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Asymptotic Behaviour of the Porous Media Equation in an Exterior Domain

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Abstract. We study the asymptotic behaviour of weak solutions $u(x, t)$ to the porous media equation in exterior domains with nontrivial boundary data which are constant in time. We prove that, when the space dimension is greater than one, this behaviour is given in the interior of the positivity set by a function, $P(x)$, which has the same value as u in the fixed boundary and such that its m-th power, $P^m(x)$, is harmonic in the exterior domain. We also prove that near the free boundary the asymptotic behaviour is given by a radial, self-similar solution of the porous media equation which is singular at the origin for all times. There is a whole family of such singular self-similar solutions. The precise one giving the asymptotic behaviour is determined through a process of matched asymptotics. We also show that the free boundary approaches a sphere as $t \to \infty$, and give the asymptotic growth rate for the radius. While it is well-known that for $n = 1$ the interface location is given by the classical 1/2-law, $|x(t)| \sim C\, t^{1/2}$, this is no more true for $n > 1$: we prove that in dimension $n \geq 3$ $|x(t)| \sim C t^{m/(n(m-1)+2)}$, and $|x(t)| \sim C t^{1/2}(\log t)^{-(m-1)/(2m)}$ for $n = 2$.
The self-similar solutions giving the asymptotic behaviour close to the free boundary are a particular instance of a larger class of compactly supported, singular, self-similar solutions of the porous media equation. We study such solutions and classify them in terms of their singular behaviour at the origin. Finally, we show that as $m \to \infty$ there is convergence to the Hele-Shaw and Stefan values.

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1. – Introduction

The aim of this paper is to study the large-time behaviour of the solution to the initial-and-boundary-value problem for the porous media equation in an exterior domain. More precisely, we consider the problem

$$(1.1) \qquad\qquad u_t = \Delta u^m \qquad\quad \text{in } Q = \Omega \times \mathbb{R}^+,$$

(1.2) $u(x, t) = g(x) \geq 0$ for $(x, t) \in S = \partial\Omega \times \mathbb{R}^+$,

(1.3) $u(x, 0) = u_0(x) \geq 0$ for $x \in \Omega$,

where $m > 1$ and Ω is the complement of a compact set with smooth boundary. The initial data $u(x, 0) = u_0(x)$ are assumed to be measurable, bounded, non-negative and compactly supported in $\overline{\Omega}$, the closure of Ω. It is well known that for any $m > 1$ equation (1.1) has the finite propagation property (more precisely, finite speed of propagation of disturbances with respect to the zero level), so that the support of the solution $u(\cdot, t)$ at time $t > 0$ is also bounded and an *interface* or *free boundary* appears to separate the regions $\{u > 0\}$ (positivity set) and $\{u = 0\}$ (zero-level set), cf. [Ar]. More important for the problem at hand is the specification of the boundary data. We consider Dirichlet data $g(x)$ which are assumed to be continuous, non-negative, non-trivial ($g(x) \not\equiv 0$) and constant in time. We can also solve the problem with variable boundary data $\psi(x, t) \geq 0$ which converge as $t \to \infty$ to a constant state, $g(x) \not\equiv 0$, without changing the asymptotic results.

The porous media equation. (PME for short) arises in a number of physical applications. Thus, the present problem can be used to describe the infiltration of water from a cylindrical reservoir into a surrounding porous medium. After the usual approximations, cf. [Be], [G4], the height of the saturated zone $z = u(x, y, t)$ obeys Boussinesq's equation, which is (1.1) with $m = 2$ in space dimension $n = 2$. We can also consider in three dimensions the laminar flow of a gas through a porous medium that surrounds a body. Then the density obeys equation (1.1) with different $m \geq 2$ depending on the law of state, [M]. In our work we consider arbitrary exponents $m > 1$ and dimensions $n > 1$ and omit the usual assumption of radial symmetry on the domain and solutions.

Problem (1.1)-(1.3) is well understood in one space dimension, where it reduces to the evolution of the PME posed in the half line $\Omega = \{x > 0\}$ with constant and positive boundary data at $x = 0$ and initial data $u_0(x)$ as prescribed above. As $t \to \infty$ the solution of this problem converges to a stationary state in the standard $xt^{-1/2}$-scale, a phenomenon that also happens for the heat equation and the Stefan problem (the scale is the same but the profiles are different, of course). The main results are as follows:

(i) For every constant $A > 0$ there exists a unique solution of the problem with boundary data $u(0, t) = A$ and initial data $u(x, 0) = 0$. This solution takes the form

(1.4) $$U(x, t) = f(y), \quad y = x\, t^{-1/2}.$$

Proofs of the result are due to Barenblatt [Ba] and Atkinson and Peletier [AtP]. Early analysis of such a solution goes back to Polubarinova-Kochina [PK].

(ii) Convergence result for general solutions: for every solution $u(x, t)$ of problem (1.1)-(1.3) with boundary data $A > 0$ and continuous, nonnegative and compactly supported initial function $u_0(x)$, Peletier [P] proved that as $t \to \infty$

(1.5) $$u(y\, t^{1/2}, t) \to f(y) \quad \text{uniformly in } y \in [0, \infty).$$

Outline of results. We describe here the asymptotic behaviour of the solutions of problem (1.1)-(1.3) in several dimensions $n > 1$, which is quite different from the case $n = 1$. Indeed, the $xt^{-1/2}$-scale is not correct. Moreover, for general non-symmetric data there is no self-similar solution which explains simultaneously the large-time behaviour of the solution and the interface of our problem. The actual behaviour can be described using the idea of *matched asymptotic expansions*. The positivity set $\Omega(t) = \{x : u(x, t) > 0\}$ expands to cover any compact subset of Ω in a finite time, and we prove that in the usual (x, t) coordinates the solution stabilizes to $P(x)$, where $Q(x) = P^m(x)$ is the solution of the exterior Dirichlet problem for the Laplacian with boundary data g^m. This is what we call the *near-field* or *inner* limit.

THREE OR MORE DIMENSIONS. In order to study what we call the *far-field limit* we introduce in dimensions $n \geq 3$ a scaling of the form

$$(1.6) \qquad y = x\, t^{-m/(n(m-1)+2)}, \qquad v = u\, t^{(n-2)/(n(m-1)+2)},$$

and prove that the scaled solution, $v(y, t)$, converges as $t \to \infty$ to a radially symmetric and self-similar solution of the porous media equation, which turns out to be singular at $x = 0$, cf. Theorem 7.1. This expansion, the *far-field* or *outer* limit, is valid close to the free boundary, in the *far-field region*. The singular self-similar solution is completely determined up to a constant, and this free constant is adjusted through suitable *matching* with the solution P which gives the behaviour in the interior of the positivity set. Observe that the scaling exponents in (1.6) cannot be extrapolated to $n = 1$, since they do not agree with (1.4).

The free boundary is one of the main features of the solution of problem (1.1)-(1.3). In the applications mentioned above it denotes the boundary of the infiltrated region or the boundary of the gas domain. Our analysis gives the asymptotic position of the free boundary as

$$(1.7) \qquad\qquad |x(t)| \sim C\, t^{m/(n(m-1)+2)} \qquad \text{if } n > 2,$$

where we use the notation $a(t) \sim b(t)$ to mean $a(t)/b(t) \to 1$ as $t \to \infty$ and $C = C(g, \Omega)$ is given in Theorem 7.2. We remark that C is independent of the initial data. We observe in particular a symmetrization effect for large times in the support of the solution and also in the solution profile far away from the fixed boundary. In the physical dimension $n = 3$ we get an interface exponent $m/(3m - 1)$, which becomes $2/5$ for $m = 2$. In all cases the exponent is less than $1/2$, the one-dimensional exponent. Let us also inform the reader that the overlapping of both expansions happens in a wide region of the form

$$\frac{1}{\varepsilon} < |x| < \varepsilon t^{m/(n(m-1)+2)}.$$

In fact, the self-similar solution which gives the outer expansion is a solution of equation (1.1) with variable boundary data on $\partial\Omega$ that converge to a constant

when the hole is circular. Hence, in this case and if the boundary data $g(x)$ are constant, *the outer expansion is a global expansion.*

DIMENSION TWO. This case offers some special features which make it more interesting and mathematically difficult. In fact, it is the border line case between the types of results described above for $n = 1$ and $n \geq 3$. The previous approach does not apply directly and has to be suitably corrected. This happens because the inner analysis predicts a constant behaviour for u for $|x| \gg 1$ as $t \to \infty$ and there is no self-similar solution which adapts to this behaviour; the closest self-similar solution has logarithmic growth in the near-field limit. It happens to be the good candidate and, as is usual in critical cases, logarithmic factors are needed to produce the desired matching. The result is that u behaves in the outer limit like

$$(1.8) \qquad u(x, t(\log t)^{(m-1)/m}) \sim \frac{1}{(\log t)^{1/m}} U\left(\frac{|x|}{t^{1/2}}\right)$$

for a certain profile U which is nonnegative, decreasing and compactly supported. The free boundary moves then like

$$(1.9) \qquad |x(t)| \sim C\, t^{1/2} (\log t)^{-(m-1)/(2m)},$$

where $C = C(g, \Omega)$ is given in Theorem 8.5.

PRECEDENTS. HELE-SHAW PROBLEM. Previous work on the asymptotic problem is due to Okrasinski and Goncerzewicz who obtained estimates on the growth of the interface by the method of subsolutions. Thus, for $n > 2$ the article [GO] shows a bound of the form $|x(t)| \geq C\, t^{1/n}$. In view of (1.7) this rate is not exact. For $n = 2$ Okrasinski [O] shows the estimate $\log |x(t)| \sim (1/2) \log t$, a weak version of (1.9).

We establish the exact asymptotic rates by means of matched asymptotics, following the recent study of the large-time behaviour of the Hele-Shaw and Stefan problems in exterior domains done by the authors in [QV]. Indeed, the similarity goes beyond the technical approach. We prove that our problem converges to the Hele-Shaw problem as $m \to \infty$, and in particular the free boundary estimates (1.7) and (1.9) become in the limit $|x(t)| \sim C\, t^{1/n}$ for $n \geq 3$, $|x(t)| \sim (t/\log t)^{1/2}$ for $n = 2$, which agree with the results of [QV]. Let us remark that a complete rigorous justification of the asymptotic expansions is provided in the paper.

STUDY OF SIMILARITY. Let us now take a closer look at the self-similar solutions used in the asymptotic description, because the study of self-similarity has an independent interest. The existence and properties of radially symmetric and self-similar solutions of the PME of the form

$$(1.10) \qquad u(x, t) = t^{-\alpha} U(|\xi|), \qquad \xi = xt^{-\beta}, \qquad \alpha(m-1) + 2\beta = 1,$$

is a widely researched subject. The main point is that there exists a way, or better several ways, of transforming the problem into a phase-plane analysis, cf.

[Ba], [J], [LOT], [AG]. For another approach see [AtP], [GP1,2], [Gi]. Solutions in these papers are nonnegative; solutions with sign changes appear in [H], [BHV] and [HV]. There are two typical ways in the literature for reducing the study of self-similar solutions of the PME to a phase-plane analysis. We present here an analysis based on the simultaneous use of both of them, and we contend that this results in a clearer understanding of the construction. Most of the solutions studied and then used in different contexts have nonsingular profiles, i.e., U is a continuous function. In our case however the compactly supported, singular, self-similar solutions which give the outer expansion of the asymptotic behaviour (after a logarithmic correction if $n = 2$) have the form (1.10) with β having the precise value

$$\beta = \frac{m}{n(m-1)+2} \equiv \beta_0,$$

which becomes $1/2$ for $n = 2$ or as $m \to 1$. These solutions have a singularity at the origin that does not depend on t,

$$u(x,t) \sim c\,|x|^{(2-n)/m} \qquad \text{for } n \geq 3,$$
$$u(x,t) \sim c\,(-\log|x|)^{1/m} \qquad \text{for } n = 2.$$

They are a particular instance of a larger class of compactly supported, singular, self-similar solutions of the form (1.10). We also investigate the existence of such solutions for $\beta \neq \beta_0$ and show that they exist for all $\beta > \beta_0/m$. The behaviour at the origin of these solutions is given by

$$u(x,t) \sim c\,t^{(\frac{\beta}{\beta_0}-1)/(m-1)}|x|^{(2-n)/m} \qquad \text{for } n \geq 3,$$
$$u(x,t) \sim c\,t^{(2\beta-1)/(m-1)}(-\log|x|)^{1/m} \qquad \text{for } n = 2.$$

This confirms that the singularity at the origin is independent of t if and only if $\beta = \beta_0$. We remark that the limit value $\beta = \beta_0/m$ corresponds to the well-known Barenblatt solutions, which are not singular at $x = 0$ for $t > 0$. As precedents, let us mention that examples of self-similar solutions of the porous media equation with a standing singularity at $x = 0$ are mentioned in [HV]. In [VV] a classification is proposed for the solutions of nonlinear parabolic equations which appear as limits of fundamental solutions. Solutions with a standing singularity appear then as one of the four possible types; they are called *razor blades*, because of their shape in the (x,t,u) representation.

ORGANIZATION OF THE PAPER. After giving some preliminary results in Section 2, we study the near-field limit in Section 3. Next we construct in Section 4 the family of self-similar solutions giving the asymptotic behaviour in the region close to the free boundary in the case $n \geq 3$. The bidimensional case is postponed to Section 5. In Section 6 we give an alternative construction and characterization of the singular solutions produced in the preceding sections,

in terms of a Cauchy problem with singular second member. The far-field limit is considered in Sections 7 ($n \geq 3$) and 8 ($n = 2$). In Section 9 we study the existence of singular self-similar solutions for $\beta \neq \beta_0$. In Section 10 we discuss the limit case $m \to \infty$. We end the paper with a section devoted to reviewing the main conclusions and listing a number of natural extensions of the methods and results of this paper.

2. – Preliminaries

We are given a bounded, simply connected domain $G \subset \mathbb{R}^n$. We will assume without loss of generality that $0 \in G$. We denote $\Omega = \mathbb{R}^n \setminus \overline{G}$ and $\partial \Omega = \Gamma$. We consider the problem

$$(2.1) \qquad \begin{cases} u_t = \Delta u^m & \text{in } Q, \\ u(x, t) = \psi(x, t) & \text{for } (x, t) \in S, \\ u(x, 0) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

where $\psi \in C(\overline{S})$, $u_0 \in L^1(\Omega)$ and ψ and u_0 are nonnegative functions. Let $Q_T = \Omega \times (0, T]$ and $S_T = \Gamma \times (0, T]$.

DEFINITION 2.1. A function u defined in $\overline{Q_T}$ is said to be a **weak solution** of problem (2.1) on $[0, T]$ if:

(i) $u \in C([0, T] : L^1(\Omega)) \cap L^\infty(Q_T)$;

(ii) for any test function $\phi \in C^{1,0}(\overline{Q_T}) \cap C^{2,1}(Q_T)$, compactly supported in $\overline{Q_T}$, with $\phi \geq 0$ in Q_T and $\phi = 0$ on S_T, u satisfies the integral identity

$$\int_\Omega u(x, t)\phi(x, t) \, dx = \iint_{Q_t} \{u(x, t)\phi_t(x, t) + u^m(x, t)\Delta\phi(x, t)\} \, dx \, dt$$

$$(2.2) \qquad\qquad + \int_\Omega u_0(x)\phi(x, 0) \, dx$$

$$\qquad\qquad - \iint_{S_t} \psi^m(x, t) \frac{\partial \phi}{\partial \nu}(x, t) \, dS(x) \, dt$$

for any $0 \leq t \leq T$. Here $\nu(x)$ is the outward-directed normal vector at $x \in \Gamma$.

A solution on $[0, \infty)$ means a solution on each $[0, T]$. If in (2.2) equality is replaced by $\leq$ ($\geq$) then we call u a weak **subsolution (supersolution)** of problem (2.2) on $[0, T]$.

The existence of solutions for problem (2.1) is proved by using a standard approximation process based on the results for the same problem in bounded domains given, for example, in [Br].

PROPOSITION 2.2. *If u is a weak subsolution of problem (2.1) with data u_0 and ψ, and $\hat{u}$ is a weak supersolution with data $\hat{u}_0$, $\hat{\psi}$, and if $\psi \leq \hat{\psi}$ on S_T, then for each $0 \leq t \leq T$,*

$$(2.3) \qquad \int_{\Omega} \{u(x, t) - \hat{u}(x, t)\}^+ \, dx \leq \int_{\Omega} \{u_0(x) - \hat{u}_0(x)\}^+ \, dx,$$

where $\{r\}^+ = \max\{r, 0\}$.

The proof is analogous to the one for one-dimensional bounded domains given in [ACP], and we omit it here. Uniqueness and comparison results follow easily from this proposition. It also gives a contraction principle in L^1.

COROLLARY 2.3. *Let u and $\hat{u}$ be weak solutions of problem (2.1) with initial functions u_0 and $\hat{u}_0$ respectively and both of them with boundary data ψ. Then*

$$\|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1.$$

We have an estimate from below for the time derivative of u which we will use when studying the near-field limit.

LEMMA 2.4. *Let u be a weak solution to problem (2.1) with initial data u_0 and boundary data $\psi(x, t) = g(x)$ independent of time. Then*

$$(2.4) \qquad\qquad u_t \geq -\frac{u}{(m - 1)t}.$$

The proof consists in applying the Maximum Principle to the PDE satisfied by $z = (m - 1)tu_t + u$. Since it is essentially the same as for the case in which the spatial domain is the whole space $\mathbb{R}^n$, we refer for the details to [V2]. Let us remark that the proof does not work if ψ depends on t unless $(m - 1)t\psi_t + \psi \geq 0$.

LEMMA 2.5. *If $u_0 \equiv 0$ and ψ does not depend on time, then $u_t \geq 0$.*

PROOF. Consider $v(x, t) = u(x, t+\tau)$. Then, $v(x, 0) = u(x, \tau) \geq 0 = u_0(x)$, and $v(x, t) = g(x) = u(x, t)$, for $x \in \Gamma$. Comparison produces the result. $\square$

3. – Near-field limit

We have the following convergence result, which describes the asymptotic behaviour as $t \to \infty$ for fixed $x \in \Omega$.

THEOREM 3.1. *Let $n \geq 2$ and $g(x) \in C(\Gamma)$, let u_0 be bounded and compactly supported in $\overline{\Omega}$ and let $g \geq 0$, $u_0 \geq 0$. Then the weak solution, u, of problem (1.1)-(1.3) converges as t tends to infinity to a stationary state $P(x)$ which is characterized as follows: $Q = P^m$ is the unique solution of the exterior Dirichlet problem for the Laplacian*

$$
(3.1) \qquad
\begin{cases}
\Delta Q = 0 & \text{if } x \in \Omega, \\
Q = g^m & \text{if } x \in \Gamma, \\
\displaystyle\lim_{|x| \to \infty} Q(x) = 0 & \text{if } n \geq 3, \\
Q(x) \text{ bounded} & \text{if } n = 2.
\end{cases}
$$

The convergence is uniform on compact subsets of $\overline{\Omega}$ for $n = 2$, uniform in $\overline{\Omega}$ for $n \geq 3$.

We need the following a priori estimate.

LEMMA 3.2. *Let u be as in Theorem 3.1. Then there is a constant $C > 0$ such that*

$$
(3.2) \qquad 0 \leq u(x, t) \leq C|x|^{(2-n)/m} \qquad \text{if } x \in \overline{\Omega}, \ t \geq 0.
$$

PROOF. The function $\hat{u}(x, t) = C|x|^{(2-n)/m}$ is a stationary solution of the porous media equation. Taking $C > 0$ big enough we have $\hat{u}(x, t) \geq g(x)$ on Γ and $\hat{u}(x, 0) \geq u_0(x)$. The result follows by comparison using sub- and super-solutions.   □

PROOF OF THEOREM 3.1. We may assume that the boundary data g are smooth. In case they are not, the result is proved by approximation.

We define $u_s(x, t) = u(x, t + s)$. It is obvious from Lemma 3.2 that

$$
0 \leq u_s(x, t) \leq C|x|^{(2-n)/m}.
$$

This estimate, combined with local regularity results (cf. [dB]), provides us with interior uniform (in s) local Hölder estimates for the family of functions $\{u_s\}$. Moreover, these estimates are valid up to the fixed boundary S, as this surface and the boundary data are smooth [Z]. Thus, Ascoli-Arzelà's theorem guarantees the existence of a sequence $\{s_n\}$ and a function $u_\infty(x, t)$ such that $u_{s_n} \to u_\infty$ uniformly on compact subsets of $\overline{\Omega} \times [0, \infty)$.

The next step is to identify the limit u_∞. It is straightforward that u_∞ satisfies the PME in the sense of distributions and that $u_\infty(x, t) = g(x)$ at the fixed boundary. First we consider the case $u_0 \equiv 0$. By Lemma 2.5 $u_t \geq 0$,

and the limit u_∞ does not depend on t. Moreover, $0 \leq u_\infty(x) \leq C|x|^{(2-n)/m}$, which means that

$$\lim_{|x|\to\infty} u_\infty^m(x) = 0 \qquad \text{if} \quad n \geq 3,$$

$$u_\infty^m \text{ is bounded} \qquad \text{if} \quad n = 2.$$

Therefore, u_∞^m is the unique solution of (3.1). The uniqueness of the limit implies that convergence is not restricted to a sequence.

For general data where $u_0 \not\equiv 0$, we use the previous results to obtain bounds from above and below. Firstly, we consider the solution $\hat{u}$ of problem (1.1)-(1.3) with the same boundary data as u, and initial data $\hat{u}_0 = 0$. Then $\hat{u}(x,t) \leq u(x,t)$. The convergence of $\hat{u}$ to P implies that on any compact set $K \subset \overline{\Omega}$ and $0 \leq t \leq T$

$$u(x, t + s_j) \geq \hat{u}(x, t + s_j) \geq P(x) - \varepsilon$$

if j is large enough, hence the desired lower bound, $u_\infty(x,t) \geq P(x)$. The upper bound is simpler. Passing to the limit in estimate (2.4) we see that for every sequence limit u_∞

$$\frac{\partial u_\infty}{\partial t} \geq 0.$$

Though the initial data $u_\infty(x,0)$ are not identically zero, the monotonicity is enough to apply the previous arguments to show that $u_\infty(x,t)$ converges to $P(x)$ as $t \to \infty$. Therefore, $u_\infty(x,t) \leq P(x)$ for all x and t in the domain. $\quad\square$

Next lemma describes the asymptotic behaviour of $P(x)$ as $|x| \to \infty$.

LEMMA 3.3. *Let $n \geq 2$. If I is the inversion mapping $I : x \mapsto x/|x|^2$, $\Omega^* = I(\Omega) \cup \{0\}$, $\mathcal{G}^*$ is the Green's function for the Laplacian in the domain Ω^* and*

(3.3) $$C_*^m(g, \Omega) = \int_{\partial\Omega^*} |x|^{2-n} g^m \left(\frac{x}{|x|^2}\right) \frac{\partial \mathcal{G}^*}{\partial \nu}(0, x)\, dS,$$

then

(3.4) $$\lim_{|x|\to\infty} |x|^{n-2} P^m(x) = C_*^m(g, \Omega) \neq 0.$$

This is a classical result for elliptic equations. A proof is given, for example, in [QV].

REMARKS. 1) If Ω is the complement of a ball $B_R(0)$, and g is a constant, $g = A$, then P is explicitly given by

$$P = A\,(R/|x|)^{(n-2)/m}.$$

When $g = A > 0$, but Ω is not a ball, we still have $P \sim A\,C(\Omega)|x|^{(2-n)/m}$.

2) There can be no uniform convergence for large $|x|$ in dimension 2 since there is a limit which is bounded away from 0 coexisting with a moving free boundary located at some large $|x|$, a consequence of the finite propagation property. The same happens of course for $n = 1$.

Finally, we can consider the problem with variable boundary data. The following result can be easily proved by approximation.

THEOREM 3.4. *The results of this section hold for all large times if we take boundary data* $\psi(x, t) \in C(S)$, $\psi \geq 0$ *which converge uniformly to* $g(x)$ *as* $t \to \infty$.

4. – A new family of self-similar solutions

In this section we construct a special one-parameter family of self-similar solutions of the porous media equation in $\mathbb{R}^n \setminus \{0\}$, which will be used later to describe the asymptotic behaviour near the free boundary of general solutions to the PME in an exterior domain. We consider first the case $n > 2$. The case $n = 2$ is slightly different, and we deal with it in Section 5. The solutions we are looking for are self-similar:

$$(4.1) \qquad \overline{u}(x, t) = t^{-\alpha} U(|\xi|), \qquad \xi = x \, t^{-\beta},$$

where, in order to satisfy equation (1.1), the similarity exponents are related by the formula

$$(4.2) \qquad \alpha(m - 1) + 2\beta = 1,$$

which allows to express α in terms of β. This latter exponent is in principle free and must be appropriately determined in order to obtain the class of solutions that describes the outer behaviour of the solutions of the general problem we are considering.

THEOREM 4.1. *Let* $n > 2$. *Precisely for the value of* β,

$$(4.3) \qquad \beta_0 = \frac{m}{n(m - 1) + 2},$$

there exists a one-parameter family of radial self-similar solutions of equation (1.1) *defined for* $\xi \neq 0$ *which are compactly supported in the space variable for all* $t > 0$ *and such that near the origin they behave like*

$$(4.4) \qquad \overline{u}_c(x, t) \sim c \, |x|^{(2-n)/m}, \quad c > 0.$$

The convergence $|x|^{(n-2)/m} \overline{u}_c(x, t) \to c$ *is uniform in sets of the form* $|x| \leq \varepsilon t^{\beta_0}$, $\varepsilon \to 0$.

Behaviour (4.4) is the precise behaviour that allows to match the solutions $\overline{u}_c$ with the outer limit of the near-field of a solution of (1.1)-(1.3) derived in the previous section. In fact, if Ω is the complement of a ball $B_R(0)$ then $\overline{u}_c$ is an approximate solution of problem (1.1)-(1.3) with boundary data $c \, R^{(2-n)/m}$, or in other words a solution of the problem with variable boundary data which converge to that constant. Let us note that since the solution is self-similar the

singularity at the origin stands for all $t > 0$, i.e., it is a razor blade singularity in the terminology used in [VV]. We recall that $\beta_0 < 1/2$ for $n > 2$ and $m > 1$.

We will see in the next section that in dimension $n = 2$ there is no self-similar solution satisfying the expected behaviour $u \sim$ constant, so that the analysis of the outer behaviour becomes more involved.

Phase plane. We are looking for a function $\bar{u}$ which is radial, compactly supported in x for all $t > 0$ and satisfies

$$u_t = \Delta u^m, \qquad x \in \mathbb{R}^n \setminus \{0\}, \quad m > 1.$$

It will be convenient to work with the *pressure* variable $v = mu^{m-1}/(m-1)$. This variable satisfies the equation

(4.5) $$v_t = (m-1)v\Delta v + |\nabla v|^2, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

We will write the self-similar solutions in the form

(4.6) $$\bar{v} = \frac{|x|^2}{t}\Phi(|\xi|), \qquad \xi = x\, t^{-\beta}.$$

Comparing with (4.1) we see that $|\xi|^2\Phi(|\xi|) = \frac{m}{m-1}U^{m-1}(|\xi|)$. For the moment we take $\beta > 0$ and review the general properties of the associated evolution for the variable Φ and fix the notations. The correct choice of β will be discussed later. We observe that the factor $|x|^2/t$ has the same dimensions as the pressure (length squared over time). Thus, Φ is adimensional. The ordinary differential equation satisfied by Φ is

(4.7) $$(m-1)\Phi|\xi|^2\Phi'' + \big((m-1)(n+3)+4\big)\Phi|\xi|\Phi'$$
$$+ \big(2n(m-1)+4\big)\Phi^2 + |\xi|^2(\Phi')^2 + \beta|\xi|\Phi' + \Phi = 0.$$

This equation can be made autonomous by introducing the new independent variable

$$\eta = \log|\xi|.$$

Written as a first-order system the resulting equations are

(4.8) $$\begin{cases} \dot{\Phi} = \Psi, \\ \dot{\Psi} = -\dfrac{(m-1)(n+2)+4}{m-1}\Psi - \dfrac{2n(m-1)+4}{m-1}\Phi \\ \qquad - \dfrac{1}{m-1}\dfrac{\Psi}{\Phi}(\Psi+\beta) - \dfrac{1}{m-1}, \end{cases}$$

where the dot denotes differentiation with respect to η. This is our first representation of the self-similar solutions. System (4.8) is singular at $\Phi = 0$. To

remove the singularity we make the nonlinear change of variable given implicitly by

$$\frac{d\tau}{d\eta} = \frac{1}{\Phi(\eta)}.$$

Then $\Phi(\tau)$ and $\Psi(\tau)$ satisfy

(4.9)
$$\begin{cases} \dfrac{d\Phi}{d\tau} = \Phi\Psi, \\[2mm] \dfrac{d\Psi}{d\tau} = -\dfrac{(m-1)(n+2)+4}{m-1}\Phi\Psi - \dfrac{2n(m-1)+4}{m-1}\Phi^2 \\[2mm] \qquad\quad - \dfrac{\Psi}{m-1}(\Psi+\beta) - \dfrac{\Phi}{m-1}. \end{cases}$$

Observe that this change of variable reverses the flow in the $\{\Phi < 0\}$ region.

We are looking for positive solutions. Thus, we will only consider the $\Phi \geq 0$ part of the $\Phi\Psi$-plane. There are two critical points there, $A = (0,0)$ and $B = (0, -\beta)$.

PROPOSITION 4.2. *The critical point* $A = (0,0)$ *is a saddle-node of system* (4.9). *The linearization of* (4.9) *around A has matrix*

$$\begin{pmatrix} 0 & 0 \\ -\dfrac{1}{m-1} & -\dfrac{\beta}{m-1} \end{pmatrix},$$

with eigenvalues $\lambda_1 = 0$ *and* $\lambda_2 = -\beta/(m-1)$ *and corresponding eigenvectors* $e_1 = (\,1 \quad -1/\beta\,)$ *and* $e_2 = (\,0 \quad 1\,)$.

PROOF. Center manifolds are tangent to e_1. Thus, we only have to consider the second and the fourth quadrants. Take a sufficiently small square $[0, \delta] \times [0, -\delta]$. It is invariant. Thus, the orbits entering it must approach A. This means that on the $\Phi \geq 0$ half-plane A is a node.

To see that A is a saddle on the $\Phi \leq 0$ half-plane we will check that the intersection of all center manifolds with this half-plane coincide. Indeed, center manifolds can be locally expressed as $\Psi = h(\Phi)$. Assume that there are two of them in $\Phi < 0$, given locally by the graphs of the functions h_1 and h_2, with $h_1(\Phi) < h_2(\Phi)$. A simple computation shows that

$$\frac{d(h_2 - h_1)}{d\Phi} > 0$$

if $\Phi \simeq 0$. Thus, it is impossible that both $h_1(0)$ and $h_2(0)$ equal 0, a contradiction.   □

PROPOSITION 4.3. *The critical point $B = (0, -\beta)$ is a saddle. The linearization of* (4.9) *around B has matrix*

$$\begin{pmatrix} -\beta & 0 \\ \dfrac{\beta((m-1)(n+2)+4)-1}{m-1} & \dfrac{\beta}{m-1} \end{pmatrix},$$

with eigenvalues $\lambda_1 = -\beta$ and $\lambda_2 = \beta/(m-1)$ and corresponding eigenvectors $e_1 = (1 \quad [1 - \beta((m-1)(n+2)+4)]/m\beta)$ and $e_2 = (0 \quad 1)$. The stable manifold corresponds to solutions with compact support $(0 < |\xi_0| < \infty)$.

PROOF. The local analysis is straightforward. The value η_0 is finite and the solution has compact support. $\qquad\square$

We are looking for solutions with compact support for $t > 0$, that is, $\Phi(\eta_0) = 0$ for a finite η_0. As is well-known, this means that the orbit enters the critical point $(\Phi, \Psi) = (0, -\beta)$. The argument is as follows: the velocity of the free boundary in the normal direction, s_n, satisfies

$$(4.10) \qquad\qquad s_n = -\nabla v \cdot \vec{n}$$

at points of the free boundary where it is smooth. For solutions of the form (4.1), (4.6) the free boundary is given by

$$(4.11) \qquad\qquad |x| = e^{\eta_0} t^\beta = \xi_0 \, t^\beta.$$

Hence (4.10) implies that $\Psi(\eta_0) = -\beta$.

Choice of β. The precise value of β which fits the asymptotic behaviour of the class of solutions studied in this paper is derived from the near-field limit of the preceding section. Thus, we assume that the solution $u(x, t)$ of problem (1.1)-(1.3) resembles a self-similar solution on compact sets of Ω far away from the fixed boundary, with $x \to \infty$ but $\xi \to 0$ as $t \to \infty$ (i.e., in the outer limit of the inner layer in the terminology of the boundary-layer theory). Thus, for $t \gg 1$ and $|x| \gg 1$ we want

$$(4.12) \qquad\qquad u^{m-1}(x, t) \sim \frac{|x|^2}{t} \Phi(|\xi|).$$

Since $u(x, t) \sim P(x) \sim C_* |x|^{(2-n)/m}$ we need

$$(4.13) \qquad\qquad \Phi(|\xi|) \sim K \, |\xi|^{-\frac{n(m-1)+2}{m}} \, t^{1 - \beta \frac{n(m-1)+2}{m}},$$

which means that β must be equal to β_0 given by (4.3), and then Φ must behave like

$$(4.14) \qquad\qquad \Phi(|\xi|) \sim K \, |\xi|^{-1/\beta_0}.$$

We will prove below that such a self-similar solution exists. In later sections we prove that it gives the desired asymptotic result.

When $\beta = \beta_0$ system (4.8) has an explicit orbit, the straight line $\Psi = -\Phi/\beta_0$. It corresponds to the stationary solutions of the porous media equation

(4.15) $u(x, t) = c |x|^{(2-n)/m}, \quad c > 0$ arbitrary.

As the explicit orbit $\Psi = -\Phi/\beta_0$ is a barrier that cannot be crossed by any other orbit, and as the origin is an attractor (remember that we are just considering the $\{\Phi > 0\}$ region), the unique orbit entering the critical point B must come from $\Psi = -\infty$. The phase plane is shown in figure 1.
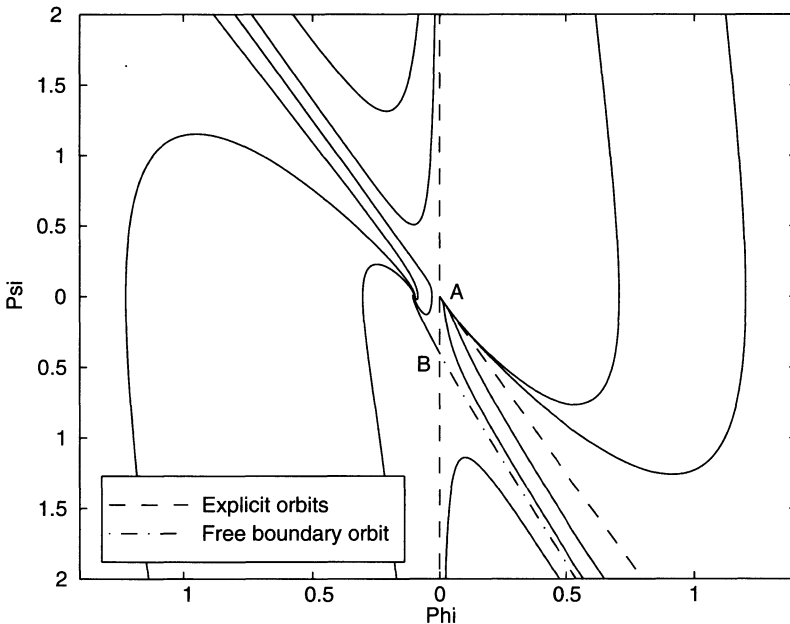
Fig. 1. $\Psi\Phi$-plane for $n = 3, m = 2$ and $\beta = \beta_0$.

New representation. Inversion. In order to have a more precise idea about where it comes from we perform an *inversion* change of variables. We now set

(4.16) $X(\eta) = \dfrac{1}{m - 1} \left(\dfrac{\Psi(\eta)}{\Phi(\eta)} + 2 \right), \qquad Y(\eta) = \dfrac{1}{(m - 1)\Phi(\eta)}.$

This kind of transformation goes back to [Ba] and [J], and is used in [H]. Observe that

$$X = \frac{1}{m - 1} \left(\frac{|\xi|\Phi'(|\xi|)}{\Phi(|\xi|)} + 2 \right) = \frac{|\xi|U'}{U}, \qquad Y = \frac{|\xi|^2}{mU^{m-1}}.$$

The resulting system is

$$(4.17) \qquad \begin{cases} \dot{X} = X(2 - n - mX) - Y(\alpha + \beta X), \\ \dot{Y} = Y(2 + (1 - m)X), \end{cases}$$

with $\alpha = (1 - 2\beta)/(m - 1)$. We are looking for positive solutions. Thus, we will only consider the $Y \geq 0$ part of the XY-plane. There are two critical points in that region, given in these variables by $C = (0, 0)$ and $D = (\frac{2-n}{m}, 0)$. Both of them coincide when $n = 2$. The local analysis of these points is straightforward.

PROPOSITION 4.4. *The linearization of* (4.17) *around* $C = (0, 0)$ *has matrix*

$$\begin{pmatrix} 2 - n & -\alpha \\ 0 & 2 \end{pmatrix},$$

with eigenvalues $\lambda_1 = 2 - n$ *and* $\lambda_2 = 2$ *and corresponding eigenvectors* $e_1 = (1 \quad 0)$ *and* $e_2 = (-\alpha/n \quad 1)$. *Thus, C is a saddle when* $n > 2$, *a repeller when* $n < 2$, *and a saddle-node for the bifurcation value* $n = 2$.

PROPOSITION 4.5. *The linearization of* (4.17) *around* $D = (\frac{2-n}{m}, 0)$ *has matrix*

$$\begin{pmatrix} n - 2 & \dfrac{1}{m - 1}\left(\dfrac{\beta}{\beta_0} - 1\right) \\ 0 & \dfrac{1}{\beta_0} \end{pmatrix},$$

with eigenvalues $\lambda_1 = n - 2$ *and* $\lambda_2 = 1/\beta_0$ *and corresponding eigenvectors* $e_1 = (1 \quad 0)$ *and* $e_2 = (\beta - \beta_0 \quad (m - 1)((2 - n)\beta_0 + 1))$. *Thus, D is a repeller when* $n > 2$, *a saddle when* $n < 2$, *and a saddle-node for the bifurcation value* $n = 2$.

Existence of the orbit for $\beta = \beta_0$, $n \geq 2$. In the new variables the explicit orbit $\Psi = -\Phi/\beta_0$ is written as $X = (2 - n)/m$, which exits D. The orbit we are looking for approaches the free boundary singularity B, which in the new variables is $(X, Y) = (-\infty, \infty)$. Thus, it must always lie to the left of the explicit orbit $X = (2 - n)/m$ and above the explicit orbit $Y = 0$. In this region $\dot{Y} > 0$. Hence, the orbit approaching B goes downwards as $\eta \to -\infty$. The null-cline $\dot{X} = 0$ is given by the pair of straight lines

$$X = \frac{2 - n}{m} \quad \text{and } Y = -(n(m - 1) + 2)X.$$

Thus, the orbit approaching B cannot go to $X = -\infty$ as $\eta \to -\infty$. As $(2 - n)/m \leq 0$ if $n \geq 2$, it comes necessarily from D. The fact that it starts at $(X, Y) = (\frac{2-n}{m}, 0)$ implies, when $n > 2$, that the corresponding solution behaves as predicted in (4.4), (4.14). Summarizing, we have the following existence and uniqueness result:
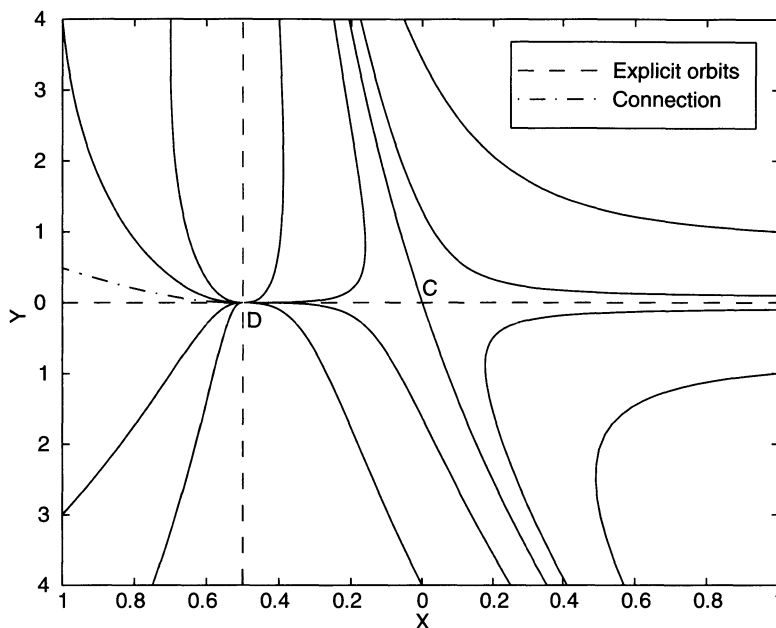
Fig. 2. XY-plane for $n = 3$, $m = 2$ and $\beta = \beta_0$.

THEOREM 4.6 (Existence of a connection). *Let $n \geq 2$ and let $\beta = \beta_0$. Then there exists a unique orbit of system* (4.8) *which joins the singular point B with the singularity D of the XY-plane.*

We recall that B represents the free boundary, and D the behaviour as $\xi \to 0$. These behaviours are given by (4.11) and (4.14) respectively when $n > 2$. We show a typical XY-plane for $n > 2$ in figure 2.

Family of solutions. By reparametrization the unique orbit gives rise to a one-parameter family of different self-similar solutions of equation (1.1) which can be obtained from one of them by just shifting η, i.e., scaling ξ. Thus, given a self-similar solution $\bar{u}$ the family of solutions

$$(4.18) \qquad \bar{u}_c(x, t) = c\,\bar{u}(x, c^{m-1}t), \quad c > 0,$$

represents all the solutions contained in the same orbit. Cf. the explicit solution (4.15). We may normalize so that $u_1(x, t) = |x|^{(2-n)/m}(1 + o(1))$ as $|x| \to 0$. It follows that the interface reads

$$(4.19) \qquad r_c(t) = r_1(c^{m-1}t) = \xi_0(c^{m-1}t)^{\beta_0}.$$

Properties. We state now two properties of $\bar{u}$ which will be useful when studying the far-field limit.

PROPOSITION 4.7. $\dfrac{\partial \overline{u}_c}{\partial t}(x, t) \geq 0$.

PROOF. A direct computation shows that

$$\overline{v}_t = -\frac{|x|^2}{t^2} \Phi(\eta) \left(1 + \beta_0 \frac{\dot{\Phi}(\eta)}{\Phi(\eta)} \right).$$

But $\dot{\Phi}(\eta) < -\Phi(\eta)/\beta_0$, from where the result is immediate.   □

PROPOSITION 4.8. *If $n > 2$, then $\overline{u}_c(x, t) \leq ca^{(2-n)/m}$ if $|x| = a$.*

PROOF. This is an immediate consequence of the monotonicity in time plus the fact that $|x|^{(n-2)/m} u_c(x, t) \to c$ as $t \to \infty$ if $|x| = a$.   □

In figure 3 we represent the profile of the self-similar solution u_c for different times. It can be seen that it approaches $P(x) = c|x|^{(2-n)/m}$ monotonically as $t \to \infty$.
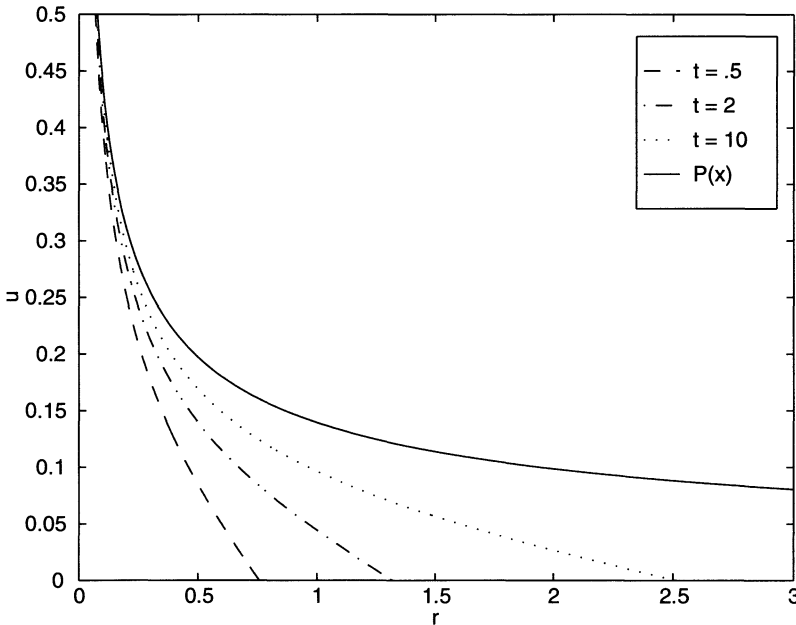

Fig. 3. Self-similar solution for $n = 3$, $m = 2$ and $c = 0.1394$ at different times.

5. – Self-similar solutions in dimension two

The phase plane analysis of the preceding section is valid for $n = 2$, but the conclusion is slightly different. Namely, the behaviour of Φ is not given by (4.14). In this section we shall determine the actual behaviour.

We take the value of β_0 corresponding to $n = 2$, which is $\beta_0 = 1/2$. This leads to solutions of the self-similar form

(5.1) $\bar{u}(x, t) = U(|\xi|), \qquad \xi = x\, t^{-1/2},$

or in pressure variable

(5.2) $\bar{v}(x, t) = |\xi|^2\, \Phi(|\xi|).$

We can make the analysis of the connection but now system (4.17) becomes

(5.3)
$$
\begin{cases}
\dot{X} = -X\left(mX + \dfrac{1}{2}Y\right), \\[2mm]
\dot{Y} = Y(2 + (1 - m)X).
\end{cases}
$$

The corresponding phase plane for $m = 2$ is shown in figure 4.
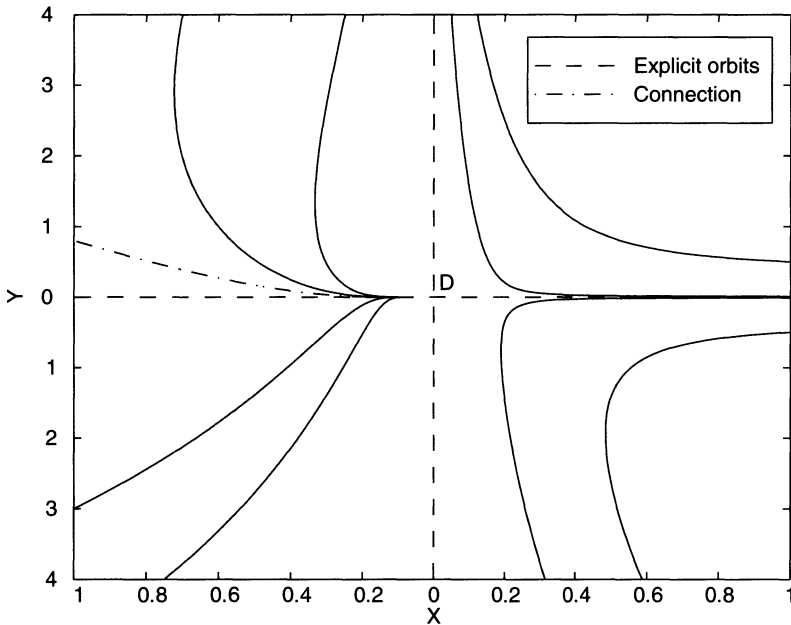
Fig. 4. XY-plane for $n = 2$, $m = 2$ and $\beta = \beta_0 = 1/2$.

The critical point D is not a repeller any more: when it joins the point C it changes its topological character and becomes a saddle-node. Let us perform a more detailed analysis of this point. There is a centre manifold $Y = h(X)$ entering the critical point. It follows from center-manifold theory (cf. [C]) that near $(0, 0)$ Y must be exponentially small compared with X, and thus that

$$\dot{X} \simeq -mX^2.$$

This implies that $X \simeq 1/(m\eta)$. If we introduce this in the equation for $\dot{Y}$ we obtain

$$\frac{\dot{Y}}{Y} \simeq 2 - \frac{m-1}{m\eta}.$$

Integration gives that

$$\log Y \simeq C + 2\eta - \frac{m-1}{m} \log |\eta|,$$

which produces the behaviour

(5.4)
$$\Phi(|\xi|) \sim K \, |\xi|^{-2}(-\log|\xi|)^{(m-1)/m}.$$

According to this formula there is a logarithmic correction with respect to the constant behaviour formally predicted by (4.14), in disagreement with the behaviour derived in formula (3.4) for the near field expansion. This difficulty is not to be solved by changing the parameter: any other value of β would have produced power differences, not just logarithmic ones. This means that β_0 is the correct value and the solution of the difficulty lies elsewhere. We summarize our result as follows.

THEOREM 5.1. *In space dimension $n = 2$ and for $\beta = \beta_0 = 1/2$ there exists a one-parameter family of radial self-similar solutions of equation* (1.1) *defined for $\xi \neq 0$, compactly supported in ξ, and such that near the origin they have a logarithmic singularity of the form*

(5.5)
$$\overline{u}_c(x,t) \sim c\,(-\log|\xi|)^{\frac{1}{m}}, \quad c > 0,$$

as $\xi = x\,t^{-1/2} \to 0$.

The convergence $\overline{u}_c(x,t) \to c(-\log|x|)^{1/m}$ is not uniform for large t since from (5.4) the precise expansion reads

$$\overline{u}_c(x,t) = c\,\left(\log\frac{1}{|x|} + \frac{1}{2}\log t\right)^{\frac{1}{m}} + \text{terms of higher order in } \xi.$$

It is clear that the self-similar solutions in both cases $n \geq 3$ and $n = 2$ share the property that the singular behaviour of u^m coincides with the behaviour of the fundamental solution of the Laplacian. We will explore this connection next.

6. – Alternative construction of the self-similar solutions

The self-similar solutions we have constructed in the preceding sections can also be characterized in terms of a Cauchy problem with singular second member as follows.

THEOREM 6.1. *Let* $n \geq 2$. *For each constant* $C > 0$ *there exists a radial, self-similar function* $\mathcal{U}(x, t; C) \geq 0$, *which is compactly supported in x for all* $t > 0$ *and satisfies*

(6.1)
$$\begin{cases} \mathcal{U}_t = \Delta \mathcal{U}^m + C\delta(x) & in \ \mathcal{D}'(\mathbb{R}^n \times (0, \infty)), \\ \mathcal{U}(x, 0) = 0 & for \ x \in \mathbb{R}^n, x \neq 0. \end{cases}$$

Such a solution is unique. In particular, the mass grows linearly with t:

(6.2)
$$\int \mathcal{U}(x, t) \, dx = Ct.$$

PROOF. We only have to put $\mathcal{U}(x, t; C) = \bar{u}_c(x, t)$, the self-similar solution constructed in the preceding sections for $n \geq 3$ and $n = 2$ respectively, to obtain the result. The constants C and c are related by the flux calculation

$$\frac{d}{dt} \int_{|x| \geq \varepsilon} \bar{u} \, dx = - \int_{|x| = \varepsilon} \frac{\partial \bar{u}^m}{\partial r} \, dS = \begin{cases} n\omega_n(n - 2) c^m & if \ n \geq 3, \\ 2\pi c^m & if \ n = 2, \end{cases}$$

which implies that

(6.3)
$$C = \begin{cases} n(n - 2)\omega_n c^m & if \ n \geq 3, \\ 2\pi c^m & if \ n = 2. \end{cases} \qquad \square$$

REMARK. The linear law (6.2) determines the correct value of β, (4.3).

Alternative construction. The solutions $\mathcal{U}$ of problem (6.1) can also be constructed using the theory of the Cauchy problem for the PME with second member posed in $\mathbb{R}^n$. Thus, the problem

(6.4)
$$\begin{cases} u_t = \Delta u^m + f(x, t) & in \ \mathcal{D}'(\mathbb{R}^n \times (0, \infty)), \\ u(x, 0) = 0 & for \ x \in \mathbb{R}^n, x \neq 0. \end{cases}$$

has a unique *mild solution* for every $f \in L^1([0, T] : L^1(\mathbb{R}^n))$. Moreover, if f is continuous and nonnegative so is the solution, which becomes a solution in the usual weak sense. There is also boundedness and continuity in the L^1 sense. Thus,

$$\|u(\cdot, t)\|_1 \leq \|u(\cdot, s)\|_1 + \int_s^t \|f(\cdot, \tau)\|_1 \, d\tau,$$

valid for all $0 \leq s < t$. The usual comparison principle applies, cf. [Bn], [BC].

In order to construct the solution $\mathcal{U}$ we use as approximate second members a family of smooth functions of the form

(6.5) $$f_k(x, t) = k^n f_1(kx),$$

where $f_1(x)$ is a smooth and nonnegative function supported in the ball of radius 1 and having integral C. We may also assume that f_1 is radially symmetric and decreasing. The corresponding solutions $u_k(x, t)$ of problem (6.4) exist, are unique and have the following properties:

• They are nonnegative, continuous, compactly supported and non-increasing as functions of $|x|$.

• They are bounded in $L^1(\mathbb{R}^n)$ for finite time, Indeed,

$$\int u_k(x, t) \, dx = Ct.$$

• We have the scaling law

(6.6) $$u_k(x, t) = k^{\frac{n-2}{m}} u_1 \left(kx, k^{\frac{n(m-1)+2}{m}} t \right),$$

which is proved by checking that the right-hand side satisfies the same equation and initial conditions as u_k.

• Pointwise bound. There is a constant $K > 0$ such that

$$u_k(x, t) \leq K |x|^{\frac{2-n}{m}}.$$

In order to prove this we regularize the harmonic function $g = |x|^{2-n}$ by replacing it inside the ball of radius 2 by a smooth concave function so that the new function $\tilde{g} \leq g$ satisfies

$$-\Delta \tilde{g} \geq c_1 > 0 \qquad \text{in } B_1(0),$$

while $\Delta \tilde{g} \leq 0$ outside. It is then clear that for some large constant K we have $K^m \Delta \tilde{g} + f_1 \leq 0$, hence $K \tilde{g}^{1/m}$ is a supersolution to the evolution equation satisfied by u_1. In view of the initial data we get the estimate for u_1. The same inequality is true for the whole family u_k because of the scaling formula. The calculations for $n = 2$ are similar with $\log(1/|x|)$ instead of $|x|^{2-n}$.

By standard compactness arguments [dB] we may pass to the limit to find a function

$$u(x, t) = \lim_{k \to \infty} u_k(x, t).$$

In principle we must pass to the limit along sequences $k_j \to \infty$ but the concentration comparison of [V1] gives us the necessary monotonicity to pass to the limit in the whole family u_k. It is not difficult to see that this function solves problem (6.1): it is a weak solution of equation (1.1) away from the origin; due to the mass estimate it must have a Dirac mass at the origin as forcing term; the initial data are also taken; it is self-similar as a consequence of formula (6.6). Hence, it coincides with the solution $\bar{u}$.

7. – Far-field limit

We are now ready to establish the behaviour of the solutions of the original problem in the outer region.

THEOREM 7.1. *Let $n > 2$. Let u be the weak solution to the porous media equation in the exterior domain Ω with initial and boundary data as in Theorem 3.1, and let $C_* = C_*(g, \Omega)$ be the constant of the near-field asymptotics given by Lemma 3.3. Then*

$$(7.1) \qquad \lim_{t \to \infty} t^{(n-2)/(n(m-1)+2)}|u(x, t) - \overline{u}(x, t; C_*)| = 0$$

uniformly on sets of the form $\{x \in \mathbb{R}^n : |x| \geq \delta t^{\beta_0}\}$, $\delta > 0$, where $\overline{u}(x, t; c) = \overline{u}_c(x, t)$ is the self-similar solution to the PME constructed in Section 4 (cf. Theorem 4.1).

PROOF. *First step: Scaling and compactness.* We define

$$(7.2) \qquad u_\lambda(x, t) = \lambda^{\alpha_0} u(\lambda^{\beta_0} x, \lambda t),$$

where α_0 is given by the relation (4.2), that is,

$$(7.3) \qquad \alpha_0 = \frac{n - 2}{n(m - 1) + 2}.$$

Using (3.2) we get that, if $\lambda^{\beta_0} x \in \Omega$, then

$$0 \leq u_\lambda(x, t) \leq C|x|^{(2-n)/m}$$

for some large constant C. Using well-known results on compactness for the porous media equation (see [dB]) we have that there are a subsequence $\{\lambda_k\}$ and a function u_∞ such that

$$u_{\lambda_k}(x, t) \to u_\infty(x, t)$$

uniformly on compact subsets of $(\mathbb{R}^n \setminus \{0\}) \times (0, \infty)$.

Second step: Identification of the limit. We take $a \gg 1$ such that $P(x) \geq (C_* - \varepsilon)|x|^{(2-n)/m}$ for all $|x| \geq a$ (Lemma 3.3). Then we take T such that $u(x, t) \geq P(x) - \varepsilon a^{(2-n)/m}$ for $|x| = a$, $t \geq T$ (Theorem 3.1). Thus we get that $u(x, t) \geq (C_* - 2\varepsilon)a^{(2-n)/m}$ if $|x| = a$, $t \geq T$. Now, $\overline{u}(x, t - T; C_* - 2\varepsilon) \leq (C_* - 2\varepsilon)a^{(2-n)/m}$ if $|x| = a$, $t \geq T$ (Proposition 4.8), and hence we finally obtain that $u(x, t) \geq \overline{u}(x, t - T; C_* - 2\varepsilon)$ if $|x| = a$, $t \geq T$. Moreover, $u(x, T) \geq 0 = \overline{u}(x, 0; C_* - 2\varepsilon)$ if $|x| \geq a$. Hence, comparing $u(x, t)$ and $\overline{u}(x, t - T; C_* - 2\varepsilon)$ for $|x| \geq a$, $t \geq T$ we conclude that

$$(7.4) \qquad u(x, t) \geq \overline{u}(x, t - T; C_* - 2\varepsilon) \qquad \text{for } |x| \geq a, \ t \geq T.$$

We now obtain a similar upper bound when the initial data are zero. In this case $u_t \geq 0$ (Lemma 2.5), and therefore $u(x,t) \leq P(x)$ for all $x \in \Omega$, $t > 0$. Taking a big enough we have

$$(7.5) \qquad u(x,t) \leq P(x) \leq (C_* + \varepsilon)|x|^{(2-n)/m} \qquad \text{for } |x| \geq a.$$

Now, $\overline{u}(x,t;C) \to C|x|^{(2-n)/m}$ as $t \to \infty$ uniformly on compact subsets of $\{|x| \geq a\}$, and thus, there is a big enough value T such that $\overline{u}(x,t;C_*+2\varepsilon) \geq (C_*+\varepsilon)a^{(2-n)/m} \geq u(x,t)$ for $|x| = a$, $t \geq T$. Moreover, as $u(x,T)$ has compact support, it is possible to take τ big enough so that $u(x,T) \leq \overline{u}(x,T+\tau;C_*+2\varepsilon)$ for $|x| \geq a$. Observe that we are using here (7.5), which is a consequence of u_0 being equal to zero. Thus, we can apply comparison to $u(x,t)$ and $\overline{u}(x,t+\tau;C_*+2\varepsilon)$ in $|x| \geq a$, $t \geq T$ (remember that $\overline{u}_c$ increases with time) to obtain

$$(7.6) \qquad u(x,t) \leq \overline{u}(x,t+\tau;C_*+2\varepsilon) \qquad \text{for } |x| \geq a, \ t \geq T.$$

Using (7.4) and (7.6) we obtain that

$$(7.7) \qquad \overline{u}\left(x,t-\frac{T}{\lambda};C_*-2\varepsilon\right) \leq u_\lambda(x,t) \leq \overline{u}\left(x,t+\frac{\tau}{\lambda};C_*+2\varepsilon\right)$$
$$\text{for } t > T/\lambda, \ |x| \geq a/\lambda^{\beta_0}.$$

Thus, letting $\lambda \to \infty$ we get $\overline{u}(x,t;C_*-2\varepsilon) \leq u_\infty(x,t) \leq \overline{u}(x,t;C_*+2\varepsilon)$. As ε is arbitrary, we finally obtain that

$$u_\infty(x,t) = \overline{u}(x,t;C_*).$$

The uniqueness of the limit implies that convergence is not restricted to a subsequence. Moreover, as the supports of the functions $\{u_\lambda\}$ and $\overline{u}$ are uniformly bounded for fixed t (cf. formula (7.7)), convergence is uniform on sets of the form $\{|x| \geq \delta\} \times \{t\}$, $t > 0$.

We consider now general initial data u_0 bounded and compactly supported in $\overline{\Omega}$. Let u and v be the solutions to problem (1.1)-(1.3) corresponding to the same boundary data, $g(x)$, and initial data $u(x,0) = u_0(x)$ and $v(x,0) = 0$ respectively. It is clear that $u(x,t) \geq v(x,t)$. Let $K = \{|x| \geq \delta\}$. Then, for λ large enough

$$\|u_\lambda(\cdot,t) - v_\lambda(\cdot,t)\|_{L^1(K)} \leq \|u_\lambda(\cdot,t) - v_\lambda(\cdot,t)\|_{L^1(\lambda^{-\beta_0}\Omega)}$$
$$\leq \lambda^{\alpha_0-n\beta_0} \int_\Omega |u(x,\lambda t) - v(x,\lambda t)| \, dx \leq \lambda^{-1}\|u_0\|_{L^1(\Omega)}.$$

Thus, letting $\lambda \to \infty$ we obtain that $\|u_\infty(\cdot,t)-\overline{u}(\cdot,t;C_*)\|_{L^1(K)} = 0$, and hence that

$$u_\infty(x,t) = \overline{u}(x,t;C_*).$$

The fact that the convergence is uniform in exterior sets (and not only locally) depends on a uniform upper bound of the supports of the u_λ, which is obtained by comparison with a large self-similar solution. Indeed, there exist $T > 0$ and $C > 0$ large enough such that

$$u(x, t) \leq \overline{u}(x, t + T; C)$$

holds. From this the required upper bound follows.

Third step: Convergence. From here on the proof is standard. A simple calculation shows that

$$|u_\lambda(\xi, 1) - \overline{u}(\xi, 1; C_*)| = \lambda^{\alpha_0}|u(\lambda^{\beta_0}\xi, \lambda) - \overline{u}(\lambda^{\beta_0}\xi, \lambda; C_*)| = \lambda^{\alpha_0}|u(x, \lambda) - \overline{u}(x, \lambda; C_*)|,$$

where we have used the notation $x = \lambda^{\beta_0}\xi$. Thus, the uniform convergence of $u_\lambda(\xi, 1)$ to $\overline{u}(\xi, 1)$ as $\lambda \to \infty$ on sets of the form $\{|\xi| \geq \delta\}$ is equivalent to the stated convergence (7.1).   □

We also prove that the free boundary approaches a sphere with a precise radius growth. This sphere coincides with the free boundary of $\overline{u}(x, t; C_*)$.

THEOREM 7.2. *Let* $m_+(t) = \max_{x \in \Gamma(t)} |x|$ *and* $m_-(t) = \min_{x \in \Gamma(t)} |x|$, *where* $\Gamma(t)$ *is the free boundary at time* t *of the solution of problem* (1.1)-(1.3) *with initial and boundary data as in Theorem 3.1. Then*

$$(7.8) \qquad \lim_{t \to \infty} \frac{m_\pm(t)}{t^{m/(n(m-1)+2)}} = \xi_0 C_*^{(m-1)\beta_0},$$

where ξ_0 *is the radius at time* $t = 1$ *of the free boundary of the function* $\overline{u}(x, t; 1)$.

PROOF. The lower bound is immediate from the convergence statement. Indeed, writing formula (7.4) in terms of supports, and using formula (4.19) to relate the support of $\overline{u}_c$ with the support of $\overline{u}_1$, we get

$$\frac{m_-(t)}{t^{\beta_0}} \geq \xi_0(C_* - 2\varepsilon)^{(m-1)\beta_0}\left(1 - \frac{T}{t}\right)^{\beta_0}.$$

The upper bound requires more effort. Let U_c be the profile of the self-similar solution u_c. Given $\varepsilon > 0$ we take δ small enough such that $\delta^{(n-2)/m}U_{C_*+\varepsilon}(\delta) \geq C_* + \frac{\varepsilon}{2}$. We know that $\overline{u}(x, t; C_*) < C_*|x|^{(2-n)/m}$ and that the functions $u_\lambda(x, 1)$ have a uniformly bounded support. Thus, using the uniform convergence on compact sets of the functions u_λ to $\overline{u}_{C_*}$, we have that $u_\lambda(x, 1) < C_*|x|^{(2-n)/m}$ for all $\lambda \geq \Lambda$, $|x| \geq \delta$. This implies that

$$(7.9) \qquad u(x, t) < C_*|x|^{(2-n)/m} \qquad \text{for } |x| \geq \delta t^{\beta_0}, \ t \geq \Lambda.$$

In particular, we have that

$$u(x, \Lambda) \leq \overline{u}(x, \Lambda + \tau; C_* + \varepsilon) \qquad \text{for } |x| \geq \delta\Lambda^{\beta_0}$$

if we take $\tau \geq 0$ big enough.

On the other hand, if $|x| = \delta t^{\beta_0}$, $t \geq \Lambda$, then $u(x, t) \leq t^{-\alpha_0} C_* \delta^{(2-n)/m}$ (cf. formula (7.9)), and

$$\bar{u}(x, t + \tau; C_* + \varepsilon) \geq \bar{u}(x, t; C_* + \varepsilon) = t^{-\alpha_0} U_{C_* + \varepsilon}(\delta) \geq t^{-\alpha_0} \left(C_* + \frac{\varepsilon}{2} \right) \delta^{(2-n)/m},$$

where we have used the monotonicity in time of $\bar{u}(x, t; C_* + \varepsilon)$. Thus,

$$u(x, t) \leq \bar{u}(x, t + \tau; C_* + \varepsilon) \qquad \text{for } |x| = \delta t^{\beta_0}, \ t \geq \Lambda.$$

Therefore, we can apply comparison to $u(x, t)$ and $\bar{u}(x, t + \tau; C_* + \varepsilon)$ for $|x| \geq \delta t^{\beta_0}$, $t \geq \Lambda$, to obtain

$$u(x, t) \leq \bar{u}(x, t + \tau; C_* + \varepsilon) \qquad \text{for } |x| \geq \delta t^{\beta_0}, \ t \geq \Lambda.$$

This implies that

$$\frac{m_+(t)}{t^{\beta_0}} \leq \xi_0 (C_* + \varepsilon)^{(m-1)\beta_0} \left(1 + \frac{\tau}{t} \right)^{\beta_0},$$

which is the required upper estimate. $\qquad \square$

REMARK. Numerical computations show that $\xi_0 = 2.1994$ when $n = 3$, $m = 2$.

THEOREM 7.3. *The results of Theorems 7.1, 7.2 hold true when we replace the boundary data $g(x)$ by a continuous and nonnegative function $\psi \in C(S)$ such that $\psi(x, t) \to g(x)$ uniformly as $t \to \infty$. The asymptotic constant C_* is still the $C_*(g, \Omega)$ defined in (3.3).*

PROOF. We replace $\psi(x, t)$ by $g_1(x) =: g(x) + \varepsilon$ (respectively $g_2(x) = \max\{g(x) - \varepsilon, 0\}$), thus obtaining solutions u_1 (respectively u_2) to which the above results apply after replacing C_* by a small perturbation. We then apply comparison to u and u_1 (respectively u and u_2) for $t \geq T$, T large enough, to conclude. $\qquad \square$

8. – Far-field limit in dimension two

Using the ideas developed in [GV] we perform the change of variables that allows to follow the evolution of the self-similar solution as a fixed profile with respect to the rescaled $x/\sqrt{t}$ variable:

$$(8.1) \qquad w(\xi, \tau) = u(\xi t^{1/2}, t), \quad \tau = \log t.$$

Then the equation reads

$$(8.2) \qquad w_\tau = \Delta w^m + \frac{1}{2}\xi \cdot \nabla w \equiv \mathcal{A}(w).$$

The self-similar solutions constructed in Section 5 are, in the new variables, just stationary solutions of (8.2). Indeed, they are given by

$$(8.3) \qquad \overline{w}_c(\xi, \tau) = \overline{u}_c(\xi t^{1/2}, t) = U_c(|\xi|).$$

Now, the different behaviour of the near-field limit of equation (1.1) and the self-similar solutions (8.3) means that the above scaling is incorrect in order to normalize the solutions of our Cauchy-Dirichlet problem for $n = 2$. Following [GV] we introduce a new rescaling

$$(8.4) \qquad w(\xi, \tau) = (\log t)^\sigma u(\xi\, t^{1/2}, t\, (\log t)^{(m-1)\sigma}), \quad \tau = \log t,$$

with logarithmic corrections in the w and t variables. Moreover, we will choose

$$(8.5) \qquad \sigma = \frac{1}{m},$$

a selection that is justified below in terms of the near-field analysis of our solutions. In any case, if u is a solution of (1.1) and we apply (8.4) then w satisfies the equation

$$(8.6) \qquad w_\tau = \Delta w^m + \frac{1}{2}\xi \cdot \nabla w + \frac{\sigma}{\tau}(w + (m-1)\Delta w^m).$$

In order to stress the fact that this equation is an asymptotically small per-turbation of (8.2) in the sense of [GV] we can write equation (8.6) in the form

$$w_\tau = \mathcal{A}(w) + \frac{1}{\tau}\mathcal{B}(w).$$

We still have to select the convenient value of σ. This is done on the basis that we want the rescaled solutions w to converge towards the self-similar solution (8.3). For this we take a fixed $x = \xi\, t^{1/2}$ and t large and use the inner behaviour of u to assert that

$$w(xt^{-1/2}, \tau) \sim \tau^\sigma\, C_*,$$

where C_* is the asymptotic constant of the inner expansion of u, cf. Section 3. On the other hand, the behaviour near zero of the self-similar solution in the new variables is $\overline{w}_c(s, \tau) \sim c(-\log |s|)^{1/m}$, $s \sim 0$. Both behaviours agree along the curve $s = xt^{-1/2}$ if and only if $\sigma = 1/m$ and also $c = 2^{1/m} C_*$. The precise convergence result is the following:

THEOREM 8.1. *Let $n = 2$ and let w be the rescaling given by (8.4) of the solution of problem (1.1)-(1.3). Then $w(\xi, \tau)$ converges to $\overline{w}_c(\xi, \tau) = U_c(|\xi|)$ as $\tau \to \infty$ uniformly on sets of the form $|\xi| \geq \delta > 0$, where $c = 2^{1/m}C_*$.*

In order to justify that the rescaled orbit of our solution, which is a solution of (8.6), converges towards a stationary solution of (8.2), more precisely, a self-similar solution of the form (8.3), we have to follow a plan consisting of three steps: (i) first, we show that the orbit $w(\xi, \tau)$ is bounded above and below, and from this it follows that it is compact in a suitable space, (ii) we pass to the limit along a sequence of delays to obtain in the limit a solution of equation (8.2), (iii) we show that such a limit is the desired self-similar solution.

I. BOUNDEDNESS. In this step we make sure that the logarithmic corrections in the scaling produce the desired effect of getting locally compact orbits. The control of the orbit is done by means of the construction of suitable upper and lower bounds for w in terms of the self-similar solutions (8.3).

LEMMA 8.2. *Let $n = 2$ and $0 < c < 2^{1/m} C_*$. There are values R and $T \gg 1$, such that the rescaled solution, w, of problem (1.1)-(1.3) satisfies*

$$(8.7) \qquad w(\xi, \tau) \geq U_c(|\xi|\, \phi_1(t)) \qquad \text{for } |\xi| \geq \frac{R}{t^{1/2}},\ t \geq T,$$

where U_c is the profile of the self-similar solution (8.3), and the correction factor $\phi_1(t)$ is given by

$$\phi_1(t) = \frac{1}{\left(1 - \frac{T}{t}\right)^{1/2}}.$$

PROOF. It is based on comparison of $w(\xi, \tau)$ with $\underline{w}_c(\xi, \tau) \equiv U_c(|\xi|\, \phi_1(t))$, which will be shown to be a suitable subsolution for the problem solved by w. Let $C_* - c\, 2^{-1/m} = 3\varepsilon$. We take R big enough such that $P(x) \geq C_* - \varepsilon$ for all $|x| \geq R$ (Lemma 3.3). Then we take T big enough so that $u(x, t) \geq P(x) - \varepsilon$ for $|x| = R$, $t \geq T$ (Theorem 3.1). Thus, $u(x, t) \geq C_* - 2\varepsilon = c\, 2^{-1/m} + \varepsilon$ if $|x| = R$, $t \geq T$. This implies that

$$w(\xi, \tau) \geq \left(\frac{c}{2^{1/m}} + \varepsilon\right) \tau^{1/m} \qquad \text{for } |\xi| = \frac{R}{t^{1/2}},\ t \geq T,\ \tau = \log t.$$

On the other hand we have

$$\lim_{\tau \to \infty} \frac{\overline{w}_c(\xi, \tau)}{\tau^{1/m}} = \frac{c}{2^{1/m}} \qquad \text{for } |\xi| = \frac{R}{t^{1/2}},$$

where we have taken into account the asymptotic behaviour of the profile U_c near the origin. Thus, we have that

$$(8.8) \qquad \overline{w}_c(\xi, \tau) \leq \left(\frac{c}{2^{1/m}} + \varepsilon\right) \tau^{1/m} \leq w(\xi, \tau) \qquad \text{for } |\xi| = \frac{R}{t^{1/2}},\ t \geq T$$

for T big enough.

Next, let us check that $\underline{w}_c$ is a subsolution for equation (8.6). As $\phi_1(t) \geq 1$, and due to the monotonicity of U_c we have that

$$\underline{w}_c(\xi, \tau) \leq \overline{w}_c(\xi, \tau).$$

Let ϕ be any smooth correction factor and $\mathcal{W}_c(\xi, \tau) = U_c(|\xi|\phi(t))$. A simple computation yields
(8.9)
$$\mathcal{W}_{c,\tau} - \mathcal{A}(\mathcal{W}_c) - \frac{1}{\tau}\mathcal{B}(\mathcal{W}_c) = G(|\xi|\phi)\left(-t\frac{\phi'}{\phi} + \frac{1}{2}\left(1 - \phi^2 - \frac{m-1}{2m\tau}\phi^2\right)\right) - \frac{U_c}{m\tau},$$

where $G(r) = -rU'_c(r)$. Observe that $G(|\xi|\phi) \geq 0$. If $\phi = \phi_1$, i.e., if $\mathcal{W}_c = \underline{w}_c$, then

$$-t\frac{\phi'_1(t)}{\phi_1(t)} + \frac{1}{2}(1 - \phi_1^2) = 0,$$

and thus

$$\underline{w}_{c,\tau} - \mathcal{A}(\underline{w}_c) - \frac{1}{\tau}\mathcal{B}(\underline{w}_c) \leq 0.$$

Hence $\underline{w}_c$ is a subsolution of equation (8.6).

Finally we consider the initial conditions at $\tau(T) = \log T$. Obviously

$$w(\xi, \tau(T)) \geq 0 = \underline{w}_c(\xi, \tau(T)) \qquad \text{for } |\xi| \geq \frac{R}{T^{1/2}}.$$

Thus, we can apply comparison to $w(\xi, \tau)$ and $\underline{w}_c(\xi, \tau)$ in $|\xi| \geq Rt^{-1/2}$, $t \geq T$, to obtain the desired estimate (8.7).   □

LEMMA 8.3. *Let $n = 2$, $c > 2^{1/m}C_*$ and $u_0(x) \equiv 0$. There are values $0 < \varepsilon < 1/2$, $T \gg 1$ and $0 < a < \sqrt{1 - 2\varepsilon}$ such that the rescaled solution, w, of problem (1.1)-(1.3) satisfies*

(8.10) $w(\xi, \tau) \leq U_c(|\xi| \phi_2(t)) \equiv \hat{w}_c(\xi, \tau)$ *for $|\xi| \geq t^{-\varepsilon}$, $t \geq T$,*

where U_c is the profile of the self-similar solution (8.3), and the correction factor $\phi_2(t)$ is given by
$$\phi_2(t) = \frac{a}{1 + \frac{1}{\log t}}.$$

REMARK 8.3.1. If $u_0 \not\equiv 0$ we still have an estimate of the form (8.10), though not for c arbitrarily close to $2^{1/m}C_*$, but only for c big enough. However, this will be enough to obtain the required compactness.

PROOF. We take ε such that $\left(\frac{C_*}{c}\right)^m < \varepsilon < \frac{1}{2}$ and then any a such that $a < \sqrt{1 - 2\varepsilon}$. We now check that $\hat{w}_c$ is a supersolution of equation (8.6) if t is large enough and $|\xi| > t^{-\varepsilon}$. In order to do this we take $\mathcal{W}_c = \hat{w}_c$ in formula (8.9), that is, $\phi = \phi_2$. An easy computation shows that

$$\frac{G(r)}{U_c(r)} = -X(\log(r)),$$

where X has the same meaning as in Sections 4 and 5. Thus,

$$\hat{w}_{c,\tau} - \mathcal{A}(\hat{w}_c) - \frac{1}{\tau}\mathcal{B}(\hat{w}_c)$$

$$= G(|\xi||\phi)\left(-t\frac{\phi_2'}{\phi_2} + \frac{1}{2}\left(1 - \phi_2^2 - \frac{m-1}{2m\tau}\phi_2^2\right) + \frac{1}{m\tau X(\log(|\xi||\phi))}\right).$$

The first term in brackets, $t\phi_2'/\phi_2$ tends to zero, the second one tends to $(1 - a^2)/2$. There is a problem with the third one since $X(\eta)$ vanishes like $1/\eta$ as $\eta \to -\infty$, precisely $X(\eta) \simeq \frac{1}{m\eta}$. But we may observe that this term becomes smaller than ε, as we are restricting ourselves to the set $|\xi| > t^{-\varepsilon}$. As $a < \sqrt{1 - 2\varepsilon}$, we conclude that $\hat{w}_{c,\tau} - \mathcal{A}(\hat{w}_c) - \frac{1}{\tau}\mathcal{B}(\hat{w}_c)$ becomes positive for $t \geq \tilde{T}$, if we take $\tilde{T}$ big enough. A careful inspection shows that we can choose $\tilde{T}$ independently of a in the specified range.

Let us now consider the boundary conditions. We recall that, as $u_0 \equiv 0$, then $u \leq P(x)$. We take $\delta = (c\varepsilon^{1/m} - C_*)/2$, and then R big enough such that $P(x) \leq C_* + \delta$ for $|x| \geq R$. Hence $w(\xi, \tau) \leq (C_* + \delta)\tau^{1/m}$ for $|\xi| \geq Rt^{-1/2}$. Observe that $|\xi| = t^{-\varepsilon} \geq Rt^{-1/2}$ if t is big enough. Now, as $\phi_2 \leq a \leq 1$, we have that $\hat{w}_c \geq U_c(|\xi|)$. On the other hand, $U_c(|\xi|) \simeq c(\varepsilon \log t)^{1/m} \simeq c\varepsilon^{1/m}\tau^{1/m}$, for $|\xi| = t^{-\varepsilon}$ as $t \to \infty$. Hence, $\lim_{\tau \to \infty} U_c(|\xi|)\tau^{-1/m} = c\varepsilon^{1/m}$. Thus, there is a value $T \geq \tilde{T}$, which does not depend on a, such that

$$\hat{w}_c(\xi, \tau) \geq U_c(|\xi|) > (c\varepsilon^{1/m} - \delta)\tau^{1/m} = (C_* + \delta)\tau^{1/m} \geq w(\xi, \tau)$$
$$\text{for } |\xi| = t^{-\varepsilon}, \ t \geq T.$$

To obtain the desired estimate (8.10) we apply comparison to $w(\xi, \tau)$ and $\hat{w}_c(\xi, \tau)$ in $|\xi| \geq t^{-\varepsilon}$, $t \geq T$. In order to have ordered initial data, $w(\xi, \tau(T)) \leq \hat{w}_c(\xi, \tau(T))$, we take a small enough. $\qquad\square$

II. CONVERGENCE. We displace the orbits $w(\xi, \tau)$ in time to construct a family of orbits

$$w_s(\xi, \tau) = w(\xi, \tau + s)$$

and use the boundedness and standard regularity results to show that this family is compact in $L_{loc}^\infty(\{\mathbb{R}^n - \{0\}\} \times (0, \infty))$. Therefore, along a subsequence s_n we get convergence to a solution w_∞ of the limit equation $w_t = \mathcal{A}w$, which is a standard scaling of the PME. The a priori bounds show that

$$(8.11) \qquad w_\infty(\xi, \tau) \geq U_{2^{1/m}C_*}(|\xi|) \qquad \text{for } |\xi| > 0, \ \tau > 0,$$

and, if $u_0 \equiv 0$, that for all $c > 2^{1/m}C_*$ there exists a constant $a = a(c)$ such that

$$(8.12) \qquad w_\infty(\xi, \tau) \leq U_c(|\xi|a) \qquad \text{for } |\xi| > 0, \ \tau > 0.$$

Combining (8.11) and (8.12) and the fact that $U_c(|\xi|a) \simeq c(-\log|\xi|)^{1/m}$ for $|\xi| \simeq 0$ it is easy to prove that

$$\lim_{|\xi| \to 0} \frac{w_\infty(\xi, \tau)}{2^{1/m} C_*(-\log|\xi|)^{1/m}} = 1.$$

Thus, $w_\infty(\xi, \tau) \simeq U_{2^{1/m}C_*}(|\xi|)$ for $|\xi| \simeq 0$. Summarizing: if $u_0 \equiv 0$, then w_∞ is a solution of (8.2) such that: (i) $w_\infty \geq U_{2^{1/m}C_*}$ and (ii) $w_\infty(\xi, \tau) \simeq U_{2^{1/m}C_*}(|\xi|)$ for $|\xi| \simeq 0$. Let us recall the key point of the limit dynamics: ε and T disappear.

III.1. IDENTIFICATION OF THE LIMIT ($u_0 \equiv 0$). We may now apply the method of [GV] to describe the ω-limit of the orbits $w(\xi, \tau)$. The appropriate metric is $L^1(\mathbb{R}^n)$ and stability is given by the L^1- contraction property, Corollary 2.3. Then Theorem 3 in [GV] asserts that the ω-limit set for solutions of (8.6) corresponding to initial data $u_0 \equiv 0$ is contained in the ω-limit set for the solutions of (8.2) greater or equal than $U_{2^{1/m}C_*}$ behaving as $U_{2^{1/m}C_*}$ for $\xi \simeq 0$. Thus, if $u_0 \equiv 0$, Theorem 8.1 is an immediate consequence of the following convergence result for the *reduced equation* (8.2).

THEOREM 8.4. *Let $n = 2$ and $\hat{c} > 0$. The ω-limit in $L^\infty_{loc}(\{\mathbb{R}^n - \{0\}\} \times (0, \infty))$ of the orbits $w(\xi, \tau)$ of equation (8.2) such that (i) $w(\xi, \tau) \geq U_{\hat{c}}(|\xi|)$ and (ii) $w(\xi, \tau) \simeq U_{\hat{c}}(|\xi|)$ for $|\xi| \simeq 0$, is the single point $U_{\hat{c}}$.*

PROOF. First we observe that there is a constant $C > 0$ such that

$$w(\xi, \tau) \leq U_C(|\xi|) \qquad \text{for } \xi \neq 0, \ \tau \geq 0.$$

We can now define for τ the optimal upper bound

$$C(\tau) = \inf\{c : \exists \ T \geq 0 \text{ such that } w(\xi, \tau) \leq U_c\left(\frac{|\xi|}{1 + \frac{T}{\tau}}\right), \forall \xi \neq 0\}.$$

It is clear that $\hat{c} \leq C(\tau) \leq C$. It is a simple consequence of the Maximum Principle that $C(\tau)$ is non-increasing. Therefore, there exists the limit

$$\overline{C} = \lim_{\tau \to \infty} C(\tau),$$

and this limit is positive and finite.

Now we consider the family of orbits $w_s(\xi, \tau) = w(\xi, \tau + s)$. By known compactness of the solutions of (8.2), there is a subsequence converging in $L^\infty_{loc}(\{\mathbb{R}^n - \{0\}\} \times (0, \infty))$. Let W be the limit of a subsequence. We have

$$U_{\hat{c}}(\xi) \leq W(\xi, \tau) \leq U_{\overline{C}}(|\xi|).$$

We shall now prove that $\overline{C} = \hat{c}$. Indeed, if $\overline{C} > \hat{c}$, then using the Strong Maximum Principle it is easy to prove that for any $T > 0$,

$$W(\xi, \tau) < U_{\overline{C}}\left(\frac{|\xi|}{1 + \frac{T}{\tau}}\right).$$

Using the behaviour of W for $|\xi| \simeq 0$, we obtain that there is a value $\varepsilon > 0$ small enough such that

$$W(\xi, \tau) < U_{\overline{C}-\varepsilon} \left(\frac{|\xi|}{1 + \frac{T}{\tau}} \right).$$

Thanks to the uniform convergence on compact sets of $\mathbb{R}^2 \setminus \{0\}$, the behaviour at the origin and the control that we have on the free boundary we can assure that

$$w_{s_j}(\xi, \tau) < U_{\overline{C}-\varepsilon} \left(\frac{|\xi|}{1 + \frac{T}{\tau}} \right)$$

for s_j large enough and some fixed τ. Thus,

$$w(\xi, \tau + s_j) < U_{\overline{C}-\varepsilon} \left(\frac{|\xi|}{1 + \frac{T}{\tau + s_j}} \right).$$

This means that

$$C(\tau + s_j) < \overline{C}.$$

By the monotonicity of $C(\tau)$ we get

$$\lim_{\tau \to \infty} C(\tau) < \overline{C}.$$

By the very definition of $\overline{C}$ this conclusion is impossible. Hence, $\overline{C} = \hat{c}$ and, therefore, $W = U_{\hat{c}}$.   □

III.2. IDENTIFICATION OF THE LIMIT (general initial data). We consider now general initial data $u_0 \in C_c(\overline{\Omega})$. Let $\tilde{u}$ be the solution to problem (1.1) corresponding to the same boundary data, $g(x)$, and initial data $\tilde{u}(x, 0) = 0$, and let $\tilde{w}$ be the corresponding rescaled solution. It is clear that $w(\xi, \tau) \geq \tilde{w}(\xi, \tau)$. Let $K = \{|\xi| \geq \delta\}$, $\hat{t} = e^{\tau + s}$ and $t' = \hat{t} \left(\log \hat{t} \right)^{\frac{m-1}{m}}$. Then, for s large enough

$$\|w_s(\cdot, \tau) - \tilde{w}_s(\cdot, \tau)\|_{L^1(K)} \leq \|w_s(\cdot, \tau) - \tilde{w}_s(\cdot, \tau)\|_{L^1(\hat{t}^{-1/2}\Omega)}$$

$$= \frac{(\tau + s)^{1/m}}{\hat{t}} \int_\Omega |u(x, t') - \tilde{u}(x, t')| \, dx \leq \frac{(\tau + s)^{1/m}}{\hat{t}} \|u_0\|_{L^1(\Omega)}.$$

Letting $s \to \infty$ we get $\|w_\infty(\cdot, \tau) - U_{2^{1/m}C_*}(|\cdot|)\|_{L^1(K)} = 0$, and hence that

$$w_\infty(\xi, \tau) = U_{2^{1/m}C_*}(|\xi|).$$

Free boundary. Let $\tilde{m}_+(\tau) = \max_{\xi \in \Gamma(\tau)} |\xi|$, $\tilde{m}_-(\tau) = \min_{\xi \in \Gamma(\tau)} |\xi|$, where $\Gamma(\tau)$ is the free boundary of the rescaled solution, w, of problem (1.1)-(1.3) at time τ. Then, following the lines of Theorem 7.2, it is easy to see that

$$\lim_{\tau \to \infty} \tilde{m}_\pm(\tau) = \xi_0 (2^{1/m} C_*)^{(m-1)/2},$$

which in the original variables reads

$$\lim_{t \to \infty} \frac{m_\pm(t (\log t)^{\frac{m-1}{m}})}{t^{\frac{1}{2}}} = \xi_0 (2^{1/m} C_*)^{(m-1)/2}.$$

Thus, we arrive to the following asymptotic result:

THEOREM 8.5. *Let $n = 2$, and let $m_{\pm}(t)$ and ξ_0 have the same meaning as in Theorem 7.2. Then*

$$\lim_{t \to \infty} \frac{m_{\pm}(t)}{t^{\frac{1}{2}} (\log t)^{-\frac{m-1}{2m}}} = \xi_0 (2^{1/m} C_*)^{(m-1)/2}.$$

REMARK 8.5.1. The results of Theorems 8.1, 8.5 hold true for variable continuous boundary data $\psi \geq 0$ such that $\psi(x, t) \to g(x)$ uniformly as $t \to \infty$.

9. – Connections for other values of β

In Sections 4 and 5 we constructed a family of compactly supported, singular, self-similar solutions of (1.1) of the form (4.1) for the specific value $\beta = \beta_0$. In this section we study the existence of such solutions for other values of $\beta > 0$. More precisely, we consider the existence of orbits connecting the critical point B of the $\Phi\Psi$-plane, representing the free boundary, and the critical point D of the XY-plane, representing a behaviour

$$(9.1) \qquad u \sim C t^{(\frac{\beta}{\beta_0} - 1)/(m-1)} |x|^{(2-n)/m} \qquad \text{if } n \neq 2.$$

For the sake of completeness we also consider connections between the critical point B and the critical point C. If $n \neq 2$, C represents the boundary behaviour

$$(9.2) \qquad u \sim C t^{(2\beta-1)/(m-1)},$$

which is not singular. If $n = 2$ then $C = D$. In this case the orbits coming out from C along a center manifold correspond to the behaviour

$$(9.3) \qquad u \sim C t^{(2\beta-1)/(m-1)} (- \log |x|)^{1/m}.$$

The exceptional orbit coming out from C along the unstable manifold shows the behaviour (9.2).

Let us remark that the behaviours (9.1) and (9.3) are independent of t if and only if $\beta = \beta_0$.

In order to make more clear the critical character of $n = 2$ we also consider values $0 < n < 2$. The phase plane analysis can be extended to non-integer dimensions and $n = 2$ is a bifurcation value. For $n \geq 2$ we prove that there are orbits connecting B and D, i.e., compactly supported, singular self-similar solutions, if and only if $\beta > \beta_0/m$. The limit value $\beta = \beta_0/m$ produces the well-known Barenblatt-Pattle solution. For $0 < n < 2$ there is such a connection only for $\beta = 1/(2m)$. It corresponds to the dipole solution. Observe that in this case the behaviour (9.1) is not singular. For $\beta > 1/(2m)$, there is a connection between B and C.

We remark that $n = 2$ is the critical value that separates the interface behaviours of the form $|x| \sim C t^{m/(n(m-1)+2)}$, valid for $n > 2$, from $|x| \sim C t^{1/2}$, valid for $n < 2$. Observe that $1/2 > \beta_0/m$ if $n \geq 2$. Thus, the *natural* scaling $1/2$ does not produce a compactly supported, self-similar solution behaving as a constant for $x = 0$. This is completely different for $0 < n < 2$. As $1/2 > 1/(2m)$, in this case we have an orbit connecting B and C. Therefore, the behaviour at the origin is given by (9.2), and it is constant.

First of all, we consider the case $n > 2$.

THEOREM 9.1. *Let* $n > 2$. *Let* γ *be the unique orbit of system* (4.8) *approaching the singular point B from* $\Phi \geq 0$ *as* $\eta \to \infty$.
(a) *If* $\beta > \beta_0/m$, $(k \equiv \alpha/\beta < n)$, *then* γ *comes out from the singularity D of the XY-plane.*
(b) *If* $\beta = \beta_0/m$, $(k = n)$, *then* γ *comes out from the singularity C of the XY-plane.*
(c) *If* $\beta < \beta_0/m$, $(k > n)$, *then* γ *does not come out either from C or from D.*

PROOF. (a) Along the half-line $X = -\beta Y$, $Y > 0$, we have $\dot{Y} > 0$ and

$$\frac{\dot{X}}{\dot{Y}} = -\beta + \frac{n\beta - \alpha}{2 + (m-1)\beta Y} > -\beta.$$

Moreover, the line $Y = 0$ is an orbit. Therefore, the region $\mathcal{F} \equiv \{Y \geq 0\} \cap \{X \geq -\beta Y\}$ is invariant. On the other hand, the orbit going out of C into the half-plane $Y \geq 0$ does so along the direction $(-\alpha/n \quad 1)$. Thus, it goes into $\mathcal{F}$ if $\alpha < n\beta$. We now remember that γ approaches B along the direction $(1 \quad \frac{1-\beta((m-1)(n+2)+4)}{m\beta})$, that is,

$$(m-1)(X + \beta Y) - 2 = \frac{\Psi + \beta}{\Phi} \simeq \frac{1 - \beta((m-1)(n+2)+4)}{m\beta}.$$

Thus, if the straight line

(9.4)
$$\begin{aligned}
Y &= -\frac{X}{\beta} + \frac{1}{(m-1)\beta} \left(\frac{1 - \beta((m-1)(n+2)+4)}{m\beta} + 2 \right) \\
&= -\frac{X}{\beta} + \frac{\beta_0 - m\beta}{m(m-1)\beta^2 \beta_0}
\end{aligned}$$

is below the line $Y = -X/\beta$, as is the case for $\beta > \beta_0/m$, then γ cannot come from C.

Hence, in order to have the result proved we just need to discard the possibility that the orbit goes to $X \to -\infty$ as $\eta \to -\infty$. The null-cline $\dot{X} = 0$ is given by

$$X(2 - n - mX) - Y(\alpha + \beta X) = 0.$$

This curve is a hyperbola, maybe degenerate, with asymptotes given by

$$X = -\frac{\alpha}{\beta} \qquad \text{and} \qquad Y = -\frac{mX}{\beta} + \frac{(2-n)\beta + m\alpha}{\beta^2}.$$

As $m > 1$, both asymptotes, and hence also the hyperbola, are above the line $Y = -X/\beta$ for $X < 0$, with $|X|$ big enough, and hence orbits which are below that line cannot approach $X = -\infty$ as $\eta \to -\infty$.

(b) In this case γ is explicit. It corresponds to the well-known Barenblatt-Pattle solutions to the PME and it comes out from the singular point C of the XY-plane.

(c) In this case we have that $\dot{Y} > 0$ and $\dot{X}/\dot{Y} < -\beta$ along the half-line $X = -\beta Y$, $Y > 0$. Therefore the region $\mathcal{H} \equiv \{Y \geq 0\} \cap \{X \leq -\beta Y\}$ is invariant. Moreover, the outgoing direction of the orbit coming out of C points towards $\mathcal{H}$. Now, the straight line (9.4), to which γ approaches, is above the line $Y = -X/\beta$. Thus, γ cannot come either from C or from D, as this latter point is below the line $Y = -X/\beta$. $\qquad\square$

When $n = 2$, $C = D$. The border case $\beta = \beta_0/m$ corresponds to a connection between the free boundary point B and the exceptional unstable fast manifold.

THEOREM 9.2. *Let $n = 2$. Let γ be the unique orbit of system* (4.8) *approaching the singular point B from $\Phi \geq 0$ as $\eta \to \infty$.*
(a) *If $\beta > \beta_0/m$, $(k < n)$, then γ comes out from the singularity C of the XY-plane along the direction $(-1 \quad 0)$ corresponding to a slow center manifold.*
(b) *If $\beta = \beta_0/m$, $(k = n)$, then γ comes out from the singularity C of the XY-plane along the direction $(-1 \quad 2m)$ corresponding to the unstable fast manifold.*
(c) *If $\beta < \beta_0/m$, $(k > n)$, then γ does not come out from the singularity C of the XY-plane.*

PROOF. (a) The region $\mathcal{F}$ is invariant (see the proof of Theorem 9.1). On the other hand, we know that the unstable manifold goes out of C along the direction $(-\alpha/2 \quad 1)$. Thus, for $\beta > \beta_0/m$ it goes into $\mathcal{F}$. We now remember that γ approaches B along the straight line (9.4), which is below $Y = -X/\beta$ for $\beta > \beta_0/m$. Thus, γ cannot come out from C along the unstable manifold. To prove the result we have to discard the possibility that the orbit goes to $X \to -\infty$ as $\eta \to -\infty$. This is done as in Theorem 9.1.

(b) If $\beta = \beta_0/m$, then γ is explicit. It corresponds to the well-known Barenblatt-Pattle solutions to the PME and comes out of C along $(-1 \quad 2m)$.

(c) In this case the region $\mathcal{H}$ is invariant (cf. proof of Theorem 9.1). Moreover, the unstable manifold going out of C into $Y \geq 0$ is contained in $\mathcal{H}$, and this is also the case for center manifolds. On the other hand, the straight line (9.4), to which γ approaches, is above the line $Y = -X/\beta$. Thus, γ cannot come from C. $\qquad\square$

If $0 < n < 2$ the critical point D does not represent a singular behaviour any more. After meeting the point C at the bifurcation value $n = 2$, D lies now to the right of C.

THEOREM 9.3. *Let $0 < n < 2$. Let γ be the unique orbit of system* (4.8) *approaching the singular point B from $\Phi \geq 0$ as $\eta \to \infty$.*
(a) *If $\beta > 1/(2m)$, $(k < 2)$, then γ comes out from the singularity C of the XY-plane.*
(b) *If $\beta = 1/(2m)$, $(k = 2)$, then γ comes out from the singularity D of the XY-plane.*
(c) *If $\beta < 1/(2m)$, $(k > 2)$, then γ does not come out either from C or from D.*

PROOF. (a) Along the half-line $X = -\beta Y + \frac{2-n}{m}$, $Y > 0$, we have $\dot{Y} > 0$ and

$$\frac{\dot{X}}{\dot{Y}} = -\beta + \frac{2\beta - \alpha}{2 + (m-1)\beta Y + \frac{(m-1)(n-2)}{m}} > -\beta.$$

Hence, the region $\mathcal{J} \equiv \{Y \geq 0\} \cap \{X \geq -\beta Y + \frac{2-n}{m}\}$ is invariant. Moreover, the orbit going out of D into $Y \geq 0$ does so along the direction $(\beta - \beta_0 \quad (m-1)((2-n)\beta_0 + 1))$. Thus, if $\beta > 1/(2m)$, it goes into $\mathcal{J}$. Now, the straight line (9.4), to which γ approaches, does not intersect $\mathcal{J}$. Thus, γ cannot come from D. The possibility that γ goes to $X \to -\infty$ as $\eta \to \infty$ is discarded as in Theorem 9.1.

(b) In this case γ is explicit. It corresponds to the well-known dipole solutions to the PME. It comes out from the singular point D on the XY-plane. In fact it is given by $Y = -2mX + 2(2 - n)$.

(c) In this case we have that $\dot{Y} > 0$ and $\dot{X}/\dot{Y} < -\beta$ on the half-line $X = -\beta Y + \frac{2-n}{m}$, $Y \geq 0$. Therefore, the region $\mathcal{K} \equiv \{Y \geq 0\} \cap \{X \leq -\beta Y + \frac{2-n}{m}\}$ is invariant. Moreover, the outgoing direction of the orbit coming out of D into $Y \geq 0$ points towards $\mathcal{K}$. Now, the straight line (9.4), to which γ approaches, does not intersect $\mathcal{K}$. Thus, γ cannot come either from D or from C, as this latter point lies in $\mathcal{K}$. $\square$

10. – The limit m → ∞: Hele-Shaw

If we take the limit $m \to \infty$ in the asymptotic rates for the growth of the free boundary, (1.7) and (1.9), we obtain precisely the rates for the growth of the free boundary for the exterior Hele-Shaw problem (cf. [QV]), that is,

$$|x| \sim Ct^{1/n} \text{ if } n > 2, \qquad |x| \sim C\left(\frac{t}{\log t}\right)^{1/2} \text{ if } n = 2.$$

This is a hint that there may be a connection between this limit and the Hele-Shaw problem. We devote this section to investigate this connection for the self-similar solutions that we constructed in sections 4 and 5, and to study the limit case $m = \infty$. This limit case is much simpler. In fact its solutions are explicit, and it throws light over the $m < \infty$ case.

The limit $m \to \infty$ for the PME has already been considered, for the case of focusing self-similar solutions, in [AGV]. If we divide by $(m - 1)$ in the pressure equation (4.5) and formally take the limit $m \to \infty$ we obtain

$$(10.1) \qquad\qquad \Delta v = 0,$$

while on the free boundary we still have the condition

$$(10.2) \qquad\qquad s_n = -\nabla v \cdot \vec{n},$$

where s_n is velocity of the interface in the normal direction $\vec{n}$. This is the well known Hele-Shaw problem. If we look for solutions of the self-similar form (4.6) for this problem we obtain that the profile Φ must satisfy

$$(10.3) \qquad\qquad |\xi|^2 \Phi'' + (n + 3)|\xi|\Phi' + 2n\Phi = 0,$$

an equation which can be made autonomous through the change of independent variable $\eta = \log |\xi|$. For $n \neq 2$ the solutions of this equation are given by

$$(10.4) \qquad \Phi(\xi) = \frac{C}{|\xi|^n} + \frac{D}{|\xi|^2}, \quad \Psi(\xi) = -\frac{nC}{|\xi|^n} - \frac{2D}{|\xi|^2},$$

and correspond to harmonic functions of the form

$$(10.5) \qquad\qquad v = Ct^{n\beta - 1}|x|^{2-n} + Dt^{2\beta - 1}.$$

Obvious changes apply to $n = 2$. Here we want to stress the similarity with the analysis of previous sections in order to see how the limit $m \to \infty$ takes place. Therefore, we write the equation as the first-order system

$$(10.6) \qquad\qquad \begin{cases} \dot{\Phi} = \Psi, \\ \dot{\Psi} = -2n\Phi - (n + 2)\Psi, \end{cases}$$

where the dot means $d/d\eta$. This linear system coincides exactly with what is obtained doing formally $m \to \infty$ in system (4.8). However, the phase portrait is a bit different. System (10.6) is not singular at $\Phi = 0$, and hence $(\Phi, \Psi) = (0, -\beta)$ is not a critical point any more. The only critical point is given by $(\Phi, \Psi) = (0, 0)$.

PROPOSITION 10.1. *The critical point $A_\infty = (0, 0)$ is an attractor. It has matrix*

$$\begin{pmatrix} 0 & 1 \\ -2n & -(n + 2) \end{pmatrix}$$

with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -n$ and corresponding eigenvectors $e_1 = (\begin{matrix} 1 & -2 \end{matrix})$ and $e_2 = (\begin{matrix} 1 & -n \end{matrix})$.

The condition at the free boundary, (10.2), implies that

$$\Phi(\eta_0) = 0, \quad \Psi(\eta_0) = -\beta$$

for a certain η_0. We are thus looking for orbits passing through $(\Phi, \Psi) = (0, -\beta)$. As we have already mentioned, this point is not critical any more.

As we are interested in solutions which are singular at the origin, we would also like to perform in this case an inversion change of variables. However, system (4.17) degenerates as $m \to \infty$, and (4.16) is not the correct change of variables for this limit. Instead we take as new variables

$$\widetilde{X}(\eta) = \frac{\Psi(\eta)}{\Phi(\eta)} + 2, \qquad \widetilde{Y}(\eta) = \frac{1}{\Phi(\eta)},$$

which are just a scaled version of X and Y of Section 4, $\widetilde{X} = (m-1)X$, $\widetilde{Y} = (m-1)Y$. In the new variables (4.17) reads

$$(10.7) \qquad \begin{cases} \dot{\widetilde{X}} = \widetilde{X}\left(2 - n - \dfrac{m}{m-1}\widetilde{X}\right) - \widetilde{Y}\left(\dfrac{1-2\beta}{m-1} + \dfrac{\beta}{m-1}\widetilde{X}\right), \\ \dot{\widetilde{Y}} = \widetilde{Y}(2 - \widetilde{X}). \end{cases}$$

Observe that $\widetilde{X} = (|\xi|\Phi'(|\xi|)/\Phi(|\xi|)) + 2$. Then (10.6) is transformed into

$$(10.8) \qquad \begin{cases} \dot{\widetilde{X}} = \widetilde{X}(2 - n - \widetilde{X}), \\ \dot{\widetilde{Y}} = \widetilde{Y}(2 - \widetilde{X}), \end{cases}$$

which is precisely the formal limit of (10.7) as $m \to \infty$. System (10.8) has two critical points, C_∞, located at $(\widetilde{X}, \widetilde{Y}) = (0, 0)$ and $D_\infty = (2-n, 0)$. Comparing them with the critical points of system (10.7), $C = (0, 0)$, $D = (\frac{m-1}{m}(2-n), 0)$ and $E = (2, -2m/\beta_0)$, we observe that $C_\infty = C$, and that D_∞ comes from D through letting $m \to \infty$. The point E, which we had not considered before, as it does not lie on the region $\widetilde{Y} \geq 0$, goes to infinity as $m \to \infty$. If $n = 2$, C_∞ and D_∞ coincide.

PROPOSITION 10.2. *The linearization of* (10.8) *around the critical point* $C_\infty = (0, 0)$ *has matrix*

$$\begin{pmatrix} 2-n & 0 \\ 0 & 2 \end{pmatrix},$$

with eigenvalues $\lambda_1 = 2 - n$ *and* $\lambda_2 = 2$ *and corresponding eigenvectors* $e_1 = (1 \quad 0)$ *and* $e_2 = (0 \quad 1)$. *Thus, if* $n > 2$, C_∞ *is a saddle, while for* $n < 2$ *it is a repeller. If* $n = 2$ *it is a saddle-node.*

PROPOSITION 10.3. *The linearization of* (10.8) *around the critical point* $D_\infty = (2-n, 0)$ *has matrix*

$$\begin{pmatrix} n-2 & 0 \\ 0 & 2 \end{pmatrix},$$

with eigenvalues $\lambda_1 = n - 2$ *and* $\lambda_2 = 2$ *and corresponding eigenvectors* $e_1 = (1 \quad 0)$ *and* $e_2 = (0 \quad 1)$. *Thus, if* $n > 2$, D_∞ *is a repeller, while it is a saddle for* $n < 2$. *If* $n = 2$ *it is a saddle-node.*

REMARK. If $n \neq 2$, then C_∞ corresponds to the behaviour

$$v \sim Ct^{2\beta-1},$$

while D_∞ represents the behaviour

$$v \sim Ct^{n\beta-1}|x|^{2-n}.$$

If $n = 2$, then $C_\infty = D_\infty$. Orbits approaching this point along a center manifold correspond to solutions with a behaviour

$$v \sim t^{2\beta-1}(-\log|x|).$$

We are looking for orbits approaching the free boundary point, B_∞, given by $(\Phi, \Psi) = (0, -\beta)$, which in our variables is $(\widetilde{X}, \widetilde{Y}) = (-\infty, \infty)$. In fact, system (10.6) can be explicitly integrated as we pointed out above, hence our work is done. Let us summarize the results.

THEOREM 10.4. *Let $n > 2$ and $\beta > 0$. The unique orbit of system* (10.6) *passing through the free boundary point $B_\infty = (0, -\beta)$ comes from the critical point D_∞ of the $\widetilde{X}\widetilde{Y}$-plane. The solutions of the Hele-Shaw problem corresponding to this orbit are given by*

$$v = Ct^{n\beta-1}\left(|x|^{2-n} - R^{2-n}(t)\right)_+, \qquad R(t) = \left(\frac{C(n-2)}{\beta}\right)^{1/n} t^\beta.$$

THEOREM 10.5. *Let $n = 2$ and $\beta > 0$. The unique orbit of system* (10.6) *passing through the free boundary point $B_\infty = (0, -\beta)$ comes out from the critical point C_∞ of the $\widetilde{X}\widetilde{Y}$-plane along a center manifold. The solutions of the Hele-Shaw problem corresponding to this orbit are given by*

$$v = Ct^{2\beta-1}\left(-\log\frac{|x|}{\left(\frac{C}{\beta}\right)^{1/2} t^\beta}\right)_+.$$

REMARK. As in the PME case, these solutions do not give the asymptotic behaviour in an exterior domain for any β. In the Hele-Shaw problem there is also a logarithmic correction (cf. [QV]).

THEOREM 10.6. *Let $n < 2$ and $\beta > 0$. The unique orbit of system* (10.6) *passing through the free boundary point $B_\infty = (0, -\beta)$ comes from the critical point C_∞ of the $\widetilde{X}\widetilde{Y}$-plane. The solutions of the Hele-Shaw problem corresponding to this orbit are given by*

$$v = Ct^{n\beta-1}\left(R^{2-n}(t) - |x|^{2-n}\right)_+, \qquad R(t) = \left(\frac{C(2-n)}{\beta}\right)^{1/n} t^\beta.$$

REMARK. In this case the solutions are not singular at the origin. Indeed, they show a behaviour $v = Ct^{2\beta-1}$.

11. – Conclusions and extensions

We have shown that the asymptotic behaviour of the solutions of the exterior problem for the porous media equation in dimensions $n \geq 2$ with fixed nonnegative and nontrivial boundary data is described by two matched expansions. The inner expansion is given by a stationary state $P(x)$ with harmonic m-power, which takes into account the details of the geometry of the domain and the form of the boundary data. On the other hand, the outer expansion, which allows to precisely locate the free boundary, is given in dimensions $n > 2$ by a radial self-similar solution of the PME which has a standing singularity at the origin, $x = 0$, $t \geq 0$. There are a number of variations of the above results that can be addressed by the methods of the paper. We discuss some of them and indicate the corresponding results.

NONINTEGER DIMENSIONS. We remark that, when we restrict ourselves to classes of radially symmetric functions, all the above assertions hold for noninteger dimensions and the analysis done for n an integer larger or equal than 3 extends to real $n > 2$. In dimension $n = 2$ the outer expansion is of the same type but for the presence of logarithmic corrections in the scaling factors, which account for the fact that there is *no* self-similar solution which approximately solves the problem even with radially symmetric domain and radial boundary data. Dimension $n = 2$ marks in this respect the change of behaviour. It can easily be shown that for radial solutions and $0 < n < 2$ the analysis produces an asymptotic behaviour similar to the one-dimensional case. Let us sketch the simple arguments: the self-similar solutions with $\beta = 1/2$ constructed in Theorem 9.3 play a special role in the proof. These solutions have the form $u_c(r, t) = U_c(rt^{-1/2})$, with $U_c(0) = c$. We start by considering solutions with initial data $u_0 \equiv 0$. Then, given an arbitrary $\varepsilon > 0$ there is a value $\tau \geq 0$ big enough such that

$$U_c(r) \leq u(rt^{1/2}, t) \leq U_{c+\varepsilon}\left(\frac{rt^{1/2}}{(t + \tau)^{1/2}}\right),$$

where c is the value of u at the fixed boundary. Convergence follows easily. In order to prove convergence for general initial data we need the following radial version of the contraction property (2.3):

$$\int_\Omega \{u(r, t) - \hat{u}(r, t)\}^+ r^{n-1} \, dr \leq \int_\Omega \{u_0(r) - \hat{u}_0(r)\}^+ r^{n-1} \, dr.$$

The proof continues as in Theorem 7.1. We leave the details to the interested reader.

INITIAL DATA: we have made the assumption of compactly supported initial data and shown that they have no influence at the asymptotic level. If the data have fast decay at infinity the result is still true. Even for data in L^1 we have

convergence in L^1 norm, thanks to the property of L^1- continuity of solutions. This will not be true if larger classes of data are chosen, a question that we leave for future research. Let us point out that there are classes of singular self-similar solutions with a different behaviour at infinity. They correspond to orbits joining the critical point $D = (\frac{2-n}{m}, 0)$ of the XY-plane with the critical point $A = (0, 0)$ of the $\Phi\Psi$-plane (in the notation of Section 4). In other words, instead of entering the point B and having a free boundary, they enter A and decay as $|\xi|$ tends to infinity. More precisely, they enter A along the eigenvector $(1 \quad -1/\beta)$, hence $\Phi \sim C|\xi|^{-1/\beta}$, which means a behaviour at infinity given by

$$u \sim Ct^\mu |x|^\delta, \qquad \mu = \frac{\frac{1}{\beta} - 1}{m - 1}, \quad \delta = \frac{2 - \frac{1}{\beta}}{m - 1}.$$

Such solutions exist for all $\beta > \beta_0/m$, that is, for $\delta > -n$. Hence these solutions are not integrable in x, as expected. For the boundary data we consider here, the correct exponent choice is $\beta = \beta_0$. An easy comparison argument shows that we may replace the condition of compact support on the initial data of our problem by the condition $u_0(x) = o(|x|^\delta)$ and the asymptotic rate is the same given in the Introduction. Of course, we lose the free boundary, which makes the observation less interesting.

LIMIT $m \to \infty$: we have shown that the asymptotic picture converges towards the asymptotics of the Hele-Shaw problem studied by the authors in [QV], where it is also proved that Hele-Shaw gives the asymptotic behaviour of the Stefan problem in dimensions $n > 1$ (but not for $n = 1$). In other words, the solutions of both the Stefan and the Hele-Shaw problem in an exterior domain have the same asymptotic behaviour, namely they converge to the solution of the Dirichlet exterior problem for the Laplacian in the interior of the positivity set, and to a singular, radial, self-similar solution of the Hele-Shaw problem close to the free boundary. Thus, in the Stefan problem we observe a phenomenon of *asymptotic simplification*, something that does not occur for the PME. Let us point out that the relative simplicity of the Hele-Shaw model allowed in [QV] for the explicit computation of the actual solutions of the exterior problem under radial symmetry. This was not possible in the present situation and it implied the need of a further development of the asymptotic tools, which makes the theory developed here more amenable to application in other contexts.

BOUNDARY DATA: the initial assumption of constancy in time was easily relaxed into uniform convergence into a stationary state $g(x)$ by approximation. Weaker convergences will pose problems which are worth studying.

More interesting is the consideration of decaying boundary data. The ideas presented in this paper have a natural application to problem (1.1)-(1.3) with variable data

(11.1) $u(x, t) = \psi(x, t)$ for $x \in \partial\Omega, \ t > 0,$

under conditions of the type

(11.2) $\psi(x, t) = O(t^{-\sigma})$ as $t \to \infty,$

using the classes of singular self-similar solutions with $\beta \neq \beta_0$ constructed in Section 9. Let us sketch some of the main developments in this case and announce the basic asymptotic result. We assume for simplicity $n > 2$ and radially symmetric data. The self-similar solutions exist for every $\beta > \beta_B \equiv \beta_0/m$, and provide us with a solution of the PME taking boundary data

$$(11.3) \qquad \psi(x, t) \sim c\, t^{-\sigma}, \qquad \beta = \beta_0(1 - \sigma(m - 1)),$$

so that they cover the range $\sigma \in (-\infty, 1/m)$. However, for $\beta = \beta_B$ the behaviour of the self-similar solution is different since it corresponds to a change in the type of connection, that now starts from the origin in the XY-plane, instead of starting at the point D. Accordingly, the boundary behaviour jumps to Barenblatt's exponent, $\sigma = n/(n(m - 1) + 2)$, which is larger than the limit of (11.3), $1/m$. Moreover, when we go down below β_B we find self-similar solutions which change sign near $x = 0$, and can thus serve as subsolutions. After a careful analysis we can state the following result.

THEOREM. *For data of the form* (11.3) *with* $\sigma \in \mathbb{R}$ *we get an interface growth of the form*

$$(11.4) \qquad \log r(t) \sim \beta \, \log t, \qquad \beta = \frac{\max\{1, \, m(1 - \sigma(m - 1))\}}{n(m - 1) + 2}.$$

When we take $\psi(x, t) = 0$, *i.e. in the limit* $\sigma \to \infty$, *we still find Barenblatt's source-type solutions with exponent* $\beta = \beta_B$.

Detailed proofs of the asymptotic behaviour of this problem will be given elsewhere, [QV2]. For $n = 1$ this problem was studied in [Ba2], [GP1,2]. A new model problem applicable to infiltration in porous strata is developed in [BV] and presents new challenges in several dimensions.

OTHER EQUATIONS AND PROBLEMS: a very natural extension concerns the Neumann problem for the PME. The technique can also be extended to other geometries, thus joining this problem with the standard Dirichlet and Cauchy problems. It can also be extended to the PME in the range $m < 1$, the so-called *fast diffusion equation*, but in this case no free boundary appears and the problem seems less interesting. The limit situation $m \to 1$ offers quite interesting limit problems in the so-called Hamilton-Jacobi scale. Several other degenerate diffusion problems are next in the line, like for example the *p-Laplacian equation*

$$(11.5) \qquad u_t = \Delta_p(u),$$

where the p-Laplacian operator is defined as

$$(11.6) \qquad \Delta_p(u) = \nabla \cdot \left(\left| \nabla u \right|^{p-2} \nabla u \right)$$

and $p > 2$ is a constant. We can also consider a more general equation, the so-called *doubly nonlinear equation*

$$(11.7) \qquad\qquad u_t = \Delta_p(u^m),$$

from which both the porous media and the p-Laplacian equations are particular cases. We can prove for this equation analogous asymptotic results and only careful attention should be paid to the special properties of the p-Laplacian operator. Let us give the results, which might interest the reader. The near-field limit is given by the stationary state $P(x)$ solving the following Dirichlet Exterior problem:

$$\begin{cases} \Delta_p(P^m) = 0 & \text{if } x \in \Omega, \\ P = g & \text{if } x \in \Gamma, \\ \lim_{|x|\to\infty} P(x) = 0 & \text{if } n > p, \\ P(x) \text{ bounded} & \text{if } n \leq p. \end{cases}$$

The behaviour of P at infinity is given by

$$\begin{cases} \lim_{|x|\to\infty} |x|^{\frac{n-p}{m(p-1)}} P(x) = C_* & \text{if } n > p, \\ \lim_{|x|\to\infty} P(x) = C_* & \text{if } n \leq p. \end{cases}$$

Thus, we see that for this problem the critical dimension is $n = p$. We introduce a scaling of the form

$$\begin{cases} y = xt^{\frac{m(p-1)}{n(m(p-1)-1)+p}}, & v = ut^{\frac{n-p}{n(m(p-1)-1)+p}} & \text{for } n > p, \\ y = xt^{1/p}, & v = u & \text{for } n < p. \end{cases}$$

The scaled solution $v(y, t)$ converges as $t \to \infty$ to a radially symmetric, self-similar solution of the doubly nonlinear equation (11.7), which turns out to be singular at the origin if $n > p$ and that has a constant behaviour at that point if $n < p$. In both cases there is a constant to be adjusted through suitable matching. The free boundary is given by

$$\begin{cases} |x| \sim Ct^{\frac{m(p-1)}{n(m(p-1)-1)+p}} & \text{for } n > p, \\ |x| \sim Ct^{1/p} & \text{for } n < p. \end{cases}$$

For the limit case $n = p$ we have, as usual, logarithmic corrections. Thus, u behaves in the outer limit like

$$u(x, t(\log t)^{\frac{m(p-1)-1}{m}}) \sim \frac{1}{(\log t)^{1/m}} U\left(\frac{|x|}{t^{1/p}}\right)$$

for a certain profile U which is nonnegative, decreasing, compactly supported and singular at the origin. The free boundary moves like

$$|x| \sim Ct^{1/p}(\log t)^{-\frac{m(p-1)-1}{pm}} \qquad \text{for } n = p.$$

Finally, problems with reaction, absorption or convection terms can be treated. The interested reader can find extensive preliminary information on asymptotic behaviour and self-similarity for many of such equations in [S4].

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