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Variational Construction of Homoclinics and Chaos in Presence of a Saddle-Saddle Equilibrium

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Abstract. We consider autonomous Lagrangian systems with two degrees of freedom, having an hyperbolic equilibrium of saddle-saddle type (that is the eingenvalues of the linearized system about the equilibrium are $\pm\lambda_1, \pm\lambda_2, \lambda_1, \lambda_2 > 0$). We assume that $\lambda_1 > \lambda_2$ and that the system possesses two homoclinic orbits. Under a nondegeneracy assumption on the homoclinics and under suitable conditions on the geometric behaviour of these homoclinics near the equilibrium we prove, by variational methods, then they give rise to an infinite family of multibump homoclinic solutions and that the topological entropy at the zero energy level is positive. A method to deal also with homoclinics satisfying a weaker nondegeneracy condition is developed and it is applied, for simplicity, when $\lambda_1 \approx \lambda_2$. An application to a perturbation of an uncoupled system is also given.

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1. – Introduction

Let us consider the following Lagrangian system

$$-\ddot{q} + \psi(q)\mathcal{J}\dot{q} + Aq = \nabla W(q) \tag{1.1}$$

where $q = (q_1, q_2) \in \mathbb{R}^2$, $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$. System (1.1) can be obtained by the following Lagrangian

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + \frac{1}{2}Aq \cdot q - \dot{q} \cdot v(q) - W(q),$$

where $v = (v_1, v_2)$ satisfies

$$\psi(q) = \partial_{q_1} v_2(q) - \partial_{q_2} v_1(q). \tag{1.2}$$

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System (1.1) admits the energy

$$\mathcal{E}(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - \frac{1}{2}Aq \cdot q + W(q)$$

as a prime integral. We shall assume

- (W1) $W \in C^2(\mathbb{R}^2, \mathbb{R})$, $W(0) = 0, \nabla W(0) = 0, D^2W(0) = 0$; for some $0 < \rho_0 < \rho_1$ (ρ_1 is specified after hypothesis $(S2)$) D^2W is L_1-Lipschitz continuous on the ball $B_0 := B(0, \rho_0)$ of center 0 and radius ρ_0 and $\overline{L}_1$-Lipschitz continuous on $B_1 := B(0, \rho_1)$;
- $(P1)$ $\psi \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies $\psi(0) = 0$ and is L_2-Lipschitz continuous (resp. $\overline{L}_2$-Lipschitz continuous) on B_0 (resp. B_1); $\nabla\psi$ is $\overline{L}_3$-Lipschitz continuous on B_1.

  By $(P1)$, we can assume (1.2), with

- $(v1)$ $v \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $v(0) = 0$, $\nabla v(0) = 0$.

Under these assumptions 0 is a hyperbolic equilibrium of (1.1) and the characteristic exponents are two couples of opposite real numbers $\pm\lambda_1, \pm\lambda_2$. In this case the equilibrium is called of saddle-saddle type. We shall assume in the sequel that

$$(S1) \qquad \lambda_1 > \lambda_2 > 0.$$

We are interested in a chaotic behaviour of the dynamics at the zero energy level.

The only other possibility for a hyperbolic equilibrium of a Hamiltonian system in a phase space of dimension 4 is the saddle-focus situation, namely when the characteristic exponents are $\pm\lambda \pm i\eta$, $\lambda, \eta > 0$. It would be the case of system (1.1) if $|\psi(0)| \in (|\lambda_1 - \lambda_2|, \lambda_1 + \lambda_2)$ (note that if $|\psi(0)| \leq |\lambda_1 - \lambda_2|$ then 0 is still a saddle-saddle equilibrium and that if $|\psi(0)| \geq \lambda_1 + \lambda_2$ the equilibrium 0 is no more hyperbolic).

The saddle-focus case has been investigated by Devaney who showed in [8] that, if the system possesses a nondegenerate (transversal) homoclinic orbit, then it is possible to embed a horseshoe – and hence a Bernoulli shift – in the dynamics of the system. This result was extended by Buffoni and Séré in [6], who relaxed the nondegeneracy condition and proved by variational methods the existence of chaos at the zero energy level under global assumptions.

These results do not apply in the saddle-saddle case.

The existence of a chaotic dynamics in presence of a saddle-saddle equilibrium has been studied by Turaev and Shil'nikov [13] and more recently by Bolotin and Rabinowitz [5] for a system on a 2-dimensional torus. In this latter paper the existence of homoclinic orbits is not assumed *a priori*, but a simple geometrical condition is given, which implies that the system possesses chaotic trajectories either at any small negative energy level or at any small positive energy level $\{\mathcal{E} = h\}$. Other results have been stated in [7] for Lagrangian systems on manifolds.

However, the chaotic trajectories which are obtained in [13] as well as in [5] or in [7] are not preserved when the energy vanishes.

The existence of a Bernoulli shift at energy level $\{\mathcal{E} = 0\}$ was studied by Holmes in [11] (see also [14]). He assumed the existence of two nondegenerate homoclinics and introduced some conditions on the way these homoclinics approach 0 which ensure, when $(S1)$ is satisfied, the existence of a horseshoe at the zero energy level. By the structural stability of the horseshoes there results chaos also on nearby energy levels $\{\mathcal{E} = h\}$, see [11].

In the present paper we deal as in [11] with the saddle-saddle case, under assumption $(S1)$. We give specific conditions, called $(H1-4)$, directly inspired to the assumptions of Holmes, which imply that the system possesses an infinite family of multibump homoclinic orbits and of solutions with infinitely many bumps, which give rise to a chaotic behaviour at the zero energy level. Furthermore we improve such results requiring for the homoclinics $\overline{q}$, $\widetilde{q}$ a nondegeneracy condition weaker than transversality. Rather than performing this relaxation in a general situation, which would require quite involved conditions, we restrict ourselves to the case when the eigenvalues are close one to each other. However we underline that the method introduced to deal with degenerate homoclinics is could be adapted to a large variety of situations where it is difficult or impossible to check the nondegeneracy assumption.

First we shall assume that

- $(S2)$ System (1.1) has 2 nondegenerate homoclinics $\overline{q}, \widetilde{q}$. "Nondegenerate" means that the unique solutions of the linearized equation at (for instance) $\overline{q}$
$$-\ddot{h} + Ah + \psi(\overline{q})\mathcal{J}\dot{h} + \nabla\psi(\overline{q}) \cdot h\mathcal{J}\dot{\overline{q}} - D^2W(\overline{q})h = 0$$
that tend to 0 as $t \to \pm\infty$ are $c\dot{\overline{q}},\ c \in \mathbb{R}$. That means that the stable and unstable manifolds to 0 intersect transversally at $(\overline{q}(t), \dot{\overline{q}}(t))$ at the zero energy level.

We can now specify the constant ρ_1 in $(W1)$: $\rho_1 > \max\{|\overline{q}|_\infty, |\widetilde{q}|_\infty\} + \rho_0$. The relaxed nondegeneracy condition is the following

- $(S2')$ System (1.1) has 2 "topologically nondegenerate" isolated homoclinics $\overline{q}, \widetilde{q}$, (see Definition 2 in Subsection 4.2).

We point out that in some situations such a condition can be checked for homoclinics obtained by variational methods which are isolated up to time translations, see for example [2], [10].

In order to get chaotic trajectories in the saddle-saddle case it is necessary to postulate the existence of (at least) two homoclinic orbits, while only one is necessary for the saddle-focus case. Even though, there exist systems with several transversal homoclinic orbits which do not have a chaotic behaviour.

Consider for example (1.1) and assume that:

$$W(q) = q_1^4 + q_2^4 \qquad \text{and} \qquad \psi(q) = 0. \tag{1.3}$$

Then the system reduces to a direct product of 1-dimensional systems. $(0, 0) \in \mathbb{R}^4$ is a saddle-saddle equilibrium with 4 transversal homoclinic trajectories but the system is integrable (another example of an integrable Hamiltonian system with several transversal homoclinic orbits is given in [9]). Thus additional assumptions are needed for chaotic behaviour. In order to obtain multibump homoclinics for system (1.1) as glued copies of $\overline{q}$ and $\widetilde{q}$, some hypotheses of geometrical nature on $\overline{q}$ and $\widetilde{q}$, similar to the ones given in [11], are required.

The results contained in this paper have already been outlined in [4]. In order to describe them we need some notations. We shall assume that $\overline{q}(\mathbb{R})$ and $\widetilde{q}(\mathbb{R})$ are not included in B_0. For $r \in (0, \rho_0/2)$ we define $\overline{T} > 0$ by $|\overline{q}(\pm\overline{T})| = r$ and $|\overline{q}(t)| < r$ for $|t| > \overline{T}$. We define in the same way $\widetilde{T}$ and we set $T = \min\{\overline{T}, \widetilde{T}\}$.

Call $(\overline{\alpha}_1, \overline{\alpha}_2) = (\overline{q}_1(-\overline{T}), \overline{q}_2(-\overline{T}))$, $(\overline{\beta}_1, \overline{\beta}_2) = (\overline{q}_1(\overline{T}), \overline{q}_2(\overline{T}))$ the extremal intersection points of $\overline{q}(\mathbb{R})$ with the circle in $\mathbb{R}^2$ of radius r; similarly we introduce $(\widetilde{\alpha}_1, \widetilde{\alpha}_2) = (\widetilde{q}_1(-\widetilde{T}), \widetilde{q}_2(-\widetilde{T}))$, $(\widetilde{\beta}_1, \widetilde{\beta}_2) = (\widetilde{q}_1(\widetilde{T}), \widetilde{q}_2(\widetilde{T}))$. Let $\overline{\omega}_u$, $\overline{\omega}_s$ be defined by

$$(\overline{\alpha}_1, \overline{\alpha}_2) = (r\cos\overline{\omega}_u, r\sin\overline{\omega}_u) \quad , \quad (\overline{\beta}_1, \overline{\beta}_2) = (r\cos\overline{\omega}_s, r\sin\overline{\omega}_s);$$

$\widetilde{\omega}_u$, $\widetilde{\omega}_s$ are defined in the same way.

We set $\Lambda = (L_1/\lambda_2^2) + (3L_2\lambda_1/\lambda_2^2)$, $\overline{\Lambda} = (\overline{L}_1/\lambda_2^2) + (3\overline{L}_2\lambda_1/\lambda_2^2) + \max\left\{|\dot{\overline{q}}|_\infty, |\dot{\widetilde{q}}|_\infty\right\}(\overline{L}_3/\lambda_2^2)$, where L_i, $\overline{L}_i$ are defined in assumptions $(W1)$, $(P1)$. Note that Λ, $\overline{\Lambda}$ do not change if the equation is modified by a time rescaling $q(t) \to q(\alpha t)$.

In the next conditions ω_u stands for $\overline{\omega}_u$ or $\widetilde{\omega}_u$ and ω_s for $\overline{\omega}_s$ or $\widetilde{\omega}_s$.

- $(H1)$ $\omega_u, \omega_s \neq n\pi/2$, $n \in \mathbb{Z}$, $\tan\omega_u \tan\omega_s < 0$ and ($\cos\overline{\omega}_u \cos\widetilde{\omega}_u < 0$ or $\cos\overline{\omega}_s \cos\widetilde{\omega}_s < 0$).
  (the above inequalities are satisfied for example if $\overline{\omega}_u \in (0, \pi/2)$, $\overline{\omega}_s \in (3\pi/2, 2\pi)$, $\widetilde{\omega}_u \in (\pi, 3\pi/2)$ and $\widetilde{\omega}_s \in (\pi/2, \pi)$);
- $(H2)$

$$\frac{\lambda_2^2}{\lambda_1^2} \frac{|\alpha_2||\beta_2| + (15\lambda_1/4\lambda_2)\Lambda r^3}{|\alpha_1||\beta_1|} \leq l\left(\frac{\lambda_1}{\lambda_2}\right) \min\left(e^{-2\frac{\lambda_1-\lambda_2}{\lambda_2}}, \left(\frac{C_1 e^{\lambda_2(T-T_{C_1})}}{18}\right)^{2\frac{\lambda_1-\lambda_2}{\lambda_2}}, \left(\frac{C_1^2}{40\overline{\Lambda} r}\right)^{\frac{\lambda_1-\lambda_2}{\lambda_2}}\right)$$

where $l(\nu) = \max_{s\in(0,1/8)}[(1-s)^2(1-s/5)/(1+s)^3]s^{\nu-1}$ and C_1 is a constant defined by (2.4), which measures the transversality of the homoclinics:

smaller is C_1 weaker is the transversality. T_{C_1} depends only on C_1 and ρ_0 and it is defined by (2.6), Section 2.

- $(H3)$

$$\frac{\lambda_2^2}{\lambda_1^2}\frac{|\alpha_2||\beta_2| + (15\lambda_1/4\lambda_2)\Lambda r^3}{|\alpha_1||\beta_1|} \leq l\left(\frac{\lambda_1}{\lambda_2}\right)\left(\frac{C_1\mathcal{M}}{36S_2 + 28\Lambda r^2}\right)^{(\lambda_1/\lambda_2)-1}$$

where $\mathcal{M} = \min\{|\overline{\alpha}_j|, |\overline{\beta}_j|, |\widetilde{\alpha}_j|, |\widetilde{\beta}_j| \ ; \ j = 1, 2\}$ and $S_2 = \max\{|\overline{\alpha}_2|, |\overline{\beta}_2|, |\widetilde{\alpha}_2|, |\widetilde{\beta}_2|\}$.

- $(H4)$ $\min(|\sin\overline{\omega}_{u,s}|, |\sin\widetilde{\omega}_{u,s}|) \geq \sqrt{\frac{\lambda_1}{\lambda_2}20\Lambda r}$, $(12\lambda_1/\lambda_2)\Lambda r \leq C_1$.

Roughly speaking the first geometric assumption $(H1)$ means that the homoclinics $\overline{q}$, $\widetilde{q}$ enter and leave the origin from different "quadrants". Note that if (1.3) holds system (1.1) does not satisfy hypothesis (H1). $(H2-3)$ quantify how small $|\tan\omega_u \tan\omega_s|$ and r must be. Note that if the system is linear (that is $W = 0$, $\psi = 0$) in the ball $B(0, \rho_0)$ then condition $(H4)$ disappears and conditions $(H2-3)$ are simplified (in $(H2-3)$, $\Lambda = 0$). Moreover if $\lambda_1/\lambda_2 \to 1$ then $l(\lambda_1/\lambda_2) \to 1$ and the second members in inequalities $(H2-3)$ tend to 1.

Before stating our first result we introduce some other notations. For $j = (j_1, \ldots, j_k) \in \{0, 1\}^k$ and for $\Theta = (\theta_1, \ldots, \theta_k)$ with $\theta_1 < \ldots < \theta_k$ we define $T_i = \overline{T}$ if $j_i = 0$ and $T_i = \widetilde{T}$ if $j_i = 1$; $d_i = (\theta_{i+1} - T_{i+1}) - (\theta_i + T_i)$ and $\overline{d} = \min_{1\leq i\leq k-1} d_i$.

THEOREM 1. *Assume* $(W1)$,$(P1)$,$(v1)$,$(S1-2)$ *and* $(H1-4)$. *Then there exist* $0 < D < J$ *such that for every* $k \in \mathbb{N}$, *for every sequence* $j = (j_1, \ldots, j_k) \in \{0, 1\}^k$ *there is* $\Theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$ *with* $d_i \in (D, J)$ *for all* $i = 1, \ldots, k-1$ *and a homoclinic solution of (1.1)* x_j *such that*

- *if* $j_i = 0$ *then on the interval* $[\theta_i - \overline{T}, \theta_i + \overline{T}]$

$$|x_j(t) - \overline{q}(t-\theta_i)| \leq \frac{r}{8}\min\left(|\cos\overline{\omega}_{u,s}|, |\cos\widetilde{\omega}_{u,s}|, |\sin\overline{\omega}_{u,s}|, |\sin\widetilde{\omega}_{u,s}|\right) = \frac{\mathcal{M}}{8},$$

- *if* $j_i = 1$ *then on the interval* $[\theta_i - \widetilde{T}, \theta_i + \widetilde{T}]$

$$|x_j(t) - \widetilde{q}(t-\theta_i)| \leq \frac{r}{8}\min\left(|\cos\overline{\omega}_{u,s}|, |\cos\widetilde{\omega}_{u,s}|, |\sin\overline{\omega}_{u,s}|, |\sin\widetilde{\omega}_{u,s}|\right) = \frac{\mathcal{M}}{8},$$

- *Outside* $(\cup_{j_i=0}[\theta_i - \overline{T}, \theta_i + \overline{T}]) \cup (\cup_{j_i=1}[\theta_i - \widetilde{T}, \theta_i + \widetilde{T}])$, $|x_i(t)| \leq 2r$.

Note that, by Theorem 1 and assumption $(H1)$, two distinct sequences $j = (j_1, \ldots, j_k)$ and $j' = (j'_1, \ldots, j'_k)$ give rise to two distinct homoclinics.

REMARK 1. **(i)** Since the distance d_i between two consecutive bumps is bounded by the constant J which is independent of the number of bumps k, by the Ascoli-Arzelá theorem there follows the existence of solutions with infinitely many bumps, see Theorem 5. In particular it implies a lower bound for the topological entropy at the zero energy level, $h^0_{\text{top}} > \log 2/(2\max\{\overline{T}, \widetilde{T}\} + J)$ and shows that the system exhibits a chaotic behaviour.

(ii) The fact that $\lambda_1 > \lambda_2$ is crucial to be able to construct multibump homoclinics.

(iii) As it will appear in the proof of Theorem 1, smaller are the quantities $(\Lambda r/|\cos\omega_u \cos\omega_s|) + |\tan\omega_u \tan\omega_s|$, $|\lambda_1 - \lambda_2|/\lambda_2$, greater is the distance between the bumps.

(iv) We do not prove the existence of multibump homoclinics in an arbitrary small neighborhood of $\overline{q}, \widetilde{q}$. Indeed in [13] it is proved that there is a neighborhood V of $\overline{q}(\mathbb{R}) \cup \widetilde{q}(\mathbb{R})$ such that the only homoclinic solutions contained in V are $\overline{q}$ and $\widetilde{q}$.

Our other results, Theorems 2, 3 and 4, resp. in Subsections 3.1, 3.2 and 4.3, are variants of Theorem 1 in special systems or when the homoclinics are degenerate.

The multibump homoclinic solutions of (1.1) will be obtained as critical points of the following action functional, which is well defined by $(W1)$ and $(v1)$ on $E = W^{1,2}(\mathbb{R}, \mathbb{R}^2)$:

$$f(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot{q}|^2 + \frac{1}{2}Aq \cdot q - \dot{q} \cdot v(q) - W(q). \tag{1.4}$$

The idea of the proofs goes as follows.

A " pseudo-critical" manifold for f, $Z_k = \{Q_\Theta \mid \Theta \in \mathbb{R}^k,\ \theta_1 < \ldots < \theta_k\}$ is constructed by gluing together translates of the homoclinics $\overline{q}(\cdot - \theta_i)$ and $\widetilde{q}(\cdot - \theta_j)$, see Section (2.1) and (2.2). Then we show that, when the bumps are sufficiently separated, that is when

$$\min_i(\theta_{i+1} - \theta_i) > \overline{D} \tag{1.5}$$

a shadowing type lemma enables to construct immersions $\mathcal{I}_k : M_k = \{\Theta \in \mathbb{R}^k \mid \min_i(\theta_{i+1} - \theta_i) > \overline{D}\} \to E$ with $\mathcal{I}_k(M_k) \approx Z_k$ such that the critical points of $g(\Theta) = f(\mathcal{I}_k(\Theta))$ gives rise to a k-bump homoclinic solutions. The geometric properties $(H1-4)$ of the homoclinics $\overline{q}$ and $\widetilde{q}$ ensure the existence of critical points of $g(\Theta)$ satisfying (1.5). We point out that $g(\Theta)$ does not possess critical points when $\min_i(\theta_{i+1} - \theta_i) \to +\infty$; therefore we need to estimate carefully the minimal distance $\overline{D}$ for which we obtain the immersions $\mathcal{I}_k$. This is done in Section 2.

For the sake of clarity we perform all the detailed computations for a system with 2 degrees of freedom, but the same method can be adapted also to study systems in dimension n, (see Remark 7) where the analytical technics based on the study of Poincaré sections are more difficult.

The paper is organized as follows. In Section 2 we perform the finite dimensional reduction for the functional f and we prove Theorem 1. In Section 3 we give examples of applications of Theorem 1 when the eigenvalues are near one each other (Theorem 2) and for a system which is a perturbation of 2-uncoupled Duffing equations (Theorem 3). In Section 4 it is shown that in the case $\lambda_1 \approx \lambda_2$ the transversality condition can be weakened assuming the topological nondegeneracy $(S2')$ (Theorem 4). Finally in Section 5 we show why the above theorems imply a chaotic dynamics (Theorem 5).

2. – Finite dimensional reduction

We shall use the following Banach spaces:

- $Y = W^{1,\infty}(\mathbb{R}, \mathbb{R}^2)$ endowed with norm $||y|| = \max\left(|y|_\infty, \frac{1}{\lambda_2}|\dot{y}|_\infty\right)$ where $|y|_\infty = \sup_{t\in\mathbb{R}} |y(t)|$.
- $E = W^{1,2}(\mathbb{R}, \mathbb{R}^2)$ endowed with scalar product $(x, y) = \sum_{j=1}^{2} \int_{\mathbb{R}} \dot{x}_j \dot{y}_j + \lambda_j^2 x_j y_j$ and associated norm $|\cdot|_E$.
- $X = \{h \in Y \mid e^{\lambda_2|t|}|h(t)|,\ e^{\lambda_2|t|}|\dot{h}(t)| \in L^\infty\}$.

Since the equilibrium 0 is hyperbolic of smaller positive characteristic exponent λ_2 by standard results (see also Lemma 2) any homoclinic solution to 0 of (1.1) belongs to X.

We have $X \subset Y \cap E$. For $A \subset X$ we shall use the notation

$$A^\perp = \{y \in Y \mid (a, y) = 0, \quad \forall a \in A\}.$$

Note that, by the exponential decay of the elements of X, $A^\perp$ is well defined and it is a closed subspace of Y.

We define the operator $S : Y \to Y$ by

$$S(y) = y - L_A(\nabla W(y) - \psi(y)\mathcal{J}\dot{y})$$

where L_A is the linear operator which assigns to h the unique solution $z = L_A h$ of

$$-\ddot{z} + Az = h \qquad \text{with} \qquad \lim_{|t|\to\infty} z(t) = 0.$$

An explicit definition of L_A is

$$(2.1) \qquad \begin{aligned} (L_A h)(t) &= \frac{(\sqrt{A})^{-1}}{2} \int_{-\infty}^{+\infty} e^{-|t-s|\sqrt{A}} h(s)ds\,, \\ \left(\frac{d}{dt} L_A h\right)(t) &= \frac{1}{2} \int_{-\infty}^{+\infty} \operatorname{sgn}(s-t)\, e^{-|t-s|\sqrt{A}} h(s)ds \end{aligned}$$

By (2.1) it is easy to see that, for all $x \in Y$,

$$(2.2) \qquad ||L_A x|| \le \frac{1}{\lambda_2^2}|x|_\infty \le \frac{1}{\lambda_2^2}||x||.$$

By (2.2) and $(W1) - (P1)$ we see that the operator S is C^1 on Y. We can also get the straightforward estimate

$$\forall \ ||y|| < \rho_1 \qquad ||dS(y)h|| \leq \Big(1 + \overline{\Lambda}||y||\Big)||h||. \tag{2.3}$$

Note also that $S(E \cap Y) \subset E \cap Y$ and that for all $q, x \in E \cap Y$

$$(S(q), x) = df(q)[x].$$

If $S(q) = 0$ and $q \in E \cap Y$ then q is a homoclinic solution to system (1.1). We can say a little bit better.

LEMMA 1. *Assume that $q \in Y$ satisfies $S(q) = 0$ and that* $\limsup_{|t|\to+\infty} \max(|q(t)|, |\dot{q}(t)|/\lambda_2) < \min(2/\Lambda, \rho_0)$. *Then q is a homoclinic solution to* (1.1).

PROOF. Let $m(t) = \max(|q(t)|, |\dot{q}(t)|/\lambda_2)$ and $c = \limsup_{|t|\to\infty} m(t)$. We assume that $c < \min(2/\Lambda, \rho_0)$ and we want to prove that $c = 0$. Provided $m(t) \leq \rho_0$ we have

$$\left|\nabla W(q(t)) - \psi(q(t))\mathcal{J}\dot{q}(t)\right| \leq \frac{1}{2}\lambda_2^2 \Lambda (m(t))^2.$$

Now easy estimates in the expression of L_A show that, if $h \in Y$, then

$$\limsup_{|t|\to\infty} \max\left(\left|L_A h(t)\right|, \frac{1}{\lambda_2}\left|\frac{d}{dt}L_A h(t)\right|\right) \leq \frac{1}{\lambda_2^2} \limsup_{|t|\to\infty} |h(t)|.$$

Therefore, since $S(q) = 0$, we get $c \leq \Lambda c^2/2$, which implies $c = 0$ by our assumption. □

REMARK 2. If q is a homoclinic solution to (1.1) then, by the characteristic exponents of the equilibrium, all $y \in Y$ which satisfies $dS(q) \cdot y = 0$ belongs to X. So the nondegeneracy condition $(S2)$ amounts to assuming that $\mathrm{Ker}\, dS(\overline{q})$ is spanned by $\dot{\overline{q}}$, where $dS(\overline{q})$ is regarded as a linear operator from Y to Y. Moreover $dS(q)$ has the form $Id + K$, where K is a compact operator on Y. In addition $dS(\overline{q})(Y) \subset \overline{Y}'$, where $\overline{Y}' = \dot{\overline{q}}^{\perp}$. Hence $dS(\overline{q})$ is a linear automorphism of $\overline{Y}'$.

We now introduce another supplemetary space $\overline{Y}''$ to $\dot{\overline{q}}$. The introduction of the above norm $||\cdot||$ and $\overline{Y}''$ instead the more natural H^1-norm and $\dot{\overline{q}}^{\perp}$ is motivated by the fact that this choice allows to obtain better estimates in hypotheses $(H2 - 4)$.

Consider $\overline{t}$ such that $|\dot{\overline{q}}(t)|$ attains its maximum at $\overline{t}$. Let $\overline{\tau}$ be some positive real number such that $|\dot{\overline{q}}(t)| \geq 3|\dot{\overline{q}}(\overline{t})|/4$ on the interval $\overline{J} = (\overline{t} - \overline{\tau}, \overline{t} + \overline{\tau})$. Let

$$\overline{a}_0 = L_A(\dot{\overline{q}}\chi_{\overline{J}}),$$

where $\chi_{\overline{J}}$ is the characteristic function of the interval $\overline{J}$. By the expression of L_A (2.1), we see that $\overline{a}_0 \in X$. We define

$$\overline{Y}'' = \overline{a}_0^{\perp} = \left\{ h \in Y \;\middle|\; \int_{\overline{t}-\overline{\tau}}^{\overline{t}+\overline{\tau}} \dot{\overline{q}}(s) \cdot h(s)\, ds = 0 \right\}.$$

$\overline{Y}''$ is a supplementary to $\dot{\overline{q}}$ and hence by Remark 2 there exist a positive constant C_0 such that for all $h \in \overline{Y}''$

$$\min_{\mu \in \mathbb{R}} ||dS(\overline{q})h - \mu \overline{a}_0|| \geq C_0 ||h||$$

(note that, due to the fact that $dS(\overline{q}) = Id + compact$, $C_0 \leq 1$). This implies that

$$\max \Big(||dS(\overline{q})h - \mu \overline{a}_0||, R|(h, \overline{a}_0)| \Big) \geq C_1 ||h||, \quad \forall (h, \mu) \in Y \times \mathbb{R}, \tag{2.4}$$

where the constant $C_1 \leq 1$ and $C_1 \to C_0$ as $R \to +\infty$. In the sequel we will fix R and assume that (2.4) holds (we can choose C_1 as close to C_0 as desired).

Now let $\overline{a} = \alpha \overline{a}_0$, where $\alpha > 0$ is chosen such that $||\overline{a}|| = C_1 + 1 + \overline{\Lambda} ||\overline{q}||$. It is easy to see by (2.3) and (2.4) that

$$\max \left(||dS(\overline{q})h - \mu \overline{a}||, R|(h, \overline{a})| \right) \geq C_1 \max(||h||, |\mu|), \quad \forall (h, \mu) \in Y \times \mathbb{R}. \tag{2.5}$$

We shall assume that also for $\widetilde{q}$ are defined the corresponding quantities $\widetilde{t}$, $\widetilde{\tau}$, $\widetilde{a}$ and that condition (2.5) holds. In the sequel we will also assume that $\max\{\overline{\tau}, \widetilde{\tau}\} < T$.

Now we define T_{C_1}. Let $\overline{T}_{C_1}$ the smallest positive time such that

$$\forall t \in \mathbb{R} \backslash [-\overline{T}_{C_1}, \overline{T}_{C_1}] \quad |\overline{q}(t)| \leq \rho_0 \quad \text{and} \quad 8\Lambda \max \left(|\overline{q}(t)|, \frac{|\dot{\overline{q}}(t)|}{\lambda_2} \right) \leq C_1. \tag{2.6}$$

We can define in the same way $\widetilde{T}_{C_1}$, and we set $T_{C_1} = \max(\overline{T}_{C_1}, \widetilde{T}_{C_1})$.

The reason for this definition will appear in the proof of Lemma 4. It is easy to see that, if $(H4)$ is satisfied, then by Lemma 2 we have $\overline{T}_{C_1} \leq \overline{T}$, $\widetilde{T}_{C_1} \leq \widetilde{T}$.

2.1. – Boundary value problems

The aim of this section is to show how solutions of the non-linear system (1.1) are approximated by solutions of the linear one $-\ddot{q} + Aq = 0$ in a sufficiently small neighborhood of the origin $B_r = \{q \in \mathbb{R}^2 \mid |q| \leq r\}$.

First we consider the linear case. The solution $q_{d,L}(t) : [0, d] \to \mathbb{R}^2$ of the linear system $-\ddot{q} + Aq = 0$ with boundary conditions $q_{d,L}(0) = \beta$ and $q_{d,L}(d) = \alpha$ is given by

$$(q_{d,L})_j(t) = \frac{\beta_j \sinh(\lambda_j(d-t)) + \alpha_j \sinh(\lambda_j t)}{\sinh(\lambda_j d)}, \quad j = 1, 2, \tag{2.7}$$

whereas the solutions $q^+_{h,L} : [0,+\infty) \to B_r$ (resp. $q^-_{h,L} : (-\infty, d] \to B_r$) of the linear system $-\ddot q + Aq = 0$ such that $\lim_{t\to+\infty} q^+_{h,L}(t) = 0$ (resp. $\lim_{t\to-\infty} q^-_{h,L}(t) = 0$) and $q^+_{h,L}(0) = \beta$ (resp. $q^-_{h,L}(d) = \alpha$) are given by

$$q^+_{h,L}(t) = e^{-t\sqrt{A}}\beta, \qquad q^-_{h,L}(t) = e^{(t-d)\sqrt{A}}\alpha. \tag{2.8}$$

We define

$$\Delta^d_L(\beta,\alpha) = \max\left\{\left|\dot q_{d,L}(0) - \dot q^+_{h,L}(0)\right|, \left|\dot q_{d,L}(d) - \dot q^-_{h,L}(d)\right|\right\}. \tag{2.9}$$

By (2.7) and (2.8) we can compute

$$(\dot q_{d,L}(0) - \dot q^+_{h,L}(0))_j = \frac{\lambda_j(\alpha_j - \beta_j e^{-\lambda_j d})}{\sinh(\lambda_j d)},$$
$$(\dot q_{d,L}(d) - \dot q^-_{h,L}(d))_j = \frac{\lambda_j(-\beta_j + \alpha_j e^{-\lambda_j d})}{\sinh(\lambda_j d)}.$$

We shall always assume that $d \geq 2/\lambda_2$. Setting $S_j = \max(|\alpha_j|, |\beta_j|)$, we deduce that

$$\Delta^d_L \leq 2\sqrt{2}\left(\frac{1+e^{-2}}{1-e^{-4}}\right)\max_{j=1,2}\left\{\lambda_j S_j \exp -(\lambda_j d)\right\}. \tag{2.10}$$

We now consider the analogous solutions of the non-linear system. Since 0 is a hyperbolic equilibrium the existence of the local stable and unstable manifolds is standard. The following lemma would follow from that but we prove it directly by a fixed point argument because we need some explicit estimates.

LEMMA 2. *For all $0 < r < r_0$ with $r_0 = \min(1/6\Lambda, \rho_0/2)$, for all $\alpha, \beta \in \mathbb{R}^2$ with $|\alpha| = |\beta| = r$ there exist unique trajectories of (1.1)*

$$q^+_h : [0,+\infty) \to B_r \quad \text{and} \quad q^-_h : (-\infty, d] \to B_r$$

such that $\lim_{t\to+\infty} q^+_h(t) = 0$, $\lim_{t\to-\infty} q^-_h(t) = 0$ and $q^+_h(0) = \beta$, $q^-_h(d) = \alpha$.
Moreover for all t we have that

$$|q^+_h(t) - q^+_{h,L}(t)| \leq \frac{2}{7} r^2 \Lambda e^{-\lambda_2 t}, \quad |\dot q^+_h(t) - \dot q^+_{h,L}(t)| \leq \frac{2}{7}\lambda_2 r^2 \Lambda e^{-\lambda_2 t},$$
$$|q^+_h(t)| \leq \frac{22}{21} r e^{-\lambda_2 t}, \quad |\dot q^+_h(t)| \leq \frac{22}{21}\lambda_2 r e^{-\lambda_2 t}$$

and the corresponding estimates for q^-_h.

The proof of Lemma 2 is given in the appendix.

Now we give a lemma, which will be used to glue together consecutive bumps, on the existence and uniqueness of orbits connecting two points α, β in a neighborhood B_r of 0. Such kind of lemma is certainly not new being deeply related with the λ-lemma. However, since we want to obtain specific estimates, we will give a proof based on a fixed point argument.

LEMMA 3. *For all* $0 < r < r_1$ *with* $r_1 = \min(1/10\Lambda, \rho_0/2)$, *for all* $\alpha, \beta \in \mathbb{R}^2$ *with* $|\beta| = |\alpha| = r$, *for all* $d > 2/\lambda_2$ *there exists a unique trajectory of* (1.1) $q_d(t)$ *such that* $q_d(0) = \beta$, $q_d(d) = \alpha$ *and* $q_d([0, d]) \subset B(0, 2r)$.

Moreover the following estimate holds:

$$\left|\left(\dot{q}_d(0) - \dot{q}_h^+(0)\right) - \left(\dot{q}_{d,L}(0) - \dot{q}_{h,L}^+(0)\right)\right|, \tag{2.11}$$
$$\left|\left(\dot{q}_d(d) - \dot{q}_h^-(d)\right) - \left(\dot{q}_{d,L}(d) - \dot{q}_{h,L}^-(d)\right)\right| < 5\lambda_2 r^2 \Lambda e^{-\lambda_2 d}$$

$$|q_d(t) - q_h^+(t)| < \frac{7}{5} r e^{\lambda_2(t-d)}, \qquad |q_d(t) - q_h^-(t)| < \frac{7}{5} r e^{-\lambda_2 t}. \tag{2.12}$$

$$|\dot{q}_d(t) - \dot{q}_h^+(t)| < \frac{14}{5}\lambda_1 r e^{\lambda_2(t-d)}, \quad |\dot{q}_d(t) - \dot{q}_h^-(t)| < \frac{14}{5}\lambda_1 r e^{-\lambda_2 t}. \tag{2.13}$$

The proof of Lemma 3 is given in the appendix. In the sequel we will call also $\gamma(\beta, \alpha, d) = q_d$ the connecting solution given by Lemma 3 and

$$e(\beta, \alpha, d) = \frac{1}{2}\int_0^d \dot{q}_d^2(t) + Aq_d(t) \cdot q_d(t)dt - \int_0^d \dot{q}_d \cdot v(q_d)dt - \int_0^d W(q_d(t))dt,$$

the value of the action on the solution $q_d = \gamma(\beta, \alpha, d)$.

2.2. – Natural constraint

As stated in the introduction our existence results are obtained by means of a finite dimensional reduction according to the following definition:

DEFINITION 1. Let M be a manifold. An immersion $\mathcal{I} : M \to Y$ such that $\mathcal{I}(M) \subset E$ is said to induce a natural constraint for the functional f if

$$\forall x \in M, \quad d(f \circ \mathcal{I})(x) = 0 \textit{ implies that } df(\mathcal{I}(x)) = 0.$$

In the sequel for $j = (j_1, \dots, j_k) \in \{0, 1\}^k$ and for $\Theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$, with $\theta_1 < \dots < \theta_k$ we will use the following notations:

$$J_i = \theta_i + (\bar{t} - \bar{\tau}, \bar{t} + \bar{\tau}) \text{ if } j_i = 0 \qquad \text{and } J_i = \theta_i + (\tilde{t} - \tilde{\tau}, \tilde{t} + \tilde{\tau}) \text{ if } j_i = 1.$$

$$u_i = \theta_i - T_i \quad \text{and } s_i = \theta_i + T_i,$$

$$d_i = (\theta_{i+1} - T_{i+1}) - (\theta_i + T_i) = u_{i+1} - s_i \text{ and } \bar{d} = \min_{1 \le i \le k-1} d_i.$$

For simplicity the dependence on $\Theta = (\theta_1, \dots, \theta_k)$ of s_i, u_i, d_i and $\overline{d}$ will remain implicit.

Fix $j = (j_1, \dots, j_k) \in \{0, 1\}^k$. Our aim is to prove the existence of a k-bump homoclinic associated to j. We now define the "pseudo-critical manifold".

Consider the k parameter family of continuous functions Q_Θ defined in the following way:

$$Q_\Theta = \begin{cases} Q^1(t) \text{ if } t \in (-\infty, s_1], \\ \gamma(Q^1(s_1), Q^2(u_2), d_1)(\cdot - s_1) \text{ if } t \in [s_1, u_2], \\ \dots \\ Q^i(t) \text{ if } t \in [u_i, s_i], \\ \gamma(Q^i(s_i), Q^{i+1}(u_{i+1}), d_i)(\cdot - s_i) \text{ if } t \in [s_i, u_{i+1}], \\ \dots \\ Q^k(t) \text{ if } t \in [u_k, +\infty) \end{cases}$$

where

$$Q^i(t) = \begin{cases} \overline{q}(\cdot - \theta_i) & \text{if} \quad j_i = 0 \\ \widetilde{q}(\cdot - \theta_i) & \text{if} \quad j_i = 1. \end{cases}$$

We recall that γ is defined in Lemma 3. The k-dimensional manifold:

$$Z_k = \{Q_\Theta, \ \ \Theta \in \mathbb{R}^k, \ \ \overline{d} > 2/\lambda_2\}$$

is a k-dimensional "pseudo-critical" manifold for f. This means that $\|S(Q_\Theta)\| \to 0$ as $\overline{d} \to +\infty$. We will give in Lemma 5 a more precise estimate.

We now show, following [3], how to build the immersions $\mathcal{I}_k$. For $(h, \mu) \in Y \times \mathbb{R}^k$ we denote $||(h, \mu)|| = \max(||h||, |\mu_1|, \dots, |\mu_k|)$. Let us define the function

$$H \ : \ \mathbb{R}^k \times Y \times \mathbb{R}^k \to Y \times \mathbb{R}^k$$

with components $H_1 \in Y$ and $H_2 \in \mathbb{R}^k$ given by:

$$H_1(\theta_1, \dots, \theta_k, h, \mu_1, \dots, \mu_k) = S(h) - \sum_{i=1}^{k} \mu_i a_i$$

$$H_2(\theta_1, \dots, \theta_k, h, \mu_1, \dots, \mu_k) = \Big(R(h - Q_\Theta, a_1), \dots, R(h - Q_\Theta, a_k)\Big)$$

where

$$a_i = \begin{cases} \overline{a}(\cdot - \theta_i) & \text{if} \quad j_i = 0 \\ \widetilde{a}(\cdot - \theta_i) & \text{if} \quad j_i = 1. \end{cases}$$

Note that H is a C^1 function of (Θ, h, μ) (Q_Θ is not a C^1 function of Θ but the function $\Theta \mapsto (Q_\Theta, a_i) = \alpha \int_{J_i} \dot{Q}^i(t) \cdot Q_\Theta(t)dt = \alpha \int_{J_i} \dot{Q}^i(t) \cdot Q^i(t)dt$ is constant.

Consider the partial derivative of H, $\partial H/\partial(h,\mu)$, evaluated at (Θ, Q_Θ, μ). It is the linear operator of $Y \times \mathbb{R}^k$ given by:

$$\frac{\partial H_1}{\partial(h,\mu)}_{|(\Theta,Q_\Theta,\mu)}[x,\eta_1,\dots,\eta_k] = dS(Q_\Theta)x - \sum_{i=1}^{k}\eta_i a_i$$
$$\frac{\partial H_2}{\partial(h,\mu)}_{|(\Theta,Q_\Theta,\mu)}[x,\eta_1,\dots,\eta_k] = \Big(R(x,a_1),\dots,R(x,a_k)\Big).$$

Note that it is of the form $Id + Compact$ and that it is independent of μ (and so we shall omit to write μ).

Following [3] there results that, provided $\overline{d}$ is great enough, $\frac{\partial H}{\partial(h,\mu)}_{|(\Theta,Q_\Theta)}$ is invertible also on the pseudo-critical manifold Z_k and the norm of the inverse satisfies a uniform bound. As said in the introduction we need in this case a specific estimate on $\overline{d}$.

LEMMA 4. *Let* $D_1 = \max(2/\lambda_2, 2(\ln(18/C_1)/\lambda_2) - 2(T - T_{C_1}))$, *and assume that* $\overline{d} \geq D_1$ *and that (H4) holds. Then, for all* $x \in Y$, *for all* $\eta = (\eta_1,\dots,\eta_k) \in \mathbb{R}^k$, *we have*

$$\left\|\frac{\partial H}{\partial(h,\mu)}_{|(\Theta,Q_\Theta)}[x,\eta]\right\| \geq \frac{C_1}{2}\|(x,\eta)\|$$

i.e.:

$$\left\|\left(dS(Q_\Theta)x - \sum_{j=1}^{k}\eta_j a_j, R(x,a_1),\dots,R(x,a_k)\right)\right\| \geq \frac{C_1}{2}\|(x,\eta_1,\dots,\eta_k)\|.$$

PROOF. We set $m_i = [(\theta_{i+1} - T_{i+1}) + (\theta_i + T_i)]/2$, $I_1 = (-\infty, m_1]$, $I_i = [m_{i-1}, m_i]$ for $2 \leq i \leq k-1$, $I_k = [m_{k-1}, +\infty)$ and $\|x\|_i = \max(\sup_{t\in I_i}|x(t)|, \sup_{t\in I_i}|\dot{x}(t)|/\lambda_2)$.

We define also the compact operators $K, \overline{K}_i, K_i$: $Y \to Y$ by

$$(Kx)(t) = L_A\mathcal{R}x, \qquad (\overline{K}_i x)(t) = L_A\overline{\mathcal{R}}_i x, \qquad (K_i x)(t) = L_A\mathcal{R}_i x,$$

where

$$\mathcal{R}x(s) = D^2W(Q_\Theta(s))x(s) - \psi(Q_\Theta(s))\mathcal{J}\dot{x}(s) - \nabla\psi(Q_\Theta(s))\cdot x\mathcal{J}\dot{Q}_\Theta(s),$$

$$\overline{\mathcal{R}}^i x(s) = D^2W(Q^i(s))x(s) - \psi(Q^i(s))\mathcal{J}\dot{x}(s) - \nabla\psi(Q^i(s))\cdot x\mathcal{J}\dot{Q}^i(s),$$

$\mathcal{R}_i x(s) = \mathcal{R}x(s)$ for $s \in \mathcal{J}_i := [\theta_i - T^i_{C_1}, \theta_i + T^i_{C_1}]$, and $\mathcal{R}_i x(s) = 0$ on $\mathbb{R}\backslash\mathcal{J}_i$. Here we use the notation $T^i_{C_1} = \overline{T}_{C_1}$ if $j_i = 0$, and $T^i_{C_1} = \widetilde{T}_{C_1}$ if $j_i = 1$.

For all $x \in Y$ we can write $dS(Q_\Theta)x = x - Kx$. We shall prove that

$$(2.14) \qquad \forall i\ \|\overline{K}_i x - K_i x\| \leq (C_1/8)\|x\|\ , \quad \|Kx - \sum_{i=1}^{k}K_i x\| \leq (C_1/8)\|x\|.$$

We first derive the lemma from (2.14). Let $(x,\eta)\in Y\times\mathbb{R}^k$; set

$$\begin{aligned}\mathcal{S} &= \left\|\left(dS(Q_\Theta)x-\sum_{j=1}^{k}\eta_j a_j,\, R(x,a_1),\dots,R(x,a_k)\right)\right\| \\ &= \max\left\{||x-Kx-\sum_{j=1}^{k}\eta_j a_j||_i,\, R|(a_i,x)|\,;\ 1\le i\le k\right\}.\end{aligned}$$

By the second inequality of (2.14)

$$\begin{aligned}\mathcal{S} &\geq \max_i\left\{\max\{||x-K_i x-\eta_i a_i||_i,\, R|(a_i,x)|\}-\sum_{j\neq i}\Big(||K_j x+\eta_j a_j||_i\Big)\right\} \\ &\quad -\frac{C_1}{8}||x||.\end{aligned} \tag{2.15}$$

We now define $z_i\in Y$ by $z_i=x$ on I_i and by $-\ddot z_i+Az_i=0$ on $(-\infty,m_{i-1})\cup(m_i,+\infty)$ $\Big((m_i,+\infty)$ if $i=1$, $(-\infty,m_{k-1})$ if $i=k\Big)$, $\lim_{t\to\pm\infty}z_i(t)=0$. By the definition of z_i and a_i it is easy to see that $(z_i,a_i)=(x,a_i)$, $||z_i||=||x||_i$ and that $||z_i-K_iz_i-\eta_ia_i||=||x-K_ix-\eta_ia_i||_i$. Moreover setting $M_i=\max\Big(||z_i-K_iz_i-\eta_ia_i||,\, R|(z_i,a_i)|\Big)$ and $M=\max_i M_i$, we know by (2.5) and (2.14) that for all i

$$M_i\geq C_1\max(||z_i||,|\eta_i|)-||K_iz_i-\overline{K}_iz_i||\geq(7C_1/8)\max(||z_i||,|\eta_i|). \tag{2.16}$$

As a consequence $M\geq(7C_1/8)\max(||x||,|\eta|)$. Now, fix i such that $M=M_i$. By (2.15) and the properties of z_i we deduce that

$$\mathcal{S}\geq M-\sum_{j\neq i}\Big(||K_jx+\eta_ja_j||_i\Big)-\frac{C_1}{8}||x|| \tag{2.17}$$

We need to estimate $||b_j||_i$ for $j\neq i$, where $b_j=K_jx+\eta a_j=K_jz_j+\eta_ja_j$. We remark that, by the definitions of a_i and of K_i, $-\ddot b_j+Ab_j=0$ on $\mathbb{R}\backslash\mathcal{J}_i$. Hence

$$\max(|b_j(t)|,|\dot b_j(t)|/\lambda_2)\leq e^{-\lambda_2 d(t,\mathcal{J}_j)}||b_j||_j.$$

So, for $i\neq j$,

$$||b_j||_i\leq e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)}e^{-\lambda_2|i-j-1|(\overline{d}+2T})||b_j||_j.$$

Now,

$$||b_j||_j\leq||z_j||_j+||z_j-K_jz_j-\eta_ja_j||_j\leq||x||+M,$$

where we have used again that $||z_j||_j = ||x||_j$. Therefore

$$\sum_{j\neq i} ||K_j x + \eta_j a_j||_i \leq e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)} \sum_{j\neq i} e^{-\lambda_2|i-j-1|(\overline{d}+2T)}(M+||x||)$$
$$\leq e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)}(1-e^{-\lambda_2(\overline{d}+2T)})^{-1}2(M+||x||)$$
$$\leq \frac{7}{3}e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)}(M+||x||),$$

because $\overline{d} \geq 2/\lambda_2$. Combining this latter estimate and (2.17), we get

$$\mathcal{S} \geq \left(1-\frac{7}{3}e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)}\right)M - \frac{7}{3}e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)}||x|| - \frac{C_1}{8}||x||$$
$$\geq \left(C_1\left[\frac{7}{8}\left(1-\frac{7}{3}e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)}\right)-\frac{1}{8}\right]-\frac{7}{3}e^{-\lambda_2((T-T_{C_1})+\overline{d}/2)}\right)||(x,\eta)||,$$

which clearly implies the result in the lemma (we use that $C_1 \leq 1$).

There remains to justify (2.14). Firstly, since $T > T_{C_1}$, $\mathcal{R}_i x(s) = \overline{\mathcal{R}}_i x(s)$ for $s \in \mathcal{J}_i$, $\mathcal{R}_i x(s) = 0$ for $s \notin \mathcal{J}_i$. Now, by $(W1)$, $(P1)$ and the definition of T_{C_1}, $|\overline{\mathcal{R}}_i x(s)| \leq \lambda_2^2(C_1/8)||x||$ for $s \notin \mathcal{J}_i$. Hence $|\overline{\mathcal{R}}_i x - \mathcal{R}_i x|_\infty \leq \lambda_2^2(C_1/8)||x||$, and by (2.2) we get the first estimate in (2.14).

Next we estimate $|\mathcal{R}x - \sum_{i=1}^k \mathcal{R}_i x|_\infty$. On $[s_i, u_i]$ $Q_\Theta = Q^i$, hence

$$\forall s \in [s_i, u_i] \quad |\mathcal{R}x - \sum_{i=1}^k \mathcal{R}_i x| = |\overline{\mathcal{R}}_i x - \mathcal{R}_i x| \leq \lambda_2^2(C_1/8)||x||.$$

So we have just to estimate $|\mathcal{R}x(s)|$ on $[s_i, u_{i+1}]$ for each $i \in \{1, \ldots, k-1\}$. Note that on $[s_i, u_{i+1}]$ $Q_\Theta = \gamma(Q^i(s_i), Q^{i+1}(u_{i+1}), d_i)$. A quite straighforward consequence of Lemmas 2 and 3 is that for $s \in [s_i, u_{i+1}]$,

$$|Q_\Theta(s)| \leq \min\left(\frac{7}{5}e^{\lambda_2(s-u_{i+1})} + \frac{22}{21}e^{-\lambda_2(s-s_i)}, \frac{7}{5}e^{-\lambda_2(s-s_i)} + \frac{22}{21}e^{-\lambda_2(s-u_{i+1})}\right) r,$$

$$|\dot{Q}_\Theta(s)| \leq \min\left(\frac{14}{5}e^{\lambda_2(s-u_{i+1})} + \frac{22}{21}e^{-\lambda_2(s-s_i)}, \frac{14}{5}e^{-\lambda_2(s-s_i)} + \frac{22}{21}e^{-\lambda_2(s-u_{i+1})}\right)\lambda_1 r.$$

Using that $d_i = u_{i+1} - s_i \geq 2/\lambda_2$ we can get

$$\max\left(|Q_\Theta(s)|, \frac{|\dot{Q}_\Theta(s)|}{\lambda_2}\right) \leq \frac{\lambda_1}{\lambda_2} r \max\left(\frac{14}{5}e^{-\lambda_2 d_i} + \frac{22}{21}, \left(\frac{14}{5}+\frac{22}{21}\right)e^{-\lambda_2 \frac{d_i}{2}}\right)$$
$$\leq \frac{3\lambda_1}{2\lambda_2} r.$$

Hence, by $(W1)$ and $(P1)$, for $s \in [s_i, u_{i+1}]$, $|\mathcal{R}x(s)| \leq (3/2)\lambda_1\lambda_2\Lambda r||x||$, and by $(H4)$ $|\mathcal{R}x - \sum_{i=1}^k \mathcal{R}_i x|_\infty \leq \lambda_2^2(C_1/8)||x||$. By (2.2) this implies the second estimate in (2.14). □

LEMMA 5. *For* $0 < r < r_1$ *and for all* $\Theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ *with* $\overline{d} > 2/\lambda_2$ *there results that*

$$||S(Q_\Theta)|| \leq \frac{9}{2\lambda_2} \max_{j=1,2} \left\{ \lambda_j S_j e^{-\lambda_j \overline{d}} \right\} + 7r^2 \Lambda e^{-\lambda_2 \overline{d}}.$$

where $S_j = \max\{|\overline{\alpha}_j|, |\overline{\beta}_j|, |\widetilde{\alpha}_j|, |\widetilde{\beta}_j|\}$.

PROOF. By construction Q_Θ solves system (1.1), except at the times $s_1, u_2, \dots$ $\dots s_{k-1}, u_k$ where $\dot{Q}_\Theta$ is discontinuous and has a jump denoted by $\Delta \dot{Q}_\Theta(u_i)$, $\Delta \dot{Q}_\Theta(s_i)$. So we have for all $y \in Y \cap E$

$$(S(Q_\Theta), y) = df(Q_\Theta)[y] = \sum_{i=1}^{k-1} y(s_i) \cdot \Delta \dot{Q}_\Theta(s_i) + y(u_{i+1}) \cdot \Delta \dot{Q}_\Theta(u_{i+1}).$$

Hence $S(Q_\Theta)$ is the element of $E \cap Y$ defined by

$$S(Q_\Theta) = \sum_{i=1}^{k-1} g(\cdot - s_i) \Delta \dot{Q}_\Theta(s_i) + g(\cdot - u_{i+1}) \Delta \dot{Q}_\Theta(u_{i+1}),$$

where $-\ddot{g} + Ag = \delta \; Id$, $\lim_{|t| \to \infty} g(t) = 0$, Id being the identity 2×2 matrix. We have

$$g(t) = \frac{1}{2} (\sqrt{A})^{-1} e^{-\sqrt{A}|t|} \quad \text{and} \quad \dot{g}(t) = -\frac{1}{2} \text{sign}(t) e^{-\sqrt{A}|t|}.$$

We set $\Delta = \max\{|\Delta \dot{Q}_\Theta(u_i)|, |\Delta \dot{Q}_\Theta(s_i)| \; ; \; 1 \leq i \leq k\}$. We have to estimate $||S(Q_\Theta)||$. Assume for example that $t \in [s_{n-1}, u_n]$. We have

$$|S(Q_\Theta)(t)| \leq \frac{\Delta}{2\lambda_2} \left(\sum_{i=1}^{n-1} (e^{-\lambda_2 |t-u_i|} + e^{-\lambda_2 |t-s_i|}) + \sum_{i=n}^{k} (e^{-\lambda_2 |t-u_i|} + e^{-\lambda_2 |t-s_i|}) \right).$$

For all $t \in [s_{n-1}, u_n]$ we have

$$\frac{1}{2\lambda_2} \sum_{i=1}^{n-1} \left(e^{-\lambda_2 (t-u_i)} + e^{-\lambda_2 (t-s_i)} \right) \leq \frac{1 + \exp -(2\lambda_2 T)}{2\lambda_2} \frac{\exp -(t - s_{n-1})\lambda_2}{1 - \exp -\lambda_2(\overline{d} + 2T)}$$
$$\leq \frac{1}{\lambda_2} \frac{\exp -(t - s_{n-1})\lambda_2}{1 - \exp -(\lambda_2 \overline{d})}.$$

In the same way we get

$$\sum_{i=n}^{k} \left(e^{\lambda_2 (t-u_i)} + e^{\lambda_2 (t-s_i)} \right) \leq \frac{1 + \exp -(2\lambda_2 T)}{2\lambda_2} \frac{\exp (t - u_n)\lambda_2}{1 - \exp -\lambda_2(\overline{d} + 2T)}$$
$$\leq \frac{1}{\lambda_2} \frac{\exp (t - u_n)\lambda_2}{1 - \exp -(\lambda_2 \overline{d})}.$$

Hence for $t \in [s_{n-1}, u_n]$, since $\overline{d} > 2/\lambda_2$ we get $|S(Q_\Theta)(t)| \leq (\Delta(1+\exp -2))/(\lambda_2(1-\exp -2))$. In the same way we can see that $|d/dt(S(Q_\Theta))(t)| \leq (\Delta(1+\exp -2))/(\lambda_2(1-\exp -2))4$. The case $t \in [u_n, s_n]$ yields the same estimates. Hence

$$||S(Q_\Theta)|| \leq \frac{\Delta(1+e^{-2})}{\lambda_2(1-e^{-2})}.$$

We now estimate Δ. By Lemma 3 we have that $|\Delta - \Delta_L| \leq 5\lambda_2 r^2 \Lambda e^{-\lambda_2 \overline{d}}$. Hence

$$||S(Q_\Theta)|| \leq \frac{(\Delta_L + (5\lambda_2 r^2 \Lambda e^{-\lambda_2 \overline{d}}))}{\lambda_2} \frac{1+e^{-2}}{1-e^{-2}}. \tag{2.18}$$

Since $u_n - s_{n-1} > \overline{d}$, $\forall n$ by (2.10) we have that:

$$\Delta_L \leq 2\sqrt{2}\frac{1+e^{-2}}{1-e^{-4}} \max_j \{\lambda_j S_j e^{-\lambda_j \overline{d}}\}. \tag{2.19}$$

By (2.19) and (2.18) we deduce the estimate of the lemma.

Now, since $\overline{d} > 2/\lambda_2$, $\lambda_1 e^{-\lambda_1 \overline{d}} \leq \lambda_2 e^{-\lambda_2 \overline{d}}$, so, if (H4) is satisfied, using that $S_1, S_2 \leq r$, and $r\Lambda \leq 1/20$ we also have

$$||S(Q_\Theta)|| \leq \frac{9}{2} r e^{-\lambda_2 \overline{d}} + 7r^2 \Lambda e^{-\lambda_2 \overline{d}} \leq 5 r e^{-\lambda_2 \overline{d}}. \tag{2.20}$$

□

In the next "shadowing type" lemma we repeat the arguments of [3] based on the contraction-mapping theorem in order to build the immersions $\mathcal{I}_k$.

LEMMA 6. *Define* $D_2 = \frac{1}{\lambda_2} \ln\left(\frac{40\overline{\Lambda} r}{C_1^2}\right)$ *and assume that (H4) is satisfied. Then* $\forall \Theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ *with* $\overline{d} > \max\{D_1, D_2\}$ *there is a* C^0 *function of* Θ, $\Theta \to w(\Theta)$ *with* $w(\Theta) \in Y$ *such that:*

- $S(Q_\Theta + w(\Theta)) \in \operatorname{span}\{a_1, \dots, a_k\} := \langle a_1, \dots, a_k \rangle$;
- $w(\Theta) \in \langle a_1, \dots, a_k \rangle^\perp$.

Moreover $\mathcal{I}_k : \Theta \to Q_\Theta + w(\Theta)$ *is a* C^1 *function and*

$$||w(\Theta)|| \leq \frac{18}{\lambda_2 C_1} \max_j \{\lambda_j S_j \exp(-\lambda_j \overline{d})\} + \frac{28}{C_1} r^2 \Lambda \exp(-\lambda_2 \overline{d}).$$

In the sequel the function $w(\Theta)$ *will be denoted also by* w_Θ.

PROOF. This proof will follow closely the one given in [3], see Lemmas 3 and 13. Therefore we shall be brief. We shall use the following abbreviation:

$$F(\Theta, w) = \frac{\partial H}{\partial(h, \mu)}_{|(\Theta, Q_\Theta + w)} \quad \in \ L(Y \times \mathbb{R}^k).$$

By Lemma 4 we know that $\forall \Theta = (\theta_1, \dots, \theta_k) \mid \overline{d} > D_1, \ ||F^{-1}(\Theta, 0)|| \le 2/C_1$.

Let $B_\delta \subset Y \times \mathbb{R}^k$ be the ball in $Y \times \mathbb{R}^k$ of center 0 and radius δ: $B_\delta = \{(w, \mu_1, \dots, \mu_k)$ such that $\max(||w||, |\mu_1|, \dots, |\mu_k|) \le \delta\}$. We have to find (w, μ) such that $H(\theta_1, \dots, \theta_k, Q_\Theta + w, \mu_1, \dots, \mu_k) = 0$. This last equation is equivalent to $\mathcal{D}(w, \mu) = (w, \mu)$ where:

$$\begin{aligned}\mathcal{D}(w, \mu) = & -F^{-1}(\Theta, 0) H(\Theta, Q_\Theta, 0) \\ & - F^{-1}(\Theta, 0)\Big(H(\Theta, Q_\Theta + w, \mu) - H(\Theta, Q_\Theta, 0) - F(\Theta, 0)[w, \mu]\Big).\end{aligned}$$

We will find $\delta > 0$ such that if $\overline{d} > \max\{D_1, D_2\}$ then

(i) $\overline{\mathcal{D}(B_\delta)} \subset B_\delta$, **(ii)** $\mathcal{D}$ is a contraction on $\overline{B_\delta}$.

It is easy to see that by $(W1)$, $(P1)$ and (2.2) $||F(\Theta, w) - F(\Theta, 0)|| < \overline{\Lambda}||w||$. As in [3] we can derive that $\forall (w, \mu) \in B_\delta$:

$$||\mathcal{D}(w, \mu)|| \le \frac{2}{C_1}||S(Q_\Theta)|| + \frac{\overline{\Lambda}}{C_1}||(w, \mu)||^2$$

Then in order to get (i) we have to solve:

$$\frac{2}{C_1}||S(Q_\Theta)|| + \frac{\overline{\Lambda}}{C_1}\delta^2 < \delta \tag{2.21}$$

A straigforward computation shows that if $||S(Q_\Theta)|| < C_1^2/8\overline{\Lambda}$ then (2.21) is satisfied for

$$\delta \in \left(C_1\frac{1 - \sqrt{1 - 8\overline{\Lambda}||S(Q_\Theta)||/C_1^2}}{2\overline{\Lambda}}, C_1\frac{1 + \sqrt{1 - 8\overline{\Lambda}||S(Q_\Theta)||/C_1^2}}{2\overline{\Lambda}}\right).$$

We now prove that also (ii) is satisfied if:

$$C_1\frac{1 - \sqrt{1 - 8\overline{\Lambda}||S(Q_\Theta)||/C_1^2}}{2\overline{\Lambda}} < \delta < \frac{C_1}{2\overline{\Lambda}}.$$

Indeed, by $(W1)$, $(P1)$ and (2.2), we have $\forall (w, \mu), (w', \mu') \in B_\delta$:

$$||\mathcal{D}(w, \mu) - \mathcal{D}(w', \mu')|| \le \frac{2}{C_1}\overline{\Lambda}\delta||(w, \mu) - (w', \mu')||$$

which implies our claim. Then in order to apply the contraction-mapping theorem take $\overline{d} > D_2$ so that $||S(Q_\Theta)|| < C_1^2/8\overline{\Lambda}$. Taking into account (2.20) the last inequality is satisfied if $\overline{d} > D_3$ as defined in the lemma. Moreover, by the previous considerations,

$$||w_\Theta|| \leq C_1 \frac{1-\sqrt{1-8\overline{\Lambda}||S(Q_\Theta)||/C_1^2}}{2\overline{\Lambda}} \leq \frac{4}{C_1}||S(Q_\Theta)||,$$

which, by Lemma 5 implies the last estimate of the lemma. The fact that $Q_\Theta + w(\Theta)$ is a C^1 function of Θ is a consequence of the Implicit function theorem applied to H. □

We define also $D_3 = \frac{1}{\lambda_2}\ln\left(\frac{40}{C_1}\right)$ so that for all $\Theta \in \mathbb{R}^k$ with $\overline{d} > \max\{D_1, D_2, D_3\}$ we have $||w(\Theta)|| < r/2$.

We define for $\overline{d} > \overline{D} = \max\{D_1, D_2, D_3\}$ the immersions $\mathcal{I}_k$

$$\mathcal{I}_k : \ M_k = \{\Theta \in \mathbb{R}^k \mid \theta_{i+1} - \theta_i > \overline{D}\} \to \mathcal{I}_k(\Theta) = Q_\Theta + w(\Theta).$$

By Lemma 6 we can prove that:

LEMMA 7. *If $\overline{d} > \overline{D}$ and (H4) is satisfied then $\mathcal{I}_k$ is a natural constraint for f.*

The proof is in the appendix.

2.3. – Critical points of $f \circ \mathcal{I}_k$ and proof of Theorem 1

We are led, in order to find k-bumps homoclinics, to look for critical points of the function $f \circ \mathcal{I}_k(\Theta) = f(Q_\Theta + w(\Theta))$. Note that, since (1.1) is autonomous, $f(Q_\Theta + w(\Theta))$ depends only on $d_1, \ldots, d_{k-1}$. Let us define

$$g(d_1, \ldots, d_{k-1}) = f(Q_\Theta + w(\Theta)).$$

By Lemma 7 a zero of the function $G : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ defined by

$$G(d_1, \ldots, d_{k-1}) = \left(\frac{\partial g(d)}{\partial d_1}, \ldots, \frac{\partial g(d)}{\partial d_{k-1}}\right)$$

gives rise to an homoclinic solution of (1.1). We will find a zero of G by means of degree theory showing in the proof of Theorem 1 that hypotheses $(H1-4)$ imply

$$|\deg(G, U, 0)| = 1$$

where

$$U = \prod_{i=1}^{k-1}(D, J) \subset \mathbb{R}^{k-1},$$

and $J > D$ are some real numbers estimated in the proof of Lemma 11.

We need some preliminary lemmas. The next one is proved in the appendix.

LEMMA 8. *For $d > 2/\lambda_2$ and $0 < r < r_1$ consider the solution q_d given by Lemma 3 and the function $e(\beta, \alpha, d)$ which is the value of the action on q_d. There results that*

$$\frac{\partial e}{\partial d}(\beta, \alpha, d) = -\mathcal{E}(d),$$

where $\mathcal{E}(d) = (\dot{q}_d^2(t) - Aq_d(t) \cdot q_d(t))/2 + W(q_d)$ is the energy of the orbit q_d.

LEMMA 9. *For all $(d_1, \dots, d_{k-1}) \in \mathbb{R}^{k-1}$ with $\overline{d} > \overline{D}$, we have:*

$$\begin{aligned} &\frac{\partial}{\partial d_i} g(d_1, \dots, d_{k-1}) \\ &= \left(\frac{\partial}{\partial d} e\right) \left(Q^i(s_i) + w_\Theta(s_i), Q^{i+1}(u_{i+1}) + w_\Theta(u_{i+1}), d_i\right). \end{aligned} \tag{2.22}$$

PROOF. We must compute

$$\frac{\partial g(d_1, \dots, d_{k-1})}{\partial d_i} = \frac{\partial f(Q_\Theta + w(\Theta))}{\partial d_i}.$$

For this purpose we consider the function of the real variable τ:

$$\begin{aligned} \sigma(\tau) &= g(d_1, \dots, d_i + \tau, \dots, d_{k-1}) \\ &= f(Q_{(\theta_1, \dots, \theta_i, \theta_{i+1}+\tau, \dots, \theta_k+\tau)} + w(\theta_1, \dots, \theta_i, \theta_{i+1} + \tau, \dots, \theta_k + \tau)) \end{aligned}$$

and we compute $\sigma'(0)$. For simplicity of notation we set $q^0 = Q_\Theta + w(\Theta)$. Let $q^\tau \in Y \cap E$ be defined as follows :

$$q^\tau = \begin{cases} q^0(s) & \text{on } (-\infty, s_i] \\ \gamma(q^0(s_i), q^0(u_{i+1}), d_i + \tau)(\cdot - s_i) & \text{on } [s_i, u_{i+1} + \tau] \\ q^0(\cdot - \tau) & \text{on } [u_{i+1} + \tau, +\infty) \end{cases}$$

Note that our notations are coherent (*i.e.* $q^\tau = q^0$ when $\tau = 0$). We will use the notation $a_i = a_{\theta_i}$, not distinguishing for simplicity between $a = \overline{a}$ and $a = \widetilde{a}$.

Since $q^0 - Q_\Theta = w(\Theta) \in \langle a_{\theta_1}, \dots, a_{\theta_k} \rangle^\perp$ and since, by the definition of a_{θ_i}, $(x, a_{\theta_i}) = \alpha \int_{J_i} x(t) \dot{Q}^i(t) dt$ we see that

$$q^\tau - Q_{(\theta_1, \dots, \theta_i, \theta_{i+1}+\tau, \dots, \theta_k+\tau)} \in \langle a_{\theta_1}, \dots, a_{\theta_i}, a_{\theta_{i+1}+\tau}, \dots, a_{\theta_k+\tau} \rangle^\perp,$$

and we can write $q^\tau - (Q_{(\theta_1, \dots, \theta_i, \theta_{i+1}+\tau, \dots, \theta_k+\tau)} + w(\theta_1, \dots, \theta_i, \theta_i + \tau, \dots, \theta_k + \tau)) = \widetilde{w}(\tau)$, where $\widetilde{w}(0) = 0$ and $\widetilde{w}(\tau) \in \langle a_{\theta_1}, \dots, a_{\theta_i}, a_{\theta_{i+1}+\tau}, \dots, a_{\theta_k+\tau} \rangle^\perp$. Since $\widetilde{w}(0) = 0$ we have that

$$\begin{aligned} 0 &= (\partial/\partial\tau)_{\tau=0} \ (\widetilde{w}(\tau), a_{\theta_i+\tau}) \\ &= (\partial \widetilde{w}(\tau)/\partial\tau, a_{\theta_i+\tau})_{\tau=0} + (\widetilde{w}(\tau), \partial_\tau \ a_{\theta_i+\tau})_{\tau=0} = (\partial \widetilde{w}(\tau)/\partial\tau, a_{\theta_i+\tau})_{|\tau=0}. \end{aligned}$$

This means that:

$$\frac{\partial \widetilde{w}(\tau)}{\partial \tau}_{|\tau=0} \in \langle a_{\theta_1}, \dots , a_{\theta_i}, a_{\theta_{i+1}}, \dots , a_{\theta_k}\rangle^{\perp}.$$

Now we can prove (2.22). Indeed, since $(S(q^0), x) = 0$ for all $x \in \langle a_{\theta_1}, \dots , a_{\theta_k}\rangle^{\perp}$,

$$\sigma'(0) = \left(S(q^0), \frac{\partial (q^\tau + \widetilde{w}(\tau))}{\partial \tau}\right)_{|\tau=0} = \left(S(q^0), \frac{\partial q^\tau}{\partial \tau}_{|\tau=0}\right) = \frac{\partial f(q^\tau)}{\partial \tau}_{|\tau=0}.$$

By the definition of q^τ we have that

$$\frac{\partial f(q^\tau)}{\partial \tau}_{|\tau=0} = \left(\frac{\partial}{\partial d} e\right)(q^0(s_i), q^0(u_{i+1}), d_i),$$

which yields (2.22). □

LEMMA 10. *For all $|\alpha|, |\beta| < r' \le r_1$, $d > 2/\lambda_2$ there results that:*

$$\left|\left(\frac{\partial}{\partial d} e\right)(\beta, \alpha, d) - \sum_{j=1}^{2} \frac{\lambda_j^2}{(\sinh(\lambda_j d))^2}\left(\alpha_j \beta_j \cosh(\lambda_j d) - \frac{(\alpha_j^2 + \beta_j^2)}{2}\right)\right| \le \frac{15}{2}\lambda_1\lambda_2\Lambda r'^3 e^{-\lambda_2 d}. \tag{2.23}$$

PROOF. If $0 < r' < r_1$, by Lemmas 2 and 3 the solutions q_h^+ and q_d are defined. We will call $q_h = q_h^+$. Since $\mathcal{E}(d)$ is a constant of the motion and the homoclinic q_h has zero energy we have

$$\mathcal{E}(d) = \frac{1}{2}(|\dot{q}_d(0)|^2 - A\beta\cdot\beta) + W(\beta) \qquad \text{and} \qquad 0 = \frac{1}{2}(|\dot{q}_h(0)|^2 - A\beta\cdot\beta) + W(\beta).$$

Hence, we obtain by substraction

$$\mathcal{E}(d) = \frac{1}{2}(|\dot{q}_d(0)|^2 - |\dot{q}_h(0)|^2). \tag{2.24}$$

Similarily for the linear system we have

$$\mathcal{E}_L(d) = \frac{1}{2}(|\dot{q}_{d,L}(0)|^2 - |\dot{q}_{h,L}(0)|^2). \tag{2.25}$$

This last expression can be computed and we get

$$\mathcal{E}_L(d) = \sum_{j=1}^{2} \frac{\lambda_j^2}{(\sinh(\lambda_j d))^2}\left(\alpha_j \beta_j \cosh(\lambda_j d) - \frac{(\alpha_j^2 + \beta_j^2)}{2}\right).$$

By (2.24) and (2.25) we have

$$\begin{aligned}|\mathcal{E}(d)-\mathcal{E}_L(d)| &= \frac{1}{2}\Big|(|\dot q_d(0)|^2-|\dot q_h(0)|^2)-(|\dot q_{d,L}(0)|^2-|\dot q_{h,L}(0)|^2)\Big|\\ &\le \frac{1}{2}\Big|\Big(\dot q_d(0)-\dot q_h(0)\Big)-\Big(\dot q_{d,L}(0)-\dot q_{h,L}(0)\Big)\Big|\Big(|\dot q_d(0)|+|\dot q_h(0)|\Big)\\ &\quad+\frac{1}{2}\Big|\dot q_{d,L}(0)-\dot q_{h,L}(0)\Big|\Big(|\dot q_d(0)-\dot q_{d,L}(0)|+|\dot q_h(0)-\dot q_{h,L}(0)|\Big).\end{aligned}$$

By Lemmas 2 and 3 we have that

$$\begin{aligned}&\Big|(\dot q_d(0)-\dot q_h(0))-(\dot q_{d,L}(0)-\dot q_{h,L}(0))\Big|\le 5\lambda_2 r'^2\Lambda e^{-\lambda_2 d}\\ &\text{and}\quad |\dot q_h(0)-\dot q_{h,L}(0)|\le\frac{2}{7}\lambda_2 r'^2\Lambda.\end{aligned}\tag{2.26}$$

For $d\lambda_2>2$ we get from (2.26) that:

$$|\dot q_d(0)-\dot q_{d,L}(0)|\le\lambda_2 r'^2\Lambda.\tag{2.27}$$

From (2.26) since $r'\Lambda\le 1/6$ and $\lambda_2<\lambda_1$ we obtain:

$$|\dot q_h(0)|\le\frac{22}{21}\lambda_1 r'.\tag{2.28}$$

The expression of $q_{d,L}$, $q_{h,L}$ in Subsection 2.1 leads to

$$|\dot q_{d,L}(0)-\dot q_{h,L}(0)|\le\frac{7}{3}r'\lambda_2 e^{-\lambda_2 d}.\tag{2.29}$$

By (2.27) and (2.7) we get

$$|\dot q_d(0)|\le\frac{4}{3}\lambda_1 r'\tag{2.30}$$

and finally by (2.26), (2.27), (2.28), (2.29) and (2.30) we get:

$$|\mathcal{E}(d)-\mathcal{E}_L(d)|\le\frac{15}{2}\lambda_1\lambda_2\Lambda r'^3 e^{-\lambda_2 d},$$

which is (2.23). □

The next lemma is the most important for the proof of Theorem 1.

LEMMA 11. *Assume* $(H1-4)$. *There exist* $\overline{D} < D < J$ *such that for all* $d = (d_1, \dots, d_{k-1}) \in U$, *for all* $i \in \{1, \dots, k-1\}$*:*

- *if* $\alpha_1^{i+1}\beta_1^i > 0$ *and* $\alpha_2^{i+1}\beta_2^i < 0$,

$$\partial_d e\Big((Q_\Theta + w(\Theta))(s_i), (Q_\Theta + w(\Theta))(u_{i+1}), J\Big) < 0$$

and

$$\partial_d e\Big((Q_\Theta + w(\Theta))(s_i), (Q_\Theta + w(\Theta))(u_{i+1}), D\Big) > 0.$$

- *if* $\alpha_1^{i+1}\beta_1^i < 0$ *and* $\alpha_2^{i+1}\beta_2^i > 0$,

$$\partial_d e\Big((Q_\Theta + w(\Theta))(s_i), (Q_\Theta + w(\Theta))(u_{i+1}), J\Big) > 0$$

and

$$\partial_d e\Big((Q_\Theta + w(\Theta))(s_i), (Q_\Theta + w(\Theta))(u_{i+1}), D\Big) < 0.$$

Estimates for D and J are given in the proof.

PROOF. By $(H1)$ the following cases can arise: $\alpha_1^{i+1}\beta_1^i > 0$ and $\alpha_2^{i+1}\beta_2^i < 0$ or $\alpha_1^{i+1}\beta_1^i < 0$ and $\alpha_2^{i+1}\beta_2^i > 0$. We first deal with the first case. Set

$$(\hat{\beta}_1^i, \hat{\beta}_2^i) = (Q_\Theta + w(\Theta))(s_i) = (\beta_1^i, \beta_2^i) + w(\Theta)(s_i)$$

and

$$(\hat{\alpha}_1^{i+1}, \hat{\alpha}_2^{i+1}) = (Q_\Theta + w(\Theta))(u_{i+1}) = (\alpha_1^{i+1}, \alpha_2^{i+1}) + w(\Theta)(u_{i+1}).$$

We will choose D large enough such that

$$\lambda_1 S_1 e^{-\lambda_1 D} \leq 2\lambda_2 S_2 e^{-\lambda_2 D}. \tag{2.31}$$

So, by Lemma 6,

$$|\hat{\alpha} - \alpha|, |\hat{\beta} - \beta| \leq ||w(\Theta)|| \leq \frac{36S_2 + 28r^2\Lambda}{C_1} e^{-\lambda_2 D}.$$

One of our conditions will be

$$\left|\frac{36S_2 + 28r^2\Lambda}{C_1} e^{-\lambda_2 D}\right| \leq \epsilon\mathcal{M}, \tag{2.32}$$

for some $\epsilon \in (0, 1/8)$ which will be chosen later on. Note that $D \geq D_3$ then holds. This implies also

$$\begin{aligned} &\hat{\alpha} \in [(1-\epsilon)\alpha, (1+\epsilon)\alpha], \ |\hat{\alpha}| \leq (1+\epsilon)r\,; \\ &\hat{\beta} \in [(1-\epsilon)\beta, (1+\epsilon)\beta], \ |\hat{\beta}| \leq (1+\epsilon)r. \end{aligned} \tag{2.33}$$

Then there results that $\hat{\alpha}_1^{i+1}\hat{\beta}_1^i > 0$ and $\hat{\alpha}_2^{i+1}\hat{\beta}_2^i < 0$. Now, by Lemma 10, we can write

$$\frac{\partial e}{\partial d}(\hat{\beta}^i, \hat{\alpha}^{i+1}, d_i) = -U_2 \frac{\cosh(\lambda_2 d_i)}{(\sinh(\lambda_2 d_i))^2} + U_1 \frac{\cosh(\lambda_1 d_i)}{(\sinh(\lambda_1 d_i))^2},$$

where

$$U_2 \leq \lambda_2^2 \left(|\hat{\alpha}_2^{i+1}||\hat{\beta}_2^i| + \frac{(\hat{\alpha}_2^{i+1})^2 + (\hat{\beta}_2^i)^2}{2\cosh(\lambda_2 d_i)} \right) + \frac{15}{4}\lambda_1\lambda_2(1+\epsilon)^3 r^3 \Lambda$$

and

$$U_1 = \lambda_1^2 \left(|\hat{\alpha}_1^{i+1}||\hat{\beta}_1^{i+1}| - \frac{(\hat{\alpha}_1^{i+1})^2 + (\hat{\beta}_1^i)^2}{2\cosh(\lambda_1 d_i)} \right).$$

By (2.33) and (2.32) we get

$$U_2 \leq \lambda_2^2(1+\epsilon)^2 \left(|\alpha_2^{i+1}||\beta_2^i| + \left((\alpha_2^{i+1})^2 + (\beta_2^{i+1})^2\right) \frac{C_1 \epsilon \mathcal{M}}{36 S_2} \right) + \frac{15}{4}\lambda_1\lambda_2(1+\epsilon)^3 r^3 \Lambda,$$

hence, since $C_1 \leq 1$, $\mathcal{M} \leq \min\{|\alpha_2^{i+1}|, |\beta_2^i|\}$ and $S_2 \geq \max\{|\alpha_2^{i+1}|, |\beta_2^i|\}$,

$$U_2 \leq \lambda_2^2(1+\epsilon)^2 \left(1 + \frac{\epsilon}{9}\right) |\alpha_2^{i+1}||\beta_2^i| + \frac{15}{4}\lambda_1\lambda_2(1+\epsilon)^3 r^3 \Lambda. \tag{2.34}$$

On the other hand, (2.32) and (2.31) imply that

$$\frac{18\lambda_1}{C_1\lambda_2} S_1 e^{-\lambda_1 D} \leq \epsilon \mathcal{M}.$$

We derive readily that

$$\frac{(\hat{\alpha}_1^{i+1})^2 + (\hat{\beta}_1^i)^2}{2\cosh(\lambda_1 d_i)} < \frac{C_1\lambda_2}{9\lambda_1}(1+\epsilon)^2 |\alpha_1^{i+1}||\beta_1^i|,$$

hence, by (2.33),

$$U_1 > \lambda_1^2(1-\epsilon)^2 \left(1 - \frac{C_1\lambda_2}{9\lambda_1}\left(\frac{1+\epsilon}{1-\epsilon}\right)^2\right) |\alpha_1^{i+1}||\beta_1^i|. \tag{2.35}$$

From (2.34), (2.35) and the fact that $\epsilon < 1/8$ we get

$$\frac{U_2}{U_1} < \frac{\lambda_2^2(1+\epsilon)^3}{\lambda_1^2(1-\epsilon)^2(1-\epsilon/5)} \left(\frac{|\alpha_2^{i+1}||\beta_2^i| + (15\lambda_1/4\lambda_2)\Lambda r^3}{|\alpha_1^{i+1}||\beta_1^i|} \right) \tag{2.36}$$

We shall take

$$D = \min_i \frac{1}{\lambda_1 - \lambda_2} \ln\left(\frac{\lambda_1^2(1-\epsilon)^2(1-\epsilon/5)}{\lambda_2^2(1+\epsilon)^3}\left(\frac{|\alpha_1^{i+1}||\beta_1^i|}{|\alpha_2^{i+1}||\beta_2^i| + (15\lambda_1/4\lambda_2)\Lambda r^3}\right)\right). \tag{2.37}$$

It is easy to see that, if $d_i = D$ and $d_j \geq D$ for all j then, by (2.36),

$$\frac{\partial e}{\partial d}(\hat{\beta}^i, \hat{\alpha}^{i+1}, d_i) > 0$$

Now, as $d_i \to +\infty$, $\partial_d e(\hat{\beta}^i, \hat{\alpha}^{i+1}, d_i) \cong -2U_2 e^{-\lambda_2 d_i}$, where

$$U_2 \geq \lambda_2^2(1-\epsilon)^2(1-\epsilon/5)|\alpha_2^{i+1}||\beta_2^i| - \frac{15}{4}\lambda_1\lambda_2(1+\epsilon)^3 r^3 \Lambda$$

(This estimate could be improved in the first case but we want to be able to extend our arguing to the second case $\alpha_2^{i+1}\beta_2^i > 0$). Hence, provided

$$|\alpha_2^{i+1}||\beta_2^i| > 20\lambda_1 r^3 \Lambda/\lambda_2, \tag{2.38}$$

we get $\partial_d e(\hat{\beta}^i, \hat{\alpha}^{i+1}, J) < 0$, for J large enough, more exactly for

$$J > \max_i \frac{1}{\lambda_1 - \lambda_2} \ln\left(\frac{\lambda_1^2}{\lambda_2^2} \frac{|\alpha_1^{i+1}||\beta_1^i|}{|\alpha_2^{i+1}||\beta_2^i| - (20\lambda_1\Lambda r^3/\lambda_2)}\right).$$

Therefore we get the desired result provided conditions (2.31), (2.32) and (2.38) are satisfied, with D defined by (2.37). Now we must choose ϵ to make condition (2.32) as weak as possible. This condition reads

$$\frac{\lambda_2^2}{\lambda_1^2} \frac{|\alpha_2||\beta_2| + (15\lambda_1/4\lambda_2)\Lambda r^3}{|\alpha_1||\beta_1|} \leq \frac{(1-\epsilon)^2(1-\epsilon/5)}{(1+\epsilon)^3} \epsilon^{(\lambda_1/\lambda_2)-1} \left(\frac{C_1 \mathcal{M}}{36 S_2 + 28\Lambda r^3}\right)^{(\lambda_1/\lambda_2)-1}. \tag{2.39}$$

Therefore we get condition $(H3)$. We get condition $(H2)$ so that Lemmas 5 and 6 are satisfied with our choice of D. The first inequality in condition $(H4)$ is just (2.38).

There remains to check that $(H2-4)$ imply that (2.31) holds true. First (by $(H2)$ and the definition of D in (2.37), $D \geq 2/\lambda_2$, hence $\lambda_1 e^{-\lambda_1 D} \leq \lambda_2 e^{-\lambda_2 D}$, and if $S_1 \leq 2S_2$ then (2.31) holds. So we shall assume that $S_1 > 2S_2$. Fix i such that

$$D = \frac{1}{\lambda_1 - \lambda_2} \ln\left(\frac{\lambda_1^2(1-\epsilon)^2(1-\epsilon/7)}{\lambda_2^2(1+\epsilon)^3}\left(\frac{|\alpha_1^{i+1}||\beta_1^i|}{|\alpha_2^{i+1}||\beta_2^i| + (15\lambda_1/4\lambda_2)\Lambda r^3}\right)\right). \tag{2.40}$$

Note that, by $(H4)$, $(\Lambda\lambda_1 r^3/\lambda_2) \le |\alpha_2^{i+1}||\beta_2^i|/20$. Combined with the fact that $\epsilon \in (0, 1/8)$, this readily implies that $e^{(\lambda_1-\lambda_2)D} \ge 2\lambda_1^2|\alpha_1^{i+1}||\beta_1^i|(5\lambda_2^2|\alpha_2^{i+1}||\beta_2^i|)^{-1}$.

Now $S_2 \le S_1/2 \le r/2$, hence $|\alpha_2^{i+1}| \le r/2$ and $|\alpha_1^{i+1}| = (r^2-(\alpha_2^{i+1})^2)^{1/2} \ge \sqrt{3}r/2$; $|\beta_1^i| \ge \sqrt{3}r/2$ as well. So we get

$$\frac{\lambda_2 S_2}{\lambda_1 S_1} e^{(\lambda_1-\lambda_2)D} \ge \frac{\lambda_2 S_2}{\lambda_1 S_1}\frac{2}{5}\frac{3r^2}{|\alpha_2^{i+1}||\beta_2^i|4} \ge \frac{3\lambda_2 r}{10\lambda_1 S_2} \ge \frac{3}{5} \ge \frac{1}{2},$$

since $S_2 \le r/2$. So (2.31) holds.

We have proved the lemma in the case where $\alpha_1^{i+1}\beta_1^i > 0$ and $\alpha_2^{i+1}\beta_2^i < 0$. The second case can be dealt with in a similar way (in fact the estimates are simpler in that case). □

We now show, using the previous lemma how to prove Theorem 1.

Proof of Theorem 1. Let J be given by Lemma 11. By Lemmas 9 and 11, g has the following property : for all $i \in \{1, \dots, k-1\}$, we have either

(P_-) : *For all* $d = (d_1, \dots, d_{k-1}) \in \overline{U}$,

$$d_i = D \Rightarrow \frac{\partial g}{\partial d_i}(d) > 0 \quad \text{and} \quad d_i = J \Rightarrow \frac{\partial g}{\partial d_i}(d) < 0,$$

or

(P_+) : *For all* $d = (d_1, \dots, d_{k-1}) \in \overline{U}$,

$$d_i = D \Rightarrow \frac{\partial g}{\partial d_i}(d) < 0 \quad \text{and} \quad d_i = J \Rightarrow \frac{\partial g}{\partial d_i}(d) > 0.$$

It can be readily seen that this property implies $|\deg(G, U, 0)| = 1$. In fact define the function $\hat{G} : \overline{U} \to \mathbb{R}^{k-1}$ as follows:

$$\hat{G}(d_1, \dots, d_{k-1}) = \left(\epsilon_1(d_1 - \frac{D+J}{2}), \dots, \epsilon_{k-1}(d_{k-1} - \frac{D+J}{2})\right),$$

where $\epsilon_i = 1$ if (P_+) is satisfied for the index i, and $\epsilon_i = -1$ if (P_-) is satisfied for the index i. Since the homotopy $G_t = (1-t)\hat{G} + tG$ for $t \in [0, 1]$ is admissible there results that $\deg(G, U, 0) = \deg(\hat{G}, U, 0) = \pm 1$ and the existence of a critical point of g in U follows. This critical point corresponds to a homoclinic, which, by (2.32) and since $\epsilon \in (0, 1/8)$, enjoys the properties given in Theorem 1. □

3. – Examples

The aim of this section is to show examples of Hamiltonian systems where the hypotheses $(H1-4)$ can be checked.

3.1. – Almost equal eigenvalues

Consider the following system

$$(S_\epsilon) \qquad -\ddot{q} + \psi(q)\mathcal{J}\dot{q} + A_\epsilon q = \nabla W(q),$$

with $A_\epsilon = \begin{pmatrix} (\lambda+\epsilon)^2 & 0 \\ 0 & (\lambda-\epsilon)^2 \end{pmatrix}$ and $\lambda > 0$. We assume that W, ψ satisfy (W1), (P1). We shall use the following assumptions:

- (A1) (S_0) has two nondegenerate homoclinics $\overline{q}$ and $\widetilde{q}$.

It can be shown that the limits as $t \to +\infty$ and as $t \to -\infty$ of $\overline{q}(t)/|\overline{q}(t)|$ (resp. $\widetilde{q}(t)/|\widetilde{q}(t)|$) do exist. Call $(\cos\overline{\omega}_s, \sin\overline{\omega}_s)$ and $(\cos\overline{\omega}_u, \sin\overline{\omega}_u)$ (resp. $(\cos\widetilde{\omega}_s, \sin\widetilde{\omega}_s)$ and $(\cos\widetilde{\omega}_u, \sin\widetilde{\omega}_u)$) these limits. The second assumption is

- (A2) $\omega_u, \omega_s \neq n\pi/2$, $n \in \mathbb{Z}$, $-1 < \tan\omega_u \tan\omega_s < 0$ and $(\cos\overline{\omega}_u \cos\widetilde{\omega}_u < 0$ or $\cos\overline{\omega}_s \cos\widetilde{\omega}_s < 0)$.

As an application of Theorem 1 we get

THEOREM 2. *Assume that (S_0) satisfies assumptions $(A1)$ and $(A2)$. Then there is $\epsilon_1 > 0$ such that, for $0 < |\epsilon| < \epsilon_1$, (S_ϵ) has a rich family of homoclinics, which induces a chaotic behaviour at the zero energy level, according to Theorem* 1. *Moreover there is $C > 0$ such that $h^0_{\text{top}} > C\epsilon$, where h^0_{top} denotes the topological entropy at the zero energy level.*

PROOF. Call $\overline{q}_0$ and $\widetilde{q}_0$ the two nondegenerate homoclinics for (S_0). By the Implicit function theorem, it is easy to get, for $|\epsilon|$ small enough, the existence of two homoclinics $\overline{q}_\epsilon$ and $\widetilde{q}_\epsilon$ for system (S_ϵ), and to see that these two homoclinics are nondegenerate, of constant of nondegeneracy $C_{1,\epsilon}$ which tends to C_1 as $\epsilon \to 0$. Moreover

$$\lim_{\epsilon\to 0} \max\left\{|\overline{q}_\epsilon - \overline{q}|_\infty, |\widetilde{q}_\epsilon - \widetilde{q}|_\infty\right\} = 0. \tag{3.1}$$

There are $r_2 > 0$ and $\epsilon_0 > 0$ such that, for $0 < r < r_2$ and $|\epsilon| < \epsilon_0$, the trajectory of $\overline{q}_\epsilon$ (resp. $\widetilde{q}_\epsilon$) crosses the circle of radius r at two points only: $\overline{\alpha}^\epsilon(r)$ and $\overline{\beta}^\epsilon(r)$ (resp. $\widetilde{\alpha}^\epsilon(r)$ and $\widetilde{\beta}^\epsilon(r)$). Moreover,

$$\lim_{r\to 0} \frac{\overline{\alpha}^0(r)}{r} = (\cos\overline{\omega}_u, \sin\overline{\omega}_u); \quad \lim_{r\to 0} \frac{\overline{\beta}^0(r)}{r} = (\cos\overline{\omega}_s, \sin\overline{\omega}_s)$$

and for all $r \in (0, r_2)$, from (3.1),

$$\lim_{\epsilon\to 0} \overline{\alpha}^\epsilon(r) = \overline{\alpha}^0(r); \quad \lim_{\epsilon\to 0} \overline{\beta}^\epsilon(r) = \overline{\beta}^0(r). \tag{3.2}$$

We have similar properties for $\widetilde{q}, \widetilde{q}_\epsilon$. By $(A2)$, there are $0 < r_3 < r_2$ and $\delta > 0$ such that $r_3 < \min\{C_1/(24\Lambda), \rho_0/2\}$ and

$$(3.3) \qquad \alpha_2^0(r_3)^2 \geq 40\frac{\lambda_1}{\lambda_2}\Lambda r_3^3 + \delta r_3^2; \quad \beta_2^0(r_3)^2 \geq 40\frac{\lambda_1}{\lambda_2}\Lambda r_3^3 + \delta r_3^2,$$

$$(3.4) \qquad |\alpha_1^0(r_3)| \geq \delta r_3; \quad |\beta_1^0(r_3)| \geq \delta r_3$$

and

$$(3.5) \qquad \frac{|\alpha_2^0(r_3)||\beta_2^0(r_3)| + (15\Lambda r_3^3/4)}{|\alpha_1^0(r_3)||\beta_1^0(r_3)|} < 1 - \delta,$$

where α_i (resp. β_i) may represent either $\overline{\alpha}_i$ or $\widetilde{\alpha}_i$ (resp. $\overline{\beta}_i$ or $\widetilde{\beta}_i$).

Now the eigenvalues of the equilibrium 0 for (S_ϵ) are $\pm\lambda_{2,\epsilon} = \pm(\lambda - |\epsilon|)$ and $\pm\lambda_{1,\epsilon} = \pm(\lambda + |\epsilon|)$, hence $\lim_{\epsilon\to 0} \lambda_{1,\epsilon}/\lambda_{2,\epsilon} = 1$.

From (3.2), (3.3) and (3.4) it is easy to see that all the second members in conditions $(H2 - 3)$ (associated to (S_ϵ)) tend to 1 as $\epsilon \to 0$. Therefore (taking $r = r_3$), there is $\epsilon_1 > 0$ such that these conditions and condition (H4) are satisfied, by (3.5), (3.2), (3.3) and (3.4), for $0 < |\epsilon| < \epsilon_1$. By Theorem 1 there is chaos at the zero energy level for $0 < |\epsilon| < \epsilon_1$. The estimate on the topological entropy follows by the results of Section 5 since the distance between two consecutive bumps is of order $1/\epsilon$ (see also the proof of the relaxed Theorem 4). □

3.2. – Perturbation of an uncoupled system

Let us consider a perturbed system of the following form

$$(3.6) \qquad \begin{aligned} -\ddot{q}_1 + \lambda_1^2 q_1 &= W_1'(q_1) + \epsilon\psi(q)\dot{q}_2 \\ -\ddot{q}_2 + \lambda_2^2 q_2 &= W_2'(q_2) - \epsilon\psi(q)\dot{q}_1 \end{aligned}$$

with $(q_1, q_2) \in \mathbb{R}^2$. We assume that $W_i(0) = W_i'(0) = W_i''(0) = 0$ for $i = 1, 2$ and that $\psi(0) = 0$. (3.6) can be written as

$$(3.7) \qquad -\ddot{q} + \epsilon\psi(q)\mathcal{J}\dot{q} + Aq = \nabla W(q),$$

where $W(q_1, q_2) = W_1(q_1) + W_2(q_2)$.

For $\epsilon = 0$ system (3.6) splits into the direct product of two 1-dimensional systems.

For the sake of simplicity we shall suppose that W_1 and W_2 are even, and that

$$\psi(q_1, -q_2) = \psi(-q_1, q_2) = \psi(q_1, q_2).$$

As a consequence, if $q = (q_1(t), q_2(t))$ is a homoclinic solution to (3.7), then $(q_1(-t), -q_2(-t))$ and $-q(t)$ are homoclinic solutions as well.

Suppose that:

$$-\ddot{q}_1 + \lambda_1^2 q_1 = W_1'(q_1) \tag{3.8}$$

possesses an homoclinic q_0. Up to a time translation, we may assume that q_0 is even. Thus, for $\epsilon = 0$, $\overline{q} = (q_0, 0)$ and $\widetilde{q} = (-q_0, 0)$ are two nondegenerate (up to time translation) homoclinic solutions of (3.6). We define

$$\Gamma = \int_{-\infty}^{+\infty} -\psi(q_0(s), 0)\dot{q}_0(s)e^{\lambda_2 s}\, ds.$$

Note that we have $|q_0(t)| + |\dot{q}_0(t)| \leq Ce^{-\lambda_1|t|}$ for some positive constant C, hence, by the properties of ψ, $|\psi(q_0, 0)\dot{q}_0(t)| \leq C'e^{-2\lambda_1|t|}$, so the integral Γ is well defined. As an application of Theorem 1 we get:

THEOREM 3. *If* $\Gamma \neq 0$, *then there is* $\epsilon_0 > 0$ *such that, for* $\epsilon \in (-\epsilon_0, 0) \cup (0, \epsilon_0)$, (3.6) *has a rich family of homoclinics and a chaotic behaviour at the zero energy level.*

Before proving this theorem we introduce h_2, defined by

$$-\ddot{h}_2 + \lambda_2^2 h_2 = -\psi(q_0, 0)\dot{q}_0 \qquad \text{and} \qquad \lim_{|t|\to+\infty} h_2(t) = 0.$$

Solving this equation we find:

$$\begin{aligned} h_2(t) &= \frac{1}{2\lambda_2}\left(\int_t^{+\infty} f(s)\exp(-\lambda_2 s)ds\right)e^{\lambda_2 t} \\ &\quad + \frac{1}{2\lambda_2}\left(\int_{-\infty}^{t} f(s)\exp(\lambda_2 s)ds\right)e^{-\lambda_2 t} \end{aligned} \tag{3.9}$$

where $f(s) = \psi(q_0(s), 0)\dot{q}_0(s)$ is an odd function. It is easy to see that $h_2(t) \sim \Gamma e^{-\lambda_2 t}/2\lambda_2$ as $t \to +\infty$ and that $h_2(t) \sim -\Gamma e^{\lambda_2 t}/2\lambda_2$ as $t \to -\infty$. We have

LEMMA 12. *There are* $\epsilon_1 > 0$ *and a non increasing function* $a(\epsilon)$ *which tends to* 0 *as* ϵ *tends to* 0 *such that, for all* $\epsilon \in (-\epsilon_1, \epsilon_1)$, (3.6) *has a homoclinic solution* $\overline{q}_\epsilon = (\overline{q}_{1,\epsilon}, \overline{q}_{2,\epsilon})$ *satisfying* $|\overline{q}_{1,\epsilon} - q_0| \leq a(\epsilon)\epsilon e^{-\lambda_2|t|}$, $|\overline{q}_{2,\epsilon} - \epsilon h_2| \leq a(\epsilon)\epsilon e^{-\lambda_2|t|}$, $\overline{q}_{1,\epsilon}(-t) = \overline{q}_{1,\epsilon}(t)$, $\overline{q}_{2,\epsilon}(-t) = -\overline{q}_{2,\epsilon}(t)$.

PROOF. This is a consequence of the Implicit function theorem. Define the Banach space

$$X' = \{q = (q_1, q_2) \in X \mid q_1(-t) = q_1(t)\ ;\ q_2(-t) = -q_2(t)\}.$$

X' is endowed with norm $||\cdot||$ defined by $||q||_1 = \max(|qe^{\lambda_2|t|}|_\infty, |\dot{q}e^{\lambda_2|t|}|_\infty/\lambda_2)$.

Let $F(q) = L_A(\nabla W(q))$ and $G(q) = L_A(\psi(q)\mathcal{J}\dot{q})$. It is easy to see, by the properties of ψ and W, that:

- F and G map X' into itself;
- $F, G : X' \to X'$ are smooth and $dF((q_0, 0)), dG((q_0, 0))$ are compact linear operators.

(3.7) is equivalent to

$$q = F(q) - \epsilon G(q). \tag{3.10}$$

Now, it is a standard fact that for $\epsilon = 0$ the linearization of (3.6) at $(q_0, 0)$ has no other homoclinic solution (v_1, v_2) that satisfies $v_1(-t) = v_1(t)$, $v_2(-t) = -v_2(t)$ than 0. Hence $\mathrm{Ker}(I - dF(q_0, 0)) = 0$ and, since $dF((q_0, 0))$ is compact, $I - dF((q_0, 0))$ is an isomorphism from X' to X'. Therefore we may apply the Implicit function theorem and we get, for all ϵ small enough in modulus, a solution $\overline{q}_\epsilon$ of (3.6). Moreover $(d/d\epsilon)_{\epsilon=0}(\overline{q}_\epsilon) = (0, h_2)$, which is the solution of the linear equation $(I - dF(q_0, 0))h = -G(q_0, 0)$. From this the estimates of the lemma follow. □

PROOF OF THEOREM 3. Without loss of generality we assume that $\Gamma < 0$ we perform the proof for $\epsilon > 0$. Then, by Lemma 12 there are ϵ_2 and $\hat{T} < 0$ such that, for $0 < \epsilon < \epsilon_2$, and $t \le \hat{T}$, $q_{2,\epsilon}(t) > 0$. We shall prove the following lemma:

LEMMA 13. *For all ω small enough there is $0 < \epsilon(\omega) < \epsilon_2$ such that, for all $\epsilon \in (0, \epsilon(\omega))$, there are $T_\epsilon \le \hat{T}$, r_ϵ such that $|\overline{q}_\epsilon(T_\epsilon)| = r_\epsilon$, $|\overline{q}_\epsilon(t)| < r_\epsilon$ for $t < T_\epsilon$, and $q_{2,\epsilon}(T_\epsilon)/q_{1,\epsilon}(T_\epsilon) = \tan\omega$. In addition $\lim_{\epsilon\to 0} T_\epsilon = -\infty$, $\lim_{\epsilon\to 0} r_\epsilon = 0$.*

PROOF. This is a consequence of Lemma 12. We may assume that ϵ_1 and ω are small enough such that $a(\epsilon_1)/(|\Gamma| - a(\epsilon_1)) < \cot\omega$. For $0 < \epsilon < \epsilon_1$ and $t \le \hat{T}$, define $f_\epsilon(t)$ by $f_\epsilon(t) = q_{1,\epsilon}(t)/q_{2,\epsilon}(t)$. This is a continuous function on $(-\infty, \hat{T}]$. Moreover, since $q_0(\hat{T}) > 0$, by Lemma 12 $\lim_{\epsilon\to 0} f_\epsilon(\hat{T}) = +\infty$. Hence there is $\epsilon(\omega) > 0$ such that, for $0 < \epsilon < \epsilon(\omega)$, $f_\epsilon(\hat{T}) > \cot\omega$. Now, $|q_0(t)| = O(e^{\lambda_1 t})$ as $t \to -\infty$. Hence, for all $0 < \epsilon < \epsilon(\omega)$, $\limsup_{t\to-\infty} |f_\epsilon(t)| \le a(\epsilon)/(|\Gamma| - a(\epsilon)) \le a(\epsilon_1)/(|\Gamma| - a(\epsilon_1)) < \cot\omega$. Hence $\{t \le \hat{T} \mid f_\epsilon(t) = \cot\omega\}$ is not empty and bounded. Let T_ϵ be its smaller element and set $r_\epsilon = |\overline{q}_\epsilon(T_\epsilon)|$. Since for all fixed $t \le \hat{T}$ $\lim_{\epsilon\to 0} f_\epsilon(t) = 0$, we must have $\lim_{\epsilon\to 0} T_\epsilon = -\infty$ and, by Lemma 12, $\lim_{\epsilon\to 0} r_\epsilon = 0$. It follows that, provided ϵ is small enough, $|\overline{q}_\epsilon|$ is strictly nondecreasing on $(-\infty, T_\epsilon]$, which yields our claim. □

Now, by the properties of W_1, W_2 and ψ, we have two homoclinic solutions to (3.6): $\overline{q}_\epsilon$ and $\widetilde{q}_\epsilon := -\overline{q}_\epsilon$. It remains to check, using the previous lemma, that for ϵ small enough, conditions $(H1 - 4)$ are satisfied. Let $\omega > 0$ be small and fixed. For $0 < \epsilon < \epsilon(\omega)$, let $r_\epsilon(\omega)$ be associated to ω. Note that $q_{1,\epsilon}(-t) = q_{1,\epsilon}(t)$ and $q_{2,\epsilon}(-t) = -q_{2,\epsilon}(t)$. Hence $\overline{q}_\epsilon$ crosses the circle of center 0 and radius $r_\epsilon(\omega)$ for the first time and the last time respectively at $\overline{\alpha} = (r_\epsilon(\omega)\cos\omega, r_\epsilon(\omega)\sin\omega)$ and $\overline{\beta} = (r_\epsilon(\omega)\cos\omega, -r_\epsilon(\omega)\sin\omega)$. For $\widetilde{q}_\epsilon$ we have $\widetilde{\alpha} = (-r_\epsilon(\omega)\cos\omega, -r_\epsilon(\omega)\sin\omega)$ and $\widetilde{\beta} = (-r_\epsilon(\omega)\cos\omega, r_\epsilon(\omega)\sin\omega)$. So it is clear that $(H1)$ is satisfied (with the notations of Section 2, $\overline{\omega}_u = \omega, \overline{\omega}_s = -\omega, \widetilde{\omega}_u = \omega + \pi, \widetilde{\omega}_s = -\omega + \pi$).

We have

$$\mathcal{Q} := \frac{\lambda_2^2}{\lambda_1^2}\frac{|\alpha_2||\beta_2| + 15(\lambda_1/4\lambda_2)\Lambda r^3}{|\alpha_1||\beta_1|} = \frac{\lambda_2^2}{\lambda_1^2}\left((\tan\omega)^2 + \frac{15\lambda_1}{4\lambda_2}\frac{\Lambda r_\epsilon(\omega)}{(\cos\omega)^2}\right).$$

Now, since, when $\epsilon = 0$, $(q_0, 0)$ is a nondegenerate (up to time translations) homoclinic orbit, for ϵ small enough $\overline{q}_\epsilon$ and $\widetilde{q}_\epsilon$ are (up to time translations) nondegenerate uniformly with respect to ϵ, and we can take for them a constant of nondegeneracy $C_1(\epsilon)$ which is bounded from below by a positive constant independent of ϵ. It follows that in condition $(H2)$ the second member is bounded from below by some constant independent of ϵ. Using $\lim_{\epsilon\to 0} r_\epsilon(\omega) = 0$, we can derive that, provided ω is smaller than some $\omega_0 > 0$, $(H2)$ is satisfied if $0 < \epsilon < \epsilon'(\omega)$.

Since $\lim_{\epsilon\to 0} r_\epsilon(\omega) = 0$, it is clear that $(H4)$ is satisfied, provided $0 < \epsilon < \epsilon''(\omega)$, with $\epsilon''(\omega) \in (0, \epsilon'(\omega))$.

The quantity which must be greater than $\mathcal{Q}$ in $(H3)$ is

$$\begin{aligned}\mathcal{B} := l(\frac{\lambda_1}{\lambda_2})\left(\frac{C_1(\epsilon)\mathcal{M}}{36S_2 + 28\Lambda r_\epsilon(\omega)^2}\right)^{\frac{\lambda_1}{\lambda_2}-1} &= l(\frac{\lambda_1}{\lambda_2})\left(\frac{C_1(\epsilon)\sin\omega}{36\sin\omega + 28\Lambda r_\epsilon(\omega)}\right)^{\frac{\lambda_1}{\lambda_2}-1} \\ &\geq l(\frac{\lambda_1}{\lambda_2})\left(\frac{C_1(\epsilon)}{36 + (7\lambda_2/5\lambda_1)}\right)^{\frac{\lambda_1}{\lambda_2}-1}\end{aligned}$$

for $0 < \epsilon < \epsilon''(\omega)$ and by $(H4)$. So, if $0 < \epsilon < \epsilon''(\omega)$, $\mathcal{B}$ is bounded from below by a positive constant independent of ϵ and ω. By the expresion of $\mathcal{Q}$ it is clear that there is $\omega > 0$ and $\epsilon_0 \in (0, \epsilon''(\omega))$ such that $\mathcal{Q} < \mathcal{B}$ (that is condition $(H3)$ is satisfied) if $0 < \epsilon < \epsilon_0$. The case with $\epsilon < 0$ can be dealt with in the same way. That completes the proof of Theorem 3.

4. – Relaxing the nondegeneracy assumption

The aim of this section is to modify the arguments of the previous section in order to show how to deal also with homoclinics which are degenerate. For simplicity, we shall restrict ourselves to the proof of the "relaxed" Theorem 2, that is Theorem 4.

4.1. – Finite dimensional reduction for degenerate homoclinic orbits

We consider a homoclinic solution q of (1.1) not necessarily nondegenerate. We assume that $|q(-T)| = |q(T)| = r$ and that $|q(t)| < r$ for all $|t| > T$.

Let $a^0 = L_A(c\dot{q}\chi_{[-T,T]})$, $c > 0$ being chosen such that $|a^0|_E = 1$. Let a^j be defined for $1 \leq j \leq p$ and satisfy $a^j = L_A e^j$, with $e^j \in L^2(\mathbb{R})$, $e^j_{\mathbb{R}\setminus[-T,T]} \equiv 0$. Moreover we shall assume that $(a^i, a^j) = \delta_{ij}$. Let $F = \langle a^0, \dots, a^p \rangle^{\perp}$ and Π be the orthogonal projection on F defined by

$$\Pi(x) = x - \sum_{j=0}^{p} (x, a^j) a^j.$$

We shall assume that there is a constant C_0' such that

$$\forall x \in F \quad ||\Pi dS(q)x|| \geq C_0' ||x||.$$

Note that Ker $dS(q)$ is finite dimensional, so one can always define a^j enjoying the above properties. For example, we can choose $(f^j)_{1 \leq j \leq p}$ such that Ker $dS(q) \subset \text{span}\{\dot{q}, f^1, \dots, f^p\}$ and set $a^j = L_A(f^j \chi_{[-T,T]})$. Moreover $(a^i, a^j) = \delta_{ij}$ for a suitable choice of $f^1, \dots, f^p$.

A simple application of the Implicit function theorem leads to the following

LEMMA 14. *There is* $\delta_0 > 0$ *and a smooth function* $w : (-\delta_0, \delta_0)^p \to Y$ *such that:*

- $w(l^1, \dots, l^p) \in F$;
- $S(q + \sum_{m=1}^{p} l^m a^m + w(l^1, \dots, l^p)) = \sum_{m=0}^{p} \alpha^m (l^1, \dots, l^p) a^m$.

Moreover $\alpha^1(l) = \dots = \alpha^p(l) = 0$ *implies* $\alpha^0(l) = 0$.

The last assertion is an easy consequence of the autonomy of the system. We shall set, for $l = (l^1, \dots, l^p)$,

$$q(\theta, l) = q_\theta + \sum_{m=1}^{p} l^m a_\theta^m + w(l)_\theta.$$

REMARK 3. By the definition of $q(\theta, l)$ and the autonomy of the system, we have $q(\theta, l) = q(0, l)_\theta$ and $S(q(\theta, l)) = \sum_{m=0}^{p} \alpha^m(l) a_\theta^m$. Moreover we have the estimate

$$||q(\theta, l) - q_\theta|| \leq \overline{C} \max_m |l^m| \leq \overline{C} \delta_0, \tag{4.1}$$

where $\overline{C}$ depends on C_0'. We can derive that $q(\theta, l)$, by the equation it satisfies, belongs to X (provided δ_0 has been chosen small enough, namely $\overline{C}\delta_0 < \min\{\rho_0, 2/\Lambda\}$).

Set $\mathcal{G}(l) = f(q(0, l)) = f(q(\theta, l))$. By the properties of $w(l)$ we can easily prove the following lemma

LEMMA 15. $(\partial \mathcal{G}/\partial l^m)(l) = \alpha^m(l)$. *As a consequence* $\mathcal{G}'(l) = 0$ *iff* $q(\theta, l)$ *is (for all* θ*) a homoclinic solution of* (1.1).

REMARK 4. Reciprocally, there is a neighborhood U of q in Y such that all the homoclinic solutions in U correspond to critical points of $\mathcal{G}$.

As in Section 2 the following estimate holds (provided R is large enough): there is $C_1' > 0$ such that for all $x \in F$

$$\max\left(\left\|dS(q)x - \sum_{m=0}^{p} \eta^m a^m\right\|, R|(a^0, x)|, \dots, R|(a^p, x)|\right) \geq C_1' \max(||x||, |\eta^0|, \dots, |\eta^p|). \tag{4.2}$$

Now assume that we have two distinct homoclinic solutions $\overline{q}$ and $\widetilde{q}$, and finite dimensional spaces $\widetilde{\overline{F}} = \langle \widetilde{\overline{a}}^0, \dots, \widetilde{\overline{a}}^p \rangle^\perp$, with the above properties.

Then, provided δ_0 is small enough, for $j = (j_1, \dots, j_k) \in \{0, 1\}^k$, for $\Theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$, $L = (l_1, \dots, l_k) \in ((-\delta_0, \delta_0)^p)^k$, with $\overline{d} > 2/\lambda_2$, we can build $Q(\Theta, L)$ in the same way as we built Q_Θ in Subsection 2.2, just substituting $q(\theta_i, l_i)$ to q_{θ_i} in the construction. Note that we keep the same T_i satisfying $|\widetilde{\overline{q}}(T_i)| = r$ so that in this case our boundary value problems may connect two points with different norms (however we know that these norms are $\leq r + \overline{C}\delta_0$).

In the spirit of Lemma 5, one can get

$$\left\| S(Q(\Theta, L)) - \sum_{i=1}^{k} \sum_{m=1}^{p} \alpha^m(l_i) a_i^m \right\| \leq K(r + |L|)e^{-\lambda_2 \overline{d}}. \tag{4.3}$$

Here is the equivalent of Lemma 6. It is not so specific, but it is enough to prove the equivalent of Theorem 2. We shall use the notations $a_i^m = \overline{a}_{\theta_i}^m$ if $j_i = 0$, $a_i^m = \widetilde{a}_{\theta_i}^m$ if $j_i = 1$.

LEMMA 16. *There are $\overline{D}_1$ and $\delta_1 > 0$ which depend on C_1', Λ, $\overline{\Lambda}$ and there exist $\overline{w}$, function of $\Theta = (\theta_1, \dots, \theta_k)$ and of $L = (l_1, \dots, l_k)$, defined for $\overline{d} \geq \overline{D}_1$ and $l_i \in (-\delta_1, \delta_1)^p$, such that $(\Theta, L) \mapsto Q(\Theta, L) + \overline{w}(\Theta, L)$ is smooth and*

- $\overline{w}(\Theta, L) \in \cap_{i=1}^k \langle a_{\theta_i}^0, \dots, a_{\theta_i}^p \rangle^\perp$;
- $S(Q(\Theta, L) + \overline{w}(\Theta, L)) = \sum_{i=1}^k \sum_{m=0}^p \alpha_i^m(\Theta, L) a_{\theta_i}^m$.

Moreover

$$||\overline{w}|| \leq K_1(r + |L|)e^{-\lambda_2 \overline{d}},$$

where $|L| = \max_{i,m} |l_i^m|$ and K_1 depends only on C_1' and $\overline{\Lambda}, \Lambda$.

The proof can be carried out in the same way as in the nondegenerate case. Set

$$g(\Theta, L) = f(Q(\Theta, L) + \overline{w}(\Theta, L)).$$

We have, by Lemmas 1 and 16

LEMMA 17. *Every critical point of g gives rise to a k-bump homoclinic solution to the system, provided δ_1 and r have been chosen small enough.*

Here again the proof does not differ from the one given in the nondegenerate case.

Finally, using the notations $\overline{\mathcal{G}}(l) = f(\overline{q}(0,l))$ and $\widetilde{\mathcal{G}}(l) = f(\widetilde{q}(0,l))$, we get:

LEMMA 18. *For all* $\Theta = (\theta_1, \dots, \theta_k)$ *with* $\overline{d} > \overline{D}_1$ *and* L *with* $|L| \le \delta_1$

$$(i)\ \frac{\partial}{\partial d_i} g(\Theta, L) = \frac{\partial}{\partial d_i} e\Big((Q(\Theta,L)+\overline{w}(\Theta,L))(s_i), (Q(\Theta,L)+\overline{w}(\Theta,L))(u_{i+1}), d_i\Big)$$

$$(ii)\ \left|\frac{\partial g}{\partial l_i^m} - \partial_{l^m}\mathcal{G}_i(l_i^1, \dots, l_i^p)\right| \le K_2(r+|L|)e^{-\lambda_2\overline{d}},$$

where K_2 *depends only on* K_1 *and* Λ, $\overline{\Lambda}$ *and* $\mathcal{G}_i = \overline{\mathcal{G}}$ *if* $j_i = 0$, $\mathcal{G}_i = \widetilde{\mathcal{G}}$ *if* $j_i = 1$.

PROOF. (i) is proved exactly in the same way as Lemma 9. For (ii), write

$$\frac{\partial}{\partial l_i^m} g = \left(S(Q(\Theta,L)+\overline{w}(\Theta,L)), \frac{\partial Q(\Theta,L)}{\partial l_i^m} + \frac{\partial \overline{w}(\Theta,L)}{\partial l_i^m}\right).$$

We have by Lemma 16 $S(Q(\Theta,L)+\overline{w}(\Theta,L)) = \sum_{i,m}\alpha_i^m(\Theta,L)a_i^m$. Since $\overline{w}(\Theta,L) \in \bigcap_{1\le i\le k}\bigcap_{0\le m\le p}\langle a_i^m\rangle^\perp$, $\partial\overline{w}/\partial l_i^m(\Theta,L)$ belongs to the same space. Hence, since supp $\partial Q(\Theta,L)/\partial l_i^m \subset [\theta_{i-1}+T_{i-1}, \theta_{i+1}-T_{i+1}]$ and supp $(-\ddot{a}_i^m + Aa_i^m) \subset [\theta_i - T_i, \theta_i + T_i]$, $(\partial g)(\partial l_i^m) =$

$$\left(\sum_{n,q}\alpha_n^q(\Theta,L)a_n^q, \frac{\partial}{\partial l_i^m}Q(\Theta,L)\right) = \sum_{q=0}^{p}\alpha_i^q(\Theta,L)\left(a_i^q, \frac{\partial}{\partial l_i^m}q^i(\theta_i,l_i)\right)$$
$$= \sum_{q=0}^{p}\alpha_i^q(\Theta,L)(a_i^q, a_i^m) = \alpha_i^m(\Theta,L).$$

We have used there the definition of $q(\theta,l)$, $\partial q(\theta_i,l_i)/\partial l_i^m = a_i^m + (\partial_{l^m} w_i(l_i))_{\theta_i}$ and $(\partial_{l^i} w_i(l_i))_{\theta_i} \in \langle a_i^0, \dots, a_i^p\rangle^\perp$, where $w_i = \overline{w}$ (resp. $w_i = \widetilde{w}$) if $j_i = 0$ (resp. $j_i = 1$). We get

$$\begin{aligned}\frac{\partial}{\partial l_i^m} g &= (S(Q(\Theta,L)) + \overline{w}(\Theta,L)), a_i^m)\\ &= (S(Q(\Theta,L), a_i^m) + O((r+|L|)e^{-\lambda_2\overline{d}})\\ &= \alpha_i^m(l_i) + O((r+|L|)e^{-\lambda_2\overline{d}})\\ &= \partial_{l^m}\mathcal{G}_i(l_i) + O((r+|L|)e^{-\lambda_2\overline{d}}),\end{aligned}$$

where we have used Lemma 16 and (4.3) in the second and the third line respectively. This is exactly (ii). □

Now we state a corollary of Lemma 18 (i) which is got from a simplified version of Lemma 10.

COROLLARY 1. *For all* (Θ, L) *with* $\overline{d} > \overline{D}_1$ *and* $|L| \le \delta_1$

$$\left|\frac{\partial}{\partial d_i} g(\Theta, L) - 2\Big(\lambda_1^2 \alpha_1^{i+1} \beta_1^i e^{-\lambda_1 d_i} + \lambda_2^2 \alpha_2^{i+1} \beta_2^i e^{-\lambda_2 d_i}\Big)\right| \le K_4 e^{-\lambda_2 d_i}\Big((r+|L|)^2 e^{-\lambda_2 \overline{d}} + (r+|L|)^3 + r|L| + |L|^2\Big), \tag{4.4}$$

where K_4 *depends only on* C_1' *and* Λ, $\overline{\Lambda}$.

PROOF. We omit the details of the proof. It is a simple consequence of Lemmas 16, 18 and 10. □

4.2. – Topological nondegeneracy

Let q be a (possibly degenerate) *isolated* homoclinic solution of (1.1). "Isolated" means here that there is a neighborhood U of q in Y such that all the homoclinics which belong to U are translates of q. Let $a = L_A(c\dot{q}\chi_{[-T,T]}) \in X$, where $c > 0$ is chosen so that $|a|_E = 1$. Let $\hat{F} = a^{\perp}$ and $\hat{\Pi} : Y \to \hat{F}$ be the projection defined by

$$\hat{\Pi}(x) = x - (x, a)a.$$

Consider $G_* : \hat{F} \to \hat{F}$, defined by

$$G_*(x) = \hat{\Pi} S(q+x) = \hat{\Pi}\Big[(q+x) - L_A\Big(\nabla W(q+x) - \psi(q+x)\mathcal{J}(\dot{q}+\dot{x})\Big)\Big] = \hat{\Pi}(S(x) - K_q(x)),$$

where

$$K_q(x) = L_A\Big[\Big(\nabla W(q+x) - \nabla W(q) - \nabla W(x)\Big) - \Big((\psi(q+x) - \psi(q))\mathcal{J}\dot{q} + (\psi(q+x) - \psi(x))\mathcal{J}\dot{x}\Big)\Big].$$

We have $K_q(0) = 0$. In addition, it is easy to see that K_q sends Y into E and that it is compact.

Note also that there is $\rho > 0$ such that $\hat{\Pi} \circ S : \hat{F} \to \hat{F}$ is a diffeomorphism from $\hat{B}(\rho)$ onto a neighborhood of 0 in $\hat{F}$ containing $\hat{B}(\rho/2)$. Let $\delta > 0$ satisfy $\hat{\Pi} K_q(\hat{B}(\delta)) \subset \hat{B}(\delta/2)$ and let $\hat{G} : \hat{B}(\delta) \to \hat{F}$ be defined by

$$\hat{G}(x) = x - (\hat{\Pi} \circ S)^{-1} \hat{\Pi} K_q(x) := x - \hat{K}_q(x). \tag{4.5}$$

We have $\hat{K}_q(0) = 0$, and $\hat{K}_q(Y) \subset E$. Hence all the zeros of $\hat{G}$ must belong to E and thus be homoclinic solutions to the system. Now q being an isolated homoclinic, 0 is an isolated zero of $\hat{G}$.

Moreover $\hat{K}_q$ is a compact operator. We can now introduce the following definition:

Definition 2. We shall say that q is a "topologically nondegenerate" homoclinic if there is $0 < \nu \leq \delta$ such that $\deg(\hat{G}, \hat{B}(\nu), 0) \neq 0$ and $\hat{G}$ has no zero in $\overline{\hat{B}(\nu)}\backslash\{0\}$.

Remark 5. We could prove without difficulty that this definition is independent of the choice of a satisfying $(a, \dot{q}) \neq 0$.

As a consequence to the hyperbolicity of the equilibrium, the solutions of the linearised system about q which belong to Y must belong to X. Therefore, if q is a nondegenerate homoclinic, $dG_*(0)$ and hence $d\hat{G}(0)$ are injective. By (4.5) $\hat{G}$ is then a local diffeomorphism about 0 and the property of "topological nondegeneracy" defined above is satisfied.

Remark 6. We point out that in certain cases, one can say that a variationally obtained isolated homoclinic is topologically nondegenerate. For instance, an isolated local minimum for f, or, under further conditions, an isolated mountain-pass critical point correspond to topologically nondegenerate homoclinics (see [2], [10] and [6]).

Now consider as in Subsection 4.1 $a^0, \ldots, a^p$ which satisfy the properties given in this subsection. We can then define the function $\mathcal{G}$ on some $(-\delta_0, \delta_0)^p$. We shall prove

Lemma 19. *Assume that the homoclinic q is isolated and topologically nondegenerate. Then 0 is an isolated critical point of $\mathcal{G}$. Moreover there is $\mu \in (0, \delta_0)$ such that* $\deg(\mathcal{G}', (-\mu, \mu)^p, 0) \neq 0$ *and $\mathcal{G}'$ has no zero in $(-\mu, \mu)^p$, except* 0.

Proof. By Remark 4 since q is an isolated homoclinic, 0 is an isolated critical point of $\mathcal{G}$ and there is μ_1 such that $\mathcal{G}'$ has no zero in $(-\mu_1, \mu_1)^p$ except 0.

Let $\hat{G}$ be the function defined above, associated to a_0. By the topological nondegeneracy property of q, there is some $\nu > 0$ such that $\deg(\hat{G}, \hat{B}(\rho), 0) \neq 0$ for all $0 < \rho < \nu$. Let $B_{\nu_2} = \{x \in F \mid ||x|| \leq \nu_2\}$. Consider for δ_2, ν_2 small enough $\varphi : (-\delta_2, \delta_2)^p \times B_{\nu_2} \to \hat{B}(\nu)$ which assigns to $(l, y) \to \varphi(l, y) = q(0, l) + y - q_0 = \sum_{j=1}^p l^j a^j + w(l) + y$. This is clearly a diffeomorphism from $(-\delta_2, \delta_2)^p \times B_{\nu_2}$ onto some neighborhood of 0 in $\hat{F}$ included in $\hat{B}(\nu)$. Now, let $\xi : (-\delta_2, \delta_2)^p \times B_{\nu_2} \to \hat{F}$ be defined by

$$\xi(l, y) = \hat{G}(\varphi(l, y)) = \varphi(l, y) + \hat{K}_q(\varphi(l, y)).$$

Since the degree is invariant under a diffeomorphism there results that

$$\deg(\xi, (-\delta_2, \delta_2)^p \times B_{\nu_2}, 0) = \deg(\hat{G}, \hat{B}(\nu), 0) \neq 0.$$

We decompose $G(\varphi(l, y)) = A(l, y) + \sum_{j=1}^p u^j(l, y) a^j$, where $(A(l, y), a^j) = 0$. For $t \in [0, 1]$, define ξ_t by

$$\xi_t(l, y) = (\Pi \circ S)^{-1} \left(A((1-t)l, y) + \sum_{j=1}^p u^j(l, (1-t)y) a^j \right).$$

ξ_t has the form $I - K_t$, where K_t is compact. Moreover, for all $l \in (-\delta_2, \delta_2)^p$, $A(l, y) = 0$ iff $y = 0$. Hence $\xi_t(l, y) = 0$ iff $y = 0$ and $u^j(l, 0) = 0$. Now $u^j(l, 0) = \alpha^j(l)$, and since q is an isolated homoclinic, it vanishes at no other point in $(-\delta_2, \delta_2)^p$ than 0. Therefore

$$\deg(\xi, (-\delta_2, \delta_2)^p \times B_{\nu_2}, 0) = \deg(\xi_1, (-\delta_2, \delta_2)^p \times B_{\nu_2}, 0).$$

Now, $A(0, \cdot)$ is a diffeomorphism from B_{ν_2} to a neighborhood of 0 in F. Let Ψ be defined on $\hat{B}(\nu_2)$ by

$$\Psi\left(y + \sum_{j=1}^{p} r_j a^j\right) = A(0, y) + \sum_{j=1}^{p} r_j a^j.$$

Since Ψ is a diffeomorphism from $B(\nu_2)$ to a neighborhood of 0 in $\hat{F}$ and we have

$$\xi_1(l, y) = (\Pi \circ S)^{-1}\left(\Psi\left(y + \sum_{j=1}^{p} \alpha^j(l) a^j\right)\right),$$

with $(\Pi \circ S)^{-1}(\Psi(0)) = 0$. Hence, setting $\tau(l, y) = \sum_{j=1}^{p} \alpha^j(l) a^j + y$ we get

$$\begin{aligned} |\deg(\xi_1, (-\delta_2, \delta_2)^p \times B_{\nu_2}, 0)| &= |\deg(\tau, (-\delta_2, \delta_2)^p \times B_{\nu_2}, 0)| \\ &= |\deg((\alpha^1, \dots, \alpha^p), (-\delta_2, \delta_2)^p, 0)|. \end{aligned}$$

Using Lemma 15, we get the result with $\mu = \min\{\delta_2, \mu_1\}$. □

4.3. – Relaxed theorem for system (S_ϵ)

We shall prove

THEOREM 4. *Assume that the hypotheses of Theorem 2 hold, with the nondegeneracy condition replaced by topological nondegeneracy. Then the same conclusion holds.*

PROOF. $\overline{q}$ and $\widetilde{q}$ may be degenerate. Since the configuration space is $\mathbb{R}^2$, $W^u(0)$ and $W^s(0)$ are 2-dimensional. Therefore dim Ker $dS(\widetilde{\overline{q}})$ cannot exceed 2. Hence in the construction of Subsection 4.1 the spaces associated to $\overline{q}$ and $\widetilde{q}$, span $\{\widetilde{\overline{a}}^0, \widetilde{\overline{a}}^1\}$, are two dimensional.

The topological nondegeneracy of $\overline{q}$ and $\widetilde{q}$ implies that, for ϵ small enough, (S_ϵ) has two homoclinics $\overline{q}_\epsilon$ and $\widetilde{q}_\epsilon$ such that

$$\lim_{\epsilon \to 0} \max\{|\overline{q}_\epsilon - \overline{q}|_\infty, |\widetilde{q}_\epsilon - \widetilde{q}|_\infty\} = 0. \tag{4.6}$$

Moreover, there is a constant C_2' independent of ϵ small such that $||dS_\epsilon(\widetilde{\overline{q}_\epsilon})x|| \geq C_2'||x||$ for $(x, \widetilde{\overline{a}}^1) = (x, \widetilde{\overline{a}}^2) = 0$. As a consequence we may define $\widetilde{\overline{q}}_\epsilon(\theta, l)$ for

$|l| \le \mu_0$. Writing $\widetilde{\overline{\mathcal{G}}}_\epsilon(l) = f_\epsilon(\widetilde{\overline{q}}_\epsilon(\theta, l))$, we know that $|\mathcal{G}'_\epsilon - \mathcal{G}'|_{L^\infty(-\mu_0,\mu_0)} \to 0$ as $\epsilon \to 0$. Note that it may occur that $\mathcal{G}_\epsilon$ has a sequence of critical points converging to 0. So we cannot say that $\overline{q}_\epsilon, \widetilde{q}_\epsilon$ are isolated homoclinic. However we know that, for all $|\mu| \le \mu_0$ there is $\overline{\epsilon}$ such that, for $|\epsilon| < \overline{\epsilon}$, all the critical points of $\mathcal{G}_\epsilon$ belong to $(-\mu, \mu)$ and $\deg(\mathcal{G}'_\epsilon, (-\mu, \mu), 0) \neq 0$.

Given k and $j = (j_1, \dots, j_k) \in \{0, 1\}^k$, we can construct $Q_\epsilon(\Theta, L)$ as well as $\overline{w}_\epsilon(\Theta, L)$ for all $L = (l_1, \dots, l_k) \in (-\mu_0, \mu_0)^k$ and for all $\Theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k$ with $\overline{d} \ge \overline{D}_1$. We can as well define $g_\epsilon(\Theta, L)$.

By the properties of the system near the equilibrium, there are $r_4 > 0$ and $\epsilon_2 > 0$ such that, for $0 < r < r_4$ and $|\epsilon| < \epsilon_2$, the trajectory of $\overline{q}_\epsilon$ (resp. $\widetilde{q}_\epsilon$) crosses the circle of radius r at two points only: $\overline{\alpha}_\epsilon(r)$ and $\overline{\beta}_\epsilon(r)$ (resp. $\widetilde{\alpha}_\epsilon(r)$ and $\widetilde{\beta}_\epsilon(r)$). Moreover,

$$\lim_{r\to 0} \frac{\overline{\alpha}_0(r)}{r} = (\cos\overline{\omega}_u, \sin\overline{\omega}_u); \quad \lim_{r\to 0} \frac{\overline{\beta}_0(r)}{r} = (\cos\overline{\omega}_s, \sin\overline{\omega}_s)$$

and for all $r \in (0, r_4)$, from (4.6),

$$\lim_{\epsilon\to 0} \overline{\alpha}_\epsilon(r) = \overline{\alpha}_0(r); \quad \lim_{\epsilon\to 0} \overline{\beta}_\epsilon(r) = \overline{\beta}_0(r). \tag{4.7}$$

We have similar properties for $\widetilde{q}, \widetilde{q}_\epsilon$. Let us define $P_1^i = \cos\omega_u^i \cos\omega_s^{i+1}$, $P_2^i = \sin\omega_u^i \sin\omega_s^{i+1}$. Lemma 18 and Corollary 1 hold (with g replaced by g_ϵ), and α, β replaced by $\alpha_\epsilon, \beta_\epsilon$ we have

$$\begin{aligned} &\left| \frac{\partial}{\partial d_i} g_\epsilon - 2r^2\left(\lambda_1^2 P_{1,\epsilon}^i e^{-\lambda_1 d_i} + \lambda_2^2 P_{2,\epsilon}^i e^{-\lambda_2 d_i}\right)\right| \\ &\le K_5(r^3 + r\nu + \nu^2 + r^2 e^{-\lambda_2 \overline{d}})e^{-\lambda_2 d_i}, \end{aligned} \tag{4.8}$$

where K_5 is independent of ϵ small and $|P_{1,\epsilon}^i - P_1^i|, |P_{2,\epsilon}^i - P_2^i| \le r(\epsilon)$ $r(\epsilon) \to 0$ as $\epsilon \to 0$. We recall that $\lambda_1 = \lambda + |\epsilon|$ and $\lambda_2 = \lambda - |\epsilon|$. We know that $P_1^i P_2^i < 0$ and that there exists $\delta > 0$ such that, for all i,

$$\frac{|P_1^i|}{|P_2^i|} \ge 1 + \delta > 1 \quad , \quad |P_1^i|, |P_2^i| \ge \delta > 0.$$

First choose $0 < r < r_5$ and $\nu_1 > 0$ such that

$$\frac{|P_1^i|}{|P_2^i| + K_5(r + \nu_1/r + \nu_1^2/r^2)} \ge 1+\delta/2 \quad , \quad |P_{1,2}^i - K_5(r+\nu_1/r+\nu_1^2/r^2)| \ge \delta/2.$$

Then set

$$D = \frac{1}{2\epsilon} \min_i \ln\left(\frac{|P_1^i|}{|P_2^i| + K_5(r + \nu_1/r + \nu_1^2/r^2)}\right) - 1 \ ;$$

$$J = \frac{1}{2\epsilon} \max_i \ln\left(\frac{|P_1^i|}{|P_2^i| - K_5(r + \nu_1/r + \nu_1^2/r^2)}\right) + 1.$$

By (4.8) it is easy to see that $\lim_{\epsilon\to 0} D = +\infty$ and that there is $\epsilon_3 > 0$ such that for all $|\epsilon| < \epsilon_3$, for all $(d_1, \dots, d_k) \in (D, J)^k$, for all $L = (l_1, \dots, l_k) \in (-\nu_1, \nu_1)^k$,

$$d_i = D \Rightarrow \text{sign}\Big(\frac{\partial}{\partial d_i} g_\epsilon\Big) = -\text{sign } P_2^i \quad , \quad d_i = J \Rightarrow \text{sign}\Big(\frac{\partial}{\partial d_i} g_\epsilon\Big) = \text{sign } P_2^i$$

and

$$|L| = \nu_1 \to |\partial_{l^m} g_\epsilon - \partial_{l^m} \mathcal{G}_m| < |\partial_{l^m} \mathcal{G}_m|/2.$$

Now arguing as in the proof of Theorem 2 and using that $\deg(\partial_{l^m}\mathcal{G}_m, (-\nu_1, \nu_1), 0) \neq 0$, we get

$$\deg(dg_\epsilon, (D, J)^k \times (-\nu_1, \nu_1)^k, 0) \neq 0,$$

which implies the desired result. □

5. – Dynamical consequences

A family of multibump solutions like the ones of Theorem 1 ensures the positivity of the topological entropy at the zero energy level $\mathcal{E}^{-1}(0)$, see also [6] and [12]. We denote by $\Phi(t, x) \in \mathbb{R}^4$ with $x = (q, \dot{q})$ the flow associated to (1.1). The definition of the topological entropy is the following:

$$h_{\text{top}} = \sup_{R,e>0} \left(\limsup_{t\to+\infty} \frac{\log s(t, e, R)}{t} \right)$$

where

$$s(t, e, R) = \max\{\text{Card}(\widetilde{E}) \mid \forall \tau \in [0, t] \ : \ \Phi(\tau, \widetilde{E}) \subset B_4(0, R),$$
$$\forall x \neq y \in \widetilde{E}, \exists \tau \in [0, t] \ : \ |\Phi(\tau, x) - \Phi(\tau, y)| \geq e > 0\}.$$

We formulate the following corollary of Theorem 1.

THEOREM 5. *Assume* $(W1)$,$(P1)$,$(\nu 1)$,$(S1-2)$ *and* $(H1-4)$. *There exist* $0 < D < J$ *such that for every sequence* $j \in \{0, 1\}^{\mathbb{Z}}$ *there is* $\Theta \in \mathbb{R}^{\mathbb{Z}}$ *with* $d_i \in (D, J)$ *and a solution* x_j *of system* (1.1) *such that*

- *if* $j_i = 0$ *then on the interval* $[\theta_i - \overline{T}, \theta_i + \overline{T}]$
$$|x_j(t) - \overline{q}(t - \theta_i)| \leq \frac{r}{8} \min(|\cos\overline{\omega}_{u,s}|, |\cos\widetilde{\omega}_{u,s}|, |\sin\overline{\omega}_{u,s}|, |\sin\widetilde{\omega}_{u,s}|),$$
- *if* $j_i = 1$ *then on the interval* $[\theta_i - \widetilde{T}, \theta_i + \widetilde{T}]$
$$|x_j(t) - \widetilde{q}(t - \theta_i)| \leq \frac{r}{8} \min(|\cos\overline{\omega}_{u,s}|, |\cos\widetilde{\omega}_{u,s}|, |\sin\overline{\omega}_{u,s}|, |\sin\widetilde{\omega}_{u,s}|),$$
- *Outside* $(\cup_{j_i=0}[\theta_i - \overline{T}, \theta_i + \overline{T}]) \cup (\cup_{j_i=1}[\theta_i - \widetilde{T}, \theta_i + \widetilde{T}])$, $|x_i(t)| \leq 2r$.

We want to estimate $s(t_k^*, e, R^*)$ with $R^* = \max\{|(\overline{q}, \dot{\overline{q}})|_\infty, |(\widetilde{q}, \dot{\widetilde{q}})|_\infty\} + r$, $t^* = (k-1)(2\max\{\overline{T}, \widetilde{T}\} + J)$. Let $A = \{j \in \{0,1\}^{\mathbb{Z}} \mid j_i = 0$ for $i < 0$ and $j_i = 0$ for $i \geq k\}$. Associate to $j \in A$ a solution x_j given by Theorem 5, for which we may assume, by the autonomy of the system, that $|x_j(0)| = r$ and $|x_j(t)| < r$, $\forall t < 0$. Consider $\widetilde{E} = \{(x_j, \dot{x}_j)(0) \mid j \in A\}$.

As an easy consequence of Theorem 5 and Hypotheses $(H1-4)$, there is $e > 0$ such that, if $j \neq j' \in A$, then there is $\tau \in [0, t_k^*]$ such that $|\Phi(\tau, x_j(0), \dot{x}_j(0)) - \Phi(\tau, x_{j'}(0), \dot{x}_{j'}(0))| \geq e$.

Hence $s(t_k^*, e, R^*) \geq \text{Card}\widetilde{E} = 2^k$. Since $1/t^* > 1/(k-1)(2\max\{\widetilde{T}, \overline{T}\} + J)$ we finally deduce that

$$h_{\text{top}}^0 > \frac{\log 2}{2\max\{\widetilde{T}, \overline{T}\} + J}.$$

REMARK 7. The above results could be generalized to systems like (1.1) with $q \in \mathbb{R}^n$, where 0 is a hyperbolic equilibrium of characteristic exponents $\pm\lambda_i$, with $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n$. The finite dimensional reduction could be performed in the same way and we would have to impose conditions similar to $(H1-4)$ in order to be able to get multibump homoclinic solutions.

6. – Appendix

We shall assume everywhere that $d \geq 2/\lambda_2$ and that $0 < r < \rho_0/2$.

PROOF OF LEMMA 2. We perform the proof for q_h^+.

The proof of the uniqueness assertion is left to the reader (in fact there is uniqueness also in the class of the functions $[0, +\infty) \to B_{2r}$.

For the existence proof let us define the Banach space:

$$Z_1 = \left\{g \in W^{1,\infty}[0, +\infty) \mid |g(t)|e^{\lambda_2 t}, |\dot{g}(t)|e^{\lambda_2 t} \in L^\infty\right\},$$

endowed with the norm

$$||g||_1 = \max\left\{\sup_{t\in[0,+\infty)} |g(t)|e^{\lambda_2 t}, \sup_{t\in[0,+\infty)} \frac{1}{\lambda_2}|\dot{g}(t)|e^{\lambda_2 t}\right\}.$$

We call $q_l = q_{h,L}^+ = e^{-t\sqrt{A}}\beta$. Our problem is equivalent to finding a fixed point in Z_1 of

$$\mathcal{F}(x) := \mathcal{L}(\nabla W(q_l + x) - \psi(q_l + x)\mathcal{J}(\dot{q}_l + \dot{x})), \tag{6.1}$$

where $\mathcal{L}$ is the operator which assigns to h the unique solution $u = \mathcal{L}h$ of the problem:

$$-\ddot{u} + Au = h \quad \text{with} \quad u(0) = 0 \text{ and } \lim_{t\to+\infty} u(t) = 0.$$

An explicit computation shows that:

$$\mathcal{L}h(t) = \frac{1}{2}(\sqrt{A})^{-1}\left[\int_t^{+\infty}\left(e^{(t-s)\sqrt{A}} - e^{-(t+s)\sqrt{A}}\right)h(s)ds + \int_0^t \left(e^{(s-t)\sqrt{A}} - e^{-(s+t)\sqrt{A}}\right)h(s)ds\right] \tag{6.2}$$

and

$$\frac{d}{dt}\mathcal{L}h(t) = \frac{1}{2}\int_t^{+\infty}\left(e^{(t-s)\sqrt{A}} + e^{-(t+s)\sqrt{A}}\right)h(s)ds - \frac{1}{2}\int_0^t \left(e^{(s-t)\sqrt{A}} - e^{-(s+t)\sqrt{A}}\right)h(s)ds.$$

Call $B_\delta = \{x \in Z_1 \mid ||x||_1 < \delta\}$. We want to solve (6.1) by means of the contraction mapping theorem in B_δ. So we want to find, for r small enough, $\delta < r$ small enough in such a way that:

(i) $\overline{\mathcal{F}(B_\delta)} \subset B_\delta$;
(ii) $\mathcal{F}$ is a contraction on $\overline{B_\delta}$.

Assume that $x \in B_\delta$. Then, by $(W1)$ and $(P1)$

$$\left|\nabla W(q_l+x) - \psi(q_l+x)\mathcal{J}(\dot{q}_l+\dot{x})\right| \le \left(\frac{1}{2}L_1(r+\delta)^2 + L_2(r+\delta)(\lambda_1 r+\lambda_2\delta)\right)e^{-2\lambda_2 t} \le \frac{\Lambda}{2}\lambda_2^2(r+\delta)^2 e^{-2\lambda_2 t}.$$

Hence, for $x \in B_\delta$,

$$|\mathcal{F}(x)(t)|,$$

$$\frac{1}{\lambda_2}\left|\frac{d}{dt}\mathcal{F}(x)(t)\right| \le \frac{\Lambda}{4}\lambda_2(r+\delta)^2\left[\int_t^{+\infty} e^{\lambda_2(t-3s)}\,ds + \int_t^{+\infty} e^{-\lambda_2(t+3s)}\,ds + \int_0^t e^{-\lambda_2(t+s)}\,ds\right].$$

We get

$$||\mathcal{F}(x)||_1 \le \frac{\Lambda}{4}(r+\delta)^2. \tag{6.3}$$

On the other side, elementary estimates give that by $(W1)$ and $(P1)$

$$\left|\nabla W(q_l+x') - \psi(q_l+x')\mathcal{J}(\dot{q}_l+\dot{x}') - \nabla W(q_l+x) + \psi(q_l+x)\mathcal{J}(\dot{q}_l+\dot{x})\right| \le \Lambda\lambda_2^2(r+\delta)||x'-x||_1 e^{-2\lambda_2 t}.$$

We easily get from this

$$||\mathcal{F}(x') - \mathcal{F}(x)||_1 \le \frac{\Lambda}{2}(r+\delta)||x'-x||_1. \tag{6.4}$$

By (6.3) and (6.4), to get (i) and (ii) it is enough that δ satisfy $\Lambda(r+\delta)^2/4 < \delta$ as well as $(r+\delta)\Lambda/2 < 1$.

We assume that $r\Lambda < 1/6$ (it will be useful to prove the next lemma). Then it can be checked that with $\delta = 2r^2\Lambda/7 \leq r/21$ the above inequalities are satisfied. Therefore equation (6.1) has a solution $q_h^+ = q_l + x$, with $||x||_1 < 2\Lambda r^2/7 < r/21$. This clearly implies the estimates of the lemma.

We must justify that $q_h^+(\mathbb{R}^+) \subset B_r$. Using the last estimate (but with a different r) and the uniqueness remark at the beginning of this proof we can get that $|\dot{q}_h^+(t) + \sqrt{A}e^{-t\sqrt{A}}\beta| \leq 2\lambda_2\Lambda|q_h^+(t)|^2/7 \leq \lambda_2|q_h^+(t)|/21$. As a consequence, $d(|q_h^+(t)|^2)/dt < 0$ for all $t \in (0, +\infty)$, and we get the claim. □

Proof of Lemma 3. We look for a solution of (1.1) of the form:

$$q_d = q_h + y_l + y \ , \quad \text{where} \quad (y_l)_j(t) = \frac{\sinh(\lambda_j t)}{\sinh(\lambda_j d)}(\alpha - q_h(d))_j$$

(we call for simplicity $q_h^+ = q_h$ the solution given by Lemma 2). As a consequence of Lemma 2 and of the assumption $d \geq 2/\lambda_2$, we have that

$$|y_l(t)| \leq \frac{6}{5} r e^{-\lambda_2(d-t)} \quad ; \quad |\dot{y}_l(t)| \leq \lambda_1 \frac{12}{5} r e^{-\lambda_2(d-t)}. \tag{6.5}$$

We define the space:

$$Z_2 = \left\{ g \in W^{1,\infty}[0,d] \ | \ \sup_{t\in[0,d]} |g(t)|e^{\lambda_2(d-t)}, \ \sup_{t\in[0,d]} |\dot{g}(t)|e^{\lambda_2(d-t)} < +\infty \right\}$$

with norm

$$||g||_2 = \max\left\{ \sup_{t\in[0,d]} |g(t)|e^{\lambda_2(d-t)}, \ \sup_{t\in[0,d]} \frac{1}{2\lambda_2}|\dot{g}(t)|e^{\lambda_2(d-t)} \right\}.$$

We have to find a solution in Z_2 of the fixed point problem

$$\begin{aligned} y = \widetilde{\mathcal{F}}(y) = \widetilde{\mathcal{L}}\Big[&\nabla W(q_h + y_l + y) - \nabla W(q_h) \\ &- \psi(q_h + y_l + y)\mathcal{J}(\dot{q}_h + \dot{y}_l + \dot{y}) + \psi(q_h)\mathcal{J}(\dot{q}_h)\Big], \end{aligned} \tag{6.6}$$

where $\widetilde{\mathcal{L}}$ is the linear operator which assigns to h the unique solution $u = \widetilde{\mathcal{L}}h$ of the problem

$$-\ddot{u} + Au = h \quad \text{with} \quad u(0) = 0 \text{ and } u(d) = 0. \tag{6.7}$$

The solution u of (6.7) is given by:

$$\begin{aligned} u_j(t) = \frac{1}{\lambda_j \sinh(\lambda_j d)}\Big[&\int_t^d h_j(s)\sinh(\lambda_j(d-s))\sinh(\lambda_j t)ds \\ &+ \int_0^t h_j(s)\sinh(\lambda_j s)\sinh(\lambda_j(d-t))ds\Big] \end{aligned} \tag{6.8}$$

and

$$\dot{u}_j(t) = \frac{1}{\sinh(\lambda_j d)}\left[\int_t^d h_j(s)\sinh(\lambda_j(d-s))\cosh(\lambda_j t)ds - \int_0^t h_j(s)\sinh(\lambda_j s)\cosh(\lambda_j(d-t))ds\right].$$

It is easy to derive from these expressions the estimate

$$|u(t)|, \frac{1}{2\lambda_2}|\dot{u}(t)| \le \frac{1}{2\lambda_2}\left[\int_0^t |h(s)|e^{\lambda_2(s-t)}\, ds + \int_t^d |h(s)|e^{\lambda_2(t-s)}\, ds\right] \tag{6.9}$$

As in the proof of Lemma 2 we have to find δ small enough such that $\overline{\widetilde{\mathcal{F}}(B_\delta)} \subset B_\delta$ and $\widetilde{\mathcal{F}}$ is a contraction on B_δ. For $y \in B_\delta$ set

$$A(t) = \nabla W(q_h + y_l + y) - \nabla W(q_h);$$
$$B(t) = \psi(q_h + y_l + y)\mathcal{J}(\dot{q}_h + \dot{y}_l + \dot{y}) - \psi(q_h)\mathcal{J}(\dot{q}_h)$$

We have by Lemma 2 and (6.5):

$$\begin{aligned}|A(t)| &\le L_1(|q_h(t)| + |y_l(t)| + |y(t)|)(|y_l(t)| + |y(t)|)\\ &\le L_1\left[\frac{22}{21}r\left(\frac{6}{5}r+\delta\right)e^{-\lambda_2 d} + \left(\frac{6}{5}r+\delta\right)^2 e^{-2\lambda_2(d-t)}\right],\end{aligned}$$

and

$$\begin{aligned}|B(t)| &\le L_2\Big[(|q_h(t)|+|y_l(t)|+|y(t)|)(|\dot{y}_l(t)|+|\dot{y}(t)|)+(|y_l(t)|+|y(t)|)|\dot{q}_h(t)|\Big]\\ &\le L_2\lambda_1\left[\frac{22}{7}r\left(\frac{6}{5}r+\delta\right)e^{-\lambda_2 d} + 2\left(\frac{6}{5}r+\delta\right)^2 e^{-2\lambda_2(d-t)}\right].\end{aligned}$$

Hence

$$|A(t)| + |B(t)| \le \Lambda\lambda_2^2\left[\frac{22}{21}r\left(\frac{6}{5}r+\delta\right)e^{-\lambda_2 d} + \left(\frac{6}{5}r+\delta\right)^2 e^{-2\lambda_2(d-t)}\right]. \tag{6.10}$$

Replacing $|h(s)|$ by $|A(s)| + |B(s)|$ in (6.9) and using (6.10), we get after easy computations:

$$||\widetilde{\mathcal{F}}(y)||_2 \le \frac{\Lambda}{2}\left[\frac{22}{21}r\left(\frac{6}{5}r+\delta\right) + \left(\frac{6}{5}r+\delta\right)^2\right]. \tag{6.11}$$

We can prove in the same way that

$$||\widetilde{\mathcal{F}}(y) - \widetilde{\mathcal{F}}(y')||_2 \le ||y-y'||_2\frac{\Lambda}{2}\left(\frac{22}{21}r + 2\left(\frac{6}{5}r+\delta\right)\right). \tag{6.12}$$

Using (6.11) and (6.12), we can see after some elementary calculus that, if $r\Lambda \le 1/10$, then $\widetilde{\mathcal{F}}$ is a contraction on B_δ, with $\delta = 2r^2\Lambda \le 1/5r$. Therefore, if $r\Lambda \le 1/10$, we get the existence of q_d, with the estimate

$$||q_d - q_h - y_l||_2 \le 2r^2\Lambda.$$

In particular,

$$|q_d(t) - q_h(t)| \le |y_l(t)| + 2r^2\Lambda e^{-\lambda_2(d-t)} \;,$$
$$|\dot{q}_d(t) - \dot{q}_h(t)| \le |\dot{y}_l(t)| + 4r^2\Lambda\lambda_2 e^{-\lambda_2(d-t)}$$

and we get (2.12) and (2.13) by (6.5). Moreover

$$|\dot{q}_d(0) - \dot{q}_h(0) - \dot{y}_l(0)| \le 4\lambda_2 r^2\Lambda e^{-\lambda_2 d},$$

hence

$$\left|(\dot{q}_d(0) - \dot{q}_h^+(0)) - (\dot{q}_{d,L}(0) - \dot{q}_{h,L}^+(0))\right| < 4\lambda_2 r^2\Lambda e^{-\lambda_2 d} + |\dot{z}(0)|,$$

where z is the solution of $-\ddot{z} + Az = 0$ with boundary conditions $z(0) = 0$ and $z(d) = q_{h,L}(d) - q_h(d)$. Using that $d \ge 2/\lambda_2$, we get by Lemma 2

$$|\dot{z}(0)| \le \max_{j=1,2}\left(\frac{\lambda_j}{\sinh(\lambda_j d)}\right)|q_h(d) - q_{h,L}(d)| \le \frac{5}{7}\lambda_2 r^2\Lambda e^{-\lambda_2 d}.$$

Estimate (2.11) follows. □

PROOF OF LEMMA 7. We first justufy that $w(\Theta) \in E$. Let $R_\Theta = Q_\Theta + w(\Theta)$. By the characterisation of w, we have $S(R_\Theta) = \sum_{i=1}^k \alpha_i(\Theta)a_i$ for some $(\alpha_1(\Theta), \dots, \alpha_k(\Theta)) \in \mathbb{R}^k$. Hence, by the definition of a_i, outside a compact interval,

$$-\ddot{R}_\Theta + \psi(R_\Theta)\mathcal{J}\dot{R}_\Theta + AR_\Theta - \nabla W(R_\Theta) = 0.$$

Now

$$\limsup_{|t|\to\infty} \max(|R_\Theta(t)|, |\dot{R}_\Theta(t)|/\lambda_2) = \limsup_{|t|\to\infty} \max(|w(\Theta)(t)|, |\dot{w}(\Theta)(t)|/\lambda_2)$$
$$< \min(2/\Lambda, \rho_0),$$

by Lemma 6. Arguing as for the proof of Lemma 1, one can derive that $\lim_{|t|\to\infty} |R_\Theta(t)| + |\dot{R}_\Theta(t)| = 0$, which implies, by the properties of the equation near the equilibrium, that $R_\Theta \in X$. Hence $w(\Theta) \in X \subset E$.

Assume that $\overline{\Theta}$ is a critical point of $f(Q_\Theta + w_\Theta)$ (equivalently $(\overline{d}_1, \dots, \overline{d}_k)$ is a critical point of $g(d)$). Then there results that:

$$\left(S(Q_{\overline{\Theta}} + w(\overline{\Theta})), \frac{\partial}{\partial\theta_j}(Q_{\overline{\Theta}} + w(\overline{\Theta}))\right) = 0 \quad \text{for} \quad j = 1, \dots, k. \tag{6.13}$$

Hence we have that

$$\begin{aligned}(6.14)\qquad &\sum_{i=1}^{k}\alpha_i(\overline{\Theta})\left(a_i,\frac{\partial}{\partial\theta_j}(Q_{\overline{\Theta}}+w(\overline{\Theta}))\right)\\ &=\sum_{i=1}^{k}\alpha_i(\overline{\Theta})\alpha\int_{J_i}\dot{Q}^i(t)\cdot\frac{\partial}{\partial\theta_j}(Q_{\overline{\Theta}}+w(\overline{\Theta}))(t)\,dt=0.\end{aligned}$$

Letting $b_{i,j}(\Theta)=\int_{J_i}\dot{Q}^i(t)\cdot\frac{\partial}{\partial\theta_j}(Q_\Theta+w(\Theta))(t)\,dt$, (6.14) yields $\sum_{i=1}^{k}\alpha_i(\overline{\Theta})b_{i,j}(\overline{\Theta})=0$. Now we show that

$$(6.15)\qquad b_{i,j}(\Theta)=\delta_{i,j}\int_{J_j}-|\dot{Q}^j(t)|^2-\dot{Q}^j(t)\cdot\dot{w}(\Theta)(t)dt$$

It appears that $(\partial Q_{\Theta|J_i}/\partial\theta_j)=-\delta_{i,j}\dot{Q}^i{}_{|J_i}\in W^{1,\infty}(J_i)$. Hence, since $(Q_\Theta+w(\Theta))$ is a C^1 function of Θ, $\partial w(\Theta)_{|J_i}/\partial\theta_j$ too is well defined in $W^{1,\infty}(J_i)$. We know that

$$(6.16)\qquad 0=\int_{J_i}\dot{Q}^i(t)\cdot w(\Theta)(t)\,dt=\int_{J_i-\Theta_i}\dot{Q}^i(t+\theta_i)\cdot w(\Theta)(t+\theta_i)\,dt.$$

Now $J_i-\theta_i$ and $\dot{Q}^i(\cdot+\theta_i)$ do not depend on Θ. Hence, deriving (6.16) with respect to θ_j we get

$$\int_{J_i}\dot{Q}^i(t)\cdot\frac{\partial}{\partial\theta_j}w(\Theta)(t)\,dt=-\delta_{i,j}\int_{J_i}\dot{Q}^i(t)\cdot\dot{w}(\Theta)(t)dt.$$

So (6.15) is proved. Therefore, if $\overline{\Theta}$ is a critical point of $f(Q_\Theta+w(\Theta))$, then $\alpha_j(\overline{\Theta})b_{j,j}(\overline{\Theta})=0$ for all j, and it is enough to check that $b_{j,j}(\overline{\Theta})\neq 0$ to conclude. But, if for example $Q^i(t)=\overline{q}(t-\overline{\theta}_i)$ then

$$|b_{j,j}(\overline{\Theta})|\geq\int_{[\bar{t}-\bar{\tau},\bar{t}+\bar{\tau}]}\frac{3}{4}|\dot{\overline{q}}(\bar{t})|\left(\frac{3}{4}|\dot{\overline{q}}(\bar{t})|-\lambda_2||w(\overline{\Theta})||\right)dt>0$$

because, for $\bar{d}\geq\overline{D}$, $||w(\overline{\Theta})||\leq r/2$ and, by Lemma 2 $|\dot{\overline{q}}(\bar{t})|\geq 21\lambda_2 r/22$. □

Proof of Lemma 8. Consider

$$\begin{aligned}(6.17)\qquad e(\beta,\alpha,d)&=\frac{1}{2}\int_0^d\dot{q}_d^2(t)+Aq_d(t)\cdot q_d(t)dt\\ &-\int_0^d\dot{q}_d\cdot v(q_d)dt-\int_0^d W(q_d(t))dt.\end{aligned}$$

We perform the change of variables $t=sd$ and set

$$\forall s\in[0,1]\ \ Q_d(s)=q_d(sd).$$

The function $Q_d(s)$ satisfies the equation:

$$
\begin{aligned}
&-\ddot{Q}_d + d^2 A Q_d + d\psi(Q_d)\mathcal{J}\dot{Q}_d = d^2\nabla W(Q_d)\\
&\text{with} \quad Q_d(0) = \beta \text{ and } Q_d(1) = \alpha.
\end{aligned} \tag{6.18}
$$

After the change of variables in (6.17) we will have

$$
\begin{aligned}
e(\beta,\alpha,d) = \frac{1}{2}\int_0^1 \frac{1}{d}\dot{Q}_d^2(s) + dAQ_d(s)\cdot Q_d(s)ds - \int_0^1 dW(Q_d(s))ds\\
- \int_0^1 \dot{Q}_d(s)\cdot v(Q_d(s))ds
\end{aligned}
$$

We now take the derivative of $e(\beta,\alpha,d)$ with respect to d. It is given by

$$
\begin{aligned}
\frac{\partial e}{\partial d}(\beta,\alpha,d) = \int_0^1 &-\frac{1}{2d^2}\dot{Q}_d^2(s) + \frac{1}{d}\dot{Q}_d(s)\cdot\partial_d\dot{Q}_d(s)\\
&+\frac{1}{2}AQ_d(s)\cdot Q_d(s) + dAQ_d(s)\cdot\partial_d Q_d(s)\\
&-\int_0^1 W(Q_d(s)) - d\nabla W(Q_d(s))\partial_d Q_d(s)ds\\
&-\frac{\partial}{\partial d}\int_0^1 \dot{Q}_d\cdot v(Q_d)ds\,.
\end{aligned} \tag{6.19}
$$

After an integration by parts in 6.19 (of the term $\dot{Q}_d(s)\cdot\partial_d\dot{Q}_d(s)/d$) using that $\partial_d Q_d(0) = 0$ and $\partial_d Q_d(1) = 0$ and putting (6.18) into (6.19) we have that

$$
\begin{aligned}
\frac{\partial e}{\partial d}(\beta,\alpha,d) = \int_0^1 &-\frac{1}{2d^2}\dot{Q}_d^2 + \frac{1}{2}AQ_d\cdot Q_d - W(Q_d)ds\\
&-\int_0^1 \psi(Q_d)\partial_d Q_d\mathcal{J}\dot{Q}_d - \frac{\partial}{\partial d}\int_0^1 v(Q_d)\dot{Q}_d ds
\end{aligned} \tag{6.20}
$$

Using that

$$
\int_0^1 \psi(Q_d)\partial_d Q_d\mathcal{J}\dot{Q}_d + \frac{\partial}{\partial d}\int_0^1 v(Q_d)\dot{Q}_d ds = 0
$$

and making the change of variable $t = sd$ in (6.20), we get

$$
\frac{\partial e}{\partial d}(\beta,\alpha,d) = -\int_0^d \left[\frac{1}{2}(\dot{q}_d^2(t) - Aq_d(t)\cdot q_d(t)) + W(q_d(t))\right]\frac{1}{d}dt.
$$

Since the integrant is nothing but the energy $(\dot{q}_d^2(t) - Aq_d(t)\cdot q_d(t))/2 + W(q_d(t)) = \mathcal{E}(d)$ which is constant we finally get that $\frac{\partial e}{\partial d}(\beta,\alpha,d) = -\mathcal{E}(d)d\frac{1}{d} = -\mathcal{E}(d)$. □

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