ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 27, nº 2 (1998), p. 309-330
<http://www.numdam.org/item?id=ASNSP_1998_4_27_2_309_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVII (1998), pp. 309-330

Integration of Monge-Ampère Equations and Surfaces with Negative Gaussian Curvature

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Abstract. We will first study the integrability condition of Monge-Ampère equations of hyperbolic type, especially of equations which describe surfaces with negative Gaussian curvature. Next, using these results, we will consider the singularities of solutions, and also of solution surfaces, of Monge-Ampère equations. The singularities of solutions do not generally coincide with those of solution surfaces. Some results of this note have been announced in [25] without proof. We will repeat some part of [25] to explain the subjects of this paper.

Mathematics Subject Classification (1991): 35L70 (primary), 53C21, 58C27, 58G17 (secondary).

1. – Introduction

In [24], we studied the singularities of solutions of real Monge-Ampère equations of hyperbolic type as follows: Let $z = z(x, y)$ be an unknown function defined for $(x, y) \in \mathbb{R}^2$, then the equation is written as

$$F(x, y, z, p, q, r, s, t) = Ar + Bs + Ct + D(rt - s^2) - E = 0 \tag{1.1}$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$. Here we assume that A, B, C, D and E are real smooth functions of (x, y, z, p, q). Our principal problems are as follows: 1) *What kinds of singularities may appear?*, and 2) *How can we extend the solutions beyond the singularities?* The best method to solve these problems is to give explicit representations of the solutions. To do so, we apply the characteristic method developed principally by D. Darboux [3] and E. Goursat [5], [6]. In Section 2, we will briefly

(*) Supported in part by the National Basic Research Program in Natural Science, Vietnam
(**) Supported by JSPS Postdoctoral Fellowship for Foreign Researchers in Japan
(***) Partially supported by Grant-in-Aid for Scientific Research (C) (No.10640219), Ministry of Education, Science and Culture (Japan)
Pervenuto alla Redazione il 13 aprile 1998.

explain it "from our point of view", because it seems to us that the method is not familiar today. In Section 3, we will study how to construct equations which are integrable in the sense of Darboux and Goursat. In Section 4, we will characterize surfaces with negative Gaussian curvature whose equations are integrable in the whole space. In Section 5, supposing the conditions which assure the integrability of (1.1), we will study the singularities of solution surfaces of (1.1). As it seems to us that we do not have any result on these problems, we think that, though we assume a little strong conditions, this is one step to construct the global theory on nonlinear hyperbolic equations.

2. – Characteristic method and intermediate integrals

In this section we will explain the characteristic method developed principally by D. Darboux [3] and E. Goursat [5], [6] "from our point of view". As it seems to us that the theory is not familiar today, we had better explain the meanings of our notations. The main idea of the method is how to reduce the solvability of (1.1) to the integration of first order partial differential equations. But, as their method is constructive, it is very useful for our purpose. Let

$$\Gamma : (x, y, z, p, q) = (x(\alpha), y(\alpha), z(\alpha), p(\alpha), q(\alpha)), \quad \alpha \in \mathbb{R}^1,$$

be a smooth curve in $\mathbb{R}^5$, and suppose that it satisfies the following "strip condition"

$$\frac{dz}{d\alpha}(\alpha) = p(\alpha)\frac{dx}{d\alpha}(\alpha) + q(\alpha)\frac{dy}{d\alpha}(\alpha). \tag{2.1}$$

As a "characteristic strip" means that one can not determine the values of the second order derivatives of solution along the strip Γ, we have the following

DEFINITION 2.1. A curve Γ concerning (x, y, z, p, q) is a "characteristic strip" if it satisfies (2.1) and

$$\det \begin{bmatrix} F_r & F_s & F_t \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{bmatrix} = F_t\dot{x}^2 - F_s\dot{x}\dot{y} + F_r\dot{y}^2 = 0 \tag{2.2}$$

where $F_t = \partial F/\partial t$, $F_s = \partial F/\partial s$, $F_r = \partial F/\partial r$, $\dot{x} = dx/d\alpha$ and $\dot{y} = dy/d\alpha$.

Denote the discriminant of (2.2) by Δ, then

$$\Delta = F_s^2 - 4F_rF_t = B^2 - 4(AC + DE).$$

If $\Delta < 0$, equation (1.1) is called to be elliptic. If $\Delta > 0$, equation (1.1) is hyperbolic. In this note, we will treat the equations of hyperbolic type. More precisely, we assume $\Delta > 0$ and also $D \neq 0$. Let λ_1 and λ_2 be the solutions of $\lambda^2 + B\lambda + (AC + DE) = 0$, then the characteristic strip satisfies the following equations:

$$\left\{\begin{array}{l} dz - p\,dx - q\,dy = 0 \\ Ddp + C\,dx + \lambda_1\,dy = 0 \\ Ddq + \lambda_2\,dx + A\,dy = 0 \end{array}\right. \tag{2.3}$$

or

$$\left\{\begin{array}{l} dz - p\,dx - q\,dy = 0 \\ Ddp + C\,dx + \lambda_2\,dy = 0 \\ Ddq + \lambda_1\,dx + A\,dy = 0 \end{array}\right. \tag{2.4}$$

Let us denote $\omega_0 = dz - p\,dx - q\,dy$, $\omega_1 = Ddp + C\,dx + \lambda_1\,dy$ and $\omega_2 = Ddq + \lambda_2\,dx + A\,dy$. Take an exterior product of ω_1 and ω_2, and substitute into their product the contact relations of second order $\{\omega_0 = 0$, $dp = r\,dx + s\,dy$ and $dq = s\,dx + r\,dy\}$. Then we get

$$\omega_1 \wedge \omega_2 = D\left\{Ar + Bs + Ct + D(rt - s^2) - E\right\} dx \wedge dy\,. \tag{2.5}$$

In a space whose dimension is greater than two, the decomposition as above is not possible in general. But, if we can decompose equation (1.1) as (2.5), we can develop the similar discussion (see [24]). Here we introduce the notion of "first integral".

DEFINITION 2.2. A function $V = V(x, y, z, p, q)$ is called "first integral" of $\{\omega_0, \omega_1, \omega_2\}$ if $dV \equiv 0 \mod \{\omega_0, \omega_1, \omega_2\}$.

REMARK. We can easily see that a function $V = V(x, y, z, p, q)$ is the "first integral" of (2.3) (or of (2.4)) if it is constant on any solution of (2.3) (or of (2.4) respectively).

G. Darboux [3] and E. Goursat [5], especially in [5], had considered equations (1.1) under the assumption that (2.3), or (2.4), has at least two independent first integrals. We denote them by u and v. Then we get the following

PROPOSITION 2.3. *Assume that $\lambda_1 \neq \lambda_2$, and that* (2.3), *or* (2.4), *has two independent first integrals $\{u, v\}$. Then we can prove that there exists a function $k = k(x, y, z, p, q) \neq 0$ satisfying*

$$du \wedge dv = k\,\omega_1 \wedge \omega_2 = k\;D\{Ar + Bs + Ct + D(rt - s^2) - E\}dx \wedge dy \tag{2.6}$$

on a submanifold on which the contact relations of second order $\{\omega_0 = 0$, $dp = r\,dx + s\,dy$ and $dq = s\,dx + r\,dy\}$ are satisfied.

If equation (1.1) is written as (2.6), it would be obvious that (2.3), or (2.4), has two independent first integrals $\{u, v\}$. If (2.3), or (2.4), has at least two independent first integrals, equation (1.1) is called to be *integrable in the sense of Monge*. But, if we may follow G. Darboux (p. 263 of [3]), it seems to us that we had better call it to be *integrable in the sense of Darboux.* Moreover, as E. Goursat had profoundly studied equations (1.1) satisfying the above condition, we would like to add the name of Goursat. By these reasons, we will call equations (1.1) with two independent first integrals to be *integrable in the sense of Darboux and Goursat.* Then the representation (2.6) gives the characterization of "Monge-Ampère equations which is integrable in the sense of Darboux and Goursat".

Let $\{u, v\}$ be two independent first integrals of (2.3). For any function g of two variables whose gradient does not vanish, $g(u, v) = 0$ is called an "intermediate integral" of (1.1).

Now we will consider the Cauchy problem for equation (1.1). Let C_0 be an initial strip defined in $\mathbb{R}^5 = \{(x, y, z, p, q)\}$. The Cauchy problem for (1.1) satisfying the initial condition C_0 is to look for a solution $z = z(x, y)$ of (1.1) which contains the strip C_0, i.e., two dimensional surface $\{(x, y, z(x, y), \partial z/\partial x(x, y), \partial z/\partial y(x, y))\}$ in $\mathbb{R}^5$ contains the strip C_0. Assume that the strip C_0 is not characteristic in the sense of Definition 2.1, then we can find an "intermediate integral" $g(u, v)$ which vanishes on C_0. Here we put $g(u, v) = f(x, y, z, p, q)$. The representation (2.6) assures that, as $du \wedge dv = 0$ on a surface $g(u, v) = 0$, a smooth solution of $f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$ satisfies equation (1.1). On the other hand, it is well-known that the Cauchy problem for first order partial differential equation admits uniquely a classical solution in a neighbourhood of the initial curve. Therefore we get the following

THEOREM 2.4 ([3], [5]). *Assume that the initial strip C_0 is not characteristic, and that it satisfies also the following condition*

$$\begin{aligned} &\dot{x}(\alpha)\frac{\partial f}{\partial q}\big(x(\alpha), y(\alpha), z(\alpha), p(\alpha), q(\alpha)\big) \\ &\quad - \dot{y}(\alpha)\frac{\partial f}{\partial p}\big(x(\alpha), y(\alpha), z(\alpha), p(\alpha), q(\alpha)\big) \neq 0 . \end{aligned} \tag{2.7}$$

Then the Cauchy problem for (1.1) *with the initial condition C_0 admits uniquely a classical solution in a neighbourhood of the initial curve.*

3. – Integrable equations in the sense of Darboux and Goursat

In this section we will consider whether or not there exist many examples which admit the integrability condition of Darboux and Goursat stated in Section 2. Suppose that partial differential equation of first order

$$f(x, y, z, p, q) = 0 \tag{3.1}$$

is given. To answer the above question, we will construct Monge-Ampère equation which accepts equation (3.1) as its "intermediate integral". Assume (grad f) $\neq 0$. Then we can locally find a function $g = g(x, y, z, p, q)$ satisfying

$$\operatorname{rank}\begin{pmatrix}\text{grad } f\\ \text{grad } g\end{pmatrix} = 2. \tag{3.2}$$

Here we take a product $df \wedge dg$ and substitute there the contact relations $\omega_0 = 0$, $dp = r\,dx + s\,dy$ and $dq = s\,dx + r\,dy$. Then we get

$$df \wedge dg = F(x, y, z, p, q, r, s, t)dx \wedge dy. \tag{3.3}$$

This representation teaches us that the equation $F = 0$ has two independent first integrals f and g. Therefore there are many equations which satisfy the integrability condition of Darboux and Goursat. But this condition is very strong. For example an equation which describes a surface with constant negative Gaussian curvature does not satisfy the above condition.

At today's point, we do not have the proof of global existence of the function $g=g(x, y, z, p, q)$ satisfying the property (3.2). Therefore the equation $F = F(x, y, z, p, q, r, s, t)=0$ has the meaning only in a domain where the function $g(x, y, z, p, q)$ is defined. But we can define the equation $F(x, y, z, p, q, r, s, t) = 0$ in the whole space for first order partial differential equations of certain types as follows.

EXAMPLE 1. Assume that f is of Hamilton-Jacobi type, i.e., $f = p + h(x, y, z, q)$. Then we choose the function g as $g = q + k(x, y, z)$. Then the equation $F = 0$ is obtained by

$$\begin{aligned} df \wedge dg &= \{(rt - s^2) + Ar + Bs + Ct - E\}dx \wedge dy \\ &= F(x, y, z, p, q, r, s, t)dx \wedge dy \end{aligned} \tag{3.4}$$

where A, B, C and E are functions of (x, y, z, p, q) uniquely determined by f and g.

EXAMPLE 2. Assume that f is quasi-linear, i.e., $f = ap + bq + c$ where a, b and c are real smooth functions of (x, y, z) and $(a, b) \neq (0, 0)$. Here we choose $g = -bp + aq + c'$ where c' is an arbitrary function of (x, y, z). Then the equation $F = 0$ is obtained by

$$\begin{aligned} df \wedge dg &= \{(a^2 + b^2)(rt - s^2) + Ar + Bs + Ct - E\}dx \wedge dy \\ &= F(x, y, z, p, q, r, s, t)dx \wedge dy \end{aligned} \tag{3.5}$$

where A, B, C and E are functions of (x, y, z, p, q) uniquely determined by f and g.

EXAMPLE 3. Any non-characteristic Cauchy problem for first order partial differential equations can be locally reduced to the following form:

$$\frac{\partial u}{\partial x} + f\left(x, y, u, \frac{\partial u}{\partial y}\right) = 0 \quad \text{in} \quad \{(x, y); x > 0, y \in \mathbb{R}^1\}, \tag{3.6}$$

$$u(0, y) = \varphi(y) \quad \text{on} \quad \{(0, y); x = 0, y \in \mathbb{R}^1\}. \tag{3.7}$$

By Example 1, we can construct Monge-Ampère equation which accepts the equation (3.6) as the intermediate integral. Corresponding to the Cauchy problem (3.6)-(3.7), we define the initial strip C_0 by

$$C_0 : (x, y, z, p, q) = (0, \alpha, \varphi(\alpha), -f(0, \alpha, \varphi(\alpha), \varphi'(\alpha)), \varphi'(\alpha)), \ \alpha \in \mathbb{R}^1 .$$

Then we can easily see that the Cauchy problem (3.6)-(3.7) is just the "intermediate integral" of the Cauchy problem for Monge-Ampère equation with the initial strip C_0.

4. – Integrability of equations which describe surfaces with negative Gaussian curvature

Let $\kappa = \kappa(x, y, z, p, q)$ be Gaussian curvature of a surface $z = z(x, y)$, then $z = z(x, y)$ satisfies the following Monge-Ampère equation:

$$rt - s^2 = \kappa(1 + p^2 + q^2)^2 . \tag{4.1}$$

We use the same notions introduced in Section 2. As we are interested in the hyperbolic case, we assume $\kappa(x, y, z, p, q) = -\gamma(x, y, z, p, q)^2$ where $\gamma(x, y, z, p, q) > 0$. In the case where κ is a negative constant, we can easily see that (4.1) does not satisfy the integrability condition of Darboux and Goursat. As we are interested in the global structure of the solution surface which satisfies equation (4.1), we will aim, in this section, to characterize equation (4.1) which is integrable in the sense of Darboux and Goursat in the whole space, that is to say, to give necessary and sufficient conditions so that the system of differential forms (2.3), or (2.4), has two independent first integrals in the whole space.

If the Gaussian curvature is negative, but not strictly negative, then we can give an example of equation which has the property mentioned in the above.

EXEMPLE 4.1. Assume that $\kappa = -c^2/(1 + p^2 + q^2)^2$ where c is a positive constant. Then equation (4.1) is written by

$$rt - s^2 = -c^2 . \tag{4.2}$$

If we may use Lemma 4.2 which will appear soon in this section, we can easily see that equation (4.2) has two independent first integrals $\{cx - q, cy + p\}$

in the whole space. Next we will construct a surface which satisfies (4.2) in the large. Let an initial strip C_0 be

$$C_0 : (x, y, z, p, q) = (0, \alpha, c\alpha^2/2, 0, c\alpha), \qquad \alpha \in R^1 . \tag{4.3}$$

Then the intermediate integral of the Cauchy problem (4.2)-(4.3) is given by $g(x, y, z, p, q) = c(x+y)+p-q = 0$. Therefore the solution of (4.2)-(4.3) is written by $z = -c(x^2 - y^2)/2$. Hence there exists a smooth surface in the large whose Gaussian curvature is equal to $-c^2/(1+p^2+q^2)^2$.

In the following of this section, we will prove the converse of Example 4.1. To explain our result, let us introduce some notations. Denote

$$\varrho(x, y, z, p, q) = (1+p^2+q^2)\gamma(x, y, z, p, q) . \tag{4.4}$$

Then, as the characteristic equation of (4.1) is equal to $\lambda^2 - \varrho^2 = 0$, we denote the solutions by $\lambda_1 = \varrho$ and $\lambda_2 = -\varrho$. Let us write

$$\omega_0 = dz - p\,dx - q\,dy, \quad \omega_1 = dp + \varrho\,dy, \quad \omega_2 = dq - \varrho\,dx .$$

Then we get

$$\omega_1 \wedge \omega_2 = \{(rt - s^2) + \varrho^2\} dx \wedge dy$$

on a submanifold on which the contact relations of second order $\{\omega_0 = 0,\ dp = r\,dx + s\,dy$ and $dq = s\,dx + r\,dy\}$ are satisfied. In [25], M. Tsuji announced that, if the function $\gamma(x, y, z, p, q)$ depends only on (p, q) and it is strictly positive, then the system of 1-form $\{\omega_0, \omega_1, \omega_2\}$ has not two independent first integrals in the whole space. As an extension of the result, we will prove the following

THEOREM 4.1. *Suppose that Gaussian curvature $k = -\gamma^2(x, y, z, p, q)$ satisfies the following conditions:*

1) $\gamma(x, y, z, p, q) > 0$,
2) $\gamma(x, y, z, p, q) \in C^2(R^5)$,
3) *For any fixed $(x_0, y_0, p_0, q_0) \in R^4$, there exists a function $g(t) \in L^1_{\mathrm{loc}}(R^1)$ such that*

$$\frac{1}{\gamma(x, y_0, z, p_0, t)} \le g(t) \tag{4.5}$$

and

$$\frac{1}{\gamma(x_0, y, z, t, q_0)} \le g(t) . \tag{4.6}$$

Then necessary and sufficient condition so that the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ has two independent first integrals in the whole space is that the function $\gamma(x, y, z, p, q)$ is equal to $c/(1+p^2+q^2)$ where c is a positive constant.

Let us introduce differential operators as follows:

$$L_1 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + \varrho\frac{\partial}{\partial q} \tag{4.7}$$

and

$$L_2 = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} - \varrho\frac{\partial}{\partial p}\,. \tag{4.8}$$

LEMMA 4.2. *Necessary and sufficient condition so that a function $u = u(x, y, z, p, q)$ is a first integral of the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ is that it satisfies*

$$L_1 u = 0 \quad and \quad L_2 u = 0\,. \tag{4.9}$$

PROOF. From the definition of the forms $\{\omega_0, \omega_1, \omega_2\}$, we have

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial p}dp + \frac{\partial u}{\partial q}dq \\ &= \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + \varrho\frac{\partial u}{\partial q}\right)dx + \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} - \varrho\frac{\partial u}{\partial p}\right)dy \\ &\quad + \frac{\partial u}{\partial z}\omega_0 + \frac{\partial u}{\partial p}\omega_1 + \frac{\partial u}{\partial q}\omega_2 \\ &= (L_1 u)dx + (L_2 u)dy + \frac{\partial u}{\partial z}\omega_0 + \frac{\partial u}{\partial p}\omega_1 + \frac{\partial u}{\partial q}\omega_2\,, \end{aligned}$$

from which we get (4.9). □

PROOF OF SUFFICIENCY OF THEOREM 4.1. Suppose $\gamma(x, y, z, p, q) = c/(1 + p^2 + q^2)$ where c is a positive constant. As $\varrho(x, y, z, p, q) = (1 + p^2 + q^2)\gamma(x, y, z, p, q) = c$, it follows

$$L_1 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + c\frac{\partial}{\partial q} \text{ and } L_2 = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} - c\frac{\partial}{\partial p}\,.$$

As we can easily verify that two functions $\{cx - q, cy + p\}$ satisfy the system of equations $L_i u = 0$ (i=1,2) in the whole space, we get the sufficiency of the above theorem. □

Next we advance to the proof of necessity of Theorem 4.1. As it is a little too long, we will prepare several lemmata. Let us put

$$\mu(x, y, z, p, q) \equiv \ln \varrho(x, y, z, p, q)\,. \tag{4.10}$$

Here we define differential operators L_3, L_4 and L_5 as follows:

$$L_3 \equiv -\frac{1}{\varrho}[L_1, L_2] = -2\frac{\partial}{\partial z} + L_1(\mu)\frac{\partial}{\partial p} + L_2(\mu)\frac{\partial}{\partial q}\,, \tag{4.11}$$

$$L_4 \equiv [L_1, L_3] = -L_1(\mu)\frac{\partial}{\partial z} + L_1^2(\mu)\frac{\partial}{\partial p} + \big(L_1 L_2(\mu) - L_3(\varrho)\big)\frac{\partial}{\partial q}\,, \tag{4.12}$$

$$L_5 \equiv [L_2, L_3] = -L_2(\mu)\frac{\partial}{\partial z} + \big(L_1 L_2(\mu) + L_3(\varrho)\big)\frac{\partial}{\partial p} - L_2^2(\mu)\frac{\partial}{\partial q}\,, \tag{4.13}$$

where $[L_i, L_j] \equiv L_i L_j - L_j L_i$ is a commutator of operators L_i and L_j.

LEMMA 4.3. *Assume that the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ has two independent first integrals in the whole space. Then $L_4 = 0$ and $L_5 = 0 \bmod \{L_1, L_2, L_3\}$, that is to say, there exist functions $a_i(x, y, z, p, q)$ and $b_i(x, y, z, p, q)$ $(i = 1, 2, 3)$ such that*

$$L_4 = a_1L_1 + a_2L_2 + a_3L_3 \text{ and } L_5 = b_1L_1 + b_2L_2 + b_3L_3 .$$

PROOF. Suppose that $u(x, y, z, p, q)$ is a first integral of the system of 1-forms $\{\omega_0, \omega_1, \omega_2\}$. From Lemma 4.2, (4.9), (4.11), (4.12) and (4.13), we see that the function u satisfies the following system of linear first order homogeneous partial differential equations:

$$\begin{cases} L_1u = 0, \\ L_2u = 0, \\ L_3u = 0, \\ L_4u = 0, \\ L_5u = 0. \end{cases} \tag{4.14}$$

Here we define a matrix G by

$$G = \begin{bmatrix} 1 & 0 & p & 0 & \varrho \\ 0 & 1 & q & -\varrho & 0 \\ 0 & 0 & -2 & L_1(\mu) & L_2(\mu) \\ 0 & 0 & -L_1(\mu) & L_1^2(\mu) & L_1L_2(\mu) - L_3(\varrho) \\ 0 & 0 & -L_2(\mu) & L_2L_1(\mu) + L_3(\varrho) & -L_2^2(\mu) \end{bmatrix} .$$

Then (4.14) means that ${}^t\mathrm{grad}\, u$ is in kernel of G where $\mathrm{grad}\, u = (\partial u/\partial x, \partial u/\partial y, \partial u/\partial z, \partial u/\partial p, \partial u/\partial q)$. As the assumption says that there exist two independent solutions of (4.14), we see rank $G \leq 3$. But, as first three rows of the matrix G are independent, we can conclude rank $G = 3$. Therefore last two rows of the matrix G must be expressed by linear combinations of first three ones. □

Here we will give a name of "condition (A)" to a set of the conditions 1), 2) and 3) appeared in Theorem 4.1, and a name of "condition (B)" to the property such that the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ has two independent first integrals in the whole space.

LEMMA 4.4. *Suppose the condition (B). Then the function $\mu = \mu(x, y, z, p, q)$ satisfies the following equations in R^5 :*

$$L_1^2(\mu) - \frac{1}{2}(L_1(\mu))^2 = 0, \tag{4.15}$$

$$L_1L_2(\mu) - \frac{1}{2}L_1(\mu)L_2(\mu) - L_3(\varrho) = 0, \tag{4.16}$$

$$L_2L_1(\mu) - \frac{1}{2}L_1(\mu)L_2(\mu) + L_3(\varrho) = 0, \tag{4.17}$$

$$L_2^2(\mu) - \frac{1}{2}(L_2(\mu))^2 = 0. \tag{4.18}$$

PROOF. Since the condition (B) is fulfilled, we see by Lemma 4.3 that there exist functions $a_i(x, y, z, p, q)$ and $b_i(x, y, z, p, q)$ $(i = 1, 2, 3)$ satisfying

$$L_4 = a_1 L_1 + a_2 L_2 + a_3 L_3 \tag{4.19}$$

and

$$L_5 = b_1 L_1 + b_2 L_2 + b_3 L_3 . \tag{4.20}$$

As the operators L_4 and L_5 do not contain $\partial/\partial x$ and $\partial/\partial y$, we get

$$a_1 = a_2 = b_1 = b_2 = 0 .$$

Hence we have by (4.11), (4.12), (4.13), (4.19) and (4.20)

$$\frac{-L_1(\mu)}{-2} = \frac{L_1^2(\mu)}{L_1(\mu)} = \frac{L_1 L_2(\mu) - L_3(\varrho)}{L_2(\mu)} , \tag{4.21}$$

$$\frac{-L_2(\mu)}{-2} = \frac{L_2 L_1(\mu) + L_3(\varrho)}{L_2(\mu)} = \frac{L_2^2(\mu)}{L_2(\mu)} , \tag{4.22}$$

from which we get (4.15)-(4.18). □

LEMMA 4.5. *Assume the condition (B), then we get $L_3(\varrho) = 0$.*

PROOF. From (4.15) and (4.16) we have

$$L_1 L_2(\mu) - L_2 L_1(\mu) - 2L_3(\varrho) = 0 .$$

On the other hand, it follows from (4.10) and (4.11)

$$L_1 L_2(\mu) - L_2 L_1(\mu) = [L_1, L_2](\mu) = -\varrho L_3(\mu) = -L_3(\varrho) .$$

Hence we get $L_3(\varrho) = 0$. □

Let us put

$$L \equiv \frac{1}{\varrho} L_1 = \frac{1}{\varrho}\frac{\partial}{\partial x} + \frac{p}{\varrho}\frac{\partial}{\partial z} + \frac{\partial}{\partial q} , \tag{4.23}$$

and denote $X(x, y, z, p, q) \equiv \ln \gamma(x, y, z, p, q)$ and $w(x, y, z, p, q) \equiv (1+p^2+q^2)L(X)$.

LEMMA 4.6. *Assume the condition (B). Then the function $w(x, y, z, p, q)$ satisfies*

$$L(w) + \frac{1}{2(1+p^2+q^2)} w^2 + \frac{2(1+p^2)}{1+p^2+q^2} = 0. \tag{4.24}$$

PROOF. Since $L_1(\mu) = (1/\varrho)L_1(\varrho)$, it follows from (4.15) and (4.23)

$$\varrho L^2(\varrho) - \frac{1}{2}\big(L(\varrho)\big)^2 = 0. \tag{4.25}$$

Using the relation (4.4) on ϱ and γ, we get

$$L(\varrho) = (1+p^2+q^2)L(\gamma) + 2q\gamma, \tag{4.26}$$
$$L^2(\varrho) = (1+p^2+q^2)L^2(\gamma) + 4qL(\gamma) + 2\gamma. \tag{4.27}$$

Substituting (4.26) and (4.27) in (4.25), we obtain

$$\left(\gamma L^2(\gamma) - \frac{1}{2}\big(L(\gamma)\big)^2\right)(1+p^2+q^2)^2 + 2q(1+p^2+q^2)\gamma\big(L(\gamma)\big) + 2\gamma^2(1+p^2) = 0.$$

This means

$$\gamma L^2(\gamma) - \frac{1}{2}\big(L(\gamma)\big)^2 + \frac{2q\gamma}{1+p^2+q^2}L(\gamma) + \frac{2\gamma^2(1+p^2)}{(1+p^2+q^2)^2} = 0. \tag{4.28}$$

As $L(X) = L(\ln\gamma) = (1/\gamma)L(\gamma)$, it follows

$$\begin{aligned} L^2(X) &= L\left(\frac{1}{\gamma}L(\gamma)\right) = L\left(\frac{1}{\gamma}\right)L(\gamma) + \frac{1}{\gamma}L^2(\gamma) \\ &= -\frac{1}{\gamma^2}\big(L(\gamma)\big)^2 + \frac{1}{\gamma}L^2(\gamma). \end{aligned} \tag{4.29}$$

Combining (4.28) and (4.29), we have

$$L^2(X) + \frac{1}{2}\big(L(X)\big)^2 + \frac{2q}{1+p^2+q^2}L(X) + \frac{2(1+p^2)}{(1+p^2+q^2)^2} = 0. \tag{4.30}$$

As $w(x, y, z, p, q) \equiv (1+p^2+q^2)L(X)$, it holds

$$L(w) = L\big((1+p^2+q^2)L(X)\big) = (1+p^2+q^2)L^2(X) + 2qL(X).$$

Substitute this into (4.30), we get (4.24). □

LEMMA 4.7. *Assume the conditions (A) and (B), then we get $w(x, y, z, p, q) = -2q$.*

PROOF. We rewrite equation (4.24) in the following form:

$$\frac{1}{\varrho}\frac{\partial w}{\partial x}+\frac{p}{\varrho}\frac{\partial w}{\partial z}+\frac{\partial w}{\partial q}+\frac{1}{2(1+p^2+q^2)}w^2+\frac{2(1+p^2)}{1+p^2+q^2}=0. \tag{4.31}$$

This is an equation defined in R^5. Let us recall (4.4) which is the definition of the function $\gamma=\gamma(x,y,z,p,q)$, and also the assumption (A). As $\gamma(x,y,z,p,q)$ is a given function in $C^2(R^5)$, we can regard the function $w=w(x,y,z,p,q)$ as a classical solution of (4.31) satisfying the following initial condition:

$$\begin{aligned} w|_{q=0} &= (1+p^2+q^2)L(\ln\gamma)|_{q=0} \\ &= \frac{(1+p^2)}{\gamma(x,y,z,p,0)}(L\gamma)|_{q=0}\equiv w_0(x,y,z,p). \end{aligned} \tag{4.32}$$

Let us solve the Cauchy problem (4.31)-(4.32). Then a system of characteristic equations is written as follows:

$$\left\{\begin{aligned} \frac{dx(t)}{dt} &= \frac{1}{(1+p^2(t)+q^2(t))\gamma(x(t),y(t),z(t),p(t),q(t))} \\ \frac{dy(t)}{dt} &= 0 \\ \frac{dz(t)}{dt} &= \frac{p(t)}{(1+p^2(t)+q^2(t))\gamma(x(t),y(t),z(t),p(t),q(t))} \\ \frac{dp(t)}{dt} &= 0 \\ \frac{dq(t)}{dt} &= 1 \\ \frac{dw(t)}{dt} &= -\frac{w^2(t)}{2(1+p^2(t)+q^2(t))}-\frac{2(1+p^2(t))}{1+p^2(t)+q^2(t)} \end{aligned}\right. \tag{4.33}$$

with the following initial conditions:

$$\begin{gathered} x(0)=x_0,\quad y(0)=y_0,\quad z(0)=z_0,\quad p(0)=p_0, \\ q(0)=0,\quad w(0)=w_0(x_0,y_0,z_0,p_0). \end{gathered}$$

We get immediately $y(t)=y_0$, $p(t)=p_0$ and $q(t)=t$. The functions $x=x(t)$ and $z=z(t)$ satisfy the following equations:

$$\left\{\begin{aligned} \frac{dx(t)}{dt} &= \frac{1}{(1+p_0^2+t^2)\gamma(x(t),y_0,z(t),p_0,t)} \\ \frac{dz(t)}{dt} &= \frac{p_0}{(1+p_0^2+t^2)\gamma(x(t),y_0,z(t),p_0,t)}, \end{aligned}\right. \tag{4.34}$$

with the initial conditions

$$x(0) = x_0, \quad z(0) = z_0 \,. \tag{4.35}$$

The assumption (A), that is to say, the conditions 1), 2) and 3) in Theorem 4.1, assures the unique existence of the solutions $x = x(t)$ and $z = z(t)$ of (4.34)-(4.35) for all $t : -\infty < t < \infty$. From the last equation in (4.33), we have

$$\frac{dw(t)}{dt} = -\frac{w^2(t)}{2(1+p_0^2+t^2)} - \frac{2(1+p_0^2)}{1+p_0^2+t^2}\,, \tag{4.36}$$

$$w(0) = w_0(x_0, y_0, z_0, p_0)\,. \tag{4.37}$$

Then we can solve the Cauchy problem (4.36)-(4.37) explicitly, that is to say, we see that the solution $w = w(t)$ is written by

$$w(t) = \frac{(1+p_0^2)(w_0(x_0, y_0, z_0, p_0) - 2t)}{(1+p_0^2) + \frac{1}{2}t w_0(x_0, y_0, z_0, p_0)}\,. \tag{4.38}$$

This means that, if $w_0(x_0, y_0, z_0, p_0)$ is not zero, then $w = w(t)$ tends to infinity in finite time. But, as $w = w(x, y, z, p, q)$ is a smooth function defined in the whole space, we can conclude $w_0(x, y, z, p) \equiv 0$. In fact, if $w_0(x, y, z, p) \not\equiv 0$, then we can find a point $(x_0, y_0, z_0, p_0) \in R^4$ such that $w_0(x_0, y_0, z_0, p_0) \neq 0$. Let us put

$$t_0 = -\frac{2(1+p_0^2)}{w_0(x_0, y_0, z_0, p_0)}\,.$$

Since the function $w(t)$ must take finite value even at $t = t_0$, it must hold

$$w_0(x_0, y_0, z_0, p_0) - 2t_0 = 0\,,$$

or

$$\frac{w_0(x_0, y_0, z_0, p_0)^2 + 4(1+p_0^2)}{w_0(x_0, y_0, z_0, p_0)} = 0\,.$$

That is a contradiction. Hence we have $w(t) = -2t$. As we see $t = q(t)$ easily, we obtain $w(x, y, z, p, q) = -2q$. □

LEMMA 4.8. *Suppose the conditions (A) and (B), then we get $L_1(\varrho) = 0$.*

PROOF. From (4.23) and Lemma 4.7, we obtain

$$\frac{1}{\varrho}L_1(\ln\gamma) = L(\ln\gamma) = L(X) = \frac{w(x, y, z, p)}{1+p^2+q^2} = -\frac{2q}{1+p^2+q^2}\,. \tag{4.39}$$

Since $\varrho = (1+p^2+q^2)\gamma$ and $L_1(\ln\gamma) = (1/\gamma)L_1(\gamma)$, it follows immediately $L_1(\gamma) = -2q\gamma^2$. Hence we have

$$\begin{aligned} L_1(\varrho) &= L_1((1+p^2+q^2)\gamma) = (1+p^2+q^2)L_1(\gamma) + \gamma L_1(1+p^2+q^2) \\ &= (1+p^2+q^2)(-2q\gamma^2) + 2\varrho q\gamma = -2\varrho q\gamma + 2\varrho q\gamma = 0\,. \end{aligned}$$ □

Analogously, repeating the same discussion for L_2, we get the following

LEMMA 4.9. *Suppose the conditions* (A) *and* (B), *then we get* $L_2(\varrho) = 0$.

LEMMA 4.10. *Suppose the conditions* (A) *and* (B), *then we see that the function* $\varrho(x, y, z, p, q)$ *doesn't depend on the variable* z, *that is to say,* $\varrho = \varrho(x, y, p, q)$.

PROOF. The definition (4.11) of L_3 gives us $L_3(\varrho) = 0$. On the other hand, using (4.10), Lemma 4.8 and Lemma 4.9, we have

$$L_3 = -2\frac{\partial}{\partial z} + L_1(\mu)\frac{\partial}{\partial p} + L_2(\mu)\frac{\partial}{\partial q} = -2\frac{\partial}{\partial z} + \frac{1}{\varrho}L_1(\varrho)\frac{\partial}{\partial p} + \frac{1}{\varrho}L_2(\varrho)\frac{\partial}{\partial q} = -2\frac{\partial}{\partial z}.$$

Hence, as $L_3(\varrho) = -2(\partial\varrho/\partial z) = 0$, we get this lemma. □

LEMMA 4.11. *Suppose the conditions* (A) *and* (B), *then it holds* $(\partial\varrho/\partial q) = 0$.

PROOF. Since the function ϱ does not depend on z, we have by Lemma 4.8

$$\frac{\partial\varrho}{\partial x} + \varrho\frac{\partial\varrho}{\partial q} = 0.$$

Suppose that there exists a point (x^0, y^0, p^0, q^0) such that

$$\frac{\partial\varrho}{\partial q}(x^0, y^0, p^0, q^0) \neq 0.$$

Consider the Cauchy problem as follows:

$$\frac{\partial\varrho}{\partial x} + \varrho\frac{\partial\varrho}{\partial q} = 0, \tag{4.40}$$

$$\varrho|_{x=x^0} = \varrho_0(q) \equiv \varrho(x^0, y^0, p^0, q). \tag{4.41}$$

Let us solve this Cauchy problem (4.40)-(4.41). Then the system of characteristic equations is written by

$$\begin{cases} \dfrac{dq}{dx} = \varrho, & \dfrac{d\varrho}{dx} = 0, \\ q(x^0) = \xi, & \varrho(x^0) = \varrho_0(\xi). \end{cases}$$

The solutions of this Cauchy problem are given by

$$q = \xi + \varrho_0(\xi)(x - x^0), \qquad \varrho = \varrho_0(\xi).$$

As $(\partial/\partial\xi)\{\xi + \varrho_0(\xi)(x - x^0)\} \neq 0$ in a neighbourhood of $x = x^0$, we can uniquely solve the equation $q = \xi + \varrho_0(\xi)(x - x^0)$ with respect to ξ in the neighbourhood of $x = x^0$, and denote it by $\xi = \xi(x, q)$. Then we get

$$\varrho(x, y^0, p^0, q) = \varrho_0(\xi(x, q)) = \varrho\big(x^0, y^0, p^0, \xi(x, q)\big).$$

From this expression we get

$$\frac{\partial\varrho}{\partial q}(x, y_0, p_0, q) = \frac{\partial\varrho_0}{\partial\xi}\cdot\frac{\partial\xi}{\partial q} = \frac{\partial\varrho_0/\partial\xi}{1 + (\partial\varrho_0/\partial\xi)(x - x^0)}.$$

When x tends to $x_0 - \{1/(\partial\varrho/\partial q)(x^0, y^0, p^0, q^0)\}$ along the line "$q = q^0 + \varrho_0(q^0)(x - x^0)$", the value $(\partial\varrho/\partial q)(x, y^0, p^0, q)$ goes to infinity. This means that $\varrho(x, y, p, q)$ is not a smooth function defined in the whole space $\mathbb{R}^5$. Hence it holds $(\partial\varrho/\partial q)(x, y, p, q) \equiv 0$. □

By the same reasoning, we get the following

LEMMA 4.12. *Suppose the conditions (A) and (B), then it holds* $(\partial\varrho/\partial p) \equiv 0$.

PROOF OF THE NECESSARY PART OF THEOREM 4.1. Substituting the results of Lemma 4.11 and Lemma 4.12 into $L_1\varrho = 0$ and $L_2\varrho = 0$ appeared in Lemma 4.8 and Lemma 4.9 respectively, we get

$$\frac{\partial\varrho}{\partial x}(x, y, p, q) = 0 \text{ and } \frac{\partial\varrho}{\partial y}(x, y, p, q) = 0 .$$

Summing up the above results, we see that $\varrho(x, y, z, p, q)$ is a positive constant. Hence we get finally

$$\gamma(x, y, z, p, q) = \frac{c}{1 + p^2 + q^2}, \quad c = \text{constant} > 0 . \qquad \square$$

5. – Solution surfaces of Monge-Ampère equations

In this section we will study the singularities of solution surfaces of Monge-Ampère equations. Here it would be better to make clear the meaning of "singularity" of surfaces, though we may write very elementary facts.

DEFINITION 5.1. A point (x^0, y^0, z^0) is called to be "singularity" of a solution $z = z(x, y)$ of (1.1) if and only if $z^0 = z(x^0, y^0)$ and $z(x, y) \notin C^2$ in a neighbourhood of (x^0, y^0).

DEFINITION 5.2. Let S be a surface in $\mathbb{R}^3$. S is regular at a point (x^0, y^0, z^0) if we can choose parameters $(\alpha, \beta) \in \mathbb{R}^2$ as follows: $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$ and $z = z(\alpha, \beta)$ satisfy the two conditions:

(i) $(x(\alpha^0, \beta^0), y(\alpha^0, \beta^0), z(\alpha^0, \beta^0)) = (x^0, y^0, z^0)$, and $x = x(\alpha, \beta), y = y(\alpha, \beta)$ and $z = z(\alpha, \beta)$ are of class C^1 in a neighbourhood of (α^0, β^0) ,

(ii) $$\operatorname{rank} \begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial z}{\partial \alpha} \\ \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = 2 .$$

DEFINITION 5.3. A point (x^0, y^0, z^0) is called to be "singularity" of a solution surface S if and only if S is not regular at the point (x^0, y^0, z^0).

Now we consider the Cauchy problem for (1.1). Let us write a initial strip C_0 as follows:

$$C_0 : (x, y, z, p, q) = \big(x_0(\alpha), y_0(\alpha), z_0(\alpha), p_0(\alpha), q_0(\alpha)\big), \alpha \in \mathbb{R}^1 .$$

We assume that all given functions are sufficiently smooth. First we will show that the singularities of solutions of (1.1) do not generally coincide with those of solution surfaces.

THEOREM 5.4. *Consider the Cauchy problem for* (1.1) *with the initial condition* C_0. *Assume the following three conditions:* I) (2.3), *or* (2.4), *has two independent first integrals in the whole space,* II) *The intermediate integral f for this Cauchy problem is written by* $f = f(x, y, z, p, q) = ap + bq - c = 0$ *where a, b, and c are smooth functions of* (x, y, z) *defined in the whole space* $\mathbb{R}^3$, *and* III) *The initial condition satisfies*

$$\operatorname{rank}\begin{pmatrix} a(x,y,z) & b(x,y,z) & c(x,y,z) \\ x_0'(\alpha) & y_0'(\alpha) & z_0'(\alpha) \end{pmatrix} = 2 \quad on \ C_0 .$$

Then the smooth solution surface of (1.1) *exists in the large, though the solution* $z = z(x, y)$ *of* (1.1) *may have singularities.*

PROOF. By Theorem 2.4, we can get a solution of the Cauchy problem for (1.1) with the initial condition C_0 by solving the following Cauchy problem

$$\begin{cases} a\partial z/\partial x + b\partial z/\partial y - c = 0 \\ z(x_0(\alpha), y_0(\alpha)) = z_0(\alpha) . \end{cases} \tag{5.1}$$

Then characteristic equations for (5.1) are written by

$$\begin{gathered} \frac{\partial x}{\partial \beta} = a, \quad \frac{\partial y}{\partial \beta} = b, \quad \frac{\partial z}{\partial \beta} = c; \\ x(0) = x_0(\alpha), \ y(0) = y_0(\alpha), \ z(0) = z_0(\alpha) . \end{gathered} \tag{5.2}$$

We denote the solutions of (5.2) by $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$ and $z = z(\alpha, \beta)$. Using the assumption (III), we can prove

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial z}{\partial \alpha} \\ \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = 2 \quad \text{for any} \ \ (\alpha, \beta) \in \mathbb{R}^2 . \tag{5.3}$$

Therefore, though the solution $z = z(x, y)$ has singularities at the points where the Jacobian $D(x, y)/D(\alpha, \beta) = 0$, the solution surface is regular even at these points. We remark that (5.3) holds in a domain where the solutions of (5.2) exist.

In the theorem, we wrote that there exists a solution surface of the Cauchy problem for (1.1) in the large. We will explain the meaning. If the solutions of (5.2) blow up at $\beta = \beta_0 \in \mathbb{R}^1$, it says that, when β tends to β_0, a point (x, y, z) goes to infinity. Therefore, we can not extend the solution beyond $\beta = \beta_0$. If we may say this in other word, this means thet, even if the definition domain of (x, y, z) in $\mathbb{R}^2 = \{(\alpha, \beta)\}$ may be bounded, the solution surface does not remain bounded. This is the meaning of "global existence of the solution surface" written in this theorem. □

Next we will give the case where the singularities of solutions of (1.1) may coincide with singularities of the solution surfaces.

THEOREM 5.5. *Consider the Cauchy problem for* (1.1) *with the initial condition* C_0. *Assume the following three conditions:* I) (2.3), *or* (2.4), *has two independent first integrals in the whole space,* II) *The intermediate integral* $f = f(x, y, z, p, q)$ *for this Cauchy problem is of Hamilton-Jacobi type, and* III) *The initial strip* C_0 *satisfies the condition* (2.7). *Then* $z = z(x, y)$ *is singular at a point* (x^0, y^0) *if and only if the solution surface* $\{(x, y, z); z = z(x, y)\}$ *is not regular at a point* (x^0, y^0, z^0) *where* $z^0 = z(x^0, y^0)$.

PROOF. By Theorem 2.4, we can get a solution of the Cauchy problem for (1.1) with the initial condition C_0 by solving the following Cauchy problem

$$(5.4)\qquad \begin{cases} f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0 \\ z(x_0(\alpha), y_0(\alpha)) = z_0(\alpha), \ \dfrac{\partial z}{\partial x}(x_0(\alpha), y_0(\alpha)) = p_0(\alpha), \\ \dfrac{\partial z}{\partial y}(x_0(\alpha), \ y_0(\alpha)) = q_0(\alpha)\,. \end{cases}$$

Then characteristic differential equations for (5.4) are written as follows:

$$(5.5)\qquad \begin{cases} \dfrac{dx}{d\beta} = \dfrac{\partial f}{\partial p}(x, y, z, p, q), \quad \dfrac{dy}{d\beta} = \dfrac{\partial f}{\partial q}(x, y, z, p, q) \\ \dfrac{dz}{d\beta} = p\dfrac{\partial f}{\partial p}(x, y, z, p, q) + q\dfrac{\partial f}{\partial q}(x, y, z, p, q) \\ \dfrac{dp}{d\beta} = -\dfrac{\partial f}{\partial x}(x, y, z, p, q) - p\dfrac{\partial f}{\partial z}(x, y, z, p, q) \\ \dfrac{dq}{d\beta} = -\dfrac{\partial f}{\partial y}(x, y, z, p, q) - q\dfrac{\partial f}{\partial z}(x, y, z, p, q)\,, \end{cases}$$

$$(5.6)\qquad \begin{aligned} &x(0) = x(\alpha), y(0) = y(\alpha), z(0) = z(\alpha), \\ &p(0) = p(\alpha), q(0) = q(\alpha), \alpha \in \mathbb{R}^1\,. \end{aligned}$$

We denote the solutions of (5.5)-(5.6) by $x = x(\alpha, \beta)$, $y = y(\alpha, \beta)$, $z = z(\alpha, \beta)$, $p = p(\alpha, \beta)$ and $q = q(\alpha, \beta)$. The assumption so that the intermediate integral $f = f(x, y, z, p, q) = 0$ is of Hamilton-Jacobi type means the global solvability of (5.5)-(5.6). This is the definition of "equations of Hamilton-Jacobi type". See M. Tsuji [23], or final "Remark" given at the end of this section. As $\omega_0 = 0$ on the solution surface, we have

$$(5.7)\quad \operatorname{rank}\begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial z}{\partial \alpha} \\ \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} & \dfrac{\partial z}{\partial \beta} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial y}{\partial \alpha} \\ \dfrac{\partial x}{\partial \beta} & \dfrac{\partial y}{\partial \beta} \end{pmatrix} \quad \text{for any } (\alpha, \beta) \in \mathbb{R}^2\,.$$

It has been proved in M. Tsuji [23] that the solution $z = z(x, y)$ is not in C^2 in neighbourhoods of the points where the Jacobian $D(x, y)/D(\alpha, \beta) = 0$. Moreover equation (5.7) means that the solution surface is also not regular at the points where the Jacobian vanishes. Summing up these results, we can get the conclusion of this theorem. □

Concerning the solution surface $S = \{(x, y, z); z = z(x, y)\}$, the problems which we are interested in are as follows: I) What kinds of singularities may appear?, and II) Can we extend the solution surface beyond the singularities? For the problem II), we have two directions. After the appearance of singularities, the solution $z = z(x, y)$ takes in general sevaral values. One way is to introduce a physical point of view. Then a solution must be single-valued. For this aim, we cut off some parts of solution so that it could become a single-valued weak or generalized solution satisfying the entropy condition for equations of conservation law or the semi-concavity condition for Hamilton-Jacobi equations. By this procedure, the singularities may appear in the solutions. See [22], [23], [10], [11], [12], [18] and [19]. The another way is to consider the above problem from geometric point of view. Then we must accept multi-valued solutions. As Monge-Ampère equations appear often in geometric problems, we should take here the second approach. This means that, without cutting off some part of solution surfaces, we should accept the whole part of solution surfaces and consider the singularities of surfaces in the meaning of Definition 5.3.

To state our results, we introduce a smooth mapping H from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$H(\alpha, \beta) = \big(x(\alpha, \beta), y(\alpha, \beta)\big), \quad (\alpha, \beta) \in \mathbb{R}^2,$$

where $x(\alpha, \beta)$ and $y(\alpha, \beta)$ are the solutions of (5.4)-(5.5). Let us write $\Sigma = \{(\alpha, \beta) \in \mathbb{R}^2; D(x, y)/D(\alpha, \beta) = 0\}$ and $H(\Sigma) = \Gamma$. Then we get the following

THEOREM 5.6. *Under the same assumptions as Theorem 5.5, we add the hypothesis such that the singularities of the mapping H are fold and cusp points only. Then the curve Γ becomes piecewise smooth and the solution surface has the singularities along the curve Γ. Moreover we can uniquely extend the solution surface beyond the singularities in the space of C^1-functions which are of class C^2 except on piecewise smooth curves.*

The canonical forms of cusp and fold points are obtained by H. Whitney [27]. The uniqueness of the extension of solution surfaces beyond the singularities follows from Theorem 4.6 in [24].

REMARK. Let us explain the meaning of "Hamilton-Jacobi type" used in Theorem 5.5 and 5.6. In M. Tsuji [23], we have studied the differences between Hamilton-Jacobi equations and equations of conservation law, under the assumption that $f(x, y, z, p, q)$ is smooth. Our conclusion is that the most characteristic property of Hamilton-Jacobi equations is the global solvability of the Cauchy problem for (5.4). On the other hand, if $f = 0$ is quasi-linear, the solutions $p(\alpha, \beta)$ and $q(\alpha, \beta)$ tend to infinity when the Jacobian $D(x, y)/D(\alpha, \beta)$ vanishes. Therefore, in the above theorem, "Hamilton-Jacobi type" means the global solvability of the Cauchy problem for (5.4). Recently S. Izumiya [11] gave the geometric characterization of Hamilton-Jacobi equations and quasi-linear partial differential equations of first order.

6. – Remarks on surfaces with negative Gaussian curvature

Let κ be Gaussian curvature of the surface $z = z(x, y)$, then $z = z(x, y)$ satisfies equation (4.1). We use the same notations used in Section 4. Let us recall the classical theorem due to D. Hilbert as follows:

THEOREM 6.1 (D. Hilbert [8]). *A surface S in $\mathbb{R}^3$ with constant negative Gaussian curvature has singular points.*

Therefore, when we may extend the classical solution of (4.1), the singularities may appear in general. But, if the Gaussian curvature is not strictly negative, there exists a surface in the large whose Gaussian curvature is negative. See Example 4.1 given in Section 4.

As the generalization of Hilbert's theorem [8], N. V. Efimov proved the following

THEOREM 6.2 (N. V. Efimov [4]). *No surface can be immersed in $\mathbb{R}^3$ so as to be complete in the induced Riemannian metric, with strictly negative Gaussian curvature.*

We write $\lambda_1 = (1 + p^2 + q^2)(-\kappa)^{1/2}$, $\lambda_2 = -\lambda_1$, $\omega_1 = dp + \lambda_1 dy$ and $\omega_2 = dq + \lambda_2 dx$. Then we have

$$\omega_1 \wedge \omega_2 = \{(rt - s^2) - \kappa(1 + p^2 + q^2)^2\} dx \wedge dy\,.$$

on a submanifold where the contact relations of second order $\{\omega_0 = dz - p\,dx - q\,dy = 0,\quad dp = r\,dx + s\,dy,\ dq = s\,dx + r\,dy\}$ are satisfied. Therefore equation (4.1) is obtained by the product of ω_1 and ω_2 on a submanifold where the contact relations of second order are satisfied. Then, as a corollary of Theorem 4.1, we get

THEOREM 6.3. *Assume that Gaussian curvature κ is strictly negative, then the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ does not have two independent first integrals defined in the whole space.*

This theorem does not deny the possibility so that the system of one forms $\{\omega_0, \omega_1, \omega_2\}$ admits *locally* two independent first integrals. See the following Example 6.4.

EXAMPLE 6.4. Let Ω be a bounded and open set in $\mathbb{R}^5 = \{(x, y, z, p, q) \in \mathbb{R}^5\}$, and suppose

$$\kappa = -\frac{1}{(1 + p^2 + q^2)^2} \text{ in } \Omega\,.$$

Next we extend the function $\kappa = \kappa(x, y, z, p, q)$ so that it is smooth and strictly negative in the whole space $\mathbb{R}^5$. Then we can easily see that equation

$$(rt - s^2) - \kappa(x, y, z, p, q)(1 + p^2 + q^2)^2 = 0$$

has two independent first integrals $\{x - q, y + p\}$ in the domain Ω. But Theorem 4.1 means that it does not have two independent first integrals defined in the whole space.

Therefore we can not apply our preceding method to solve (4.1). Then our problem is how we can get a family of characteristic strips. We can obtain it by solving a system of first order partial differential equations as follows:

$$(6.1) \qquad \begin{cases} \dfrac{\partial p}{\partial \alpha} + \lambda_1 \dfrac{\partial y}{\partial \alpha} = 0 \\ \dfrac{\partial q}{\partial \alpha} - \lambda_1 \dfrac{\partial x}{\partial \alpha} = 0 \\ \dfrac{\partial p}{\partial \beta} - \lambda_1 \dfrac{\partial y}{\partial \beta} = 0 \\ \dfrac{\partial q}{\partial \beta} + \lambda_1 \dfrac{\partial x}{\partial \beta} = 0 . \end{cases}$$

The local solvability of (6.1) is already proved by H. Lewy [14] and J. Hadamard [7]. But, to develop the global theory, we would have to consider the global behaviour of the solutions of (6.1). For certain nonlinear wave equations, we can get global solutions of (6.1). Though we have written our idea a little in M. Tsuji [26], we will soon publish detailed paper on this subject.

Finally we would like to give some comments on M. Kossowski [13]. He constructed local solutions of (1.1) by the method which is almost similar to the characteristic method. Then the biggest problem is how to get the family of characteristic strips. In [13], he could obtain it by solving certain system of equations which is numbered as (7) in [13]. His equation (7) in [13] is corresponding to (6.1) in our case. As he assumed the analyticity on equations (1.1), he could solve it by Cauchy-Kowalewski theorem. As we consider (1.1) in C^∞-space, we need the condition of hyperbolicity on (1.1).

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