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Liouville-Gelfand Type Problems for the N-Laplacian on Bounded Domains of $\mathbb{R}^N$

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Abstract. In this article it is used minimax methods to establish some results of existence and multiplicity of solutions for the N-Laplacian equation on bounded domains of $\mathbb{R}^N$, with Dirichlet boundary conditions, when the nonlinearity has exponential growth. The subcritical and critical cases are considered.

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1. – Introduction

In this article, we study the existence and multiplicity of solutions for the following quasilinear elliptic problem

$$(P)_\lambda \qquad \begin{cases} -\Delta_N u = -\operatorname{div}(|\nabla u|^{N-2}\nabla u) = \lambda f(x,u), & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N (N \geq 2)$ with boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, and the nonlinearity $f(x,s)$ satisfies

(f_1) $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x,0) > 0$, for every $x \in \Omega$,

and the growth condition

$(f)_{\alpha_0}$ There exists $\alpha_0 \geq 0$ such that

$$\lim_{s\to\infty} \frac{|f(x,s)|}{\exp(\alpha s^{N/N-1})} = \begin{cases} 0, & \forall\, \alpha > \alpha_0, \text{ unif. on } \bar{\Omega}, \\ +\infty, & \forall\, \alpha < \alpha_0, \text{ unif. on } \bar{\Omega}. \end{cases}$$

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In the literature [1], [10], [11], $f(x,s)$ is said to have subcritical or critical growth when $\alpha_0 = 0$ or $\alpha_0 > 0$, respectively. We note that such notion is motivated by Trudinger-Moser estimates [17], [23] which provide

$$\exp(\alpha|u|^{N/N-1}) \in L^1(\Omega), \ \forall \ u \in W_0^{1,N}(\Omega), \ \forall \ \alpha > 0, \tag{1.1}$$

and

$$\sup_{\|u\|_{W_0^{1,N}} \leq 1} \int_\Omega \exp(\alpha|u|^{N/N-1})\,dx \leq C(N) \in \mathbb{R}, \ \forall \ \alpha \leq \alpha_N = N w_{N-1}^{\frac{1}{N-1}}, \tag{1.2}$$

where w_k is the volume of S^k. We also observe that a typical and relevant case to be considered for problem $(P)_\lambda$ is given by $f(x,s) = \exp(\alpha_0 s^{N/N-1})$.

In our first result, we establish the existence of a solution for $(P)_\lambda$ when $\lambda > 0$ is sufficiently small,

THEOREM 1.1. *Suppose $f(x,s)$ satisfies (f_1) and $(f)_{\alpha_0}$. Then, there exists $\bar{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least one solution for every $\lambda \in (0, \bar{\lambda})$.*

To obtain the existence of a second solution for problem $(P)_\lambda$ in the subcritical case, we assume that $f(x,s)$ satisfies

(f_2) There are constants $\theta > N$ and $R > 0$ such that

$$0 < \theta F(x,s) \leq s f(x,s), \ \forall \ x \in \bar{\Omega}, \ \ s \geq R.$$

THEOREM 1.2 (Second solution: Subcritical case). *Suppose $f(x,s)$ satisfies (f_1), (f_2) and $(f)_{\alpha_0}$, with $\alpha_0 = 0$. Then, there exists $\bar{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least two solutions for every $\lambda \in (0, \bar{\lambda})$.*

Note that (f_2) is the version of the famous Ambrosetti-Rabinowitz condition [3] for the N-Laplacian. It implies, in particular, that $f(x,s)/s^N \to \infty$, as $s \to \infty$, uniformly on $\bar{\Omega}$.

In our next result, we provide the existence of two solutions for $(P)_\lambda$ when $f(x,s)$ has critical growth. In that case, we shall need to suppose a stronger version of condition (f_2),

$(\hat{f}_2)$ For every $\theta > N$, there exists $R(\theta) > 0$ such that

$$0 < \theta F(x,s) \leq s f(x,s), \forall \ x \in \bar{\Omega}, \ \ s \geq R(\theta).$$

Assuming the following further restriction on the growth of $f(x,s)$,

(f_3) There exists a non-empty open set $\hat{\Omega} \subset \Omega$ such that

$$\lim_{s\to\infty} \inf_{x\in\hat{\Omega}} \frac{f(x,s)}{\exp(\alpha_0 s^{N/N-1})} = \infty,$$

we obtain

THEOREM 1.3 (Second solution: Critical case). *Suppose $f(x,s)$ satisfies (f_1), $(\hat{f}_2)$, (f_3) and $(f)_{\alpha_0}$, with $\alpha_0 > 0$. Then, there exists $\overline{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least two solutions for every $\lambda \in (0, \overline{\lambda})$.*

Exploiting the convexity of the primitive $F(x,s)$, in our final result we are able to consider a weaker version of (f_3), obtaining the same conclusion of Theorem 1.3. More specifically, we suppose

$(\hat{f}_3)$ There exist a non-empty open set $\hat{\Omega} \subset \Omega$

$$\lim_{s\to\infty} \inf_{x\in\hat{\Omega}} \frac{f(x,s)s^\beta}{\exp(\alpha_0 s^{N/N-1})} = \infty,$$

where $\beta = \frac{1}{2(N-1)}$ if $N = 3$, and $\beta = \frac{1}{N-1}$ otherwise.

(f_4) $F(x,.)$ is convex on $[0,\infty)$ for every $x \in \hat{\Omega} \subset \Omega$, $\hat{\Omega}$ given by $(\hat{f}_3)$,

THEOREM 1.4 (Second solution: Convex Critical case). *Suppose $f(x,s)$ satisfies (f_1), $(\hat{f}_2)$, $(\hat{f}_3)$, (f_4) and $(f)_{\alpha_0}$, with $\alpha_0 > 0$. Then, there exists $\overline{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least two solutions for every $\lambda \in (0, \overline{\lambda})$.*

We observe that Theorem 1.4 establishes the existence of two solutions of $(P)_\lambda$ for $\lambda > 0$ sufficiently small when $f(x,s) = \exp(\alpha_0 s^{N/N-1})$.

As it is well known, the classical Liouville-Gelfand problem is given by

$$(LG)_\lambda \qquad \begin{cases} -\Delta u = \lambda e^u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with boundary $\partial\Omega$, and $\lambda > 0$ is a real parameter. First considered by Liouville [16], for the case $N = 1$, and afterwards by Bratu [4], for $N = 2$, and Gelfand [13], for $N \geq 1$, this problem has been extensively studied during the last three decades (See [7], [8], [12] and references therein). As observed in [12], problem $(LG)_\lambda$ is of great relevance since it appears in mathematical models associated with astrophysical phenomena and to problems in combustion reactions.

In [8], Crandall and Rabinowitz used bifurcation theory to establish the existence of one solution for problem $(LG)_\lambda$, for $\lambda > 0$ sufficiently small, and a nonlinearity $f(x,s)$ replacing e^s. In [8] no growth restriction on $f(x,s)$ is assumed. To obtain such result, those authors assume $f(x,s) \in C^3(\Omega \times \mathbb{R}, \mathbb{R})$, $f_s(x,0) > 0$ and $f_{ss}(x,s) > 0$, for every $x \in \Omega$ and $s > 0$ Supposing that $f(x,s)$ has a subcritical growth, they show that this solution is a local minimum for the associated functional. Then, using critical point theory, they are able to prove the existence of a second solution. We note that Theorems 1.3 and 1.4 improve the last mentioned result of [8] when $N = 2$ since they allow $f(x,s)$ to have critical growth. In particular, we may consider $f(x,s) = e^{s^2}$.

In [12], Garcia Azorero and Peral Alonso proved the existence of solutions for $(LG)_\lambda$, with $\lambda > 0$ sufficiently small, when the Laplacian is replaced by a p-Laplacian operator. The nonexistence of solutions for $(LG)_\lambda$ for this more general class of operators, when $\lambda > 0$ is sufficiently large, was also established in [12]. We should also mention the article by Clément, Figueiredo and Mitidieri [7], where the exact number of solutions for an operator more general that the p-Laplacian is established when Ω is an open ball of $\mathbb{R}^N$. In [7], it is not assumed any growth restriction on $f(x,s)$.

We note that the solutions mentioned in Theorems 1.1-1.4 are weak solutions of $(P)_\lambda$ (See [19]). We also observe that in this article, we use minimax methods to derive such solutions.

To prove Theorem 1.1, we first provide an abstract result that establishes the existence of a critical point for a functional of class C^1 defined on a real Banach space assuming a version of the famous Palais-Smale condition for the weak topology (See Definitions and Proposition 2.2 in Section 2). Motivated by the argument used in [18], we prove that the associated functional satisfies such condition under hypotheses (f_1) and $(f)_{\alpha_0}$. Taking $\lambda > 0$ sufficiently small, we are able to apply the mentioned abstract result. In our proof of Theorem 1.2, we use condition (f_2) to verify that the associated functional satisfies the Palais-Smale condition. As in [8], this provides the existence of a second solution for $(P)_\lambda$ via the Mountain Pass Theorem [3].

In the proofs of Theorems 1.3 and 1.4, we argue by contradiction, assuming that Theorem 1.1 provides the only possible solution of $(P)_\lambda$. This assumption and condition $(\hat{f}_2)$ allow us to use the argument of Brezis and Nirenberg [6] and a result of Lions [15] to verify that the associated functional satisfies the Palais-Smale condition on a given interval of the real line.We use conditions (f_3) and $(\hat{f}_3)$, respectively, to establish that the level associated with the Mountain Pass Theorem belongs to this interval. As in the proof of Theorem 1.2, that implies the existence of a second solution.

Finally, we should mention that the existence of a nonzero solution for $(P)_\lambda$ when $f(x,0) \equiv 0$ has been intensively studied in recent years (See [1], [2], [10], [11] and references therein) . As it is shown in [1] (See also [10]), when $f(x,s) \geq 0$, for $s \geq 0$, a weaker version of (f_3) may be considered. We also observe that our method may be used to improve such results since in those articles a stronger version of $(\hat{f}_2)$ is assumed. Condition (f_3) can also be used in that setting to study the case where $f(x,s)$ may assume negative values.

The article is organized in the following way: In Section 2, we introduce the notion of Palais-Smale condition for the weak topology and establish two abstract results which are used to prove our results. There, we also recall the variational framework associated with $(P)_\lambda$ and state a version of Trudinger-Moser inequality (1.2) for $W^{1,N}(\Omega)$ when Ω is an open ball in $\mathbb{R}^N$. In section 2, we also state a result by Lions [15] that will be used, via contradiction, to verify $(PS)_c$, for c below a given level, when condition $(f)_{\alpha_0}$ holds with $\alpha_0 > 0$. In Section 3, we prove the weak version of Palais-Smale condition for the associated functional. In Section 4, we prove Theorems 1.1 and 1.2. In Section 5 we establish the estimates that are used to prove Theorem 1.3. In Section 6, we prove Theorem 1.3. In Section 7, we establish the estimates for the associated functional when conditions $(\hat{f}_3)$ and (f_4) are assumed. There, we also present the proof of Theorem 1.4. In Appendix A, we prove the Trudinger-Moser inequality mentioned in Section 2. Finally, in Appendix B, we prove an inequality for vector fields on $\mathbb{R}^N$, used in Section 7 to establish the necessary estimates.

2. – Preliminaries

Given E a real Banach space and Φ a functional of class C^1 on E, we recall that Φ satisfies Palais-Smale condition at level $c \in \mathbb{R}$ [Denoted $(PS)_c$] on an open set $\mathcal{O} \subset E$ if every sequence $(u_n) \subset \mathcal{O}$ for which (i) $\Phi(u_n) \to c$ and (ii) $\Phi'(u_n) \to 0$, as $n \to \infty$, possesses a converging subsequence. We also observe that Φ satisfies $(PS)_c$ if it satisfies $(\mathrm{PS})_c$ on E, and we say that Φ satisfies (PS) when it satisfies $(PS)_c$ for every $c \in \mathbb{R}$. Finally, we note that every sequence $(u_n) \subset E$ satisfying (i) and (ii) is called a Palais-Smale [(PS)] sequence.

To establish the existence of a critical point when the functional is bounded from below on a closed convex subsets of E, we introduce a version of the Palais-Smale condition for the weak topology.

Definition 2.1. Given $c \in \mathbb{R}$, we say that $\Phi \in C^1(E, \mathbb{R})$ satisfies the $(\mathrm{wPS})_c$ on $A \subset E$ if every sequence $(u_n) \subset A$ for which $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a critical point of Φ. We say that Φ satisfies (wPS) on A if Φ satisfies $(\mathrm{wPS})_c$ on A, for every $c \in \mathbb{R}$. When Φ satisfies (wPS) on E, we simply say that Φ satisfies (wPS).

Assuming

(Φ_1) There exist a closed bounded set $A \subset E$, constants $\gamma \leq b \in \mathbb{R}$, and $u_0 \in \overset{\circ}{A}$ such that

(i) $\Phi(u) \geq \gamma, \ \forall\ u \in A,$
(ii) $\Phi(u) \geq b \geq \Phi(u_0), \ \forall\ u \in \partial A,$

we define

$$c_1 = \inf_{u \in A} \Phi(u). \tag{2.1}$$

The following abstract result provides a critical point for Φ under conditions (Φ_1) and (wPS).

PROPOSITION 2.2. *Let E be a real Banach space. Suppose $\Phi \in C^1(E, \mathbb{R})$ satisfies (Φ_1), with A a closed bounded convex subset of E. Then, Φ possesses a critical point $u \in A$ provided it satisfies $(wPS)_{c_1}$ on A.*

PROOF. Arguing by contradiction, we suppose that Φ does not have a critical point $u \in A$. Under this assumption, we claim that Φ satisfies $(PS)_{c_1}$ on $\overset{\circ}{A}$. Effectively, given a sequence $(u_n) \subset \overset{\circ}{A}$ such that $\Phi(u_n) \to c_1$ and $\Phi'(u_n) \to 0$, as $n \to \infty$, by $(\text{wPS})_{c_1}$, (u_n) possesses a subsequence converging weakly to a critical point u. Furthermore, $u \in A$ since A is a closed convex subset of E. This contradicts our assumption and proves the claim.

We note that $\gamma \leq c_1 \leq \Phi(u_0)$. If $c_1 = \Phi(u_0)$, the conclusion is immediate. Thus, we may assume $c_1 < \Phi(u_0) \leq b$. In this case, we take $0 < \overline{\varepsilon} < \Phi(u_0) - c_1$. Then, we argue as in Proposition 2.7 of [19], using a local version of the Deformation Lemma [21], to obtain a contradiction with the definition of c_1. Proposition 2.2 is proved. □

REMARK 2.3. When Φ satisfies $(PS)_{c_1}$ on $\overset{\circ}{A}$, the second part of the proof of Proposition 2.2 shows that actually Φ possesses a local minimum $u \in \overset{\circ}{A}$ such that $\Phi(u) = c_1$.

Taking $b \in \mathbb{R}$ and A, given by (Φ_1), we consider

(Φ_2) There exists $e \in E \setminus A$ such that

$$\Phi(e) \leq b \leq \Phi(u), \quad \forall\ u \in \partial A,$$

and we define

$$c_2 = \inf_{g \in \Gamma} \max_{u \in g} \Phi(u) \geq b,$$

where

$$\Gamma = \{g \in C([0,1], E); \quad g(0) = u_0, \quad g(1) = e\}.$$

As a consequence of Proposition 2.2, Remark 2.3 and the argument employed in [21], we obtain the following version of the Mountain Pass Theorem [3].

PROPOSITION 2.4. *Let E be a real Banach space. Suppose $\Phi \in C^1(E, \mathbb{R})$ satisfies (Φ_1), with A closed and convex subset of E, and (Φ_2). Then, Φ possesses at least two critical points provided it satisfies $(PS)_c$, for every $c \leq c_2$.*

PROOF. By Proposition 2.2 and Remark 2.3, Φ possesses a local minimum $u_1 \in \overset{\circ}{A}$ such that $\Phi(u_1) = c_1$. Furthermore, if Φ does not have any critical point on ∂A, we may invoke the local version of the Deformation Lemma [21] one more time to obtain a neighbourhood V of u_0 and $\epsilon > 0$ such that $u_0 \in V$, $e \notin V$ and

$$c_1 \leq \max\{\Phi(u_0), \Phi(e)\} < \inf_{u \in \partial A} \Phi(u) + \epsilon \ \leq \inf_{u \in \partial V} \Phi(u) \leq c_2.$$

Consequently, by the Mountain Pass Theorem [3], c_2 is a critical value of Φ. The proposition is proved. □

Observe that when $c_1 = c_2$, by the above proof, Φ must have a critical point $u \in \partial A$ such that $\Phi(u) = c_1$.

Now, we recall the variational framework associated with problem $(P)_\lambda$. Considering the Sobolev space $W_0^{1,N}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^N \, dx\right)^{1/N}, \quad \forall\ u \in W_0^{1,N}(\Omega),$$

the functional associated with $(P)_\lambda$ $I_\lambda : W_0^{1,N}(\Omega) \to \mathbb{R}$ is given by

$$I_\lambda(u) = \frac{1}{N}\int_\Omega |\nabla u|^N \, dx - \lambda \int_\Omega F(x,u)\, dx, \quad \forall\ u \in W_0^{1,N}(\Omega), \tag{2.2}$$

where we assume $f(x,s) = f(x,0)$, for every $x \in \bar{\Omega}$, $s < 0$, and we take $F(x,s) = \int_0^s f(x,t)\,dt$, for $x \in \bar{\Omega}$, $s \in \mathbb{R}$. Under the hypothesis $(f)_{\alpha_0}$, the functional I_λ is well defined and belongs to $C^1(W_0^{1,N}(\Omega), \mathbb{R})$ (See [1], [10]). Furthermore,

$$I'_\lambda(u)v = \int_\Omega |\nabla u|^{N-2}\nabla u \nabla v \, dx - \lambda \int_\Omega f(x,u)v\, dx, \quad \forall\ u, v \in W_0^{1,N}(\Omega).$$

Thus, every critical point of I_λ is a weak solution of $(P)_\lambda$.

We also remark that if $f(x,s)$ satisfies conditions (f_1) and $(f)_{\alpha_0}$, then, for every $\beta > \alpha_0$, there exists $C = C(\beta) > 0$ such that

$$\max\{|f(x,s)|, |F(x,s)|\} \le C\exp(\beta |s|^{\frac{N}{N-1}}), \quad \forall\ x \in \bar{\Omega}, \quad s \ge 0. \tag{2.3}$$

As a direct consequence of (1.1) and (2.3), we obtain that $F(x,u(x)) \in L^1(\Omega)$ and $f(x,u(x)) \in L^q(\Omega)$, for every $q \ge 1$, whenever $u \in W_0^{1,N}(\Omega)$.

The following lemma establishes a version of Trudinger-Moser inequality (1.2) for $W^{1,N}(\Omega)$ when Ω is an open ball in $\mathbb{R}^N$.

LEMMA 2.5. *Let $B(x_0, R)$ be an open ball in $\mathbb{R}^N$ with radius $R > 0$ and center $x_0 \in \mathbb{R}^N$. Then, there exist constants $\hat{\alpha} = \hat{\alpha}(N) > 0$ and $C(N,R) > 0$ such that*

$$\int_{B(x_0,R)} \exp(\hat{\alpha}|u|^{N/N-1})\, dx \le C(N,R),$$

for every $u \in W^{1,N}(B(x_0,R))$ such that $\|u\|_{W^{1,N}(B(x_0,R))} \le 1$.

PROOF. For the sake of completeness, we present the proof of Lemma 2.5 in Appendix A. □

Finally, we state a theorem due to Lions [15] which will be essential to verify, via contradiction, that the functional I_λ satisfies $(PS)_c$, for c below a given level, when $f(x,s)$ satisfies the critical growth condition.

THEOREM 2.6. *Let $\{u_n \in W_0^{1,N}(\Omega) \mid \|u_n\| = 1\}$ be a sequence in $W_0^{1,N}(\Omega)$ converging weakly to a nonzero function u. Then, for every $0 < p < (1-\|u\|^N)^{\frac{-1}{N-1}}$, we have*

$$\sup_{n\in\mathbb{N}} \int_\Omega \exp\left(p\alpha_N |u_n|^{\frac{N}{N-1}}\right) dx < \infty.$$

3. – (wPS) condition

In this section, we shall prove a technical result that will be used to establish (wPS) condition for the functional $I_\lambda(u)$, defined by (2.2), when the nonlinearity $f(x,s)$ satisfies the critical growth condition,

(f_5) There exist $\alpha, C > 0$ such that

$$|f(x,s)| \le C \exp\left(\alpha |s|^{\frac{N}{N-1}}\right), \quad \forall\ x \in \bar{\Omega}, \quad s \in \mathbb{R}.$$

Our objective is to verify that any bounded sequence $(u_n) \subset W_0^{1,N}(\Omega)$ such that $I'_\lambda(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a solution of $(P)_\lambda$. Such result provides (wPS) condition for the functional I_λ.

Considering that next result is independent of the parameter $\lambda > 0$, we denote by (P) and I the problem $(P)_\lambda$ and the functional I_λ, respectively.

The proof of the following proposition is based on the argument used in [18] for the Neumann problem (See also [10]).

PROPOSITION 3.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Suppose $f(x,s) \in C(\bar{\Omega}\times\mathbb{R}, \mathbb{R})$ satisfies (f_5). Then, any bounded sequence $(u_n) \subset W_0^{1,N}(\Omega)$ such that $I'(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a solution of (P).*

REMARK 3.2 (i) Note that Proposition 3.1 generalizes to the N-Laplacian a well known fact for the Laplacian operator on $\Omega \subset \mathbb{R}^N$, $N > 2$, when the nonlinearity $f(x,s)$ satisfies the polynomial critical growth condition. (ii) We also observe that in Proposition 3.1 it is not assumed that (u_n) is a Palais-Smale sequence since $I(u_n)$ may be unbounded. (iii) Finally, we note that in [22], we prove a similar result for the p-Laplacian on $\Omega = \mathbb{R}^N$.

The proof of Proposition 3.1 will be carried out in a series of steps. First, by the Sobolev Embedding Theorem, Banach-Alaoglu Theorem and the characterization of $C(\bar{\Omega})^*$, given by the Riesz Representation Theorem [20], we may suppose that (u_n) has a subsequence, still denoted by (u_n), and there exist

$u \in W_0^{1,N}(\Omega)$ and $\mu \in \mathcal{M}(\bar{\Omega})$, the space of regular Borel measure on $\bar{\Omega}$, such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } W_0^{1,N}(\Omega), \\ |\nabla u_n|^N \rightharpoonup \mu, & \text{weakly* in } \mathcal{M}(\bar{\Omega}), \\ u_n \to u, & \text{strongly in } L^p(\Omega), \ 1 \le p < \infty, \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \\ |u_n(x)| \le h_p(x), & \text{a.e. in } \Omega, \text{ where } h_p \in L^p(\Omega), \ 1 \le p < \infty. \end{cases} \tag{3.1}$$

Now, we fix $0 < \sigma < \infty$ such that $\alpha\sigma^{\frac{N}{N-1}} < \hat{\alpha}$, with $\hat{\alpha}$ given by Lemma 2.5. Setting $\Omega_\sigma = \{x \in \Omega \mid \mu(x) \ge \sigma\}$, we have that Ω_σ is a finite set since μ is a bounded nonnegative measure on $\bar{\Omega}$. Furthermore,

LEMMA 3.3. *Let $K \subset (\Omega \setminus \Omega_\sigma)$ be a compact set. Then, there exist $q > 1$ and $M = M(K) > 0$ such that*

$$\int_K |f(x, u_n(x))|^q \, dx \le M, \ \forall\, n \in \mathbb{N}.$$

PROOF. To prove such result, we take $q > 1$ such that $\alpha q \sigma^{\frac{N}{N-1}} < \hat{\alpha}$ and consider $r_1 = \text{dist}(K, \partial\Omega \cup \Omega_\sigma) > 0$, the distance between K and $\partial\Omega \cup \Omega_\sigma$. For every $x \in K$, there exists $0 < r_x < r_1$ such that

$$\mu(B(x, 2r_x)) + \|u\|^N_{L^N(B(x,2r_x))} < \sigma^N. \tag{3.2}$$

Using the compactness of K, we find $j \in \mathbb{N}$ so that

$$K \subset \bigcup_{i=1}^{j} B(x_i, r_{x_i}) \equiv \bigcup_{i=1}^{j} B_i. \tag{3.3}$$

Applying (3.1) and (3.2), we find $n_0 \in \mathbb{N}$ such that

$$\|u_n\|^N_{W^{1,N}(B_i)} \le \sigma^N, \ \forall\, n \ge n_0, \ 1 \le i \le j.$$

Consequently, from Lemma 2.5, (f_5), (3.3) and our choice of q, there exists $M > 0$ such that

$$\int_K |f(x, u_n(x))|^q dx \le \sum_{i=1}^{N} \int_{B_i} \exp\left(\hat{\alpha}\left(\frac{|u_n(x)|}{\|u_n\|_{W^{1,N}(B_i)}}\right)^{\frac{N}{N-1}}\right) dx \le M,$$

for every $n \ge n_0$. This proves the lemma. □

LEMMA 3.4. *Let $K \subset (\Omega \setminus \Omega_\sigma)$ be a compact set. Then, $\nabla u_n \to \nabla u$, strongly in $(L^N(K))^N$, as $n \to \infty$.*

PROOF. Taking $\psi \in C_0^\infty(\Omega \setminus \Omega_\sigma)$ such that $\psi \equiv 1$, on K, and $0 \le \psi \le 1$, and considering that

$$(|a|^{N-2}a - |b|^{N-2}b) \cdot (a-b) \ge 2^{2-N}|a-b|^N, \ \forall\ a, b \in \mathbb{R}^N, \tag{3.4}$$

we obtain

$$\begin{aligned} 2^{2-N}\|\nabla u_n - \nabla u\|^N_{L^N(K)} &\le \int_\Omega \left[(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u).(\nabla u_n - \nabla u)\right]\psi\, dx \\ &= \int_\Omega \left[|\nabla u_n|^N \psi - |\nabla u_n|^{N-2}(\nabla u_n.\nabla u)\psi \right. \\ &\quad \left. - |\nabla u|^{N-2}(\nabla u.\nabla(u_n - u))\psi\right] dx. \end{aligned} \tag{3.5}$$

As $I'(u_n) \to 0$, as $n \to \infty$, we have

$$\int_\Omega \left[|\nabla u_n|^{N-2}\left((\nabla u_n.\nabla u)\psi + (\nabla u_n.\nabla\psi)u\right) - \psi f(x,u_n)u\right] dx = o(1), \tag{3.6}$$

as $n \to \infty$. Moreover, since (ψu_n) is a bounded sequence in $W_0^{1,N}(\Omega)$, we also have

$$\int_\Omega \left[|\nabla u_n|^N \psi + |\nabla u_n|^{N-2}(\nabla u_n.\nabla\psi)u_n - \psi f(x,u_n)u_n\right] dx = o(1), \tag{3.7}$$

as $n \to \infty$. Combining (3.5)-(3.7), we obtain

$$\begin{aligned} 2^{2-N}\|\nabla u_n - \nabla u\|^N_{L^N(K)} &\le \int_\Omega \psi f(x,u_n)(u_n - u)\, dx + \int_\Omega |\nabla u_n|^{N-2}(u - u_n)(\nabla u_n.\nabla\psi)\, dx \\ &\quad + \int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla(u - u_n))\psi\, dx + o(1), \text{ as } n \to \infty. \end{aligned}$$

Applying Lemma 3.3, for the compact set $\text{supp}\psi \subset (\Omega\setminus\Omega_\sigma)$, and using Hölder's inequality, we get

$$\begin{aligned} &2^{2-N}\|\nabla u_n - \nabla u\|^N_{L^N(K)} \\ &\le \|\psi\|_{L^\infty(\Omega)} M^{\frac{1}{q}} \|u_n - u\|_{L^{\frac{q}{q-1}}(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)} \|\nabla u_n\|^{N-1}_{L^N(\Omega)} \|u - u_n\|_{L^N(\Omega)} \\ &\quad + \int_\Omega |\nabla u|^{N-2}\psi(\nabla u.\nabla(u - u_n))\, dx + o(1), \text{ as } n \to \infty. \end{aligned}$$

The hypothesis that $(u_n) \subset W_0^{1,N}(\Omega)$ is bounded and (3.1) show that $\nabla u_n \to \nabla u$, strongly in $(L^N(K))^N$, as desired. The lemma is proved. □

As a direct consequence of Lemma 3.4, we have

COROLLARY 3.5. *The sequence* $(u_n) \subset W_0^{1,N}(\Omega)$ *possesses a subsequence* (u_{n_i}) *satisfying* $\nabla u_{n_i}(x) \to \nabla u(x)$, *for almost every* $x \in \Omega$.

The following Lemma shows that $I'(u)$ restricted to $W_0^{1,N}(\Omega \setminus \Omega_\sigma)$ is the null operator.

LEMMA 3.6.

$$(I'(u), \phi) = \int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla\phi)\, dx - \int_\Omega f(x,u)\phi\, dx = 0, \tag{3.8}$$

for every $\phi \in C_0^\infty(\Omega \setminus \Omega_\sigma)$.

PROOF. Given $\phi \in C_0^\infty(\Omega \setminus \Omega_\sigma)$, by Hölder's inequality and the fact that $(u_n) \subset W_0^{1,N}(\Omega)$ is a bounded sequence, we have that $(|\nabla u_{n_i}|^{N-2}\nabla u_{n_i}.\nabla\phi)$ is a family of uniformly integrable functions in $L^1(\Omega)$. Thus, by Vitali's Theorem [20] and Corollary 3.5, we get

$$\int_\Omega |\nabla u_{n_i}|^{N-2}(\nabla u_{n_i}.\nabla\phi)\, dx \to \int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla\phi)\, dx, \text{ as } i \to \infty. \tag{3.9}$$

We also assert that

$$\int_\Omega f(x,u_{n_i})\phi\, dx \to \int_\Omega f(x,u)\phi\, dx, \text{ as } i \to \infty, \tag{3.10}$$

for every $\phi \in C_0^\infty(\Omega \setminus \Omega_\sigma)$. Effectively, by Lemma 3.3, there exist $q > 1$ and $M_1 > 0$ so that

$$\int_{K_2} |f(x,u_n)|^q\, dx \le M_1, \tag{3.10}$$

where $K_2 = \text{supp}\phi$. Given $\epsilon > 0$, from (3.1) and Egoroff's Theorem, there exists $E \subset \Omega$ such that $|E| < \epsilon$ and $u_n(x) \to u(x)$, uniformly on $(\Omega \setminus E)$. Using Hölder's inequality, (3.11), and (f_5), we get $M_2 > 0$ such that

$$\left|\int_\Omega (f(x,u_n) - f(x,u))\phi\, dx\right| \le \int_{\Omega\setminus E} |f(x,u_n) - f(x,u)||\phi|\, dx + M_2\epsilon^{\frac{q-1}{q}}.$$

As $\epsilon > 0$ can be chosen arbitrarily small and $f(x,u_n(x)) \to f(x,u(x))$, uniformly on $\Omega \setminus E$, we derive (3.10). Now, we use (3.9), (3.10) and the fact that $I'(u_n) \to 0$, as $n \to \infty$, to verify that (3.8) holds. □

In the following, we conclude the proof of Proposition 3.1. In view of (1.1), (f_5) and the density of $C_0^\infty(\Omega)$ in $W_0^{1,N}(\Omega)$, it suffices to show that relation (3.8) holds for every $\phi \in C_0^\infty(\Omega)$.

Given $\phi \in C_0^\infty(\Omega)$ such that $\text{supp}\phi \cap \Omega_\sigma \ne \emptyset$, we take $K = \text{supp}\phi$, $\hat{\Omega}_\sigma = \Omega_\sigma \cap K = \{y_1, \ldots, y_l\}$, $1 \le l \le j$, and $r_1 > 0$ such that $2r_1 < |y_i - y_m|$, $i \ne m$, and $2r_1 < \text{dist}(K, \partial\Omega)$. We consider, $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$, on $[0, 1]$, and $\psi \equiv 0$, on $[2, \infty)$, and we define

$$\psi_{i,r}(x) = \psi\left(\frac{|x - y_i|}{r}\right), \ \forall\ x \in \Omega,\ 1 \leq i \leq l,\ 0 < r < r_1.$$

We also set $\psi_{l+1,r}(x) = 1 - \sum_{i=1}^{l} \psi_{i,r}(x)$, for every $x \in \Omega$. Hence, $\phi(x) \equiv \sum_{i=1}^{l+1} \phi\psi_{i,r}(x)$ and $\phi\psi_{l+1,r} \in C_0^\infty(\Omega \setminus \Omega_\sigma)$. From Lemma 3.6, we have

$$(I'(u), \phi) = \sum_{i=1}^{l} \int_\Omega |\nabla u|^{N-2}(\nabla u, \nabla(\phi\psi_{i,r}))\,dx - \sum_{i=1}^{l} \int_\Omega f(x, u)\phi\psi_{i,r}\,dx, \ \forall\ 0 < r < r_1. \tag{3.12}$$

Applying Hölder's inequality, for every $1 \leq i \leq l$, we get

$$\begin{aligned} &\left|\int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla(\phi\psi_{i,r}))\,dx\right| \\ &\leq \left[\int_{B_i} |\nabla u|^N\,dx\right]^{\frac{N-1}{N}} \Big[\|\nabla\phi\|_{L^\infty(\Omega)}\|\psi_{i,r}\|_{L^N(B_i)} \\ &\quad + \|\phi\|_{L^\infty(\Omega)}\|\nabla\psi_{i,r}\|_{L^N(B_i)}\Big], \end{aligned} \tag{3.13}$$

where $B_i \equiv B(y_i, 2r)$, $0 < r < r_1$. On the other hand, from the first Trudinger-Moser inequality (1.1) and (f_5), we find $M_3 > 0$ such that, for every $1 \leq i \leq l$,

$$\left|\int_\Omega f(x, u)\phi\psi_{i,r}\,dx\right| \leq M_3\|\phi\|_{L^\infty(\Omega)}|B_i|^{\frac{N-1}{2N}}\|\psi_{i,r}\|_{L^N(\Omega)}. \tag{3.14}$$

We use our definition of $\psi_{i,r}$ to get $M_4 > 0$ so that

$$\|\psi_{i,r}\|_{W^{1,N}(B_i)} \leq M_4, \ \forall\ 0 < r < r_1,\ 1 \leq i \leq l.$$

Consequently, given $\epsilon > 0$, by Lebesgue's Dominated Convergence Theorem, (3.13) and (3.14), we find $0 < r_2 < r_1$ so that

$$\begin{cases} \left|\int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla(\phi\psi_{i,r}))\,dx\right| < \epsilon, \\ \left|\int_\Omega f(x, u)\phi\psi_{i,r}\,dx\right| < \epsilon, \ \forall\ 1 \leq i \leq l,\ 0 < r < r_2. \end{cases} \tag{3.15}$$

for every $0 < r < r_2$, $1 \leq i \leq l$. From (3.12), (3.15) and the fact that $\epsilon > 0$ can be chosen arbitrarily small, we obtain that (3.8) holds for every $\phi \in C_0^\infty(\Omega)$. This concludes the proof of Proposition 3.1. □

As a direct consequence of Proposition 3.1, we have the following results:

COROLLARY 3.7. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Suppose that $f(x,s) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies (f_5). Then, I satisfies* (wPS) *on A, for every bounded set $A \subset W_0^{1,N}(\Omega)$.*

COROLLARY 3.8. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Suppose that $f(x,s) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies (f_5). Then, I satisfies* (wPS) *provided every* (PS) *sequence associated with I possesses a bounded subsequence.*

4. – Theorems 1.1 and 1.2

In this section, we apply the abstract results described in Section 2 to prove Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. The weak solution of problem (P_λ) will be established with the aid of Proposition 2.2. For this, it suffices to verify that I_λ, for $\lambda > 0$ sufficiently small, satisfies (Φ_1) and $(\text{wPS})_{c_1}$ on the closure of $B(0,\rho)$, denoted by $B[0,\rho]$, for some appropriate value of $\rho > 0$.

Given $\beta > \alpha_0$, we take $\rho \in (0, (\frac{\alpha_N}{\beta})^{\frac{N-1}{N}})$ and use (2.3) to obtain $C_1 > 0$ such that

$$
\begin{aligned}
I_\lambda(u) &\geq \frac{1}{N}||u||^N - \lambda C_1 \int_\Omega \exp(\beta |u|^{\frac{N}{N-1}})\, dx \\
&= \frac{1}{N}||u||^N - \lambda C_1 \int_\Omega \exp\left(\beta ||u||^{\frac{N}{N-1}} \left(\frac{|u|}{||u||}\right)^{\frac{N}{N-1}}\right) dx,
\end{aligned}
$$

for every $u \in W_0^{1,N}(\Omega)$ such that $\|u\| \leq \rho$. Hence, by Trudinger-Moser inequality (1.2), we find $C_2(N) > 0$ such that

$$
I_\lambda(u) \geq \frac{1}{N}\|u\|^N - \lambda C_2(N),
$$

for every $u \in B[0,\rho]$.

Taking $\bar{\lambda} = N^{-1} C_2(N)^{-1} \rho^N$, $u_0 = 0$, $\gamma = -\bar{\lambda} C_2(N)$, $b = 0$, and considering $c_\lambda = c_1$, c_1 given by (2.1), we have that I_λ satisfies condition (Φ_1), for every $0 < \lambda < \bar{\lambda}$.

Finally, we observe that conditions (f_1), $(f)_{\alpha_0}$ and Corollary 3.7 imply that I_λ satisfies (wPS) condition on $B[0,\rho]$. Theorem 1.1 is proved. □

Before proving Theorem 1.2, we note that, from (f_1) and (f_2), there exists a constant $C > 0$ such that

$$
F(x,s) \geq C|s|^\theta - C, \quad \forall\, x \in \bar{\Omega},\ s \geq 0. \tag{4.1}
$$

PROOF OF THEOREM 1.2. Considering $\bar{\lambda} > 0$, given in the proof of Theorem 1.1, we have that the functional I_λ satisfies (Φ_1), for every $\lambda \in (0, \bar{\lambda})$. Thus, by Proposition 2.4 , it suffices to verify that I_λ satisfies (Φ_2) and (PS) for such values of λ.

Choosing $u \in W_0^{1,N}(\Omega) \setminus \{0\}$ such that $u(x) > 0$, for every $x \in \Omega$, from (4.1), we obtain

$$I_\lambda(tu) \leq \frac{t^N}{N}||u||^N - \lambda C t^\theta \int_\Omega u^\theta \, dx + C|\Omega|.$$

Therefore, $I_\lambda(tu) \to -\infty$, as $t \to +\infty$, since $C > 0$ and $\theta > N$. Consequently, I_λ satisfies Φ_2.

Now, we shall verify that I_λ satisfies (PS). Let $(u_n) \subset W_0^{1,N}(\Omega)$ be a sequence such that $(I_\lambda(u_n)) \subset \mathbb{R}$ is bounded, and $I'_\lambda(u_n) \to 0$, as $n \to \infty$, i.e,

$$\left| \frac{1}{N} \int_\Omega |\nabla u_n|^N \, dx - \lambda \int_\Omega F(x, u_n) \, dx \right| \leq C < \infty \ , \ \ \forall \ n \in \mathbb{N}, \tag{4.2}$$

and

$$\left| \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla v \, dx - \lambda \int_\Omega f(x, u_n) v \, dx \right| \leq \varepsilon_n ||v||, \tag{4.3}$$

for every $v \in W_0^{1,N}(\Omega)$, where $\varepsilon_n \to 0$, as $n \to \infty$. Taking $\theta > N$, given by (f_2), we use (4.2) and (4.3) to get

$$\int_\Omega |\nabla u_n|^N \, dx - \lambda \int_\Omega (\theta F(x, u_n) - f(x, u_n) u_n) \, dx \leq C + \varepsilon_n ||u_n||.$$

From this inequality, (f_2), and our definition of $f(x, s)$ for $s \leq 0$, we conclude that (u_n) is a bounded sequence in $W_0^{1,N}(\Omega)$. Consequently, we may assume that

$$u_n \rightharpoonup u \ \text{ weakly in } \ W_0^{1,N}(\Omega), \quad u_n \to u \ \text{ strongly in } \ L^q(\Omega), \ \forall \ q > 1.$$

From (4.3), with $v = u_n - u$, we have

$$\lim_{n \to \infty} \left\{ \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u) \, dx - \lambda \int_\Omega f(x, u_n)(u_n - u) \, dx \right\} = 0. \tag{4.4}$$

Using Hölder's inequality, we may estimate the second integral in the above equation,

$$\left| \int_\Omega f(x, u_n)(u_n - u) \, dx \right| \leq \left(\int_\Omega |f(x, u_n)|^p \, dx \right)^{1/p} ||u_n - u||_{L^q},$$

where $p, q > 1$ are fixed with $\frac{1}{q} + \frac{1}{p} = 1$. Noting that (u_n) is a bounded sequence, we may find $\beta > \alpha_0 = 0$ such that $\beta p ||u_n||^{N/N-1} < \alpha_N$, for every $n \in \mathbb{N}$. Hence, by (2.3), we have

$$\int_\Omega |f(x, u_n)(u_n - u)| dx \leq C \left\{ \int_\Omega \exp \left(p\beta \|u_n\|^{\frac{N}{N-1}} \left(\frac{|u_n|}{\|u_n\|} \right)^{\frac{N}{N-1}} \right) \right\}^{\frac{1}{p}} \|u_n - u\|_{L^q}.$$

Thus, by Trudinger-Moser inequality (1.2), we obtain $C_2 > 0$ such that

$$\left| \int_\Omega f(x, u_n)(u_n - u)\, dx \right| \leq C_2 ||u_n - u||_{L^q}.$$

Since $u_n \to u$ strongly in $L^q(\Omega)$, from (4.4) and the above inequality, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla(u_n - u)\, dx = 0.$$

On the other hand,

$$\lim_{n\to\infty} \int_\Omega |\nabla u|^{N-2} \nabla u \nabla(u_n - u)\, dx = 0,$$

because $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$. Consequently,

$$\lim_{n\to\infty} \int (|\nabla u_n|^{N-2} \nabla u_n - |\nabla u|^{N-2} \nabla u)(\nabla u_n - \nabla u)\, dx = 0.$$

Thus, by inequality (3.4), we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^N\, dx = 0.$$

This implies that I_λ satisfies (PS) condition. Theorem 1.2 is proved. □

REMARK 4.1. As it is shown in [9] (See also [14]), any solution of $(P)_\lambda$ is in $C^{1,\alpha}(\Omega)$, for $N \geq 3$, and in $C^{2,\alpha}(\Omega)$, for $N = 2$.

5. – Estimates

We start this section with the definition of Moser functions (See [17]). Let $x_0 \in \Omega$ and $R > 0$ be such that the ball $B(x_0, R)$ of radius R centered at x_0 is contained in Ω. The Moser functions are defined for $0 < r < R$ by

$$M_r(x) = \frac{1}{w_{N-1}^{1/N}} \begin{cases} \left(\log \dfrac{R}{r}\right)^{\frac{N-1}{N}}, & if \quad 0 \le |x - x_0| \le r, \\ \dfrac{\log\left(\frac{R}{|x-x_0|}\right)}{\left(\log \frac{R}{r}\right)^{1/N}}, & if \quad r \le |x - x_0| \le R, \\ 0, & if \quad |x - x_0| \ge R. \end{cases}$$

Then, $M_r \in W_0^{1,N}(\Omega)$, $||M_r|| = 1$ and supp (M_r) is contained in $B(x_0, R)$. Considering $\hat{\Omega}$ given by (f_3), we take $x_0 \in \hat{\Omega}$ and consider the Moser sequence $M_n(x) = M_{\frac{R_n}{n}}(x)$ where $R_n = (\log n)^{\frac{1-N}{N}}$, for every $n \in \mathbb{N}$. Without loss of generality, we may suppose that $\text{supp}(M_n) \subset \hat{\Omega}$, for every $n \in \mathbb{N}$.

Taking $\bar{\lambda} > 0$ and u_λ, for $\lambda \in (0, \bar{\lambda})$, given in the proof of Theorem 1.1, we have

Proposition 5.1. *Suppose $f(x,s)$ satisfies (f_1), $(f)_{\alpha_0}$, with $\alpha_0 > 0$, and (f_3). Then, for every $\lambda \in (0, \bar{\lambda})$, there exists $n \in \mathbb{N}$ such that*

$$\max\{I_\lambda(u_\lambda + tM_n) \mid t \ge 0\} < I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

The proof of Proposition 5.1 will be carried out through the verification of several steps. First, we suppose by contradiction that, for every n, we have

$$\max\{I_\lambda(u_\lambda + tM_n) \mid t \ge 0\} \ge I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}. \tag{5.1}$$

Now, we apply the argument employed in the proof of Theorem 1.2 to conclude that $I_\lambda(u_\lambda + tM_n) \to -\infty$, as $t \to \infty$, for every $n \in \mathbb{N}$. Thus, there exists $t_n > 0$ such that

$$I_\lambda(u_\lambda + t_n M_n) = \max\{I_\lambda(u_\lambda + tM_n) \mid t \ge 0\}. \tag{5.2}$$

The following lemmas provide estimates for the value of t_n.

Lemma 5.2. *The sequence $(t_n) \subset \mathbb{R}$ is bounded.*

PROOF. Since $\frac{d}{dt}[I_\lambda(u_\lambda + tM_n)] = 0$ for $t = t_n$, it follows that

$$\int_\Omega |\nabla(u_\lambda + t_n M_n)|^{N-2}\nabla(u_\lambda + t_n M_n)\cdot\nabla M_n\,dx = \lambda\int_\Omega f(x, u_\lambda + t_n M_n)M_n\,dx.$$

Invoking Holder's inequality, we obtain

$$\|u_\lambda + t_n M_n\|^{N-1} \geq \lambda\int_\Omega f(x, u_\lambda + t_n M_n)M_n\,dx. \tag{5.3}$$

We observe that given $M > 0$, from (f_3), there exists a positive constant C such that

$$f(x,s) \geq M\exp\left(\alpha_0|s|^{N/N-1}\right) - C, \quad \forall\ s \geq 0, x \in \hat{\Omega}. \tag{5.4}$$

Thus, from (5.3)-(5.4), the definition of the function M_n and the nonnegativity of u_λ, we have

$$\|u_\lambda + t_n M_n\|^{N-1} \geq \lambda M\int_{B(x_0,R_n)} \exp\left(\alpha_0|t_n M_n|^{\frac{N}{N-1}}\right)M_n\,dx \ -\lambda C\int_{B(x_0,R_n)} M_n\,dx.$$

Using the definition of the function M_n one more time, we find $\hat{C} > 0$ such that

$$\begin{aligned}\|u_\lambda + M_n\|^{N-1} &\geq \lambda M\int_{B(x_0,\frac{R_n}{n})} \exp(\alpha_0|t_n M_n|^{\frac{N}{N-1}})M_n\,dx - \ \lambda\hat{C}R_n^N\\ &= \frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N}\exp\left[\left(\frac{\alpha_0}{\alpha_N}t_n^{\frac{N}{N-1}} - 1\right)N\log n\right]R_n^N(\log n)^{\frac{N-1}{N}}\\ &\quad - \lambda\hat{C}R_n^N.\end{aligned}$$

Hence, from the definition of R_n, we get

$$\|u_\lambda + M_n\|^{N-1} \geq \frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N}\exp\left[\left(\frac{\alpha_0}{\alpha_N}t_n^{\frac{N}{N-1}} - 1\right)N\log n\right] - \lambda\hat{C}R_n^N. \tag{5.5}$$

Since $R_n \to 0$, as $n \to \infty$, from (5.5), we conclude that $(t_n) \subset \mathbb{R}$ is a bounded sequence. Lemma 5.2 is proved. □

LEMMA 5.3. *There exist a positive constant $C = C(\lambda, \alpha_0, N)$ and $n_0 \in \mathbb{N}$ such that*

$$t_n^{N/N-1} \geq \frac{\alpha_N}{\alpha_0} - \frac{C R_n^N}{(\log n)^{1/N}}, \ \forall\, n \geq n_0.$$

PROOF. From equation (5.1),

$$I_\lambda(u_\lambda) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \leq \frac{1}{N}\int_\Omega |\nabla(u_\lambda + t_n M_n)|^N\,dx - \lambda\int_\Omega F(x, u_\lambda + t_n M_n)\,dx.$$

Hence,

$$\begin{aligned}\frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} &\leq \frac{t_n^N}{N} - \lambda\int_\Omega (F(x, u_\lambda + t_n M_n) - F(x, u_\lambda))\,dx\\ &+ \frac{1}{N}\sum_{k=1}^{N-1}\binom{N}{k} t_n^k \int_\Omega |\nabla u_\lambda|^{N-k}|\nabla M_n|^k\,dx.\end{aligned}$$

Furthermore,

$$F(x, u_\lambda + t_n M_n) - F(x, u_\lambda) = \int_0^{t_n M_n} f(x, s + u_\lambda)\,ds \geq -m t_n M_n,$$

where $m \geq 0$ is given by (f_1) and $(f)_{\alpha_0}$. Consequently,

$$\begin{aligned}\frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} &\leq \frac{t_n^N}{N} + \lambda m t_n \int_\Omega M_n dx\\ &+ \frac{1}{N}\sum_{k=1}^{N-1}\binom{N}{k} t_n^k \int_\Omega |\nabla u_\lambda|^{N-k}|\nabla M_n|^k\,dx.\end{aligned} \tag{5.6}$$

On the other hand, from the definition of the sequence (M_n), we have

$$\int_\Omega M_n\,dx = \frac{R_n^N w_{N-1}^{\frac{N-1}{N}}}{N}\left\{\frac{2(\log n)^{\frac{N-1}{N}}}{n^N} + \frac{1}{N}(1 - \frac{1}{n^N})\frac{1}{(\log n)^{1/N}}\right\} \tag{5.7}$$

$$\int_\Omega |\nabla u_\lambda|^{n-k}|\nabla M_n|^k\,dx \leq C(N, n, \lambda, k)\frac{1}{(\log n)^{k/N}}, \tag{5.8}$$

where $C(N, n, \lambda, k) = \frac{R_n^{Nk} w_{N-1}^{(1-\frac{k}{N})}}{(N-k)}(1 - \frac{1}{n^{N-k}})||\nabla u_\lambda||_{L^\infty}^{N-k}$.

Using (5.6)-(5.8) and Lemma 5.2, we find a constant $C > 0$ such that

$$t_n^{N/N-1} \geq \left(\frac{\alpha_N}{\alpha_0}\right)\left(1 - \frac{C(\frac{\alpha_N}{\alpha_0})^{N-1} R_n^N}{(\log N)^{1/n}}\right)^{1/(N-1)}.$$

A direct application of Mean Value Theorem to the function $h(s) = (1-s)^{1/(N-1)}$ on the above relation provides the conclusion of Lemma 5.3. □

Now, we shall use Lemmas 5.2 and 5.3 to derive the desired contradiction. From (5.5), Lemma 5.3 and the definition of R_n, we obtain

$$\|u_\lambda + t_n M_n\|^{N-1} \geq \frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N} \exp\left(-\frac{\alpha_0 C N}{\alpha_N}\right) - \lambda \hat{C} R_n^N.$$

Thus,

$$\frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N} \exp\left(-\frac{\alpha_0 C N}{\alpha_N}\right) \leq (\|u_\lambda\| + t_n)^{N-1} + \lambda \hat{C} R_n^N.$$

But, this contradicts Lemma 5.2, since M can be arbitrarily chosen and $R_n \to 0$, as $n \to \infty$. Proposition 5.1 is proved. □

6. – Theorem 1.3

In this section, after the verification of some preliminary results, we prove Theorem 1.3.

LEMMA 6.1. *Suppose $f(x,s)$ satisfies (f_1), $(\hat{f_2})$ and $(f)_{\alpha_0}$. Then, any* (PS) *sequence $(u_n) \subset W_0^{1,N}(\Omega)$ associated with I_λ possesses a subsequence (u_{n_i}) converging weakly in $W_0^{1,N}(\Omega)$ to a solution u of $(P)_\lambda$. Furthermore,*

$$\int_\Omega F(x, u_{n_i}(x))\,dx \to \int_\Omega F(x, u(x))\,dx, \text{ as } n \to \infty.$$

REMARK 6.2. We note that Lemma 6.1 also holds when $f(x,s)$ satisfies $(\hat{f_2})$ and $(f)_{\alpha_0}$ for $s \leq -R(\theta)$, and $s \leq 0$, respectively.

PROOF. Consider a sequence $(u_n) \subset W_0^{1,N}(\Omega)$ such that

$$\begin{cases} I_\lambda(u_n) \to c, \\ I'_\lambda(u_n) \to 0, \text{ as } n \to \infty. \end{cases} \tag{6.1}$$

Arguing as in Section 4, we obtain that (u_n) is a bounded sequence. Therefore, by Proposition 3.1, there exists a subsequence, that we continue to denote by (u_n), converging weakly in $W_0^{1,N}(\Omega)$ to a solution u of $(P)_\lambda$. Moreover, we may assume that $u_n(x) \to u(x)$, for almost every $x \in \Omega$. From (6.1) and (f_1), we get

$$\|u_n^-\|^N \leq \|u_n^-\|^N + \lambda \int_\Omega f(x,0) u_n^-(x)\,dx \leq \|I'_\lambda(u_n)\| \|u_n^-\| \to 0, \tag{6.2}$$

as $n \to \infty$. Hence, $u_n(x) \to u(x) \geq 0$, as $n \to \infty$, for almost every $x \in \Omega$. Now, we fix $\theta_1 > N$, and we consider $R_1 = R(\theta_1) > 0$ given by $(\hat{f}_2)$. From (6.1) and $(\hat{f}_2)$, we find $M_1 > 0$ such that

$$\int_{\{|u_n(x)| \geq R_1\}} \left[\frac{1}{\theta_1} f(x, u_n(x)) u_n(x) - F(x, u_n(x)) \right] dx \leq M_1. \tag{6.3}$$

Observing that $|\{x \in \Omega \mid u_n(x) \leq -R_1\}| \to 0$, as $n \to \infty$, from (2.3), (6.2), (6.3) and Hölder's inequality, we have

$$\int_{\{u_n^+(x) \geq R_1\}} \left[\frac{1}{\theta_1} f(x, u_n^+(x)) u_n^+(x) - F(x, u_n^+(x)) \right] dx \leq M_1. \tag{6.4}$$

Given $\epsilon > 0$, we take $\theta_2 > \theta_1$ such that $\frac{\theta_1 M_1}{\theta_2 - \theta_1} \leq \epsilon$ and $R_2 > \max\{R_1, R(\theta_2)\}$, $R(\theta_2)$ given by $(\hat{f}_2)$. Applying (6.4) and $(\hat{f}_2)$, we obtain

$$\int_{\{u_n^+(x) \geq R_2\}} |F(x, u_n^+(x))| \, dx \leq \epsilon. \tag{6.5}$$

Applying Egoroff's Theorem, we find $E \subset \Omega$ such that $|E| < \epsilon$ and $u_n(x) \to u(x)$, as $n \to \infty$, uniformly on $(\Omega \setminus E)$. Hence, from (2.3) and (6.2), we have

$$\begin{aligned} &\left| \int_\Omega [F(x, u_n(x)) - F(x, u(x))] \, dx \right| \\ &\leq \int_E |F(x, u_n^+(x))| \, dx + \int_E |F(x, u(x))| \, dx + o(1), \quad \text{as } n \to \infty. \end{aligned} \tag{6.6}$$

Fixed $q > 1$, we use (2.3) and Hölder's inequality to get $M_2 > 0$ such that

$$\int_E |F(x, u(x))| \, dx \leq M_2 \epsilon^{\frac{1}{q}}. \tag{6.7}$$

From (6.5), (6.7) and Lesbegue's Dominated Convergence Theorem, we have

$$\begin{aligned} \int_E |F(x, u_n^+(x))| \, dx &\leq \epsilon + \int_{E \cap \{0 \leq u_n(x) \leq R_2\}} |F(x, u_n^+(x))| \, dx \\ &\leq \epsilon + \int_{E \cap \{0 \leq u_n(x) \leq R_2\}} |F(x, u(x))| \, dx + o(1) \\ &\leq \epsilon + M_2 \epsilon^{\frac{1}{q}} + o(1), \quad \text{as } n \to \infty. \end{aligned}$$

The above inequality, (6.6), (6.7) and the fact that $\epsilon > 0$ can be chosen arbitrarily provide the conclusion of the proof of Lemma 6.1. □

Considering $c_\lambda = I_\lambda(u_\lambda) + \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$, with u_λ given by the proof of Theorem 1.1, we shall verify that I_λ satisfies (PS) condition below the level c_λ, whenever we suppose that $u = u_\lambda$ is the only possible solution of $(P)_\lambda$.

LEMMA 6.3. *Suppose* $f(x,s)$ *satisfies* (f_1), (f_{α_0}), *with* $\alpha_0 > 0$, *and* $(\hat{f}_2)$. *Assume that* u_λ *is the only possible solution of* $(P)_\lambda$, *for* $0 < \lambda < \bar{\lambda}$. *Then,* I_λ *satisfies* $(\mathrm{PS})_c$, *for every* $c < c_\lambda = I_\lambda(u_\lambda) + \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}$.

PROOF. Let $(u_n) \subset W_0^{1,N}(\Omega)$ be a sequence such that

$$\begin{cases} I_\lambda(u_n) \to c < c_\lambda, \\ I'_\lambda(u_n) \to 0, \ \text{as}, \ n \to \infty. \end{cases} \tag{6.8}$$

Since u_λ is the only solution of $(P)_\lambda$, by Lemma 6.1, we may assume that $u_n \rightharpoonup u_\lambda$, as $n \to \infty$, weakly in $W_0^{1,N}(\Omega)$ and

$$\int_\Omega F(x,u_n(x))\,dx \to \int_\Omega F(x,u_\lambda(x))\,dx, \ \text{as } n \to \infty. \tag{6.9}$$

From (6.8) and (6.9), we have

$$\|u_n\|^N \to N\left(c + \lambda \int_\Omega F(x,u_\lambda(x))\,dx\right), \ \text{as } n \to \infty. \tag{6.10}$$

Taking $v_n = \frac{u_n}{\|u_n\|}$, we get that

$$v_n \rightharpoonup v = \frac{u_\lambda}{[N(c+d)]^{\frac{1}{N}}},$$

where $d = \lambda \int_\Omega F(x,u_\lambda(x))\,dx$. Considering $\beta > \alpha_0$ such that

$$c < I_\lambda(u_\lambda) + \frac{1}{N}\left(\frac{\alpha_N}{\beta}\right)^{N-1}, \tag{6.11}$$

by (2.3), we find $q > 1$ and $C > 0$ so that

$$|f(x,s)|^q \le C \exp\left(\beta |s|^{\frac{N}{N-1}}\right), \ \forall\ x \in \Omega, \ s \in \mathbb{R}.$$

Thus,

$$\int_\Omega |f(x,u_n(x))|^q\,dx \le C \int_\Omega \exp\left(\beta \|u_n\|^{\frac{N}{N-1}} |v_n(x)|^{\frac{N}{N-1}}\right) dx, \tag{6.12}$$

for every $n \in \mathbb{N}$. On the other hand, by (6.11),

$$1 - \|v\|^N < \left(\frac{\alpha_N}{\beta}\right)^{N-1} \frac{1}{N(c+d)}.$$

Consequently, from (6.10), there exists $p > 0$ such that

$$\frac{\beta}{\alpha_N}\|u_n\|^{\frac{N}{N-1}} < p < \left(1 - \|v\|^N\right)^{\frac{-1}{N-1}}.$$

Hence, by Theorem 2.6 and (6.12), there exists $M > 0$ such that

$$\int_\Omega |f(x, u_n(x))|^q \, dx \leq M, \;\; \forall \; n \in \mathbb{N}.$$

Applying Egoroff's Theorem, the above inequaltiy and the argument employed in the proof of Proposition 3.1, we obtain

$$\int_\Omega f(x, u_n(x))u_n(x)\, dx \to \int_\Omega f(x, u_\lambda(x))u_\lambda(x)\, dx, \;\; \text{as } n \to \infty.$$

Therefore, by (6.8),

$$\|u_n\|^N \to \lambda \int_\Omega f(x, u_\lambda(x))u_\lambda(x)\, dx = \|u_\lambda\|^N.$$

The Lemma 6.3 is proved. □

Now, we may conclude the proof of Theorem 1.3. Arguing by contradiction, we suppose that u_λ, for $0 < \lambda < \bar{\lambda}$, is the only possible solution of $(P)_\lambda$. By Lemma 6.3, I_λ satisfies $(PS)_c$ for every $c < c_\lambda$. Furthermore, by the argument employed in the proof of Theorem 1.1, I_λ satisfies (Φ_1) on $B[0, \rho]$, for $\rho > 0$ sufficiently small. Hence, Proposition 2.2 and Remark 2.3 imply $u_\lambda \in \overset{\circ}{B}[0, \rho]$. Invoking Propositions 5.1 and 2.4 and Lemma 6.3, we conclude that I_λ possesses at least two critical points. However, this contradicts the fact that u_λ is the only critical point of I_λ. Theorem 1.3 is proved. □

7. – Theorem 1.4

In this section we establish a proof of Theorem 1.4. The key ingredient is the verification of Proposition 5.1 under conditions $(\hat{f}_3)$ and (f_4). To obtain such result we exploit the convexity of the function $F(x, s)$ and the fact that u_λ, for $\lambda \in (0, \bar{\lambda})$, is a solution of $(P)_\lambda$.

First, we state a basic result that will be used in our estimates.

LEMMA 7.1. *Let $a, b \in \mathbb{R}^N$, $N \geq 2$, and $\langle.,.\rangle$ the standard scalar product in $\mathbb{R}^N$. Then, there exists a nonnegative polynomial $p_N(x, y)$ ($p_2 \equiv 0$) such that*

$$|a + b|^N \leq |a|^N + N|a|^{N-2}\langle a, b\rangle + |b|^N + p_N(|a|, |b|). \tag{7.1}$$

Furthermore, the smallest exponent of the variable y of $p_N(x, y)$ is $3/2$ for $N = 3$ and 2 for $N \geq 4$, and the greatest exponent of y is strictly smaller than N.

PROOF. We present a proof of Lemma 7.1 in Appendix B. □

Now, we are ready to establish the version of Proposition 5.1. Consider β, $\hat{\Omega}$ given by $(\hat{f}_3)$. Let $x_0 \in \hat{\Omega}$ and the Moser sequence associated $M_n = M_{\frac{R_n}{n}}$, where $R_n = (\log n)^{-\frac{(N-1)(1-\beta)}{N^2}}$ if $N \geq 3$, and $R_n = R$ if $N = 2$, where $R > 0$ is chosen so that $B(x_0, R) \subset \hat{\Omega}$.

PROPOSITION 7.2. *Suppose $f(x,s)$ satisfies (f_1), $(f)_{\alpha_0}$, with $\alpha_0 > 0$, $(\hat{f}_3)$, and (f_4). Then, for every $\lambda \in (0, \bar{\lambda})$, there exists $n \in \mathbb{N}$ such that*

$$\max\{I_\lambda(u_\lambda + tM_n) \mid t \geq 0\} < I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

Arguing as in the proof of Proposition 5.1, we suppose by contradiction that for every $n \in \mathbb{N}$, (5.1) holds. As before, there exists $t_n \in \mathbb{R}$ satisfying equation (5.2). The following two results are versions of Lemmas 5.2 and 5.3 for this new situation.

LEMMA 7.3. *The sequence $(t_n) \subset \mathbb{R}$ is bounded.*

PROOF. Arguing as in the proof of Lemma 5.2, we have that equation (5.3) must hold. By (f_1) and (f_4), for every $x \in \hat{\Omega}$, the function $f(x, .)$ is positive on $[0, \infty)$ and nondecreasing. Thus, from (5.3),

$$\|u_\lambda + t_n M_n\|^{N-1} \geq \lambda \int_{B(x_0, \frac{R_n}{n})} f(x, t_n M_n) M_n \, dx. \tag{7.2}$$

Now, by $(\hat{f}_3)$, given $M > 0$ there exists $R_M > 0$ such that

$$s^\beta f(x,s) \geq M \exp\left(\alpha_o s^{\frac{N}{N-1}}\right), \quad \forall \ s \geq R_M, x \in \hat{\Omega}. \tag{7.3}$$

Consequently, by the definition of M_n, for n sufficiently large, we get

$$t_n \|u_\lambda + t_n M_n\|^{N-1} \geq \lambda M w_{N-1}^{\frac{N-1}{N}} \exp\left[\left(\frac{\alpha_0}{\alpha_N} t_n^{\frac{N}{N-1}} - 1 + \frac{\log R_n^N}{N \log n} + \frac{(N-1)(1-\beta)\log(\log n)}{N \log n}\right) N \log n\right].$$

Now, from definition of R_n, we have

$$\frac{\log R_n^N}{N \log n} = \frac{(\beta - 1)(N-1)\log(\log n)}{N \log n} \to 0, \quad \text{as } n \to \infty.$$

Thus, we conclude that $(t_n) \subset \mathbb{R}$ is a bounded sequence. The lemma is proved. □

LEMMA 7.4. *There exist* $n_0 > 0$, *and a positive constant* $C(\lambda, \alpha_0, N) \geq 0$ $(C(\lambda, \alpha_0, 2) = 0)$ *such that*

$$t_n^{N/N-1} \geq \frac{\alpha_N}{\alpha_0} - \frac{CR_n^N}{(\log n)^{\gamma/N}}, \quad \forall\, n \geq n_0,$$

where $\gamma = 3/2$ *if* $N = 3$, *and* $\gamma = 2$ *if* $N \geq 4$.

PROOF. From equations (5.1)-(5.2), we have

$$I_\lambda(u_\lambda) + \frac{1}{N}\left(\frac{\alpha_0}{\alpha_N})^{N-1} \leq I_\lambda(u_\lambda + t_n M_n\right).$$

Consequently,

$$\begin{aligned}\frac{1}{N}\left(\frac{\alpha_0}{\alpha_N}\right)^{N-1} &\leq \frac{1}{N}\int_\Omega (|\nabla u_\lambda + \nabla(t_n M_n)|^N - |\nabla u_\lambda|^N)\,dx \\ &\quad - \lambda\int_\Omega (F(x, u_\lambda + t_n M_n) - F(x, u_\lambda))\,dx.\end{aligned} \tag{7.4}$$

Using Lemma 7.1 with $a = \nabla u_\lambda(x)$, and $b = \nabla(t_n M_n(x))$, we have

$$\begin{aligned}&\int_\Omega \left(|\nabla u_\lambda + \nabla(t_n M_N)|^N - |\nabla u_\lambda|^N\right) dx \\ &\leq \int_\Omega \left(N|\nabla u_\lambda|^{N-2}\nabla u_\lambda \nabla(t_n M_n) + |\nabla(t_n M_n)|^N + p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|)\right) dx.\end{aligned} \tag{7.5}$$

From (7.4), (7.5), $\|M_n\| = 1$, and the fact that u_λ is a solution of $(P)_\lambda$, we obtain

$$\begin{aligned}\frac{1}{N}\left(\frac{\alpha_0}{\alpha_N}\right)^{N-1} \leq \frac{t_n^N}{N} &- \lambda\int_\Omega \Big[\left(F(x, u_\lambda + t_n M_n) - F(x, u_\lambda) - f(x, u_\lambda)t_n M_n\right) \\ &+ p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|)\Big]\,dx.\end{aligned} \tag{7.6}$$

Hence, from (7.6) and (f_4), we get

$$\frac{1}{N}\left(\frac{\alpha_0}{\alpha_N}\right)^{N-1} \leq \frac{t_n^N}{N} + \int_\Omega p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|)\,dx. \tag{7.7}$$

In the particular case $N = 2$, from Lemma 7.1, we have that $p_2 = 0$. From (7.7), we obtain

$$t_n^{\frac{N}{N-1}} \geq \frac{\alpha_0}{\alpha_N}.$$

Thus, it suffices to consider $N \geq 3$. Using the definition of the function M_n, we obtain the following estimates

$$(7.8) \quad \int_\Omega |\nabla u_\lambda|^l |\nabla M_n|^k \, dx \leq \|\nabla u_\lambda\|^l_{L^\infty(\Omega)} \frac{R_n^N w_{N-1}^{\frac{N-k}{N}}}{(N-k)(\log n)^{\frac{k}{N}}}, \ \forall l \geq 0, \ 1 \leq k < N.$$

Now, from (7.8), Lemma 7.3, and the definition of the polynomial $p_N(x, y)$ (See Lemma 7.1.), there exists a positive constant C such that

$$(7.9) \qquad \int_\Omega p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|) \, dx \leq \frac{C R_n^N}{(\log n)^{\gamma/N}},$$

where $\gamma = 3/2$, for $N = 3$, and $\gamma = 2$, for $N \geq 4$. Hence, from (7.7), (7.9), we have

$$\frac{t_n^N}{N} \geq \frac{1}{N}\left(\frac{\alpha_0}{\alpha_N}\right)^{N-1} - \frac{C R_n^N}{(\log n)^{\frac{\gamma}{N}}}.$$

Arguing as in the proof of Lemma 5.2, we get the conclusion of Lemma 7.4. □

To prove Proposition 7.2, we use Lemmas 7.3 and 7.4 to derive the desired contradiction. From (7.7) and (7.9), for n sufficiently large, we have

$$t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M \int_{B(x_0, \frac{R_n}{N})} \exp(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}) M_n^{1-\beta} \, dx.$$

Using the definition of M_n and Lemma 7.4, we get

$$(7.10) \quad t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M w_{N-1}^{\frac{N-(1-\beta)}{N}} \exp\left(\frac{-NCR_n^N \log n}{(\log n)^{\gamma/N}}\right) R_n^N (\log n)^{\frac{(N-1)(1-\beta)}{N}},$$

for $N \geq 3$, and

$$(7.11) \quad \begin{aligned} t_n^\beta \|u_\lambda + t_n M_n\| &\geq \lambda M w_{N-1}^{\frac{N-1}{N}} R^N \exp[(\frac{\alpha_0}{\alpha_N} t_n^{\frac{N}{N-1}} - 1) N \log n] \\ &\geq \lambda M w_{N-1}^{\frac{N-1}{N}} R^N, \end{aligned}$$

for $N = 2$.

From the definition of R_n, we obtain

$$(7.12) \qquad R_n^N (\log n)^{\frac{(N-1)(1-\beta)}{N}} = 1, \quad \text{and} \quad \frac{R_n^N \log n}{(\log n)^{\gamma/N}} = 1.$$

From (7.10) or (7.11) and (7.12), we have a contradiction because the left hand sides of (7.10) and (7.11) are bounded and M can be chosen arbitrarily large. This proves Proposition 7.2. □

Finally, we observe that the proof of Theorem 1.4 follows the same argument employed in the proof of Theorem 1.3, with Proposition 7.2 replacing Proposition 5.1.

8. – Appendix A

In this Appendix, we prove Lemma 2.5. First, we note that, without loss of generality, we may suppose $B(x_0, R) = B(0, R) \equiv B_R$.

Setting $u_M = \frac{1}{B_R}\int_{B_R} u(x)\,dx$, we may apply Lemma 7.16 in [14] to find $C = C(N) > 0$ such that

$$|u(x) - u_M| \le C(N)\int_{B_R} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\,dy, \text{ a.e. in } B_R.$$

Taking $v(x) = u(x) - u_M$, $h \in L^p(B_R)$, $p > 1$, $q = \frac{p}{p-1}$, and we use Hölder's inequality, as in [23], to obtain

$$\int_{B_R} |h(x)||v(x)|\,dx$$

$$\le C(N)\left[\iint_{B_R\times B_R} \frac{|h(x)|}{|x-y|^{N-\frac{1}{q}}}\,dxdy\right]^{\frac{N-1}{N}}\left[\iint_{B_R\times B_R} \frac{|\nabla u(x)|^N|h(x)|}{|x-y|^{\frac{N-1}{q}}}\,dxdy\right]^{\frac{1}{N}}.$$

Observing that the diameter of B_R is equal to $2R$, we get a constant $C_1(N) > 0$ such that

$$\iint_{B_R\times B_R} \frac{|h(x)|}{|x-y|^{N-\frac{1}{q}}}\,dx\,dy \le C_1(N)q\|h\|_{L^p(B_R)}R^{\frac{N+1}{q}}.$$

Applying Hölder's inequality one more time, we find $C_2(N) > 0$ such that

$$\int_{B_R} \frac{|h(x)|}{|x-y|^{\frac{N-1}{q}}}\,dx \le C_2(N)\|h\|_{L^p(B_R)}R^{\frac{1}{q}}.$$

Combining the above inequalities, we find $C_3(N) > 0$ such that

$$\int_{B_R} |h(x)||v(x)|\,dx \le C_3(N)q^{\frac{N-1}{N}}R^{\frac{N}{q}}\|h\|_{L^p(B_R)}\|\nabla u\|_{L^N(B_R)},$$

for every $h \in L^p(B_R)$. Therefore,

$$\|v\|_{L^q(B_R)} \le C_3(N)q^{\frac{N-1}{N}}R^{\frac{N}{q}}\|\nabla u\|_{L^N(B_R)},$$

for every $q > 1$. Consequently, there exists $C_4(N) > 0$ so that

$$\int_{B_R} |u - u_M|^{\frac{Nq}{N-1}}\,dx \le C_4(N)q^q R^N,$$

whenever $u \in W^{1,N}(B_R)$, $\|u\|_{W^{1,N}(B_R)} \leq 1$. Now, we use the power series expansion of $\psi(t) = e^t$ and the above inequality to derive

$$\int_{B_R} \exp(\alpha|u-u_M|^{\frac{N}{N-1}})\,dx \leq |B_R|+\alpha \int_{B_R} |u-u_M|^{\frac{N}{N-1}}\,dx+R^N \sum_{q=2}^{\infty} \frac{\alpha^q C_4(N)^q q^q}{q!},$$

if $\|u\|_{W^{1,N}(B_R)} \leq 1$. Hence, there exist $\hat{\alpha} = \hat{\alpha}(N) > 0$, and $C_5(N) > 0$ such that

$$\int_{B_R} \exp\left(\hat{\alpha} 2^{\frac{1}{N}} |u - u_M|^{\frac{N}{N-1}}\right) dx \leq C_5(N) R^N + \hat{\alpha} 2^{\frac{1}{N-1}} \|u - u_M\|^{\frac{N}{N-1}}_{L^{\frac{N}{N-1}}(B_R)}.$$

Since

$$|u_M| \leq C_6(N) R^{-1} \|u\|^{L^N(B_R)}, \ \forall\ u \in W^{1,N}(B_R),$$

for some $C_6(N) > 0$, we may use the convexity of the function $\psi(t) = t^{\frac{N}{N-1}}$ to obtain $C(N, R) > 0$ such that

$$\int_{B_R} \exp\left(\hat{\alpha}|u|^{\frac{N}{N-1}}\right) dx \leq C(N, R),$$

for every $u \in W^{1,N}(B_R)$ satisfying $\|u\|_{W^{1,N}(B_R)} \leq 1$. Lemma 2.5 is proved. □

9. – Appendix B

In this Appendix we prove Lemma 7.1. First, we establish an inequality that will be necessary in the sequel.

LEMMA 9.1. *Let x, y be real numbers with $x > 0$ and $x + y \geq 0$. Consider $k = \frac{N}{2}$, where $N \in \mathbb{N}$, and $N \geq 3$. Then, there exist nonnegative constants C_1, C_2 such that*

$$(x + y)^k \leq x^k + kx^{k-1}y + |y|^k + C_1 x|y|^{k-1} + C_2 x^{k-2} y^2.$$

Furthermore, $C_1 = C_2 = 0$ if $N = 3$ and 4, and $C_1 = 0$ when $N = 5$.

PROOF. Since $x > 0$ and $(x + y)^k = x^k(1 + yx^{-1})^k$, it suffices to consider $(1 + z)^k$ for every $z \geq -1$.

(i) Case $N = 3$. Let $g(z) = 1 + \frac{3}{2}z + |z|^{\frac{3}{2}} - (1 + z)^{\frac{3}{2}}$, for every $z \geq -1$. We must show that the function g is nonnegative. Direct calculation shows that $g'(z) \geq 0$ for every $z \geq 0$, and $g(0) = 0$. When $z \in [-1, 0]$, we consider $r = |z|$. Thus, $g(z) = h(r) = 1 - \frac{3}{2}r + r^{\frac{3}{2}} - (1 - r)^{\frac{3}{2}}$, and $h'(r) \geq 0$, and $h(0) = 0$. Hence, $g(z) \geq 0$ for every $z \geq -1$.

(ii) Case $N = 4$. The proof is immediate.
(iii) Case $N = 5$. Consider the polynomial function:

$$P(z) = (1+z)^{\frac{5}{2}} - (1 + \frac{5}{2}z + |z|^{\frac{5}{2}}).$$

By L'Hospital's Theorem, we have

$$\lim_{|z|\to 0} \frac{P(z)}{z^2} = \frac{15}{8}.$$

Moreover, by Mean Value Theorem, we get

$$\lim_{|z|\to \infty} \frac{P(z)}{z^2} = 0.$$

Consequently, there exists a nonnegative constant C such that

$$(1+z)^k \leq 1 + \frac{5}{2}z + |z|^{\frac{5}{2}} + Cz^2, \ \forall \ z \in \mathbb{R}.$$

(iv) Case $N \geq 6$. Consider the polynomial function:

$$P_k(z) = (1+z)^k - (1 + kz + |z|^k), \text{ where } k = \frac{N}{2}, \text{ and } z \geq -1.$$

Arguing as above, we have

$$\lim_{|z|\to 0} \frac{P_k(z)}{z^2} = \frac{k(k-1)}{2},$$

and

$$\lim_{|z|\to \infty} \frac{P_k(z)}{|z|^{k-1}} = k.$$

Consequently, there exist nonegative constants C_1, C_2 such that

$$(1+z)^k \leq 1 + kz + |z|^k + C_1|z|^{k-1} + C_2z^2, \ \forall \ z \in \mathbb{R}.$$

Lemma 7.3 is proved. □

PROOF OF LEMMA 7.1. The proof is immediate when $N = 2$. Thus, it suffices to verify the lemma for $N \geq 3$. Writing

$$|a+b|^N = (|a+b|^2)^{N/2} = (|a|^2 + 2\langle a, b\rangle + |b|^2)^{N/2},$$

and using Lemma 9.1 with $x = |a|^2$, and $y = 2\langle a, b\rangle + |b|^2$, we have

$$|a+b|^N \le |a|^N + N|a|^{N-2}\langle a, b\rangle + \frac{N}{2}|a|^{N-2}|b|^2 + (2|a||b| + |b|^2)^{N/2}$$
$$+2^{(N-2)/2}C_1|a|^2(2|a||b|)^{(N-2)/2} + 2^{(N-2)/2}C_1|a|^2|b|^{N-2}$$
$$+4C_2|a|^{N-2}|b|^2 + 4C_2|a|^{N-3}|b|^3 + C_2|a|^{N-4}|b|^4.$$

Applying Lemma 9.1 one more time, we obtain

$$|a+b|^N \le |a|^N + N|a|^{N-2}\langle a, b\rangle + |b|^N + p_N(|a|, |b|),$$

where

$$p_N(|a|, |b|) = \left(\frac{N}{2} + 4C_2\right)|a|^{N-2}|b|^2 + N2^{(N-4)/2}|a|^{(N-2)/2}|b|^{(N+2)/2}$$
$$+ 2^{N-2}C_1|a|^{(N+2)/2}|b|^{(N-2)/2} + 2^{(N-2)/2}C_1|a|^2|b|^{N-2}$$
$$+ 2^{N/2}|a|^{N/2}|b|^{N/2} + 2^{(N-4)/2}C_2|a|^{(N-4)/2}|b|^{(N+4)/2}$$
$$+ 2C_1|a||b|^{N-1} + 4C_2|a|^{N-3}|b|^3 + C_2|a|^{N-4}|b|^4.$$

Finally, since $C_1 = C_2 = 0$ if $N = 3$ and 5, and $C_1 = 0$ when $N = 5$, from the definition of p_N we conclude that the smallest exponent of $|b|$ is 3/2, for $N = 3$, and 2, for $N \ge 4$, and the greatest exponent of $|b|$ is strictly smaller than N. Lemma 7.1 is proved. □

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