^{-1}(b)$ , also denoted  $E_b$ , is the *fibre over  $b$* .

A *cross-section of  $\xi$*  is a function  $f: B \rightarrow E$  such that  $f(b) \in E_b$  for each  $b \in B$ . We say that  $f$  *passes through  $x$*  if  $x \in f(B)$ . If for each  $x \in E$  there exists a continuous cross-section of  $\xi$  passing through  $x$ , we say that  $\xi$  *has enough continuous cross-sections* or is *full*.

A *Banach bundle over  $B$*  is a bundle  $\xi = (E, B, \pi)$  over  $B$ , together with operations of addition and scalar multiplication, and norms making each fibre  $E_b$  into a Banach space, and satisfying the following conditions:

- (A1) the function  $E \ni x \mapsto \|x\| \in \mathbb{R}_+$  is continuous;
- (A2) the operation  $+$  is continuous as a function on  $\{(x, y) \in E^2 \mid \pi(x) = \pi(y)\}$  to  $E$ ;
- (A3) for each  $\lambda \in \mathbb{F}$ , the mapping  $E \ni x \mapsto \lambda \cdot x \in E$  is continuous;
- (A4) if  $b \in B$  and  $\{x_\alpha\}_{\alpha \in A}$  is any net in  $E$  such that  $\|x_\alpha\| \rightarrow 0$  and  $\pi(x_\alpha) \rightarrow b$  in  $B$ , then  $x_\alpha \rightarrow 0_b$  in  $E$ .

Here  $+$ ,  $\cdot$ , and  $\|\cdot\|$  are the operations of addition, scalar multiplication, and norm in each fibre  $E_b$ , and  $0_b$  denotes the origin of  $E_b$ .

A related concept is that of a *loose Banach bundle*. A loose Banach bundle differs from an ordinary Banach bundle in that it satisfies a less constraining postulate than (A1), namely the condition that function  $E \ni x \mapsto \|x\| \in \mathbb{R}_+$  be upper semi-continuous (that is to say,  $\{b \in B \mid \|x\| < \delta\}$  should be open in  $B$  for every  $\delta \in \mathbb{R}_+^*$ ). Banach bundles and loose Banach bundles are also known as Banach bundles in the sense of Fell and in the sense of Hofmann, respectively, after J. M. G. Fell and K. H. Hofmann who introduced and extensively studied the respective types of Banach bundle (see [14], [21], [22], [23]). In this paper, no use of loose Banach bundles will be made.

For a detailed exposition of the theory of Banach bundles the reader is referred to [16].

### 5.2. – A class of Banach bundles

It seems instructive to begin a discussion of Banach bundles with an example. We shall construct a class of Banach bundles having enough continuous cross-sections. Some members of this class will intervene in the derivation of a generalisation of the Trotter-Kato theorem to be presented in the next section.

EXAMPLE 5.1. Let  $\Sigma$  be a topological Hausdorff space. For each  $\sigma \in \Sigma$ , let  $E_\sigma$  be a Banach space, and, for each pair  $(\sigma, \tau) \in \Sigma^2$ , let  $\rho_\sigma^\tau$  be an operator in  $\mathcal{L}(E_\tau, E_\sigma)$  such that the following conditions hold:

- (B1)  $\rho_\sigma^\sigma = \text{id}_{E_\sigma}$  for each  $\sigma \in \Sigma$ ;
- (B2)  $\lim_{\mu \rightarrow \sigma} \|\rho_\mu^\sigma \rho_\sigma^\tau x - \rho_\mu^\tau x\| = 0$  for each pair  $(\sigma, \tau) \in \Sigma^2$  and each  $x \in E_\tau$ ;
- (B3)  $\lim_{\sigma \rightarrow \tau} \|\rho_\sigma^\tau x\| = \|x\|$  for each  $\tau \in \Sigma$  and each  $x \in E_\tau$ .

Here the norms involved are understood to be the norms associated with the appropriate Banach spaces. Let  $E$  be the disjoint union  $\bigcup_{\sigma \in \Sigma} E_\sigma$ , and let  $\pi: E \rightarrow \Sigma$  be the mapping defined by the requirement that  $x \in E_{\pi(x)}$  for all  $x \in E$ . We shall convert  $E$  into a total space of a Banach bundle over  $\Sigma$  by introducing on  $E$  a suitable topology.

For each  $x \in E$ , let  $\mathfrak{B}(x)$  be the family of all subsets of  $E$  of the form

$$B(x, U, \epsilon) = \left\{ y \in E \mid \pi(y) \in U, \|y - \rho_{\pi(y)}^{\pi(x)} x\| < \epsilon \right\},$$

where  $U \subset \Sigma$  runs over all open neighbourhoods of  $\pi(x)$ , and  $\epsilon$  runs over all positive numbers. Let  $\mathfrak{T}$  be the family of all unions of elements of  $\bigcup_{x \in E} \mathfrak{B}(x)$ . We claim that  $\mathfrak{T}$  is a topology of  $E$  such that, for each  $x \in E$ ,  $\mathfrak{B}(x)$  is a base for  $\mathfrak{T}$  at  $x$ .

Indeed, in view of (B1), if  $x \in E$ ,  $U \subset \Sigma$  is an open neighbourhood of  $\pi(x)$ , and  $\epsilon \in \mathbb{R}_+^*$ , then  $x \in B(x, U, \epsilon)$ . Furthermore, if  $x \in E$  and  $B(x, U, \epsilon), B(x, V, \delta) \in \mathfrak{B}(x)$ , then  $B(x, W, \eta) \subset B(x, U, \epsilon) \cap B(x, V, \delta)$  provided  $W \subset \Sigma$  is an open neighbourhood of  $\pi(x)$  with  $W \subset U \cap V$ , and  $\eta \in \mathbb{R}_+^*$  satisfies  $\eta < \min\{\epsilon, \delta\}$ . It remains to show that if  $x \in E$ ,  $B(x, U, \epsilon) \in \mathfrak{B}(x)$ , and  $y \in B(x, U, \epsilon)$ , then there exists  $B(y, V, \delta) \in \mathfrak{B}(y)$  such that  $B(y, V, \delta) \subset B(x, U, \epsilon)$ . Let  $x \in E$ ,  $B(x, U, \epsilon) \in \mathfrak{B}(x)$  and  $y \in B(x, U, \epsilon)$ . Choose  $\epsilon' \in \mathbb{R}_+^*$  so that

$$\|y - \rho_{\pi(y)}^{\pi(x)} x\| < \epsilon' < \epsilon$$

and set  $\delta = \frac{1}{2}(\epsilon - \epsilon')$ . By (B2) and (B3), there is an open neighbourhood  $V \subset U$  of  $\pi(y)$  such that

$$\|\rho_{\pi(z)}^{\pi(y)} (y - \rho_{\pi(y)}^{\pi(x)} x)\| < \epsilon' \quad \text{and} \quad \|\rho_{\pi(z)}^{\pi(y)} \rho_{\pi(y)}^{\pi(x)} x - \rho_{\pi(z)}^{\pi(x)} x\| < \delta$$

for all  $z \in \pi^{-1}(V)$ . Now, if  $z \in \pi^{-1}(V)$  is such that  $\|z - \rho_{\pi(z)}^{\pi(y)} y\| < \delta$ , then

$$\begin{aligned} \|z - \rho_{\pi(z)}^{\pi(x)} x\| &\leq \|z - \rho_{\pi(z)}^{\pi(y)} y\| + \|\rho_{\pi(z)}^{\pi(y)} (y - \rho_{\pi(y)}^{\pi(x)} x)\| + \|\rho_{\pi(z)}^{\pi(y)} \rho_{\pi(y)}^{\pi(x)} x - \rho_{\pi(z)}^{\pi(x)} x\| \\ &< \delta + \epsilon' + \delta = \epsilon, \end{aligned}$$

showing that  $B(y, V, \delta) \subset B(x, U, \epsilon)$ . The claim is established.

It is easily seen that  $\pi$  is continuous and open with respect to  $\mathfrak{T}$ , and that all the postulates defining a Banach bundle are met. Thus  $(E, \Sigma, \pi)$  is a Banach bundle over  $\Sigma$ . Hereafter it will be denoted  $\eta$ .

A moment's reflection reveals that a function  $f: \Sigma \rightarrow E$  is a continuous cross-section of  $\eta$  if  $f(\sigma) \in E_\sigma$  and  $\lim_{\tau \rightarrow \sigma} \|\rho_\tau^\sigma f(\sigma) - f(\tau)\| = 0$  for all  $\sigma \in \Sigma$ . It is easily seen that, for each  $x \in E$ , the function  $\rho_x$  given by

$$(5.1) \quad \rho_x(\sigma) = \rho_\sigma^{\pi(x)} x \quad (\sigma \in \Sigma)$$

is a continuous section of  $\eta$  passing through  $x$ . Therefore  $\eta$  has enough continuous cross-sections.

We remark that, in accordance with a fundamental result of A. Douady and L. dal Soglio-Hérault [12], any Banach bundle over a space that is either locally compact or paracompact has enough continuous cross-sections.

**5.3. – Bounded Banach bundle maps**

Let  $\xi = (E, B, \pi)$  a Banach bundle over  $B$ . Given a subset  $M \subset B$  and a cross-section  $f$  of  $\xi$ , let

$$\|f\|_M = \sup_{b \in M} \|f(b)\|.$$

Abbreviate  $\|f\|_B$  to  $\|f\|$ . Let  $\Gamma(\xi)$  be the set of all continuous cross-sections of  $\xi$ . Evidently,  $\Gamma(\xi)$  is a vector space when addition and scalar multiplication are carried out pointwise on  $B$ . The zero element of  $\Gamma(\xi)$  is the zero cross-section  $b \mapsto 0_b$ . A cross-section  $f$  of  $\xi$  is said to be bounded if  $\|f\| < +\infty$ . Let  $\Gamma_b(\xi)$  be the space of all bounded continuous cross-sections of  $\xi$ . Clearly,  $\Gamma_b(\xi) \ni f \mapsto \|f\| \in \mathbb{R}_+$  is a norm under which  $\Gamma_b(\xi)$  is complete (cf. [15, Chap. II, Sec. 13.13]). If for each  $x \in E$  there exists a bounded continuous cross-section of  $\xi$  passing through  $x$ , we say that  $\xi$  has enough bounded continuous cross-sections.

PROPOSITION 5.2. *If  $\xi = (E, B, \pi)$  is a full Banach bundle over a regular topological space  $B$ , then  $\xi$  has enough bounded continuous cross-sections.*

PROOF. Let  $\xi = (E, B, \pi)$  be a full Banach bundle over a regular topological space  $B$ . Given  $x \in E$ , let  $f$  be a continuous cross-section of  $\xi$  passing through  $x$ . By the continuity of the norm, which is guaranteed by postulate (A1), there is an open neighbourhood  $U \subset B$  of  $\pi(x)$  such that  $\|f\|_U < +\infty$ . By the regularity of  $B$ , there is a continuous function  $\psi: B \rightarrow [0, 1]$  such that  $\psi(\pi(x)) = 1$  and  $\psi(B \setminus U) = \{0\}$ . It is now clear that  $\psi f$  is a bounded continuous cross-section of  $\xi$  passing through  $x$ . □

A continuous mapping  $S: E \rightarrow E$  is a *bounded Banach bundle map over  $B$*  if the following conditions are fulfilled:

- (i) for each  $b \in B$ ,  $S(E_b) \subset E_b$  and the restriction  $S_b$  of  $S$  to  $E_b$  is a continuous linear operator;
- (ii)  $\|S\| := \sup_{b \in B} \|S_b\| < +\infty$ .

Let  $\mathcal{L}(\xi)$  be the set of all bounded Banach bundle maps over  $B$ . With addition, multiplication, and scalar multiplication carried out pointwise on each fibre  $E_b$ ,

with the mapping  $b \mapsto \mathbf{0}_b$  as the zero element (here  $\mathbf{0}_b$  denotes the zero operator in  $\mathcal{L}(E_b)$ ), and with the norm  $\mathcal{L}(\xi) \ni S \mapsto \|S\| \in \mathbb{R}_+$ ,  $\mathcal{L}(\xi)$  is a Banach algebra.

Bounded Banach bundle maps are most often constructed by applying the following simple result (cf. [16, Chap. II, Sec. 13.16]):

**PROPOSITION 5.3.** *Let  $\xi = (E, B, \pi)$  be a Banach bundle over  $B$ . Let  $B \ni b \mapsto S_b \in \mathcal{L}(E_b)$  be a mapping satisfying the following conditions:*

- (i)  $\sup_{b \in B} \|S_b\| < +\infty$ ;
- (ii) *there exists a set  $\mathcal{F} \subset \Gamma(\xi)$  such that:*
  - (a) *for each  $b \in B$ ,  $\{f(b) \mid f \in \mathcal{F}\}$  has a dense linear span in  $E_b$ ;*
  - (b) *for each  $f \in \mathcal{F}$ , the mapping  $b \mapsto S_b f(b)$  is a continuous cross-section of  $\xi$ .*

*Then the mapping  $S: E \rightarrow E$  defined by*

$$Sx = S_{\pi(x)}x \quad (x \in E)$$

*is a bounded Banach bundle map over  $B$  and  $\|S\| = \sup_{b \in B} \|S_b\|$ .*

Let  $S \in \mathcal{L}(\xi)$ . Since  $S \circ f \in \Gamma_b(\xi)$  whenever  $f \in \Gamma_b(\xi)$ , setting

$$S_*f = S \circ f \quad (f \in \Gamma_b(\xi))$$

defines a linear operator  $S_*$  in  $\mathcal{L}(\Gamma_b(\xi))$ . Clearly,  $\|S_*\| \leq \|S\|$ .

**PROPOSITION 5.4.** *If  $\xi = (E, B, \pi)$  is a full Banach bundle over a regular topological space  $B$ , then  $\|S_*\| = \|S\|$  for all  $S \in \mathcal{L}(\xi)$ .*

**PROOF.** Let  $\xi = (E, B, \pi)$  be a full Banach bundle over a regular topological space  $B$  and let  $S \in \mathcal{L}(\xi)$ . We have to show that, for each  $b \in B$ ,

$$(5.2) \quad \|S_b\| \leq \|S_*\|.$$

Fixing  $b \in B$  arbitrarily, let  $x \in E_b$ . By Proposition 5.2, there is a bounded continuous cross-section  $f$  passing through  $x$ . By the continuity of the norm ensured by (A1), for each  $\epsilon \in \mathbb{R}_+^\bullet$  there is an open neighbourhood  $U \subset B$  of  $x$  such that  $|\|f\|_U - \|x\|| < \epsilon$ . By the regularity of  $B$ , there is a continuous function  $\psi: B \rightarrow [0, 1]$  such that  $\psi(\pi(x)) = 1$  and  $\psi(B \setminus U) = \{0\}$ . Clearly,  $\psi f$  is a bounded continuous cross-section of  $\xi$  passing through  $x$ , and

$$\|S_b x\| = \|(S_*(\psi f))(b)\| \leq \|S_*\| \|\psi f\| \leq \|S_*\| \|f\|_U \leq \|S_*\| (\|x\| + \epsilon).$$

The arbitrariness of  $\epsilon$  implies that  $\|S_b x\| \leq \|S_*\| \|x\|$ , which in turn establishes (5.2).  $\square$

**5.4. – Banach sub-bundles**

Given two Banach bundles  $\xi = (E, B, \pi)$  and  $\zeta = (F, B, \sigma)$  over  $B$ ,  $\zeta$  is said to be a *Banach sub-bundle* of  $\xi$  if the following conditions hold:

- (i)  $F_b \subset E_b$  for each  $b \in B$  (so that  $F \subset E$ );
- (ii)  $\sigma = \pi \upharpoonright F$ , where  $\pi \upharpoonright F$  denotes the restriction of  $\pi$  to  $F$ ;
- (iii)  $F$  has the relativised topology of  $E$ .

Let  $\xi = (E, B, \pi)$  be a Banach bundle over  $B$ , and, for each  $b \in B$ , let  $F_b$  be a closed linear subspace of the fibre  $E_b$ . Let  $F = \bigcup_{b \in B} F_b$  carry the relativised topology of  $E$ . Then  $\zeta = (F, B, \pi \upharpoonright F)$  satisfies all the postulates defining a Banach bundle except possibly the condition that  $\pi \upharpoonright F$  be open. If  $\pi \upharpoonright F$  is open, then  $\zeta$  is a Banach bundle over  $B$  and also a Banach sub-bundle of  $\xi$ .

The following result often proves useful in establishing the openness of  $\pi \upharpoonright F$ :

**PROPOSITION 5.5.** *Let  $\mathcal{F}$  be a subset of  $\Gamma(\xi)$  such that:*

- (a)  $f(b) \in F_b$  for each  $f \in \mathcal{F}$  and each  $b \in B$ ;
- (b)  $\{f(b) : f \in \mathcal{F}\}$  is norm dense in  $F_b$  for each  $b \in B$ .

*Then  $\pi \upharpoonright F$  is open.*

**PROOF.** Let  $U$  be an open subset of  $F$ . It suffices to show that, for each  $x \in U$ , there exists an open neighbourhood  $V \subset B$  of  $\pi(x)$  such that  $\pi(U) \supset V$ . Fix  $x \in U$  arbitrarily. Since  $\{f(\pi(x)) \mid f \in \mathcal{F}\}$  is dense in  $F_{\pi(x)}$  and the topology of  $E$  relativised to  $E_{\pi(x)}$  coincides with the norm topology of  $E_{\pi(x)}$ , there exists  $f \in \mathcal{F}$  such that  $f(\pi(x)) \in U$ . By the continuity of  $f$ , there exists an open neighbourhood  $V \subset B$  of  $\pi(x)$  such that  $f(V) \subset U$ . Since  $\pi(f(V)) = V$ , we have  $V \subset \pi(U)$ , and so  $V$  turns out to be the desired neighbourhood. □

**5.5. – Representations on Banach bundles and on spaces of cross-sections**

Let  $\mathbf{A}$  be a Banach algebra and let  $\xi = (E, B, \pi)$  be a Banach bundle over  $B$ . A *bounded representation of  $\mathbf{A}$  on  $\xi$*  is a mapping  $\phi : \mathbf{A} \rightarrow \mathcal{L}(\xi)$  satisfying the following conditions:

- (i) for each  $b \in B$ , the mapping  $\phi_b$  defined by

$$\phi_b(a) = (\phi(a))_b \quad (a \in \mathbf{A})$$

is a bounded representation of  $\mathbf{A}$  on  $E_b$ ;

- (ii)  $\|\phi\| := \sup_{b \in B} \|\phi_b\| < +\infty$ .

Any bounded representation  $\phi$  of  $\mathbf{A}$  on  $\xi$  gives rise to a bounded representation  $\phi_*$  of  $\mathbf{A}$  on  $\Gamma_b(\xi)$  defined by

$$\phi_*(a)f = (\phi(a))_*f \quad (a \in \mathbf{A}, f \in \Gamma_b(\xi)).$$

Clearly, the inequality  $\|\phi_*\| \leq \|\phi\|$  holds. The following result can be immediately deduced from Proposition 5.4:



PROPOSITION 5.6. *If  $\xi = (E, B, \pi)$  is a full Banach bundle over a regular topological space  $B$ , then  $\|\phi_*\| = \|\phi\|$  for any bounded representation  $\phi$  of a Banach algebra on  $\xi$ .*

Suppose now that  $\mathbf{A}$  is a Banach algebra with a left bounded approximate identity. Let  $\xi = (E, B, \pi)$  be a Banach bundle over  $B$ , and let  $\phi$  be a bounded representation of  $\mathbf{A}$  on  $\xi$ . Let  $S = \bigcup_{b \in B} \mathcal{R}_{\phi_b}$  carry the relativised topology of  $E$ . When  $\pi \upharpoonright S$  is open,  $S$  is a total space of a Banach bundle  $(S, B, \pi \upharpoonright S)$  over  $B$ , which we shall denote  $\xi_\phi$ .

In many cases arising in applications,  $\pi \upharpoonright S$  is open because of the following result:

PROPOSITION 5.7. *If  $\xi$  has enough bounded continuous cross-sections, then  $\pi \upharpoonright S$  is open.*

PROOF. Let

$$\mathcal{F} = \{f \in \Gamma_b(\xi) : f = \phi_*(a)g \text{ for some } a \in \mathbf{A} \text{ and some } g \in \Gamma_b(\xi)\}.$$

It suffices to show that  $\mathcal{F}$  satisfies conditions (a) and (b) from Proposition 5.5.

Clearly, if  $f \in \mathcal{F}$ , then  $f(b) \in \mathcal{R}_{\phi_b}$  for all  $b \in B$ , so (a) is satisfied.

Let  $x \in \mathcal{R}_{\phi_{\pi(x)}}$  and let  $f$  be a bounded continuous cross-section passing through  $x$ . For each  $\alpha \in A$ ,  $\phi_*(e_\alpha)f$  is a member of  $\mathcal{F}$  and, moreover,  $\lim_{\alpha \in A} (\phi_*(e_\alpha)f)(\pi(x)) = x$ . Therefore (b) is satisfied too.  $\square$

Let the notation  $\Gamma_b(\xi_\phi)$  apply irrespectively of whether or not  $\pi \upharpoonright S$  is open. Set

$$\Gamma_b(\xi_\phi) = \{f \in \Gamma_b(\xi) \mid f(b) \in \mathcal{R}_{\phi_b} \text{ for all } b \in B\}.$$

When  $\pi \upharpoonright S$  is open and  $\xi_\phi$  is a genuine Banach bundle, the new definition coincides with the old one.

The study of  $\Gamma_b(\xi_\phi)$  will occupy us for the rest of this subsection.

THEOREM 5.8. *With  $\mathbf{A}$ ,  $\xi$ , and  $\phi$  as above, we have*

$$(5.3) \quad \mathcal{R}_{\phi_*} \subset \Gamma_b(\xi_\phi)$$

and, for each  $S \in \text{Mul}_l(\mathbf{A})$ , each  $f \in \mathcal{R}_{\phi_*}$ , and each  $b \in B$ ,

$$(5.4) \quad (\widetilde{\phi_*}(S)f)(b) = \widetilde{\phi_b}(S)f(b).$$

PROOF. Inclusion (5.3) is obvious.

To prove (5.4), fix  $f \in \mathcal{R}_{\phi_*}$  and  $b \in B$  arbitrarily. First take  $L_a$  with  $a \in \mathbf{A}$  as  $S$ . Since  $\widetilde{\phi_*}(L_a) = L_{\phi_*(a)}$  and likewise  $\widetilde{\phi_b}(L_a) = L_{\phi_b(a)}$ , we have

$$(\widetilde{\phi_*}(L_a)f)(b) = (\phi_*(a)f)(b) = \phi_b(a)f(b) = \widetilde{\phi_b}(L_a)f(b).$$

Next consider an arbitrary  $S \in \text{Mul}_l(\mathbf{A})$ . Let  $\{a_\alpha\}_{\alpha \in A}$  be a net in  $\mathbf{A}$  such that  $\{L_{a_\alpha}\}_{\alpha \in A}$  converges to  $S$  in the strong operator topology of  $\text{Mul}_l(\mathbf{A})$ .

Theorem 2.4 ensures that  $\widetilde{\phi}_*(S)f = \lim_{\alpha \in A} \widetilde{\phi}_*(L_{a_\alpha})f$  and  $\widetilde{\phi}_b(S)f(b) = \lim_{\alpha \in A} \widetilde{\phi}_b(L_{a_\alpha})f(b)$ . Now, using (5.4) in the special case just considered and taking into account the continuity of the mapping  $\Gamma_b(\xi) \ni g \mapsto g(b) \in E$ , we conclude that

$$\begin{aligned} (\widetilde{\phi}_*(S)f)(b) &= (\lim_{\alpha \in A} \widetilde{\phi}_*(L_{a_\alpha})f)(b) = \lim_{\alpha \in A} (\widetilde{\phi}_*(L_{a_\alpha})f)(b) = \lim_{\alpha \in A} \widetilde{\phi}_b(L_{a_\alpha})f(b) \\ &= \widetilde{\phi}_b(S)f(b). \end{aligned}$$

Thus (5.4) is proved in full generality. □

The following example shows that inclusion (5.3) can be proper.

EXAMPLE 5.9. Equip  $\mathbb{N}$  and  $\mathbb{C}$  with their usual topologies. Let  $E = \mathbb{N} \times \mathbb{C}$ , and let  $\pi_1$  and  $\pi_2$  be the canonical projections from  $E$  onto  $\mathbb{N}$  and  $\mathbb{C}$ , respectively. Clearly,  $\xi = (E, \mathbb{N}, \pi_1)$  is a Banach bundle over  $\mathbb{N}$  with constant fibre  $\mathbb{C}$ . Any cross-section  $f$  of  $E$  is automatically continuous and can be written as

$$f(n) = (n, (\pi_2 \circ f)(n)) \quad (n \in \mathbb{N}).$$

Let  $l^\infty$  be the space of all bounded complex-valued sequences, and let  $c_0$  be the space of all complex-valued sequences converging to zero. It is plain that

$$(5.5) \quad \Gamma_b(\xi) = \{f \in \Gamma(\xi) \mid \pi_2 \circ f \in l^\infty\}.$$

Let  $L^1(\mathbb{T})$  be the algebra of all (classes of) Lebesgue integrable functions on the circle group  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$  with convolution as product. For each  $n \in \mathbb{N}$ , define a representation  $\phi_n$  of  $L^1(\mathbb{T})$  on  $\mathbb{C}$  by

$$\phi_n(a)s = \widehat{a}(n)s \quad (a \in L^1(\mathbb{T}), s \in \mathbb{C})$$

where  $\widehat{a}(n)$  is the  $n$ th Fourier coefficient of  $a$  given by

$$\widehat{a}(n) = \int_{\mathbb{T}} a(t)t^{-n} dt.$$

It is clear that  $\|\phi_n\| \leq 1$  for all  $n \in \mathbb{N}$ . Given  $a \in L^1(\mathbb{T})$ , define a mapping  $\phi(a): \xi \rightarrow \xi$  by

$$\phi(a)x = (\pi_1(x), \phi_{\pi_1(x)}(a)\pi_2(x)) \quad (x \in E).$$

Applying Proposition 5.3 with  $\mathcal{F}$  equal to  $\Gamma_b(\xi)$ , we find that each  $\phi(a)$  is a Banach bundle map over  $\mathbb{N}$ . Thus the mapping  $\phi: a \mapsto \phi(a)$  is a bounded representation of  $L^1(\mathbb{T})$  on  $\xi$ . It induces a bounded representation  $\phi_*$  of  $L^1(\mathbb{T})$  on  $\Gamma_b(\xi)$ . A moment's consideration reveals that

$$(5.6) \quad \{f \in \Gamma_b(\xi) \mid \pi_2 \circ f \in c_0\} \subset \mathcal{R}_{\phi_*}.$$

We next show that the converse inclusion

$$(5.7) \quad \mathcal{R}_{\phi_*} \subset \{f \in \Gamma_b(\xi) \mid \pi_2 \circ f \in c_0\}$$

also holds true. Let  $M(\mathbb{T})$  be the algebra of all bounded Borel measures on  $\mathbb{T}$  with convolution as multiplication. For each  $\mu \in M(\mathbb{T})$ , let  $T_\mu$  be the operator in  $\mathcal{L}(L^1(\mathbb{T}))$  given by

$$T_\mu a = \mu * a \quad (a \in L^1(\mathbb{T})).$$

By Wendel's theorem mentioned earlier,  $\text{Mul}(L^1(\mathbb{T}))$  coincides with the set  $\{T_\mu \mid \mu \in M(\mathbb{T})\}$ . It is easily verified that, for each  $n \in \mathbb{N}$  and each  $t \in \mathbb{T}$ ,

$$\phi_n(T_{\delta_t} a) = t^n \phi_n(a) \quad (a \in L^1(\mathbb{T})),$$

whence

$$(5.8) \quad \widetilde{\phi}_n(T_{\delta_t})s = t^n s \quad (s \in \mathbb{C}).$$

Now, by the analogue of Theorem 3.3 for the algebra  $L^1(\mathbb{T})$  (see [24, pp. 606-607]), the mapping  $t \mapsto \widetilde{\phi}_*(T_{\delta_t})$  is a strongly continuous representation of  $\mathbb{T}$  on  $\mathcal{R}_{\phi_*}$ , and so, if  $f \in \mathcal{R}_{\phi_*}$ , then

$$\lim_{t \rightarrow 1} \|\widetilde{\phi}_*(T_{\delta_t})f - f\| = 0.$$

This together with (5.8) yields

$$\limsup_{t \rightarrow 1} \sup_{n \in \mathbb{N}} |(\pi_2 \circ f)(n)| |t^n - 1| = 0.$$

In particular

$$\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} |(\pi_2 \circ f)(n)| |e^{\pi i n/m} - 1| = 0.$$

But, if  $n = m$ , then  $|e^{\pi i n/m} - 1| = 2$ , and so  $\pi_2 \circ f \in c_0$ , proving (5.7).

In view of (5.6) and (5.7),

$$\mathcal{R}_{\phi_*} = \{f \in \Gamma_b(\xi) \mid \pi_2 \circ f \in c_0\}.$$

Comparing this equality with (5.5), we see that in the present setting containment (5.3) is proper.

With future applications in mind, we now indicate one instance when inclusion (5.3) becomes equality. To this end, we first prepare the following:

PROPOSITION 5.10. *Let  $\{e_i\}_{i \in I}$  be a subnet of the approximate identity  $\{e_\alpha\}_{\alpha \in A}$  of  $\mathbf{A}$  and let  $\{x_i\}_{i \in I}$  be an  $I$ -indexed net in  $E$  converging to an element  $x$  of  $\mathcal{R}_{\phi_\pi(x)}$ . Then*

$$\lim_{i \in I} \|\phi(e_i)x_i - x_i\| = 0.$$

PROOF. Fix  $\epsilon \in \mathbb{R}_+^\bullet$  arbitrarily. Since  $x \in \mathcal{R}_{\phi_\pi(x)}$ , it follows from (2.4) that there exists  $a \in \mathbf{A}$  such that

$$(5.9) \quad \|\phi(a)x - x\| < \epsilon.$$

Since the mapping  $E \ni y \mapsto \phi(a)y \in E$  is continuous, it follows from (A2) and (A3) that also the mapping  $E \ni y \mapsto \phi(a)y - y \in E$  is continuous. In view of (5.9) and (A1), there exists  $\iota_1$  such that, for each  $\iota$  with  $\iota_1 \prec \iota$ ,

$$(5.10) \quad \|\phi(a)x_\iota - x_\iota\| < \epsilon.$$

Hence, for each  $\iota$  with  $\iota_1 \prec \iota$ ,

$$(5.11) \quad \|\phi(e_\iota a)x_\iota - \phi(e_\iota)x_\iota\| \leq \|\phi_{\pi(x_\iota)}(e_\iota)\| \|\phi(a)x_\iota - x_\iota\| \leq \|\phi\| K \epsilon,$$

where, of course,  $K = \sup_{\alpha \in A} \|e_\alpha\|$ . Let  $\iota_2$  be such that, for each  $\iota$  with  $\iota_2 \prec \iota$ ,

$$\|e_\iota a - a\| < \epsilon.$$

In view of (A1), there exists  $\iota_3$  such that, for each  $\iota$  with  $\iota_3 \prec \iota$ ,

$$\|x_\iota\| < \|x\| + \epsilon.$$

Let  $\iota_4$  be a common successor of  $\iota_2$  and  $\iota_3$ . Clearly, if  $\iota$  is such that  $\iota_4 \prec \iota$ , then

$$(5.12) \quad \|\phi(e_\iota a)x_\iota - \phi(a)x_\iota\| \leq \|\phi\| \|e_\iota a - a\| \|x_\iota\| \leq \|\phi\| \epsilon (\|x\| + \epsilon).$$

Let  $\iota_5$  be a common successor of  $\iota_1$  and  $\iota_4$ . Combining (5.10), (5.11) and (5.12), we see that, for each  $\iota$  with  $\iota_5 \prec \iota$ ,

$$\begin{aligned} \|\phi(e_\iota)x_\iota - x_\iota\| &\leq \|\phi(a)x_\iota - x_\iota\| + \|\phi(e_\iota a)x_\iota - \phi(a)x_\iota\| \\ &\quad + \|\phi(e_\iota a)x_\iota - \phi(e_\iota)x_\iota\| \\ &< \epsilon(1 + \|\phi\| (\|x\| + K + \epsilon)), \end{aligned}$$

whence the result. □

THEOREM 5.11. *If  $B$  is compact, then*

$$(5.13) \quad \mathcal{R}_{\phi_*} = \Gamma_b(\xi_\phi).$$

PROOF. In view of Theorem 5.8, we need only to show that  $\Gamma_b(\xi_\phi) \subset \mathcal{R}_{\phi_*}$ . Suppose on the contrary that there is  $f \in \Gamma_b(\xi_\phi) \setminus \mathcal{R}_{\phi_*}$ . Then there exists a positive number  $\epsilon$ , a subset  $A' \subset A$  cofinal in  $A$ , and a net  $\{b_\alpha\}_{\alpha \in A'}$  in  $B$  such that

$$(5.14) \quad \|(\phi_*(e_\alpha)f)(b_\alpha) - f(b_\alpha)\| = \|\phi(e_\alpha)f(b_\alpha) - f(b_\alpha)\| \geq \epsilon$$

for each  $\alpha \in A'$ . Since  $B$  is compact, the net  $\{b_\alpha\}_{\alpha \in A'}$  has a subnet converging to a point  $b \in B$ . More specifically, there exists a directed set  $I$  and a mapping  $k: I \rightarrow A'$  such that  $k$  is non-decreasing (i.e.  $\iota_1 < \iota_2$  implies  $k(\iota_1) < k(\iota_2)$ ),  $k(I)$  is cofinal in  $A'$ , and the net  $\{b_{k(\iota)}\}_{\iota \in I}$  converges to  $b$ . For each  $\iota \in I$ , put  $x_\iota = f(b_{k(\iota)})$ ,  $x = f(b)$ , and  $e_\iota = e_{k(\iota)}$ . Since  $f$  is continuous,  $\{x_\iota\}_{\iota \in I}$  converges to  $x$ . Moreover, by (5.14), we have  $\|\phi(e_\iota)x_\iota - x_\iota\| \geq \epsilon$  for each  $\iota \in I$ . But this is impossible in view of Proposition 5.10. The result follows.  $\square$

### 5.6. – An abstract Trotter-Kato theorem

Let  $\xi = (E, B, \pi)$  be a Banach bundle over  $B$ . A family  $\{\rho_x\}_{x \in E}$  of continuous cross-sections of  $\xi$  is termed *admissible* if the following condition is satisfied: for every  $x \in E$  and every  $\epsilon \in \mathbb{R}_+^*$  there exists an open neighbourhood  $V \subset B$  of  $\pi(x)$  and a positive number  $\delta$  such that if  $y \in E_{\pi(x)}$  satisfies  $\|x - y\| < \delta$ , then  $\|\rho_x - \rho_y\|_V < \epsilon$ . Note that any Banach bundle with an admissible family of cross-sections is full.

EXAMPLE 5.12. Let  $\Sigma$  be a locally compact space, and let  $\eta = (E, \Sigma, \pi)$  be the Banach bundle over  $\Sigma$  from the class constructed in Example 5.1. For each  $x \in E$ , let  $\rho_x$  be the continuous cross-section of  $\eta$  given by (5.1). We contend that the family  $\{\rho_x\}_{x \in \Sigma}$  is admissible. Indeed, for each  $x \in E$ , the function  $\sigma \mapsto \|\rho_x(\sigma)\|$  is continuous and hence bounded on every compact subset of  $\Sigma$ . In particular, for every  $\sigma \in \Sigma$  and every open relatively compact neighbourhood  $V \subset \Sigma$  of  $\sigma$ , we have  $\sup_{\tau \in V} \|\rho_\tau^\sigma x\| < +\infty$  whatever  $x \in E_\sigma$ . By the Banach-Steinhaus theorem,  $L_{\sigma,V} := \sup_{\tau \in V} \|\rho_\tau^\sigma\|$  is finite. Accordingly, for every  $\epsilon \in \mathbb{R}_+^*$ , every  $\sigma \in \Sigma$ , and every relatively compact open neighbourhood  $V \subset \Sigma$  of  $\sigma$ , if we let  $\delta = \epsilon/L_{\sigma,V}$ , then, for all  $x, y \in E_\sigma$  with  $\|x - y\| < \delta$ ,

$$\|\rho_x - \rho_y\|_V \leq L_{\sigma,V} \|x - y\| < \epsilon.$$

This establishes the contention.

PROPOSITION 5.13. *Let  $\xi = (E, B, \pi)$  be a Banach bundle, and let  $\{\rho_x\}_{x \in E}$  be an admissible family of continuous cross-sections of  $\xi$ . Then, for each compact subset  $\mathcal{K}$  of  $\Gamma_b(\xi)$  and each  $b \in B$ ,*

$$(5.15) \quad \limsup_{c \rightarrow b, f \in \mathcal{K}} \|f(c) - \rho_{f(b)}(c)\| = 0.$$

PROOF. If  $f \in \Gamma(\xi)$ , then

$$\lim_{c \rightarrow b} f(c) = f(b) = \lim_{c \rightarrow b} \rho_{f(b)}(c),$$

and so

$$(5.16) \quad \lim_{c \rightarrow b} \|f(c) - \rho_{f(b)}(c)\| = 0.$$

Fix  $\epsilon \in \mathbb{R}_+^*$  arbitrarily. Let  $f \in \mathcal{K}$ . In view of (5.16), there exists an open neighbourhood  $U \subset B$  of  $b$  such that

$$\|f - \rho_{f(b)}\|_U < \epsilon/3.$$

Since  $\{\rho_x\}_{x \in E}$  is admissible, there exists an open neighbourhood  $V \subset B$  of  $b$  and a positive number  $\delta$  such that if  $y \in E_b$  satisfies  $\|f(b) - y\| < \delta$ , then  $\|\rho_{f(b)} - \rho_y\|_V < \epsilon/3$ . Let  $\eta = \min(\delta, \epsilon/3)$ , and let

$$\mathcal{U}(f) = \{g \in \mathcal{K} \mid \|g - f\| < \eta\}.$$

Clearly,  $\mathcal{U}(f)$  is an open neighbourhood of  $f$  in  $\mathcal{K}$ . If  $g \in \mathcal{U}(f)$ , then  $\|f(b) - g(b)\| < \eta \leq \delta$ , and so  $\|\rho_{f(b)} - \rho_{g(b)}\|_V < \epsilon/3$ . Thus, letting  $W = U \cap V$ , we have

$$\begin{aligned} \|g - \rho_{g(b)}\|_W &\leq \|g - f\|_W + \|f - \rho_{f(b)}\|_W + \|\rho_{f(b)} - \rho_{g(b)}\|_W \\ &\leq \|g - f\| + \|f - \rho_{f(b)}\|_U + \|\rho_{f(b)} - \rho_{g(b)}\|_V < \epsilon. \end{aligned}$$

Since  $\mathcal{K}$  is compact, we can cover  $\mathcal{K}$  with a finite number of the  $\mathcal{U}(f)$ , say  $U(f_1), \dots, U(f_k)$ . For each  $i \in \{1, \dots, k\}$ , let  $W_i$  be an open neighbourhood of  $b$  such that

$$\|g - \rho_{g(b)}\|_{W_i} < \epsilon$$

for all  $g \in \mathcal{U}(f_i)$ . Let  $Z = \bigcap_{i=1}^n W_i$ . Obviously,

$$\|g - \rho_{g(b)}\|_Z < \epsilon$$

for all  $g \in \mathcal{K}$ , which completes the proof. □

We are now in a position to state the main result of this section.

**THEOREM 5.14 (Abstract Trotter-Kato Theorem).** *Let  $\mathbf{A}$  be a Banach algebra with a left bounded approximate identity, let  $\xi = (E, B, \pi)$  be a Banach bundle, and let  $\{\rho_x\}_{x \in E}$  be an admissible family of continuous cross-sections of  $\xi$ . Then, for each  $f \in \Gamma_b(\xi)$ , each subset  $S \subset \text{Mul}_l(\mathbf{A})$  that is compact in the strong operator topology, and each  $b \in B$ ,*

$$(5.17) \quad \limsup_{c \rightarrow b} \sup_{S \in \mathcal{S}} \|\widetilde{\phi}_c(S)f(c) - \rho_{\widetilde{\phi}_b(S)f(b)}(c)\| = 0.$$

PROOF. Endow  $\text{Mul}_l(\mathbf{A})$  with the strong operator topology. Let  $f \in \Gamma_b(\xi_\phi)$ , let  $S$  be a compact subset  $\text{Mul}_l(\mathbf{A})$ , and let  $c \in B$ . According to Theorem 2.4, the mapping  $\text{Mul}_l(\mathbf{A}) \ni S \mapsto \widetilde{\phi}_*(S)f \in \Gamma_b(\xi_\phi)$  is continuous. Therefore  $\widetilde{\phi}_*(S)f$  is compact.

In view of Proposition 5.13,

$$\limsup_{c \rightarrow b} \sup_{S \in \mathcal{S}} \left\| (\widetilde{\phi}_*(S)f)(c) - \rho_{(\widetilde{\phi}_*(S)f)(b)}(c) \right\| = 0.$$

Now, to complete the proof, it suffices to note that, in view of Theorem 5.8,

$$(\widetilde{\phi}_*(S)f)(d) = \widetilde{\phi}_d(S)f(d)$$

for each  $S \in \mathcal{S}$  and each  $d \in B$ . □

## 6. – A generalisation of the Trotter-Kato theorem

In this section, we establish a result that generalises simultaneously the classical version and Kisyński's version of the Trotter-Kato theorem. While this generalisation will draw on the abstract Trotter-Kato theorem from the foregoing section, it will be formulated in a bundle-free way.

### 6.1. – Classical and bundle-theoretic set-ups

We start by describing the set-up within which the Trotter-Kato theorem was originally developed. This set-up was first adopted by H. P. Trotter [42] and was later exploited by T. G. Kurtz [29], [30] and other authors.

Let  $\{\infty\}$  be a singleton set disjoint from  $\mathbb{N}$ , and let  $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ . An *approximation system* is a pair of sequences  $(\{E_n\}_{n \in \overline{\mathbb{N}}}, \{P_n\}_{n \in \mathbb{N}})$ , where, for each  $n \in \overline{\mathbb{N}}$ ,  $E_n$  is a Banach space, and, for each  $n \in \mathbb{N}$ ,  $P_n$  is an operator in  $\mathcal{L}(E_\infty, E_n)$  such that

$$(6.1) \quad \lim_{n \rightarrow \infty} \|P_n x\| = \|x\| \quad (x \in E_\infty).$$

The norms involved here are, of course, the norms associated with the appropriate Banach spaces. The spaces  $E_n$  ( $n \in \mathbb{N}$ ) can be viewed as successive approximations of the space  $E_\infty$ . The approximation process is then governed by the operators  $P_n$  ( $n \in \mathbb{N}$ ). Note that, in view of the Banach-Steinhaus theorem, condition (6.1) entails the finiteness of  $\sup_{n \in \mathbb{N}} \|P_n\|$ . We say that a sequence  $\{x_n\}_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} E_n$  converges to  $x \in E_\infty$  and write  $x = \lim_{n \rightarrow \infty} x_n$  if

$$\lim_{n \rightarrow \infty} \|P_n x - x_n\| = 0.$$

Denote by  $c(\{E_n\}_{n \in \bar{\mathbb{N}}})$  the space of convergent sequences in  $\prod_{n \in \bar{\mathbb{N}}} E_n$  appended by their respective limits; that is,

$$c(\{E_n\}_{n \in \bar{\mathbb{N}}}) = \left\{ \{x_n\}_{n \in \bar{\mathbb{N}}} \in \prod_{n \in \bar{\mathbb{N}}} E_n \mid \lim_{n \rightarrow \infty} \|P_n x_\infty - x_n\| = 0 \right\}.$$

Equipped with the norm  $\|\{x_n\}_{n \in \bar{\mathbb{N}}}\| = \sup_{n \in \bar{\mathbb{N}}} \|x_n\|$ ,  $c(\{E_n\}_{n \in \bar{\mathbb{N}}})$  is a Banach space.

We now reformulate the above definitions in the language of Banach bundles. We first note that  $\bar{\mathbb{N}}$  can be converted into a compact space by enriching it with the topology uniquely determined by the following conditions: 1° all singleton sets contained in  $\mathbb{N}$  are open; 2° the subsets of  $\bar{\mathbb{N}}$  whose complements are finite subsets of  $\mathbb{N}$  form a neighbourhood basis of  $\infty$ . From now on we shall assume that  $\bar{\mathbb{N}}$  is endowed with this topology. When  $\mathbb{N}$  is equipped with the discrete topology making it a locally compact space,  $\bar{\mathbb{N}}$  is the Alexandroff (one-point) compactification of  $\mathbb{N}$ .

Let  $(\{E_n\}_{n \in \bar{\mathbb{N}}}, \{P_n\}_{n \in \mathbb{N}})$  be an approximation system. For each pair  $(m, n) \in \bar{\mathbb{N}}^2$ , define an operator  $\rho_n^m$  in  $\mathcal{L}(E_m, E_n)$  by

$$(6.2) \quad \rho_n^m = \begin{cases} \text{id}_{E_m} & \text{if } n = m, \\ 0 & \text{if } m \in \mathbb{N} \text{ and } n \neq m, \\ P_n & \text{if } m = \infty \text{ and } n \neq m. \end{cases}$$

A straightforward verification shows that the family  $\{\rho_n^m\}_{(m,n) \in \bar{\mathbb{N}}^2}$  satisfies conditions (B1-B3) from Example 5.1, the fulfillment of condition (B3) being a consequence of (6.1). As shown in Example 5.1, associated with  $\{\rho_n^m\}_{(m,n) \in \bar{\mathbb{N}}^2}$  is a Banach bundle  $\eta = (E, \bar{\mathbb{N}}, \pi)$  over  $\bar{\mathbb{N}}$ . The compactness of the base  $\bar{\mathbb{N}}$  implies that every continuous cross-section of  $\eta$  is bounded. In view of the characterisation of continuous cross-sections of  $\eta$  given in Example 5.1, any continuous cross-sections of  $\eta$  can be identified with a sequence in  $c(\{E_n\}_{n \in \bar{\mathbb{N}}})$  and *vice versa*. It is also immediately seen that the norms of  $\Gamma_b(\eta)$  and  $c(\{E_n\}_{n \in \bar{\mathbb{N}}})$  coincide, and so the Banach spaces  $\Gamma_b(\eta)$  and  $c(\{E_n\}_{n \in \bar{\mathbb{N}}})$  are isometric. From now on we shall simply identify  $\Gamma_b(\eta)$  with  $c(\{E_n\}_{n \in \bar{\mathbb{N}}})$ .

### 6.2. – Representations on $c(\{E_n\}_{n \in \bar{\mathbb{N}}})$

Throughout the rest of the paper  $(\{E_n\}_{n \in \bar{\mathbb{N}}}, \{P_n\}_{n \in \mathbb{N}})$  will be a fixed approximation system.

Let  $\mathbf{A}$  be a Banach algebra with a left bounded approximate identity. Suppose that, for each  $n \in \bar{\mathbb{N}}$ ,  $\phi_n$  is a continuous representation of  $\mathbf{A}$  on  $E_n$  such that the following conditions are satisfied:

- (i)  $\|\phi\| := \sup_{n \in \bar{\mathbb{N}}} \|\phi_n\| < +\infty$ ;
- (ii)  $\phi_\infty(a)x = \lim_{n \rightarrow \infty} \phi_n(a)P_n x$  for each  $a \in \mathbf{A}$  and each  $x \in E_\infty$ .



Given  $a \in \mathbf{A}$  and  $x \in \eta$ , set

$$\phi(a)x = \phi_{\pi(x)}(a)x.$$

We claim that, for each  $a \in \mathbf{A}$ ,  $\phi(a): \eta \rightarrow \eta$  is a bounded Banach bundle map over  $\overline{\mathbb{N}}$ . To establish the claim, we apply Proposition 5.3 taking for  $\mathcal{F}$  the set  $\{\rho_x \mid x \in E\}$ , where the  $\rho_x$  are given by (5.1). All that we need to check is that  $\{\phi_n(a)\rho_x(n)\}_{n \in \overline{\mathbb{N}}}$  is in  $c(\{E_n\}_{n \in \overline{\mathbb{N}}})$  for every  $x \in E$ . Note that, given  $x \in E$ , if  $\pi(x) \in \mathbb{N}$ , then

$$\rho_x(n) = \begin{cases} x & \text{if } n = \pi(x), \\ 0 & \text{if } n \in \overline{\mathbb{N}} \setminus \{\pi(x)\}, \end{cases}$$

and if  $\pi(x) = \infty$ , then

$$\rho_x(n) = \begin{cases} x & \text{if } n = \pi(x), \\ P_n x & \text{if } n \in \overline{\mathbb{N}} \setminus \{\pi(x)\}. \end{cases}$$

Thus, if  $\pi(x) \in \mathbb{N}$ , then  $\phi_n(a)\rho_x(n) = 0$  for all  $n \in \overline{\mathbb{N}} \setminus \{\pi(x)\}$ , and so  $\{\phi_n(a)\rho_x(n)\}_{n \in \overline{\mathbb{N}}}$  is in  $c(\{E_n\}_{n \in \overline{\mathbb{N}}})$ ; and if  $\pi(x) = \infty$ , then the sequence  $\{\phi_n(a)P_n x\}_{n \in \mathbb{N}}$  converges to  $\phi_\infty(a)x$ , and so again  $\{\phi_n(a)\rho_x(n)\}_{n \in \overline{\mathbb{N}}}$  is in  $c(\{E_n\}_{n \in \overline{\mathbb{N}}})$ , as was to be verified.

It is clear that the mapping  $\phi: a \mapsto \phi(a)$  is a bounded representation of  $\mathbf{A}$  on  $\eta$ . In view of the identification of  $\Gamma_b(\eta)$  with  $c(\{E_n\}_{n \in \overline{\mathbb{N}}})$ , the corresponding representation  $\phi_*$  can be written as

$$\phi_*(a)(\{x_n\}_{n \in \overline{\mathbb{N}}}) = \{\phi_n(a)x_n\}_{n \in \overline{\mathbb{N}}}$$

for all  $a \in \mathbf{A}$  and all  $\{x_n\}_{n \in \overline{\mathbb{N}}} \in c(\{E_n\}_{n \in \overline{\mathbb{N}}})$ . Since  $\eta$  is full and  $\overline{\mathbb{N}}$  is compact, it follows from Proposition 5.6 that

$$(6.3) \quad \|\phi_*\| = \|\phi\|.$$

Another consequence of  $\eta$  being full and  $\overline{\mathbb{N}}$  being compact is, in view of Proposition 5.2, that  $\eta$  has enough bounded continuous sections. Furthermore, by Proposition 5.7,  $\eta_{\phi_*}$  is a genuine Banach bundle and a sub-bundle of  $\eta$ . Finally, Theorem 5.11 guarantees that  $\mathcal{R}_{\phi_*} = \Gamma_b(\eta_{\phi_*})$ , or equivalently

$$(6.4) \quad \mathcal{R}_{\phi_*} = \left\{ \{x_n\}_{n \in \overline{\mathbb{N}}} \in c(\{E_n\}_{n \in \overline{\mathbb{N}}}) \mid x_n \in \mathcal{R}_{\phi_n} \text{ for all } n \in \overline{\mathbb{N}} \right\}.$$

### 6.3. – A generalised Trotter-Kato theorem

Suppose that, for each  $n \in \mathbb{N}$ ,  $R_n = \{R_{\lambda,n}\}_{\lambda \in (w, +\infty)}$  is a pseudo-resolvent in  $\mathcal{L}(E_n)$  such that the following conditions are fulfilled:

(R1)  $c_R := \sup_{n \in \mathbb{N}} c_{R_n} < +\infty$ ;

(R2) for each  $\lambda \in (w, +\infty)$  and each  $x \in E_\infty$ , the sequence  $\{R_{\lambda,n}P_n x\}_{n \in \mathbb{N}}$  is convergent.

For each  $\lambda \in (w, +\infty)$  and each  $x \in E_\infty$ , set

$$(6.5) \quad R_{\lambda, \infty}x = \lim_{n \rightarrow \infty} R_{\lambda, n}P_nx.$$

It is easily verified that  $R_\infty = \{R_{\lambda, \infty}\}_{\lambda \in (w, +\infty)}$  is a pseudo-resolvent in  $\mathcal{L}(E_\infty)$  with  $c_{R_\infty} \leq c_R$ .

In view of Theorem 1.1, corresponding to each  $R_n$  ( $n \in \overline{\mathbb{N}}$ ) there is a continuous representation  $\phi_n$  of  $L_w^1(\mathbb{R}_+)$  such that  $R_{\lambda, n} = \phi_n(\epsilon_{-\lambda})$  for all  $\lambda \in (w, +\infty)$  and  $\|\phi_n\| = c_{R_n}$ . Continuing to denote  $\sup_{n \in \overline{\mathbb{N}}} \|\phi_n\|$  by  $\|\phi\|$ , we clearly have

$$(6.6) \quad \|\phi\| = c_R.$$

Now (6.5) can be rewritten as

$$\lim_{n \rightarrow \infty} \phi_n(\epsilon_{-\lambda})P_nx = \phi_\infty(\epsilon_{-\lambda})x$$

for all  $x \in E_\infty$  and all  $\lambda \in (w, +\infty)$ . Since, on account of (6.6), the mappings  $\phi_n$  ( $n \in \overline{\mathbb{N}}$ ) are equibounded and the set  $\{\epsilon_{-\lambda} \mid \lambda \in (w, +\infty)\}$  is linearly dense in  $L_w^1(\mathbb{R}_+)$ , the last equality extends to

$$\lim_{n \rightarrow \infty} \phi_n(f)P_nx = \phi_\infty(f)x$$

for all  $x \in E_\infty$  and all  $f \in L_w^1(\mathbb{R}_+)$ .

Applying the material from the previous subsection, we can now construct a bounded representation  $\phi: f \mapsto \phi(f)$  of  $L_w^1(\mathbb{R}_+)$  on  $\eta$ , and can subsequently form the associated bounded representation  $\phi_*$  of  $L_w^1(\mathbb{R}_+)$  on  $c(\{E_n\}_{n \in \overline{\mathbb{N}}})$  defined by

$$\phi_*(f)(\{x_n\}_{n \in \overline{\mathbb{N}}}) = \{\phi_n(f)x_n\}_{n \in \overline{\mathbb{N}}}$$

for all  $f \in L_w^1(\mathbb{R}_+)$  and all  $\{x_n\}_{n \in \overline{\mathbb{N}}} \in c(\{E_n\}_{n \in \overline{\mathbb{N}}})$ . In view of (6.3) and (6.6), we have  $\|\phi_*\| = c_R$ . Taking into account (6.4) and the fact that  $\mathcal{R}_{R_n} = \mathcal{R}_{\phi_n}$  for each  $n \in \overline{\mathbb{N}}$ , we can represent  $\mathcal{R}_{\phi_*}$  as

$$(6.7) \quad \mathcal{R}_{\phi_*} = \left\{ \{x_n\}_{n \in \overline{\mathbb{N}}} \in c(\{E_n\}_{n \in \overline{\mathbb{N}}}) \mid x_n \in \mathcal{R}_{R_n} \text{ for each } n \in \overline{\mathbb{N}} \right\}.$$

In accordance with Theorem 4.1, corresponding to each  $\phi_n$  ( $n \in \overline{\mathbb{N}}$ ) there is a strongly continuous semigroup  $\{S_{t, n}\}_{t \in \mathbb{R}_+}$  on  $\mathcal{R}_{R_n}$  enjoying the following properties:

- (i)  $S_{t, n}\phi_n(f)x = \phi_n(T_{\delta_t}f)x$  for  $t \in \mathbb{R}_+$ ,  $f \in L_w^1(\mathbb{R}_+)$ , and  $x \in \mathcal{R}_{R_n}$ ;
- (ii)  $\|S_{t, n}\| \leq c_{R_n}e^{wt}$  for  $t \in \mathbb{R}_+$ ;
- (iii)  $R_{\lambda, n}x = \int_{\mathbb{R}_+} e^{-\lambda t} S_{t, n}x dt$  for  $\lambda \in (w, +\infty)$  and  $x \in \mathcal{R}_{R_n}$ .

Recall that  $S_{t, n} = \tilde{\phi}_n(T_{\delta_t})$  for each  $n \in \overline{\mathbb{N}}$  and each  $t \in \mathbb{R}_+$ . As we already know, the mapping  $t \mapsto T_{\delta_t}$  is continuous under the strong operator topology on

$\text{Mul}(L_w^1(\mathbb{R}_+))$ . Therefore, for each  $\tau \in \mathbb{R}_+$ ,  $\{T_{\delta_s} : s \in [0, \tau]\}$  is a compact subset of  $\text{Mul}(L_w^1(\mathbb{R}_+))$ . From Example 5.12 we know that  $\{\rho_x\}_{x \in \Sigma}$  is an admissible family of cross-sections of  $\eta$ . Combining these observations with Theorem 5.14, we finally conclude that, for each  $\{x_n\}_{n \in \mathbb{N}} \in \mathcal{R}_{\phi_*}$  and each  $\tau \in \mathbb{R}_+$ ,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, \tau]} \|S_{t,n}x_n - P_n S_{t,\infty}x_\infty\| = 0.$$

We have thus proved the following result:

**THEOREM 6.1 (Generalised Trotter-Kato Theorem).** *Let  $(\{E_n\}_{n \in \mathbb{N}}, \{P_n\}_{n \in \mathbb{N}})$  be an approximation system. Suppose that, for each  $n \in \mathbb{N}$ ,  $R_n = \{R_{\lambda,n}\}_{\lambda \in (w, +\infty)}$  is a pseudo-resolvent in  $\mathcal{L}(E_n)$  such that conditions (R1) and (R2) hold. Let  $R_{\lambda,\infty}$  be a pseudo-resolvent in  $\mathcal{L}(E_\infty)$  defined by (6.5). For each  $n \in \mathbb{N}$ , let  $\phi_n$  be the representation of  $L_w^1(\mathbb{R}_+)$  associated with  $R_{\lambda,\infty}$  and let  $\{S_{t,n}\}_{t \in \mathbb{R}_+}$  be the semigroup associated with  $\phi_n$ . Then, for each  $\tau \in \mathbb{R}_+$  and each  $\{x_n\}_{n \in \mathbb{N}} \in \mathcal{R}_{\phi_*}$ , where  $\mathcal{R}_{\phi_*}$  is as in (6.7),*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, \tau]} \|S_{t,n}x_n - P_n S_{t,\infty}x_\infty\| = 0.$$

We bring the paper to an end by pointing out the relation between Theorem 6.1 and the classical version and Kisyński's version of the Trotter-Kato theorem. The classical statement is a special instance of Theorem 6.1 concerning the case in which, for each  $n \in \mathbb{N}$ , the pseudo-resolvent  $R_n$  has a dense image in the respective Banach space  $E_n$ , and hence is a resolvent of the generator of  $\{S_{t,n}\}_{t \in \mathbb{R}_+}$  (recall the discussion following the proof of Theorem 4.1). Kisyński's result is in turn a special instance of Theorem 6.1 treating the case in which all the spaces  $E_n$  ( $n \in \mathbb{N}$ ) coincide.

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