

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 27, n° 1 (1998), p. 1-26

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On the Schauder Estimates of Solutions to Parabolic Equations

QING HAN

Abstract. In this paper we give a priori estimates on asymptotic polynomials of solutions to parabolic differential equations at any points. This leads to a pointwise version of Schauder estimates. The result in this paper improves the classical Schauder estimates in a way that the estimates of solutions and their derivatives at one point depend on the coefficient and nonhomogeneous terms at this particular point.

Mathematics Subject Classification (1991): 35B05, 35B45.

Schauder estimates play an important role in the theory of parabolic equations, on which are based the existence and regularity of solutions. It was first proved, for the constant coefficient equations, by writing solutions explicitly in terms of fundamental solutions and analyzing the corresponding singular integrals. The general case can be recovered by a technique of freezing coefficients. For Hölder norm estimates on higher order derivatives we need to differentiate equations to get equations for the derivatives. Basically a priori estimates, or regularities, of solutions in a set depend on properties of coefficients and nonhomogeneous terms in the same set.

In this paper we will use the idea of comparing solutions with polynomials to obtain Schauder estimates, especially on higher order derivatives. We do not need to differentiate the equations to get equations for derivatives. Hence we need very weak assumptions on coefficients. Specifically in order to obtain regularity and a priori estimates on derivatives of solutions at one point, we only need assumptions on coefficients and nonhomogeneous terms at this particular point. We will carry out our discussion for parabolic equations of arbitrary order.

The idea of comparing solutions with polynomials was used by Caffarelli in [4] to discuss fully nonlinear elliptic equations. It was generalized to the

parabolic case by Wang in [18]. They derived Schauder estimates for viscosity solutions to equations of the second order by comparing solutions with quadratic polynomials. Such a comparison was made in a neighborhood of a point by the maximum principle. Hence regularities of solutions at one point is determined basically by properties of coefficients and nonhomogeneous terms at the same point. They proved, among others, that the second order derivatives of solutions are Hölder continuous at one point if coefficients and nonhomogeneous terms are Hölder continuous only at that point. Their arguments are mainly for the second order equations since the maximum principle is essentially employed.

In order to generalize this idea to higher order equations we need to construct polynomials by other methods. In [3], Bers proved that solutions to homogeneous linear elliptic equations, which vanish at 0 with the order d , are asymptotic to nonzero homogeneous polynomials of degree d . Such a result was generalized in [2] to solutions to parabolic equations of the second order. It is reasonable to think that derivatives of solution at 0 of the order not exceeding d are zero, that derivatives of the order d are given by coefficients of such polynomials and that whether the derivatives of the order d are Hölder continuous are determined by the error terms. In order to obtain Schauder estimates on solutions at this particular point we need to derive a priori estimates on asymptotic polynomials and error terms.

As a crucial step we need to obtain a priori estimates on solutions themselves with respect to their vanishing order. Such a result, in its simplest form, states as follows. Suppose u is a solution to some homogeneous linear parabolic equation of general order. If for some integer d ,

$$\limsup_{(x,t) \rightarrow 0} \frac{|u(x,t)|}{|(x,t)|^d} < \infty$$

then there holds

$$\sup_{|(x,t)| < 1} \frac{|u(x,t)|}{|(x,t)|^d} \leq C|u|_{L^\infty(Q_1)}$$

where C is a constant depending only on coefficients of equations and the integer d . We should emphasize that in our discussion we are interested only in a priori estimates. We do not assume solutions satisfy the unique continuation. However it is the best case, from the point of view of a priori estimates, that solutions vanish at infinite order at some point, since all “derivatives” vanish at this point. Due to the similar reason, our Schauder estimates on higher order derivatives of solutions at a particular point begin with the order equal to the vanishing order, since “derivatives” of the order less than the vanishing order equal to zero.

The method is based on singular integral estimates. With the help of the fundamental solutions we express solutions, including derivatives of solutions of arbitrary order, by singular integrals. There is no need to differentiate equations. Hence it is only needed the behavior of coefficients and nonhomogeneous terms at a particular point. For precise statements see Theorem 3.1 and Theorem 3.2.

Not only do we have a priori estimates on higher order derivatives, we may also prove the continuity of these derivatives. Since “derivatives” of solutions are given by coefficients of polynomials, we just need to prove the convergence of such polynomials.

The method in this paper, based on fundamental solutions and singular integral estimates, is general in nature. It can be applied to more general setting. In this paper we will confine ourselves in the discussion of a single linear equation of the parabolic type.

The motivation for results in the current paper originates from the discussion of nodal sets of nonsmooth solutions. We would like to obtain results on the geometric structure and measure estimates of the nodal sets, where solutions vanish, and the singular nodal sets, where solutions and their derivatives of any orders not exceeding the order of equations vanish. For equations of the second order the singular nodal sets are just critical nodal sets. In order to do this we need to blow up solutions at nodal points. If solutions do not vanish at infinite order the blow-up limits are homogeneous polynomials, which are called the leading polynomials of solutions and whose degrees are called the vanishing orders of solutions. Hence locally solutions can be viewed as perturbations of polynomials, although no analyticity or smoothness is required. Since nodal sets of polynomials are easy to analyze, we want to argue that locally the nodal sets of solutions, in both geometric structure and measure estimates, do not change significantly compared with nodal sets of the leading polynomials, at least for almost all nodal points with respect to some appropriate Hausdorff dimension. In other words the nodal sets of leading polynomials represent locally the nodal sets of solutions. We also want to get similar conclusions for singular nodal sets, although situation there is more complicated. In order to achieve these we need the a priori estimates on the leading polynomials and the error terms. Such a priori estimates, stated as Theorem 2.2, are the simplest forms of Schauder estimates. General forms are given in Theorems 3.1 and 3.2. In this paper we only discuss such estimates. We will pursue the applications on nodal sets elsewhere.

The paper is written as follows. In Section 1, we will discuss solutions to parabolic equations with constant coefficients. The a priori estimates for such solutions play an important role in the following section. In Section 2 we will prove the a priori estimates on the asymptotic polynomials and error terms. In fact it gives the simplest form of the Schauder estimates. In Section 3 we will prove general pointwise Schauder estimates by induction.

1. – Solutions to constant coefficient equations

Throughout this paper we fix m as an even integer.

A function $f(x, t)$ is p -homogeneous of p -degree d if for any $\lambda > 0$ and

$(x, t) \in \mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}$, there holds

$$f(\lambda x, \lambda^m t) = \lambda^d f(x, t) .$$

A polynomial $P(x, t)$ is of p -degree at most d , nonnegative integer, if it can be decomposed into a sum of p -homogeneous polynomials, whose p -degree of p -homogeneity is at most d .

For any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ and any $r > 0$ we define the p -ball

$$Q_r(x_0, t_0) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}; |x - x_0| < r, -r^m < t - t_0 < 0\} .$$

Correspondingly we define the p -norm for $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ as

$$|(x, t)| = (|x|^m + |t|)^{\frac{1}{m}} .$$

We may check easily that

$$|(x, t) + (y, s)| \leq |(x, t)| + |(y, s)| .$$

We define $W_p^{m,1}(Q_1)$ as the Sobolev space of functions whose x -derivatives up to m -th order and t -derivative of first order belong to $L^p(Q_1)$.

Suppose $\{a_\nu\}$ is a collection of constants for any multi-index $\nu \in \mathbb{Z}_+^n$ with $|\nu| = m$. The equation

$$(1.1) \quad Lu \equiv \frac{\partial u}{\partial t} - \sum_{|\nu|=m} a_\nu D^\nu u = 0$$

is parabolic if

$$(1.2) \quad (-1)^{\frac{m}{2}-1} \sum_{|\nu|=m} a_\nu \xi^\nu \geq \lambda \quad \text{for any } \xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$$

for some positive constant λ . We also assume

$$(1.3) \quad \sum_{|\nu|=m} |a_\nu| \leq \kappa$$

for some positive constant κ .

Then there exists a fundamental solution $\Gamma(x, t)$ for $t > 0$ such that

$$L\Gamma = 0 .$$

In fact, by Fourier transformation, we have the explicit expression

$$\Gamma(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left\{ i\xi \cdot x + (-1)^{\frac{m}{2}} t \sum_{|\nu|=m} a_\nu \xi^\nu \right\} d\xi .$$

We also have the following estimates

$$(1.4) \quad |D_x^\mu D_t^l \Gamma(x, t)| \leq \frac{C}{t^{\frac{n+|\mu|+ml}{m}}} \exp \left\{ -c \left(\frac{|x|^m}{t} \right)^{\frac{1}{m-1}} \right\} \quad \text{for } t > 0$$

where C and c are constants depending only on n, m, λ, κ and $|\mu| + lm$. For a proof see [8]. We always define $\Gamma(x, t) \equiv 0$ for $t \leq 0$.

For $f \in L^p(Q_1)$, for some $p > 1$, define

$$u(x, t) = \int_{Q_1} \Gamma(x - y, t - s) f(y, s) dy ds .$$

Then $u \in W_p^{m,1}(Q_1)$ and satisfies $Lu = f$. Moreover by Calderon-Zygmund decomposition for parabolic equations and Marcinkiewicz interpolation there holds the following estimate

$$\|D^m u\|_{L^p(Q_1)} \leq C \|f\|_{L^p(Q_1)}$$

where C is a positive constant depending only on n, m, λ and κ . For detailed analysis see [7] or [13].

Based on this estimate, with the following Lemma 1.1, there exists interior $W_p^{m,1}$ estimates for solution u of the equation

$$Lu = f .$$

We first discuss some basic properties of solutions to homogeneous equations.

LEMMA 1.1. *Suppose u is a solution of (1.1). Then for any multi-index $\mu \in \mathbb{Z}_+^n$ and any nonnegative integer l there holds*

$$\left| D_x^\mu D_t^l u(x, t) \right| \leq \frac{C}{(R - |(x, t)|)^{|\mu|+ml}} \max_{Q_R} |u| \quad \text{for any } (x, t) \in Q_{\frac{R}{2}}$$

where C is a constant depending only on n, m, λ, κ and $|\mu| + ml$.

PROOF. The proof is standard, based on induction. \square

LEMMA 1.2. *Suppose u is a solution of (1.1). Then for any nonnegative integer d there exist p -homogeneous polynomials P_i with p -degree i , $i = 0, 1, \dots, d$, such that*

$$|u(x, t) - \sum_{i=0}^d P_i(x, t)| \leq C \frac{|(x, t)|^{d+1}}{(R - |(x, t)|)^{d+1}} \max_{Q_R} |u| \quad \text{for any } (x, t) \in Q_{\frac{R}{2}}$$

where the polynomial P_i satisfies $LP_i = 0$ for $i = 0, 1, \dots, d$. In fact P_i is given by

$$P_i(x, t) = \sum_{|\mu|+m=l=i} D_{x,t}^{\mu,l} u(0, 0) \frac{x^\mu t^l}{\mu!l!}.$$

PROOF. The proof is straightforward. By Taylor expansion, we have

$$\begin{aligned} u(x, t) &= \sum_{i=0}^d \sum_{|\mu|+l=i} (D^{\mu,l} u)(0, 0) \frac{x^\mu t^l}{\mu!l!} + \sum_{|\mu|+l=d+1} (D^{\mu,l} u)(\xi(x, t)x, \eta(x, t)t) \frac{x^\mu t^l}{\mu!l!} \\ &= \sum_{i=0}^d P_i(x, t) + R_d(x, t) \end{aligned}$$

with

$$\begin{aligned} R_d(x, t) &= \sum_{i=0}^d \sum_{\substack{|\mu|+l=i \\ |\mu|+m \geq d+1}} D^{\mu,l} u(0, 0) \frac{x^\mu t^l}{\mu!l!} \\ &\quad + \sum_{|\mu|+l=d+1} D^{\mu,l} u(\xi(x, t)x, \eta(x, t)t) \frac{x^\mu t^l}{\mu!l!} \end{aligned}$$

where $\xi(x, t)$, $\eta(x, t)$ are positive numbers less than 1. By Lemma 1.1 we have the desired estimate on R_d .

Direct calculation yields that each P_i is a solution of equation (1.1). In fact, we have

$$\frac{\partial}{\partial t} P_i(x, t) = \sum_{\substack{|\mu|+m=l=i \\ l \geq 1}} D^{\mu,l} u(0, 0) \frac{x^\mu t^{l-1}}{\mu!(l-1)!} = \sum_{|\mu|+m+m=l=i} D^{\mu,l+1} u(0, 0) \frac{x^\mu t^l}{\mu!l!}$$

and for any $\nu \in \mathbb{Z}_+^n$ with $|\nu| = m$

$$\frac{\partial^m}{\partial x^\nu} P_i(x, t) = \sum_{\substack{|\mu|+m=l=i \\ \mu_k \geq \nu_k}} D^{\mu,l} u(0, 0) \frac{x^{\mu-\nu} t^l}{(\mu-\nu)!l!} = \sum_{|\mu|+m+m=l=i} D^{\mu+\nu,l} u(0, 0) \frac{x^\mu t^l}{\mu!l!}.$$

Therefore

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \sum_{|\nu|=m} a_\nu D_x^\nu \right) P_i(x, t) \\ &= \sum_{|\mu|+m+m=l=i} \left\{ D^{\mu,l+1} u(0, 0) - \sum_{|\nu|=m} a_\nu D^{\mu+\nu,l} u(0, 0) \right\} \frac{x^\mu t^l}{\mu!l!} = 0 \end{aligned}$$

since $D^{\mu,l} u$ is also a solution. \square

REMARK. In both Lemma 1.1 and Lemma 1.2 the sup-norm at the right side may be replaced by L^p -norm for $p > 1 + n/m$.

The following result plays an important role in the subsequent discussion.

LEMMA 1.3. *Suppose $f \in L^p(Q_1)$, $p > 1 + n/m$, satisfies*

$$(1.5) \quad \|f\|_{L^p(Q_r)} \leq \gamma r^{d-m+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq 1$$

for some positive constants γ , $\alpha \in (0, 1)$, and some integer $d \geq m$. Then there exists a function $u \in W_p^{m,1}(Q_1)$ such that

$$(1.6) \quad Lu = f \quad \text{in } Q_1$$

and

$$(1.7) \quad \sum_{i=0}^m r^i \|D^i u\|_{L^p(Q_r)} \leq C \gamma r^{d+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq 1$$

where $C > 0$ is constant depending only on $n, m, p, d, \lambda, \alpha$ and κ .

REMARK. In general Lemma 1.3 does not hold if $\alpha = 0$ or $\alpha = 1$.

PROOF. Without loss of generality we assume $f(x, t) = 0$ for $|(x, t)| > 1$. Let Γ be the fundamental solution of L . We set

$$w(x, t) = \int_{|(y,s)| < 1} \Gamma(x-y, t-s) f(y, s) dy ds.$$

Then w satisfies

$$Lw = f \quad \text{in } Q_1$$

and

$$(1.8) \quad \|w\|_{W_p^{m,1}(Q_1)} \leq C \|f\|_{L^p(Q_1)} \leq C \gamma.$$

For each $(y, s) \neq 0$, consider the Taylor expansion of $\Gamma(x-y, t-s)$ at $(x, t) = 0$. For each nonnegative integer k , let Γ_k denote the p -homogeneous k -th order terms, i.e.,

$$\Gamma_k(x, y; t, s) = \sum_{|\mu|+m_l=k} D_x^\mu D_t^l \Gamma(-y, -s) \frac{x^\mu t^l}{\mu! l!}.$$

Lemma 1.2 implies

$$L\Gamma_k(\cdot, y; \cdot, s) = 0 \quad \text{in } Q_1.$$

By (1.4) we also have

$$(1.9) \quad |D_x^\mu D_t^l \Gamma(x, t)| \leq \frac{C}{|(x, t)|^{n+|\mu|+m_l}}$$

where C depends only on n, m, λ, κ and $|\mu| + ml$. Set

$$(1.10) \quad v(x, t) = \int_{|(y,s)| < 1} \sum_{k=0}^d \Gamma_k(x, y; t, s) f(y, s) dy ds.$$

Then v is a polynomial of p -degree $\leq d$ and satisfies $Lv = 0$. We may show by (1.5) that

$$(1.11) \quad |v(x, t)| \leq C\gamma \quad \text{in } Q_1(0).$$

The proof is based on the direct calculation. In fact, by setting $1/p' = 1 - 1/p$, we have for $(x, t) \in Q_1(0)$

$$\begin{aligned} |v(x, t)| &\leq C \sum_{k=0}^d \int_{|(y,t)| < 1} \frac{|f(y, s)|}{|(y, s)|^{n+k}} dy ds \\ &\leq C \sum_{k=0}^d \sum_{i=1}^{\infty} \int_{\frac{1}{2^i} < |(y,s)| < \frac{1}{2^{i-1}}} \frac{|f(y, s)|}{|(y, s)|^{n+k}} dy ds \\ &\leq C \sum_{k=0}^d \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right)^{-(n+k)} \int_{|(y,s)| < \frac{1}{2^{i-1}}} |f(y, s)| dy ds \\ &\leq C \sum_{k=0}^d \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right)^{\frac{n+m}{p'} - (n+k)} \left(\int_{|(y,s)| < \frac{1}{2^{i-1}}} |f(y, s)|^p dy ds \right)^{\frac{1}{p}} \\ &\leq C\gamma \sum_{k=0}^d \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{\frac{n+m}{p'} - (n+k)} \left(\frac{1}{2^i}\right)^{d-m+\alpha + \frac{n+m}{p}} \\ &= C\gamma \sum_{k=0}^d \sum_{i=0}^{\infty} \left(\frac{1}{2^i}\right)^{d-k+\alpha} = C\gamma \sum_{k=0}^d \sum_{i=0}^{\infty} \left(\frac{1}{2^{d-k+\alpha}}\right)^i \leq C\gamma. \end{aligned}$$

Now we set

$$(1.12) \quad \begin{aligned} u(x, t) &= w(x, t) - v(x, t) \\ &= \int_{|(y,s)| < 1} \left\{ \Gamma(x - y, t - s) - \sum_{k=0}^d \Gamma_k(x, y; t, s) \right\} f(y, s) dy ds. \end{aligned}$$

Obviously $Lu = f$. We will show that u vanishes with the order at least $d + \alpha$ at $(0, 0)$. In fact we will prove that

$$(1.13) \quad |u(x, t)| \leq C\gamma |(x, t)|^{d+\alpha} \quad \text{for } |(x, t)| < \frac{1}{2}.$$

Fix $0 < |(x, t)| < \frac{1}{2}$. Split the integral (1.12) into three parts:

$$\begin{aligned} I_1 &= \int_{|(y,s)| < 2|(x,t)|} \Gamma(x-y, t-s) f(y, s) dy ds \\ I_2 &= - \int_{|(y,s)| < 2|(x,t)|} \sum_{k=0}^d \Gamma_k(x, y; t, s) f(y, s) dy ds \\ I_3 &= \int_{2|(x,t)| < |(y,s)| < 1} \left[\Gamma(x-y, t-s) - \sum_{k=0}^d \Gamma_k(x, y; t, s) \right] f(y, s) dy ds . \end{aligned}$$

Again denote $p' = \frac{p}{p-1}$. Then by Hölder inequality

$$\begin{aligned} |I_1| &\leq \left(\int_{|(y,s)| < 2|(x,t)|} \frac{C}{|(x-y, t-s)|^{np'}} dy ds \right)^{\frac{1}{p'}} \left(\int_{|(y,s)| < 2|(x,t)|} |f(y, s)|^p dy ds \right)^{\frac{1}{p}} \\ &\leq \left(\int_{|(y,s)| < 3|(x,t)|} \frac{C}{|(y, s)|^{np'}} dy ds \right)^{\frac{1}{p'}} \cdot \left(\int_{|(y,s)| < 2|(x,t)|} |f(y, s)|^p dy \right)^{\frac{1}{p}} \\ &\leq C\gamma |(x, t)|^{\frac{n+m}{p'} - n} \cdot |(x, t)|^{d-m+\alpha + \frac{n+m}{p}} = C\gamma |(x, t)|^{d+\alpha} , \end{aligned}$$

provided $p > 1 + \frac{n}{m}$. For I_2 we use similar method as that for (1.11). By (1.9) we have

$$\begin{aligned} |I_2| &\leq C \sum_{k=0}^d |(x, t)|^k \int_{|(y,s)| < 2|(x,t)|} \frac{|f(y, s)|}{|(y, s)|^{n+k}} dy ds \\ &\leq C \sum_{k=0}^d |(x, t)|^k \sum_{i=0}^{\infty} \int_{\frac{|(x,t)|}{2^i} < |(y,s)| < \frac{2|(x,t)|}{2^i}} \frac{|f(y, s)|}{|(y, s)|^{n+k}} dy ds \\ &\leq C\gamma \sum_{k=0}^d |(x, t)|^k \sum_{i=0}^{\infty} \left(\frac{|(x, t)|}{2^i} \right)^{d-k+\alpha} \\ &= C\gamma |(x, t)|^{d+\alpha} \sum_{k=0}^d \sum_{i=0}^{\infty} \left(\frac{1}{2^{d-k+\alpha}} \right)^i \\ &\leq C\gamma |(x, t)|^{d+\alpha} . \end{aligned}$$

Last, for (x, t) and (y, s) satisfying $2|(x, t)| < |(y, s)|$, by the proof of Lemma 1.2,

we have

$$\begin{aligned}
\Gamma(x-y, t-s) &= \sum_{k=0}^d \Gamma_k(x, y; t, s) \\
&= \sum_{i=0}^d \sum_{\substack{|\mu|+l=i \\ |\mu|+ml \geq d+1}} D^{\mu, l} \Gamma(-y, -s) \frac{x^\mu t^l}{\mu! l!} \\
&\quad + \sum_{|\mu|+l=d+1} D^{\mu, l} u(\xi x - y, \eta t - s) \frac{x^\mu t^l}{\mu! l!}
\end{aligned}$$

where $\xi = \xi(x, t; y, s)$ and $\eta = \eta(x, t; y, s)$ are positive numbers less than 1. By (1.9) it is bounded by

$$\begin{aligned}
&\sum_{i=0}^d \sum_{\substack{|\mu|+l=i \\ |\mu|+ml \geq d+1}} \frac{1}{|(y, s)|^{n+|\mu|+ml}} \frac{|x|^\mu |t|^l}{\mu! l!} \\
&\quad + \sum_{|\mu|+l=d+1} \frac{1}{|(\xi x - y, \eta t - s)|^{n+|\mu|+ml}} \frac{|x|^\mu |t|^l}{\mu! l!} \\
&\leq C \sum_{j=d+1}^{m(d+1)} \left\{ \frac{1}{|(y, s)|^{n+j}} + \frac{1}{|(\xi x - y, \eta t - s)|^{n+j}} \right\} |(x, t)|^j.
\end{aligned}$$

Note $2|(\xi x - y, \eta t - s)| > |(y, s)|$ since $2|(x, t)| < |(y, s)|$. Therefore we get

$$\left| \Gamma(x-y, t-s) - \sum_{k=0}^d \Gamma_k(x, y; t, s) \right| \leq C \sum_{j=d+1}^{m(d+1)} \frac{|(x, t)|^j}{|(y, s)|^{n+j}}.$$

For any $(x, t) \in Q_{1/2}$ there exists an integer M such that $2^{M-1}|(x, t)| \leq 1 < 2^M|(x, t)|$. Then we have

$$\begin{aligned}
|I_3| &\leq \sum_{j=d+1}^{m(d+1)} |(x, t)|^j \int_{2|(x, t)| < |(y, s)| < 1} \frac{|f(y, s)|}{|(y, s)|^{n+j}} dy ds \\
&\leq \sum_{j=d+1}^{m(d+1)} |(x, t)|^j \sum_{i=1}^{M-1} \int_{2^i|(x, t)| < |(y, s)| < 2^{i+1}|(x, t)|} \frac{|f(y, s)|}{|(y, s)|^{n+j}} dy ds \\
&\leq C\gamma \sum_{j=d+1}^{m(d+1)} |(x, t)|^j \sum_{i=1}^{M-1} (2^i |(x, t)|)^{d-j+\alpha} \\
&= C\gamma |(x, t)|^{d+\alpha} \sum_{j=d+1}^{m(d+1)} \sum_{i=1}^{M-1} \frac{1}{(2^{j-d-\alpha})^i} \\
&\leq C\gamma |(x, t)|^{d+\alpha}.
\end{aligned}$$

This finishes the proof of (1.13). Since $Lu = f$ in $Q_1(0)$, interior estimates imply

$$(1.14) \quad \sum_{i=0}^m r^i \|D^i u\|_{L^p(Q_r)} \leq C\gamma r^{d+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq \frac{1}{4}.$$

By (1.8) and (1.11), the estimate (1.14) can be extended to Q_1 . \square

COROLLARY 1.4. *Suppose $f \in L^p(Q_1)$, $p > 1 + n/m$, satisfies*

$$\|f\|_{L^p(Q_r)} \leq \gamma r^{d-m+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq 1$$

for some positive constants $\gamma > 0$, $\alpha \in (0, 1)$, and some integer $d \geq m$. For any solution $u \in W_p^{m,1}(Q_1)$ of $Lu = f$ there exists a polynomial P_d of p -degree $\leq d$ with $LP_d = 0$ such that

$$|u(x, t) - P_d(x, t)| \leq C(\gamma + \|u\|_{L^p(Q_1)}) |(x, t)|^{d+\alpha} \quad \text{for any } (x, t) \in Q_{\frac{1}{2}}$$

where $C > 0$ is constant depending only on $n, m, p, d, \lambda, \alpha$ and κ .

PROOF. By Lemma 1.3 there exists a $v \in W_p^{m,1}(Q_1)$ with $Lv = f$ such that

$$|v(x, t)| \leq C\gamma |(x, t)|^{d+\alpha} \quad \text{for any } (x, t) \in Q_{\frac{1}{2}}$$

and

$$\|v\|_{L^p(Q_1)} \leq C\gamma.$$

Note $L(u - v) = 0$. By Lemma 1.2, we may write

$$u(x, t) - v(x, t) = P_d(x, t) + R_d(x, t)$$

where P_d is a polynomial of p -degree $\leq d$ and R_d satisfies

$$|R_d(x, t)| \leq C\|u - v\|_{L^p(Q_1)} |(x, t)|^{d+1} \leq C(\gamma + \|u\|_{L^p(Q_1)}) |(x, t)|^{d+1}$$

for any $(x, t) \in Q_{\frac{1}{2}}$. Now we have $u = P_d + v + R_d$ and $u - P_d$ has the required estimate. \square

REMARK. Both Lemma 1.3 and Corollary 1.4 hold for any integer d with $0 \leq d < m$.

2. – A priori estimates on leading polynomials and error terms

Now we may control the leading polynomials and error terms for solutions of general parabolic equations in a uniform way. Roughly speaking if solution u vanishes with order d at 0, then “derivatives” of order up to $d - 1$ vanish at 0 and “derivatives” of order d and error term can be estimated uniformly.

Suppose that L is an m -th order homogeneous parabolic linear operator in $Q_1(0) \subset \mathbb{R}^n \times \mathbb{R}$ given by

$$(2.1) \quad L \equiv \frac{\partial}{\partial t} u - \sum_{|\nu|=0}^m a_\nu(x, t) D_x^\nu$$

where the coefficients verify the following assumptions:

$$(2.2) \quad (-1)^{\frac{m}{2}-1} \sum_{|\nu|=m} a_\nu(0, 0) \xi^\nu \geq \lambda \quad \forall \xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n, ;$$

$$(2.3) \quad \sum_{|\nu|=0}^m |a_\nu(x, t)| \leq \kappa, \quad \forall (x, t) \in Q_1(0) ;$$

and

$$(2.4) \quad \sum_{|\nu|=m} |a_\nu(x, t) - a_\nu(0, 0)| \leq \omega(|(x, t)|), \quad \forall (x, t) \in Q_1(0)$$

for some positive constants λ, κ and some continuous and increasing function $\omega: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\omega(r) \rightarrow 0$ as $r \rightarrow 0+$.

We should emphasize that our assumptions (2.2) and (2.4) are made only at the origin.

We first prove an interior estimate with balls centered only at origin.

LEMMA 2.1. *Let L be an m -th order parabolic operator in $Q_1(0)$ with the form (2.1) satisfying (2.2)-(2.4) and u a $W_p^{m,1}$ solution of $Lu = f$ in $Q_1(0)$ for some $f \in L^p(Q_1)$ with $p > 1 + \frac{n}{m}$. Then there holds the following estimate for any $r \leq R$*

$$\sum_{i=1}^m r^i \|D_x^i u\|_{L^p(Q_r)} \leq C (\|u\|_{L^p(Q_{2r})} + r^m \|f\|_{L^p(Q_{2r})})$$

where C and R are positive constants depending only on λ, κ and ω .

PROOF. Set $L(0) = \partial_t - \sum_{|\nu|=m} a_\nu(0, 0) D_x^\nu$. Then we may write the equation as

$$L(0)u = \sum_{|\nu|=m} (a_\nu - a_\nu(0, 0)) D_x^\nu u + \sum_{|\nu|<m} a_\nu D_x^\nu u + f.$$

By introducing cut-off functions we have for any $0 < r < R \leq 1$

$$\begin{aligned} \|D_x^m u\|_{L^p(Q_r)} &\leq C \{ (\omega(R) + \varepsilon) \|D_x^m u\|_{L^p(Q_R)} \\ &\quad + \frac{C(\varepsilon)}{(R-r)^m} \|u\|_{L^p(Q_R)} + \|f\|_{L^p(Q_R)} \} \end{aligned}$$

where ε is an arbitrary positive number and the constant C depends only on λ, κ and ω . Choose ε such that $C\varepsilon = 1/4$. Then for any R such that $C\omega(R) \leq 1/4$, we get for any $0 < r \leq R$

$$\begin{aligned} \|D_x^m u\|_{L^p(Q_r)} &\leq \frac{1}{2} \|D_x^m u\|_{L^p(Q_R)} \\ &\quad + C \left\{ \frac{1}{(R-r)^m} \|u\|_{L^p(Q_R)} + \|f\|_{L^p(Q_R)} \right\}. \end{aligned}$$

By a standard iteration we get for any $r \leq R$

$$\|D_x^m u\|_{L^p(Q_r)} \leq C \left\{ \frac{1}{(R-r)^m} \|u\|_{L^p(Q_R)} + \|f\|_{L^p(Q_R)} \right\}.$$

Hence for any $r > 0$ with $C\omega(2r) < 1/4$ we obtain

$$r^m \|D_x^m u\|_{L^p(Q_r)} \leq C (\|u\|_{L^p(Q_{2r})} + r^m \|f\|_{L^p(Q_{2r})}).$$

Lemma 2.1 follows from the interpolation. \square

For subsequent results we need more assumptions on leading coefficients. We assume, in addition,

$$(2.5) \quad \sum_{|\nu|=m} \|a_\nu - a_\nu(0, 0)\|_{L^p(Q_r)} \leq Kr^{\alpha + \frac{n+m}{p}}, \quad \forall r \leq 1$$

for some positive constants K and $\alpha < 1$.

THEOREM 2.2. *Let L be an m -th order parabolic operator in $Q_1(0)$ with the form (2.1) satisfying (2.2)-(2.5) and u a $W_p^{m,1}$ solution of $Lu = f$ in $Q_1(0)$ for some $f \in L^p(Q_1)$ with $p > 1 + \frac{n}{m}$. Suppose, for some p -homogeneous polynomial Q of p -degree $d - m$, f satisfies*

$$(2.6) \quad \|f - Q\|_{L^p(Q_r)} \leq \gamma r^{d-m+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq 1$$

for some integer $d \geq m$ and $\gamma > 0$. Then if

$$(2.7) \quad \limsup_{r \rightarrow 0} \frac{1}{r^{d-1+\beta+\frac{n+m}{p}}} \|u\|_{L^p(Q_r)} < \infty,$$

for some $\beta \in (0, 1]$ there holds

$$(2.8) \quad \|u\|_{L^p(Q_r)} \leq C(\|u\|_{L^p(Q_1(0))} + \gamma + \|Q\|_{L^p(Q_1(0))})r^{d+\frac{n+m}{p}} \quad \text{for } r \leq 1$$

where C is a constant depending only on $n, p, m, d, \lambda, \kappa, \alpha$ and K . Moreover there exists a p -homogeneous polynomial P of p -degree d such that

$$(2.9) \quad \frac{\partial}{\partial t} P - \sum_{|v|=m} a_v(0, 0) D_x^v P = Q \quad \text{in } \mathbb{R}^n \times \mathbb{R},$$

$$(2.10) \quad |P(x, t)| \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)})|(x, t)|^d \quad \text{in } Q_1(0)$$

$$(2.11) \quad |u(x, t) - P(x, t)| \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)})|(x, t)|^{d+\alpha} \quad \text{in } Q_R(0)$$

and

$$(2.12) \quad \sum_{i=1}^m r^i \|D_x^i(u - P)\|_{L^p(Q_r(0))} \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)})r^{d+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq R$$

where C and R are constants depending only on $n, p, m, d, \lambda, \alpha, \kappa, K$ and ω .

PROOF. We prove Theorem 2.2 in two steps.

STEP 1. Existence of the p -homogeneous polynomial P .

We set for nonnegative integer k

$$(2.13) \quad c_k = \sup_{r \leq 1} \frac{\|u\|_{L^p(Q_r)}}{r^{d-1+\beta+k\alpha_1+\frac{n+m}{p}}}$$

for some $0 < \alpha_1 \leq \alpha$. We will prove $c_k < \infty$ as long as $\beta + k\alpha_1 \leq 1$. By (2.7) we know $c_0 < \infty$.

Lemma 2.1 implies that for any $r \leq R$

$$(2.14) \quad \begin{aligned} \sum_{i=1}^m r^i \|D_x^i u\|_{L^p(Q_r)} &\leq C(\|u\|_{L^p(Q_{2r})} + r^m \|f\|_{L^p(Q_{2r})}) \\ &\leq C\{c_0 r^{d-1+\beta+\frac{n+m}{p}} + (\gamma + \|Q\|_{L^p(Q_1)})r^{d+\frac{n+m}{p}}\} \\ &\leq C(c_0 + \gamma + \|Q\|_{L^p(Q_1)})r^{d-1+\beta+\frac{n+m}{p}}. \end{aligned}$$

Set $\tilde{L} = \partial_t - \sum_{|v|=m} a_v D_x^v$ and $L(0) = \partial_t - \sum_{|v|=m} a_v(0, 0) D_x^v$. We write the equation as

$$\tilde{L}u = Q + \sum_{|v|<m} a_v D_x^v u + (f - Q) \equiv Q + \phi.$$

By taking the L^p -norm in $Q_r(0)$ and combining (2.14) we get

$$(2.15) \quad \begin{aligned} \|\phi\|_{L^p(Q_r)} &\leq C(r^{1-m}\|u\|_{L^p(Q_{2r})} + r\|f\|_{L^p(Q_r)} + \|f - Q\|_{L^p(Q_{2r})}) \\ &\leq C(c_0 + \gamma + \|Q\|_{L^p(Q_1)})r^{d-m+\min(\beta,\alpha)+\frac{n+m}{p}} \end{aligned}$$

for any $r \leq R$.

Take a p -homogeneous polynomial P_1 of p -degree d such that $L(0)P_1 = Q$. Then we have

$$\begin{aligned} \tilde{L}(u - P_1) &= (L(0) - \tilde{L})P_1 + \phi \\ &= \sum_{|v|=m} (a_v - a_v(0,0))D^v P_1 + \phi. \end{aligned}$$

By assumption (2.5), we get for any $r \leq 1$

$$\sum_{|v|=m} \|(a_v - a_v(0,0))D^v P_1\|_{L^p(Q_r)} \leq C\|P_1\|_{L^p(Q_1)}r^{d-m+\alpha+\frac{n+m}{p}}$$

where C depends only on n, d, m, p and K . Therefore $u - P_1$ satisfies

$$\tilde{L}(u - P_1) = \tilde{\phi} \quad \text{in } Q_1$$

with

$$(2.16) \quad \|\tilde{\phi}\|_{L^p(Q_r)} \leq C(c_0 + \gamma + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)})r^{d-m+\min(\beta,\alpha)+\frac{n+m}{p}}$$

for any $r \leq R$. It is obvious that (2.16) also holds for any q to replace p with $1 < q < p$. By (2.14) we get for any $r \leq R$

$$\begin{aligned} &\sum_{i=1}^m r^i \|D_x^i(u - P_1)\|_{L^p(Q_r)} \\ &\leq C(c_0 + \gamma + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)})r^{d-1+\beta+\frac{n+m}{p}}. \end{aligned}$$

We may write the equation as

$$(2.17) \quad \begin{aligned} L(0)(u - P_1) &= (L(0) - \tilde{L})(u - P_1) + \tilde{\phi} \\ &= \sum_{|v|=m} (a_v - a_v(0,0))D^v(u - P_1) + \tilde{\phi}. \end{aligned}$$

For any q with $1 + \frac{n}{m} < q < p$ we have

$$\begin{aligned} &\sum_{|v|=m} \|(a_v - a_v(0,0))D^v(u - P_1)\|_{L^q(Q_r)} \\ &\leq \sum_{|v|=m} \|a_v - a_v(0,0)\|_{L^{\frac{pq}{p-q}}(Q_r)} \|D^m(u - P_1)\|_{L^p(Q_r)} \\ &\leq C \sum_{|v|=m} \|a_v - a_v(0,0)\|_{L^{\frac{pq}{p-q}}(Q_r)} \\ &\quad \cdot (c_0 + \gamma + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)})r^{d-m-1+\beta+\frac{n+m}{p}}. \end{aligned}$$

If $p > 2(1 + \frac{n}{m})$ we take $q = p/2 > 1 + \frac{n}{m}$. Then $\frac{pq}{p-q} = p$. Hence we have

$$\begin{aligned} & \sum_{|\nu|=m} \|(a_\nu - a_\nu(0, 0)) D^\nu(u - P_1)\|_{L^q(Q_r)} \\ & \leq C(c_0 + \gamma + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}) r^{d-m-1+\beta+\alpha+\frac{n+m}{q}}. \end{aligned}$$

If $p \leq 2(1 + \frac{n}{m})$ we may take any q with $1 + \frac{n}{m} < q < p$. Then $\frac{pq}{p-q} > p$. Hence

$$\begin{aligned} \sum_{|\nu|=m} \|a_\nu - a_\nu(0, 0)\|_{L^{\frac{pq}{p-q}}(Q_r)} & \leq C \sum_{|\nu|=m} \|a_\nu - a_\nu(0, 0)\|_{L^p(Q_r)}^{\frac{p-q}{q}} \\ & \leq Cr^{(\alpha+\frac{n+m}{p})(\frac{p}{q}-1)}. \end{aligned}$$

This implies

$$\begin{aligned} & \sum_{|\nu|=m} \|(a_\nu - a_\nu(0, 0)) D^\nu(u - P_1)\|_{L^q(Q_r)} \\ & \leq C(c_0 + \gamma + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}) r^{d-m-1+\beta+\frac{\alpha(p-q)}{q}+\frac{n+m}{q}}. \end{aligned}$$

In both cases we conclude for some $\alpha_1 \leq \alpha$ and some q with $1 + \frac{n}{m} < q < p$

$$(2.18) \quad \begin{aligned} & \sum_{|\nu|=m} \|(a_\nu - a_\nu(0, 0)) D^\nu \psi\|_{L^q(Q_r)} \\ & \leq C(c_0 + \gamma + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}) r^{d-m-1+\beta+\alpha_1+\frac{n+m}{q}} \end{aligned}$$

for any $r \leq R$.

If $\alpha_1 + \beta < 1$, we apply Corollary 1.4 to $u - P_1$ with p replaced by q . By (2.16)-(2.18) there exists a polynomial P_0 with p -degree $\leq d - 1$ such that

$$\begin{aligned} & |u(x, t) - P_1(x, t) - P_0(x, t)| \\ & \leq C \{c_0 + \gamma + \|Q\|_{L^p(Q_1)} + \|u\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}\} |(x, t)|^{d-1+\beta+\alpha_1} \end{aligned}$$

or

$$\begin{aligned} & |u(x, t) - P_0(x, t)| \\ & \leq C \{c_0 + \gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}\} |(x, t)|^{d-1+\beta+\alpha_1} \end{aligned}$$

for any $(x, t) \in Q_R$. By (2.7) this implies that $P_0 \equiv 0$ and

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d-1+\beta+\alpha_1+\frac{n+m}{p}}} \|u\|_{L^p(Q_r)} < \infty.$$

Hence $c_1 < \infty$. This is an improvement, compared with (2.7). We may repeat the above argument. We may assume, by choosing a smaller β if necessary, that

$$k\alpha_1 + \beta < 1 < (k+1)\alpha_1 + \beta = \alpha_0$$

for some nonnegative integer k . By repeating the above argument k times we obtain

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d-1+k\beta+\alpha_1+\frac{n+m}{p}}} \|u\|_{L^p(Q_r)} < \infty.$$

Then we get instead of (2.18)

$$\begin{aligned} & \sum_{|v|=m} \|(a_v - a_v(0, 0)) D^v \psi\|_{L^q(Q_r)} \\ & \leq C(c_k + \gamma + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}) r^{d-m+\alpha_0+\frac{n+m}{q}} \end{aligned}$$

for any $r \leq R$. By Corollary 1.4 again, there exists a polynomial P_2 of p -degree $\leq d$ with $L(0)P_2 = 0$ such that

$$\begin{aligned} & |u(x, t) - P_1(x, t) - P_2(x, t)| \\ & \leq C \{c_0 + \gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}\} |(x, t)|^{d+\alpha_0} \end{aligned}$$

for any $(x, t) \in Q_R$. Set $P = P_1 + P_2$. Then $L(0)P = Q$ and (2.7) implies that P is p -homogeneous of p -degree d . In particular we have

$$(2.19) \quad \limsup_{r \rightarrow 0} \frac{1}{r^{d+\frac{n+m}{p}}} \|u\|_{L^p(Q_r)} < \infty.$$

Now set

$$\tilde{c} = \sup_{r \leq 0} \frac{1}{r^{d+\frac{n+m}{p}}} \|u\|_{L^p(Q_r)} < \infty.$$

By essentially the same argument we obtain

$$(2.20) \quad \begin{aligned} & |u(x, t) - P(x, t)| \\ & \leq C \{\tilde{c} + \gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)} + \|P_1\|_{L^p(Q_1)}\} |(x, t)|^{d+\alpha} \end{aligned}$$

for any $(x, t) \in Q_R$.

STEP 2. Estimates of P and $u - P$.

We will prove the required estimates under the additional assumption

$$(2.21) \quad \sum_{|v|=m} |a_v(x, t) - a_v(0)| + \sum_{|v|<m} |a_v(x, t)| \leq \eta \quad \text{in } Q_1$$

for some small $\eta > 0$, depending only on $n, p, m, d, \lambda, \kappa, \alpha, K$ and ω . The general case can be recovered by a simple transformation $(x, t) \rightarrow (Rx, R^m t)$ for an appropriate $R \in (0, 1)$.

Set $\psi = u - P$ and

$$(2.22) \quad \delta = \sup_{r \leq 1} \frac{\|\psi\|_{L^p(Q_r)}}{r^{d+\alpha+\frac{n+m}{p}}}.$$

This is finite by (2.20). Then we may write equation as

$$L\psi = f - LP = (f - Q) + (L(0) - L)P$$

since $L(0)P = Q$. By Lemma 2.1 we have for any $r \leq R$

$$(2.23) \quad \begin{aligned} & \sum_{i=1}^m r^i \|D^i \psi\|_{L^p(Q_r)} \\ & \leq C(\|\psi\|_{L^p(Q_{2r})} + r^m \|f - Q\|_{L^p(Q_{2r})} + r^m \|(L(0) - L)P\|_{L^p(Q_{2r})}) \\ & \leq C(\delta + \gamma + \|P\|_{L^p(Q_1)}) r^{d+\alpha+\frac{n+m}{p}}. \end{aligned}$$

We write the equation as

$$L(0)\psi = \sum_{|v|=m} (a_v - a_v(0)) D_x^v \psi + \sum_{|v|<m} a_v D_x^v \psi + (f - Q) + (L(0) - L)P \equiv F.$$

Then we have by (2.21) for any $r \leq R$

$$(2.24) \quad \|F\|_{L^p(Q_r)} \leq C(\eta\delta + \gamma + \|P\|_{L^p(Q_1)}) r^{d-m+\alpha+\frac{n}{p}}.$$

Now we may apply Corollary 1.4 to obtain a polynomial \tilde{P} of p -degree $\leq d$ such that

$$\|\psi - \tilde{P}\|_{L^p(Q_r)} \leq C(\eta\delta + \gamma + \|\psi\|_{L^p(Q_1)} + \|P\|_{L^p(Q_1)}) r^{d+\alpha+\frac{n+m}{p}} \quad \text{for } r \leq R.$$

Condition (2.22) implies $\tilde{P} \equiv 0$. Hence we have

$$\|\psi\|_{L^p(Q_r)} \leq C(\eta\delta + \gamma + \|\psi\|_{L^p(Q_1)} + \|P\|_{L^p(Q_1)}) r^{d+\alpha+\frac{n+m}{p}} \quad \text{for } r \leq R.$$

It is obviously true for $R \leq r \leq 1$. By taking the supremum over $r \in (0, 1]$ we get

$$\delta \leq C(\eta\delta + \gamma + \|\psi\|_{L^p(Q_1)} + \|P\|_{L^p(Q_1)}).$$

If η is small such that $C\eta \leq 1/2$, we have

$$\delta \leq C(\gamma + \|P\|_{L^p(Q_1)} + \|\psi\|_{L^p(Q_1)})$$

or

$$\|\psi\|_{L^p(Q_r)} \leq C(\gamma + \|P\|_{L^p(Q_1)} + \|\psi\|_{L^p(Q_1)}) r^{d+\alpha+\frac{n+m}{p}} \quad \text{for } r \leq 1.$$

By (2.24) we get

$$\|F\|_{L^p(Q_r)} \leq C(\gamma + \|P\|_{L^p(Q_1)} + \|\psi\|_{L^p(Q_1)})r^{d-m+\alpha+\frac{n}{p}}.$$

Hence Corollary 1.4 implies

$$(2.25) \quad |\psi(x, t)| \leq C(\gamma + \|P\|_{L^p(Q_1)} + \|\psi\|_{L^p(Q_1)})|(x, t)|^{d+\alpha} \quad \text{in } Q_R.$$

By definition of ψ we have

$$|P(x, t)| \leq |u(x, t)| + C(\gamma + \|u\|_{L^p(Q_1)} + \|P\|_{L^\infty(Q_1)})|(x, t)|^{d+\alpha} \quad \text{in } Q_R.$$

Again interior estimates imply that

$$|u(x, t)| \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)}) \quad \text{in } Q_R.$$

We then obtain

$$|P(x, t)| \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)}) + C\|P\|_{L^\infty(Q_1)}|(x, t)|^{d+\alpha} \quad \text{in } Q_R.$$

Suppose P restricted in $\{(e_x, e_t) \in \mathbb{R}^n \times \mathbb{R}; |(e_x, e_t)| = 1\}$ attains its maximum at (e_x, e_t) . Choose $x = |(x, t)|e_x$ and $t = |(x, t)|^m e_t$. Then we get by p -homogeneity of P

$$\begin{aligned} & |P|_{L^\infty(Q_1)}|(x, t)|^d \\ & \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)}) + C\|P\|_{L^\infty(Q_1)}|(x, t)|^{d+\alpha}. \end{aligned}$$

Choose (x, t) small we obtain

$$|P|_{L^\infty(Q_1)} \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)})$$

or

$$|P(x, t)| \leq C(\gamma + \|u\|_{L^p(Q_1)} + \|Q\|_{L^p(Q_1)})|(x, t)|^d.$$

This is (2.10). With (2.25) we get (2.11). (2.12) follows from interior estimates. \square

REMARK. We can prove a similar result for integer $d < m$. Except $f \in L^p(Q_1)$ we do not need any extra assumptions on f . Then (2.9) holds with $Q \equiv 0$ and (2.10)-(2.12) hold with $\gamma = \|f\|_{L^p(Q_1)}$.

REMARK. Theorem 2.2 still holds if instead of (2.3) a_ν satisfies some appropriate integral condition for $|\nu| \leq m - 1$.

REMARK We can also compare leading polynomials of two solutions. In the following we take $i = 1, 2$. Suppose L_i is an m -th order parabolic operator as in Theorem 2.2 and u_i is a $W_p^{m,1}$ solution of $L_i u_i = f_i$ in $Q_1(0)$ for some

$f_i \in L^p(Q_1)$ with $p > 1 + \frac{n}{m}$. Suppose, for some p -homogeneous polynomial Q_i of p -degree $d - m$, f_i satisfies

$$\|f_i - Q_i\|_{L^p(Q_r)} \leq \gamma_i r^{d-m+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq 1$$

for some positive constants $\gamma_i > 0$. We assume

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d+\frac{n+m}{p}}} \|u_i\|_{L^p(Q_r)} < \infty$$

and P_i is the p -homogeneous polynomial of p -degree d given by Theorem 2.2.

We subtract two equations to get

$$(2.26) \quad L_2(u_2 - u_1) = (L_1 - L_2)u_1 + (f_2 - f_1) \equiv F.$$

We write $u_1 = P_1 + \psi_1$. By (2.10)-(2.12) we have

$$|P_1(x, t)| \leq C(\gamma_1 + \|u_1\|_{L^p(Q_1)} + \|Q_1\|_{L^p(Q_1)})|(x, t)|^d \quad \text{in } Q_1(0)$$

and

$$\begin{aligned} & \sum_{i=0}^m r^i \|D_x^i \psi_1\|_{L^p(Q_r(0))} \\ & \leq C(\gamma_1 + \|u_1\|_{L^p(Q_1)} + \|Q_1\|_{L^p(Q_1)})r^{d+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq R. \end{aligned}$$

The function F begin with p -homogeneous polynomial of p -degree $d - m$

$$Q = \sum_{|v|=m} (a_{2,v}(0) - a_{1,v}(0))D_x^v P_1 + (Q_2 - Q_1)$$

and the error term has the estimate

$$\begin{aligned} \|F - Q\|_{L^p(Q_r)} & \leq |L_1 - L_2| \left(\sum_{i=0}^m r^i \|D_x^i \psi_1\|_{L^p(Q_r(0))} + \sum_{i=0}^{m-1} r^i \|D_x^i P_1\|_{L^p(Q_r(0))} \right) \\ & \quad + \|(f_1 - Q_1) - (f_2 - Q_2)\|_{L^p(Q_r)} \end{aligned}$$

where $|L_1 - L_2|$ denotes the maximal difference of corresponding coefficients of L_1 and L_2 . Set γ such that

$$\|(f_1 - Q_1) - (f_2 - Q_2)\|_{L^p(Q_r)} \leq \gamma r^{d-m+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq 1.$$

Hence we have

$$\begin{aligned} \|F - Q\|_{L^p(Q_r)} & \leq \{\gamma + C|L_1 - L_2|(\gamma_1 + \|u_1\|_{L^p(Q_1)} \\ & \quad + \|Q_1\|_{L^p(Q_1)})\}r^{d-m+\alpha+\frac{n+m}{p}} \quad \text{for any } r \leq R. \end{aligned}$$

We may apply Theorem 2.2 to the equation (2.26) to obtain

$$|P_1(x, t) - P_2(x, t)| \leq C_* |(x, t)|^d \quad \text{in } Q_1(0)$$

and

$$|(u_1(x, t) - P_1(x, t)) - (u_2(x, t) - P_2(x, t))| \leq C_* |(x, t)|^{d+\alpha} \quad \text{in } Q_{\frac{R}{2}}(0)$$

where C_* satisfies

$$C_* \leq C \{ \gamma + \|u_1 - u_2\|_{L^p(Q_1)} + \|Q_1 - Q_2\|_{L^p(Q_1)} \\ + |L_1 - L_2|(\gamma_1 + \|u_1\|_{L^p(Q_1)} + \|Q_1\|_{L^p(Q_1)}) \}$$

where C is a constant depending only on $n, p, m, d, \lambda, \alpha, \kappa, K$ and ω .

Hence we have the following result.

COROLLARY 2.3. *Let $\{L_i\}_{i=0}^\infty$ be a sequence of m -th order parabolic operators in $Q_1(0)$ with the form (2.1) satisfying (2.2)-(2.5) and $\{u_i\}_{i=0}^\infty$ a sequence of $W_p^{m,1}$ functions such that each u_i is a solution of $L_i u_i = f_i$ in $Q_1(0)$ for some $f_i \in L^p(Q_1)$ with $p > 1 + \frac{n}{m}$. Suppose that $L_i \rightarrow L_0$ as $i \rightarrow \infty$ in the sense that the corresponding coefficients converge in the sup-norm and that, for a sequence of p -homogeneous polynomial $\{Q_i\}_{i=0}^\infty$ of p -degree $d - m$, f_i satisfies*

$$\sup_{r \geq 1} \frac{1}{r^{d-m+\alpha+\frac{n+m}{p}}} \|f_0 - Q_0\|_{L^p(Q_r)} < \infty$$

and

$$\|Q_i - Q_0\|_{L^p(Q_1)} + \sup_{r \geq 1} \frac{1}{r^{d-m+\alpha+\frac{n+m}{p}}} \|(f_i - Q_i) - (f_0 - Q_0)\|_{L^p(Q_r)} \rightarrow 0 \\ \text{as } i \rightarrow \infty$$

for some integer $d \geq m$. If

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d+\frac{n+m}{p}}} \|u_i\|_{L^p(Q_r)} < \infty \quad \text{for any } i = 1, 2, \dots,$$

and

$$u_i \rightarrow u_0 \quad \text{as } i \rightarrow \infty \text{ in } L^p(Q_1)$$

then there holds

$$\sup_{|(x,t)| < \frac{1}{2}} \frac{1}{|(x,t)|^d} |u_i(x,t) - u_0(x,t)| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Moreover if $\{P_i\}_{i=0}^\infty$ is the sequence of p -homogeneous polynomials of p -degree d for $\{u_i\}_{i=0}^\infty$ as in Theorem 2.2 then

$$|P_i - P_0|_{L^\infty(Q_1)} + \sup_{|(x,t)| < \frac{1}{2}} \frac{1}{|(x,t)|^{d+\alpha}} |(u_i(x,t) - P_i(x,t)) - (u_0(x,t) - P_0(x,t))| \\ \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

3. – Schauder estimates

Before we state our main estimates we first introduce some terminology.

DEFINITION. Let u be an L^p function in Q_1 for $1 \leq p \leq \infty$. For $\alpha \in (0, 1)$ and nonnegative integer d , u is $C^{d+\alpha, \frac{d+\alpha}{m}}$ at 0 in L^p sense, $u \in C_{L^p}^{d+\alpha, \frac{d+\alpha}{m}}(0)$, if for some polynomial P of p -degree not exceeding d ,

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d+\alpha+\frac{n+m}{p}}} \|u - P\|_{L^p(Q_r)} < \infty.$$

Note in the above definition, the polynomial P is unique if they exist. We may also define the corresponding semi-norms as follows:

$$[u]_{C_{L^p}^{i, \frac{i}{m}}}(0) = \sum_{|\nu|+m\ell=i} |D_x^\nu D_t^\ell P(0)| \quad \text{for } i = 0, 1, \dots, d$$

$$[u]_{C_{L^p}^{d+\alpha, \frac{d+\alpha}{m}}}(0) = \sup_{0 < r \leq 1} \frac{1}{r^{d+\alpha+\frac{n+m}{p}}} \|u - P\|_{L^p(Q_r)}.$$

We may also define the following norm

$$|u|_{C_{L^p}^{d+\alpha, \frac{d+\alpha}{m}}}(0) = \sum_{i=0}^d [u]_{C_{L^p}^{i, \frac{i}{m}}}(0) + [u]_{C_{L^p}^{d+\alpha, \frac{d+\alpha}{m}}}(0).$$

For brevity we use the notation $C^{d+\alpha}$ instead of $C^{d+\alpha, \frac{d+\alpha}{m}}$.

Now we may state the pointwise Schauder estimates. The operators are as discussed in Section 2. Suppose that L is an m -th order homogeneous parabolic linear operator in $Q_1(0) \subset \mathbb{R}^n \times \mathbb{R}$ given by

$$(3.1) \quad L \equiv \frac{\partial}{\partial t} u - \sum_{|\nu|=0}^m a_\nu(x, t) D_x^\nu$$

where the coefficients verify the following assumptions:

$$(3.2) \quad (-1)^{\frac{m}{2}-1} \sum_{|\nu|=m} a_\nu(0, 0) \xi^\nu \geq \lambda \quad \forall \xi \in \mathbb{S}^{n-1} \subset \mathbb{R}^n, \quad (x, t) \in Q_1(0);$$

$$(3.3) \quad \sum_{|\nu|=0}^m |a_\nu(x, t)| \leq \kappa, \quad \forall (x, t) \in Q_1(0);$$

and

$$(3.4) \quad \sum_{|\nu|=m} |a_\nu(x, t) - a_\nu(0, 0)| \leq \omega(|(x, t)|) \quad \forall (x, t) \in Q_1(0)$$

for some positive constants λ, κ and some increasing continuous function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\omega(0) = 0$.

We state a special case first.

THEOREM 3.1. *Let L be an m -th order parabolic operator in $Q_1(0)$ with the form (3.1) satisfying (3.2)-(3.4) and u a $W_p^{m,1}$ solution of $Lu = f$ in $Q_1(0)$ for some $f \in L^p(Q_1)$ with $p > 1 + \frac{n}{m}$. Suppose that $d \geq m$ is a nonnegative integer such that*

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d+\frac{n+m}{p}}} \|u\|_{L^p(Q_r)} < \infty,$$

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d-m+\frac{n+m}{p}}} \|f\|_{L^p(Q_r)} < \infty.$$

If, for some $\alpha \in (0, 1)$ and some integer $l \geq 0$, $f \in C_{LP}^{d-m+l+\alpha}(0)$ and $a_\nu \in C_{LP}^{l-m+|\nu|+\alpha}(0)$ for any $|\nu| \geq \max\{m-l, 0\}$, then $u \in C_{L^\infty}^{d+l+\alpha}(0)$. Moreover there holds the following estimate

$$\sum_{i=d}^{d+l} [u]_{C_{L^\infty}^i(0)} + [u]_{C_{L^\infty}^{d+l,\alpha}(0)} \leq C \left\{ \|u\|_{L^p(Q_1)} + \sum_{i=d-m}^{d-m+l} [f]_{C_{LP}^i(0)} + [f]_{C_{LP}^{d-m+l+\alpha}(0)} \right\}$$

where C depends only on $n, m, p, d, l, \lambda, \kappa, \alpha, \omega$ and $[a_\nu]_{C_{LP}^i(0)}$ for $i = d-m, \dots, d-m+l$ and $[a_\nu]_{C_{LP}^{l-m+|\nu|+\alpha}(0)}$ for $|\nu| \geq m-l$.

PROOF. We will prove Theorem 3.1 by induction on l . For $l = 0$, it is Theorem A2.2. Note that (2.5) is equivalent to $a_\nu \in C_{LP}^\alpha(0)$ for $|\nu| = m$. For illustration we prove for $l = 1$. By assumptions there exist p -homogeneous polynomials Q_{d-m} and Q_{d-m+1} of p -degrees $d-m$ and $d-m+1$ respectively, p -homogeneous polynomials $a_\nu^{(i)}$ of p -degree i for $|\nu| \geq m-i$, $i = 0, 1$, such that

$$\|f - Q_{d-m} - Q_{d-m+1}\|_{L^p(Q_r)} \leq [f]_{C_{LP}^{d-m+1+\alpha}(0)} r^{d-m+1+\alpha+\frac{n+m}{p}},$$

$$\|a_\nu - a_\nu(0) - a_\nu^{(1)}\|_{L^p(Q_r)} \leq [a_\nu]_{C_{LP}^{1+\alpha}(0)} r^{1+\alpha+\frac{n+m}{p}}, \quad |\nu| = m,$$

$$\|a_\nu - a_\nu^{(0)}\|_{L^p(Q_r)} \leq [a_\nu]_{C_{LP}^\alpha(0)} r^{\alpha+\frac{n+m}{p}}, \quad |\nu| = m-1,$$

for any $r \leq 1$. Write $\phi = f - Q_{d-m} - Q_{d-m+1}$. By Theorem 2.2, there exists a homogeneous polynomial P_d satisfying (2.9) to (2.12). Set $\psi = u - P_d$. Then ψ satisfies the equation

$$L\psi = Q_{d-m+1} + \phi + \sum_{|\nu|=m} (a_\nu - a_\nu(0)) D^\nu P_d + \sum_{|\nu|<m} a_\nu D^\nu P_d \equiv \tilde{f}.$$

It is easy to see that the polynomial

$$\tilde{Q} = Q_{d-m+1} + \sum_{|\nu|=m} a_\nu^{(1)} D^\nu P_d + \sum_{|\nu|=m-1} a_\nu^{(0)} D^\nu P_d$$

is p -homogeneous with p -degree $d - m + 1$ and that

$$\|\tilde{f} - \tilde{Q}\|_{L^p(Q_r)} \leq \{[f]_{C_{L^p}^{d-m+1+\alpha}(0)} + C\|P_d\|_{L^p(Q_1)}\} r^{d-m+1+\alpha+\frac{n+m}{p}}$$

for any $r \leq 1$. By (2.12) we have

$$\limsup_{r \rightarrow 0} \frac{1}{r^{d+\alpha+\frac{n+m}{p}}} \|\psi\|_{L^p(Q_r)} < \infty.$$

We may apply Theorem 2.2 to ψ with d replaced by $d + 1$. Hence there exists a p -homogeneous polynomial P_{d+1} of p -degree $d + 1$ satisfying

$$\frac{\partial}{\partial t} P_{d+1} - \sum_{|\nu|=m} a_\nu(0, 0) D_x^\nu P_{d+1} = \tilde{Q}$$

and

$$\begin{aligned} |P_{d+1}(x, t)| &\leq C_* |x, t|^{d+1} && \text{for } (x, t) \in Q_{R/2} \\ |\psi(x, t) - P_{d+1}(x, t)| &\leq C_* |x, t|^{d+1+\alpha} && \text{for } (x, t) \in Q_{R/2} \end{aligned}$$

where

$$C_* \leq C([f]_{C_{L^p}^{d-m+1+\alpha}(0)} + \|\psi\|_{L^p(Q_R)} + \|\tilde{P}_d\|_{L^p(Q_1)} + \|\tilde{Q}\|_{L^p(Q_1)}).$$

By expression for \tilde{Q} and estimates on ψ and P_d there holds

$$\begin{aligned} C_* &\leq C([f]_{C_{L^p}^{d-m+1+\alpha}(0)} + [f]_{C_{L^p}^{d-m+\alpha}(0)} \\ &\quad + \|Q_{d-m}\|_{L^p(Q_{\frac{1}{2}}(0))} + \|Q_{d-m+1}\|_{L^p(Q_1(0))} + \|u\|_{L^p(Q_1(0))}) \\ &\leq C(\|u\|_{L^p(Q_1(0))} + [f]_{C_{L^p}^{d-m}(0)} + [f]_{C_{L^p}^{d-m+1}(0)} + [f]_{C_{L^p}^{d-m+1+\alpha}(0)}). \end{aligned}$$

This finishes the proof for $l = 1$. □

THEOREM 3.2. *Let L be an m -th order parabolic operator in $Q_1(0)$ with the form (3.1) satisfying (3.2)-(3.4) and u a $W_p^{m,1}$ solution of $Lu = f$ in $Q_1(0)$ for some $f \in L^p(Q_1)$ with $p > 1 + \frac{n}{m}$. If, for some $\alpha \in (0, 1)$ and some integer $d \geq m$, $f \in C_{L^p}^{d-m+\alpha}(0)$ and $a_\nu \in C_{L^p}^{d-m+\alpha}(0)$ for any $|\nu| = 0, 1, \dots, m$, then $u \in C_{L^\infty}^{d+\alpha}(0)$. Moreover there holds the following estimate*

$$|u|_{C_{L^\infty}^{d+\alpha}(0)} \leq C\{\|u\|_{L^p(Q_1)} + |f|_{C_{L^p}^{d-m+\alpha}(0)}\}$$

where C depends only on $n, m, p, d, \lambda, \kappa, \alpha, \omega$ and $[a_\nu]_{C_{L^p}^{d-m+\alpha}(0)}$ for $|\nu| = 0, 1, \dots, m$.

PROOF. It is similar to that of Theorem 3.1. \square

REMARK. Both Theorem 3.1 and Theorem 3.2 hold for integer d with $1 \leq d \leq m - 1$. No extra assumptions are needed for coefficients a_ν and nonhomogeneous term f . Only $\|f\|_{L^p(Q_1)}$ appears in the right side of the estimates.

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