

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 26, n° 4 (1998), p. 623-661

http://www.numdam.org/item?id=ASNSP_1998_4_26_4_623_0

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On the Birkhoff Normal Form of a Completely Integrable Hamiltonian System Near a Fixed Point with Resonance

THOMAS KAPPELER* – YUJI KODAMA* – ANDRAS NÉMETHI**

Abstract. We consider an integrable Hamiltonian system with a real analytic Hamiltonian H near an elliptic fixed point P . If H has a simple resonance and admits a semisimple Hessian at P we show that there exists a real analytic change of coordinates which brings the Hamiltonian into normal form. In the new coordinates, the level sets of the system are analyzed in terms of the nature of the simple resonance.

Mathematics Subject Classification (1991): 58F07 (primary), 58F18, 58F36 (secondary).

1. – Introduction and summary of the results

In this paper we are concerned with the normal form of a completely integrable Hamiltonian system near an equilibrium point. Let $H = H(z)$ be an analytic function, $H: U \rightarrow \mathbb{C}$, defined on an open neighborhood U of the origin in \mathbb{C}^{2n} . Assume that H has a power series expansion near the origin $z = (x, y) = 0 \in \mathbb{C}^{2n}$ of the form $H = \sum_{j=1}^n \lambda_j x_j y_j + O(|z|^3)$ where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

Normal form theory for Hamiltonian systems was first studied by Birkhoff (cf. [Mo]). He proved that in the case where $\lambda_1, \dots, \lambda_n$ are rationally independent (i.e. in the nonresonant case), there exists a formal canonical coordinate transformation, $z = \varphi(\zeta) = \zeta + O(|\zeta|^2)$, so that $H \circ \varphi$ is a formal power series,

$$\sum_{\alpha \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}} c_\alpha \xi^\alpha \eta^\alpha \quad (\text{Birkhoff normal form})$$

with $\zeta = (\xi, \eta) \in \mathbb{C}^{2n}$. Later Siegel [Si] showed that the power series which

* Supported in part by NSF.

** Supported in part by an OSU Seed Grant.

Pervenuto alla Redazione il 10 giugno 1996 e in forma definitiva il 28 gennaio 1998.

define the coordinate transformation φ are generically divergent. Notice that if these power series are convergent, then the Hamiltonian system is in fact integrable with $I_j := \xi_j \eta_j$ ($1 \leq j \leq n$) being functionally independent integrals which Poisson commute. The converse is also true: first results concerning a convergent Birkhoff normal form of an integrable Hamiltonian system near a nonresonant fixed point are due to Vey [Ve] (cf. also [E1] where results in the C^∞ -case were proved) and were later substantially improved by Ito [It1].

For the purpose of classification it is useful to generalize the concept of Birkhoff normal form to Hamiltonian systems near a resonant fixed point and one might ask again if an integrable Hamiltonian system has a Birkhoff normal form near a resonant fixed point.

The only results so far in this direction are due to Ito [It2] and concern a special case of simple resonance. In this paper we treat the general case of a simple resonance. To state our results we introduce the following notation:

Let $G_j = G_j(z)$ ($1 \leq j \leq n$) be holomorphic functions, $G_j: U \rightarrow \mathbb{C}$, defined for $z = (x_k, y_k)_{1 \leq k \leq n}$ in an open neighborhood U of the origin in \mathbb{C}^{2n} such that G_1, \dots, G_n pairwise Poisson commute, i.e. $\{G_i, G_j\} = \sum_{k=1}^n \frac{\partial G_i}{\partial x_k} \frac{\partial G_j}{\partial y_k} - \frac{\partial G_i}{\partial y_k} \frac{\partial G_j}{\partial x_k} = 0$, are elements in \mathcal{M}^2 and have the property that dG_1, \dots, dG_n are generically linearly independent. Here \mathcal{M}^2 denotes the vector space of germs of analytic functions f at 0, which vanish up to first order at 0 ($f(0) = 0$, $\partial_{x_k} f(0) = 0$, $\partial_{y_k} f(0) = 0$, $1 \leq k \leq n$). Let \mathcal{A} be the algebra defined by

$$(1.1) \quad \mathcal{A} := \{f \in \mathcal{M}^2 \mid \{f, G_j\} \equiv 0, \quad 1 \leq j \leq n\}.$$

Then \mathcal{A} is Abelian ($\{f, g\} = 0$ for all $f, g \in \mathcal{A}$), and has the property that $h \in \mathcal{A}$ if there exists $g \in \mathcal{A}$ with $g \neq 0$ and $h \cdot g \in \mathcal{A}$.

Following Ito [It2], let \mathcal{P}_m denote the vector space over \mathbb{C} of all homogeneous polynomials of degree m in $2n$ variables with complex coefficients. Then \mathcal{P}_2 is a Lie algebra under the Poisson bracket $\{\cdot, \cdot\}$. The map $\mathcal{P}_2 \rightarrow sp(n, \mathbb{C})$ associating to $f \in \mathcal{P}_2$ the $2n \times 2n$ matrix $\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} \partial_x^2 f & \partial_{xy}^2 f \\ \partial_{xy}^2 f & \partial_y^2 f \end{pmatrix}$ is a Lie algebra isomorphism. Here $sp(n, \mathbb{C})$ is the Lie algebra of the group $Sp(n, \mathbb{C})$ of $2n \times 2n$ symplectic matrices. As $sp(n, \mathbb{C})$ is semisimple, \mathcal{P}_2 is semisimple and therefore admits a Jordan decomposition: for $f \in \mathcal{P}_2$, we write $f = f_s + f_{\text{nil}}$ where $f_s = \Pi_s f$ is the projection of f on its semisimple part and $f_{\text{nil}} = \Pi_{\text{nil}} f$ is the projection of f on its nilpotent part, i.e. $\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} \partial_x^2 f_s & \partial_{xy}^2 f_s \\ \partial_{xy}^2 f_s & \partial_y^2 f_s \end{pmatrix}$ is a semisimple and $\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} \partial_x^2 f_{\text{nil}} & \partial_{xy}^2 f_{\text{nil}} \\ \partial_{xy}^2 f_{\text{nil}} & \partial_y^2 f_{\text{nil}} \end{pmatrix}$ is a nilpotent matrix. Notice that $\{f_s, f_{\text{nil}}\} = 0$ and one can find a linear symplectic change of coordinates so that $f_s = \sum_{j=1}^n \lambda_j x_j y_j$. The numbers $\pm \lambda_1, \dots, \pm \lambda_n$ coincide with the spectrum of $\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \begin{pmatrix} \partial_x^2 f_s & \partial_{xy}^2 f_s \\ \partial_{xy}^2 f_s & \partial_y^2 f_s \end{pmatrix}$ and are therefore independent of the choice of coordinates. Denote by Λ_f the sublattice of \mathbb{Z}^n defined by

$$\Lambda_f := \{\alpha \in \mathbb{Z}^n \mid \langle \alpha, \lambda \rangle = 0\}$$

where $\lambda := (\lambda_1, \dots, \lambda_n)$ and $\langle \alpha, \lambda \rangle = \sum_{j=1}^n \alpha_j \lambda_j$. We say that Λ_f is the resonance lattice associated to $f \in \mathcal{P}_2$.

For the algebra \mathcal{A} above denote by \mathcal{A}_s and \mathcal{A}_{nil} the semisimple respectively nilpotent part of the projection \mathcal{A}_2 of \mathcal{A} onto \mathcal{P}_2 .

DEFINITION. (i) \mathcal{A} is said to be *nonresonant* at 0 if there exists $H \in \mathcal{A}$ so that the resonance lattice Λ_{H_s} associated to the semisimple part $H_s \in \mathcal{P}_2$ of H^2 (or of H , for short) is trivial, i.e. $\Lambda_{H_s} = \{0\}$.

(ii) \mathcal{A} is said to have a *simple resonance* at 0 if there exists $\mu \in \mathbb{Z}^n \setminus \{0\}$ with $|\mu| := \sum_1^n |\mu_j| \geq 2$ such that for every f in \mathcal{A} , $\{k\mu \mid k \in \mathbb{Z}\} \subset \Lambda_{f_s}$ and for some $H \in \mathcal{A}$, $\Lambda_{H_s} = \{k\mu \mid k \in \mathbb{Z}\}$. The vector μ is called a *prime resonance vector* of \mathcal{A} and is uniquely determined up to sign. We say that $\Lambda_{\mathcal{A}} := \{k\mu \mid k \in \mathbb{Z}\}$ is the resonance lattice of \mathcal{A} . It is a lattice with $\dim \Lambda_{\mathcal{A}} = 1$.

Let $\mu \in \mathbb{Z}^n \setminus \{0\}$ be a prime resonance vector. Then we can choose a basis of \mathbb{Z}^n , $(\rho^{(j)})_{1 \leq j \leq n}$, so that $\langle \rho^{(j)}, \mu \rangle = \delta_{jn}$. In particular, the $n \times n$ matrix whose columns are given by the $\rho^{(j)}$'s is unimodular (i.e. in $GL(n, \mathbb{Z})$) and $\rho^{(1)}, \dots, \rho^{(n-1)}$ is a basis (over \mathbb{Z}) of the $n - 1$ dimensional sublattice $\{\xi \in \mathbb{Z}^n \mid \langle \xi, \mu \rangle = 0\}$ of \mathbb{Z}^n . Introduce $\tau_j := \sum_{k=1}^n \rho_k^{(j)} x_k y_k$ ($1 \leq j \leq n$) as well as $\tau_{n+1} = x^{\mu^+} y^{\mu^-}$, $\tau_{n+2} = x^{\mu^-} y^{\mu^+}$ where $\mu^- := \mu^+ - \mu$ and $\mu^+ = (\mu_k^+)_{1 \leq k \leq n}$ is given by $\mu_k^+ := \mu_k$ if $\mu_k \geq 0$ and $\mu_k^+ := 0$ if $\mu_k < 0$.

In the first part of this paper (Section 2) we prove the following

THEOREM 1.1. *Assume that \mathcal{A} has a simple resonance at 0 and let $\mu \in \mathbb{Z}^n \setminus \{0\}$ be a prime resonant vector of \mathcal{A} (thus, in particular, $|\mu| \geq 2$). Then there exists an analytic, symplectic change of coordinates φ in a neighborhood of 0 in \mathbb{C}^{2n} , so that with respect to the new coordinates, \mathcal{A} has the following properties:*

- (1) $\tau_j \in \mathcal{A}$ ($1 \leq j \leq n - 1$);
- (2) any element f in \mathcal{A} has a convergent power series expansion in $\tau_1, \dots, \tau_{n+2}$.

REMARK 1.1. The special case of Theorem 1.1 where $\mu = (\mu_1, \mu_2, 0, \dots, 0) \in \mathbb{Z}^n \setminus \{0\}$ is due to Ito [It2]. In the same paper he also considers the case where $\mu = (1, 0, \dots, 0)$ [It2, Theorem 3] which is not included in the formulation of Theorem 1.1.

To prove his results Ito uses a rapidly convergent iteration procedure and it turns out that the same procedure can be applied to prove Theorem 1.1.

REMARK 1.2. As in [It2, Theorem 2], there is an analogous result to Theorem 1.1 for an algebra \mathcal{A} of germs of real analytic functions at 0, generated by real analytic integrals G_1, \dots, G_n which Poisson commute. We say that \mathcal{A} is elliptic if 0 is an elliptic fixed point (i.e. for $H \in \mathcal{A}$ arbitrary, the spectrum of $\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} d^2 H_s$ is purely imaginary where $d^2 H_s$ denotes the Hessian of H_s).

THEOREM 1.1'. *Assume that \mathcal{A} is an algebra of germs of real analytic functions at 0, generated by functionally independent integrals G_1, \dots, G_n which Poisson commute. Further assume that $\mathcal{A}_{\text{nil}} = 0$ and that \mathcal{A} is elliptic and has a simple resonance. Let μ be a prime resonance vector, $|\mu| \geq 2$.*

Then there exists a real analytic, symplectic change of coordinates φ in a neighborhood of 0 in \mathbb{R}^{2n} , so that with respect to the new coordinates, (\hat{x}, \hat{y}) , \mathcal{A} has the following properties:

- (1) $\hat{\tau}_j := \sum_{k=1}^n \rho_k^{(j)} (\hat{x}_k^2 + \hat{y}_k^2) / 2 \in \mathcal{A}$ ($1 \leq j \leq n - 1$);
- (2) any element f in \mathcal{A} has a convergent power series expansion in $\hat{\tau}_1, \dots, \hat{\tau}_n, \Re \hat{\tau}_{n+1}, \Im \hat{\tau}_{n+1}$ where $\hat{\tau}_n := \sum_{k=1}^n \rho_k^{(n)} (\hat{x}_k^2 + \hat{y}_k^2) / 2$ and $\hat{\tau}_{n+1} := \prod_{k=1}^n (\hat{x}_k + i \hat{y}_k)^{\mu_k^+} (\hat{x}_k - i \hat{y}_k)^{\mu_k^-}$.

Notice that in view of Corollary 1.2 below, $\mathcal{A}_{nil} \neq 0$ implies that $|\mu| = 2$. Further we remark that the case where \mathcal{A} is elliptic with $\mathcal{A}_{nil} \neq 0$ and has a simple resonance has been treated in [It2] (cf. [Ar1, Appendix 6] for a classification of quadratic Hamiltonians).

Let us contrast Theorem 1.1' with the corresponding one for integrable systems without resonances which is due to Vey [Ve] and, in a generalized version, to Ito [It1]. This result asserts that there exist Birkhoff coordinates (\hat{x}, \hat{y}) near 0, i.e. coordinates whose associated symplectic polar coordinates, given by $I_k := (\hat{x}_k^2 + \hat{y}_k^2) / 2, \phi_k := \arctan(\hat{y}_k / \hat{x}_k)$ ($1 \leq k \leq n$), are action-angle variables for the integrable Hamiltonian system under consideration. The Hamiltonian equations, when expressed in action-angle coordinates, take a particularly easy form,

$$\dot{\phi}_k = \frac{\partial H}{\partial I_k}; \quad \dot{I}_k = 0 \quad (1 \leq k \leq n)$$

and any conserved quantity which is real analytic near 0, has a convergent power series expansion in I_1, \dots, I_n . In the case of an integrable system with a simple resonance, the Hamiltonian equations are — inevitably — more complicated. Theorem 1.1' provides coordinates (\hat{x}, \hat{y}) for which the Hamiltonian equations take a relatively simple form.

REMARK 1.3. The coordinates which have the properties stated in Theorem 1.1 are not unique. One verifies that a symplectic transformation provided by a Hamiltonian flow whose Hamiltonian has a power series expansion in $\tau_1, \dots, \tau_{n+2}$, leads to new coordinates with the same properties as stated in Theorem 1.1. However, given a power series expansion in $\tau_1, \dots, \tau_{n+2}$ of an element $f \in \mathcal{A}$, one verifies that the coefficients corresponding to the monomials in $\tau_1, \dots, \tau_{n-1}$ only, are independent of the choice of coordinates.

REMARK 1.4. One might ask if a result similar to the one of Theorem 1.1 is true if \mathcal{A} has multiple resonances, i.e., a resonance lattice with $R = \dim \Lambda_{\mathcal{A}} \geq 2$. Even in the case where the resonances are decoupled (i.e. $\Lambda_{\mathcal{A}}$ has a basis $\mu^{(1)}, \dots, \mu^{(R)}$ such that $\text{supp} \mu^{(i)} \cap \text{supp} \mu^{(j)} = \emptyset$ for $i \neq j$) it turns out that the method of proof used for Theorem 1.1 breaks down in general (cf. Appendix A).

As an immediate consequence of Theorem 1.1 we obtain the following

COROLLARY 1.2. (i) If \mathcal{A} is nonresonant, then $\mathcal{A}_{nil} = 0$ and $\dim \mathcal{A}_s = n$.

(ii) If \mathcal{A} has a simple resonance then $\dim \mathcal{A}_s = n - 1$. If for a prime resonance vector $\mu, |\mu| \geq 3$, then $\mathcal{A}_{nil} = \{0\}$.

PROOF. (i) By Ito's result [It1], there exist coordinates x_j, y_j ($1 \leq j \leq n$) in a neighborhood of 0 in \mathbb{C}^{2n} so that, when expressed in these coordinates, \mathcal{A} consists of power series in $\tau_j = x_j y_j$ ($1 \leq j \leq n$) which converge in a neighborhood of $0 \in \mathbb{C}^{2n}$. Therefore, $\dim \mathcal{A}_s = n$ and $\mathcal{A}_{\text{nil}} = \{0\}$.

(ii) By Theorem 1.1, there exist coordinates x_k, y_k ($1 \leq k \leq n$) in a neighborhood of 0 in \mathbb{C}^{2n} so that, when expressed in these coordinates, any element in \mathcal{A} has a convergent power series expansion in $\tau_1, \dots, \tau_{n+2}$ where $\tau_j := \sum_{k=1}^n \rho_k^{(j)} x_k y_k$ ($1 \leq j \leq n$), $\tau_{n+1} = x^{\mu^+} y^{\mu^-}$ and $\tau_{n+2} = x^{\mu^-} y^{\mu^+}$ are defined as above.

Moreover, Theorem 1.1 guarantees that the functions, $\tau_1, \dots, \tau_{n-1}$, are elements in \mathcal{A} . This implies that $\dim \mathcal{A}_s = n - 1$, as otherwise τ_n would be also in \mathcal{A}_s and one would conclude that \mathcal{A} is nonresonant at 0. If, in addition, $|\mu| \geq 3$, then for any power series f in $\tau_1, \dots, \tau_{n+2}$, $f_{\text{nil}} = 0$ and thus in particular $\mathcal{A}_{\text{nil}} = \{0\}$. \square

In the second part of this paper (Section 3), we make a detailed analysis of the level sets $M_c := \{(\hat{x}, \hat{y}) \in (\mathbb{R}^{2n}, 0) \mid G_j = c_j (1 \leq j \leq n)\}$ for real integrable systems with $c = (c_1, \dots, c_n)$, and study the fibration provided by these level sets. Here (\hat{x}, \hat{y}) are the coordinates provided by Theorem 1.1', $G_j = \hat{\tau}_j$ ($1 \leq j \leq n-1$) and $(\mathbb{R}^{2n}, 0)$ denotes a neighborhood of 0 invariant under the flows of the Hamiltonian vectorfields corresponding to $\hat{\tau}_j$, $1 \leq j \leq n-1$ (cf. Section 3). G_n can be expressed as a power series in $\hat{\tau}_1, \dots, \hat{\tau}_n, \Re \hat{\tau}_{n+1}, \Im \hat{\tau}_{n+1}$ (which, due to the resonance assumption, does not contain a term linear in $\hat{\tau}_n$), and is such that the G_j 's generate \mathcal{A} . In particular, we prove that if the prime resonance vector μ oscillates (i.e. μ has negative and positive components), then, for generic c , M_c is a disjoint union of tori of dimension n (cf. Proposition 3.4). If μ is nonnegative ($\mu_j \geq 0$ for $1 \leq j \leq n$), then, for generic c sufficiently small and generic \mathcal{A} (i.e. generic G_n), M_c has one connected component diffeomorphic to $(S^1)^{n-1} \times (0, 1)$ in case $|\mu| = 2$ or 3 and is a disjoint union of tori of dimension n if $|\mu| \geq 5$ (cf. Proposition 3.5). In Subsection 3.2, we study nongeneric level sets and in Subsection 3.3 we analyze the fibration provided by the level sets. In Appendix B we analyze the level sets M_c for complex systems.

Concerning the second part, somewhat related results can be found in [Fo] as well as in [CB] (cf. [Du]) where, in connection with the question of global action-angle variables, one can find a discussion of the monodromy of the fibration $F : M \rightarrow B$, with M denoting the phase space and fibers being Liouville tori. For a generic class of integrable systems of two degrees of freedom, Fomenko [Fo] studies — in particular classifies — generic regular energy surfaces and their fibrations where the fibers are, up to singularities, Liouville tori for the systems considered and extends some of his results to generic systems of arbitrary many degrees. Our analysis is concerned with the study of the foliation by level sets — not necessarily tori — of an integrable system near a singular point with a simple resonance and is of a local nature.

2. – Proof of Theorem 1.1

In this section we provide a proof of Theorem 1.1. As we follow Ito's method of proof, we present only an outline, emphasizing the parts which are different. Throughout this section we use the notation introduced in Section 1 and assume that the assumptions of Theorem 1.1 hold.

2.1. – Preliminaries

Choose H in \mathcal{A} so that $\Lambda_{H_s} = \Lambda_{\mathcal{A}}$ where H_s denotes the semisimple part of H ,

$$(2.1) \quad H_s := \sum_{j=1}^n \lambda_j x_j y_j.$$

For a power series $f = \sum_{\alpha, \beta \in \mathbb{Z}_{\geq 0}^n} c_{\alpha\beta} x^\alpha y^\beta$ at the origin we use the notation

$$(2.2) \quad f = f^d + f^{d+1} + f^{d+2} + \dots$$

where f^j ($j \geq d$) is a homogeneous polynomial of degree j with $d = \text{degree}(f^d) \geq 0$. We refer to f^d as the lowest order part of f . A power series f is said to be in H_s -normal form (or Birkhoff normal form) if

$$\{H_s, f\} = \sum_{k=1}^n \frac{\partial H_s}{\partial x_k} \frac{\partial f}{\partial y_k} - \frac{\partial H_s}{\partial y_k} \frac{\partial f}{\partial x_k} \equiv 0.$$

It is said to be in H_s -normal form up to order $d + d_1$ if $f^d + \dots + f^{d+d_1}$ is in H_s -normal form. Notice that a power series f which is in H_s -normal form can be considered as a power series in $(n+2)$ variables $\tau_1, \dots, \tau_{n+2}$. Moreover, as

$$(2.3) \quad \tau_{n+1} \tau_{n+2} = x^{\mu^+ + \mu^-} y^{\mu^+ + \mu^-} = \prod_{j=1}^n (x_j y_j)^{|\mu_j|},$$

is a function of $\tau = (\tau_1, \dots, \tau_n)$, f is of the form

$$(2.4) \quad f(z) = f_1(\tau, \tau_{n+1}) + f_2(\tau, \tau_{n+2})$$

where $f_j(\tau, \tau_{n+j})$ are power series in τ_1, \dots, τ_n and τ_{n+j} ($j = 1, 2$). Alternatively, f can be considered as a Laurent series in $\tau_1, \dots, \tau_{n+1}$, eliminating τ_{n+2} in $f_2(\tau, \tau_{n+2})$ by using (2.3).

For the remainder of all of Section 2, given a power series f in H_s -normal form, we denote by $\frac{\partial f}{\partial \tau_j}$ ($1 \leq j \leq n+1$) the partial derivative of f with respect to τ_j when f is considered as a Laurent series in $\tau_1, \dots, \tau_{n+1}$.

Using that

$$(2.5) \quad \begin{aligned} \{\tau_i, \tau_j\} &= 0 \quad (1 \leq i, j \leq n), & \{\tau_i, \tau_{n+1}\} &= 0 \quad (1 \leq i \leq n-1), \\ & & \{\tau_n, \tau_{n+1}\} &= -\tau_{n+1} \end{aligned}$$

the Poisson bracket $\{f, g\}$ of power series f, g in H_s -normal form can be computed as

$$(2.6) \quad \{f, g\} = -\tau_{n+1} \left(\frac{\partial f}{\partial \tau_n} \frac{\partial g}{\partial \tau_{n+1}} - \frac{\partial f}{\partial \tau_{n+1}} \frac{\partial g}{\partial \tau_n} \right).$$

2.2. – Construction of a formal coordinate transformation

In this subsection, we construct the transformation φ formally. Introduce the projection $\Pi_N f$ of a power series f onto its H_s -normal part and define $\Pi_R f := f - \Pi_N f$. For any convergent power series f denote by $X_f^t := \exp(tX_f)$ the flow corresponding to the Hamiltonian vector field X_f . The coordinate transformation φ of Theorem 1.1 is constructed from a sequence of transformations each of which is a Hamiltonian flow obtained in a well known fashion:

PROPOSITION 2.1. *Let $K = K^2 + K^3 + \dots$ be a power series with $K_s = H_s \equiv \sum_{j=1}^n \lambda_j x_j y_j$. Assume that K is in H_s -normal form up to order $1 + d$ ($d \geq 1$). Then there exists a unique polynomial W of the form*

$$(2.7) \quad W = W^{d+2} + \dots + W^{2d+1}$$

with $\Pi_N W = 0$ such that $K \circ X_W^{t=1}$ is in H_s -normal form up to order $1 + 2d$. (The flow $X_W^t(\zeta)$ exists for $|t| \leq 1$ for ζ in a sufficiently small neighborhood of 0.)

Applying Proposition 2.1 successively for $H = H^2 + H^3 + \dots$, one obtains

COROLLARY 2.2. *There exists a sequence of symplectic coordinate transformations φ_j ($j \geq 0$), $\varphi_j = X_{W_j}^{t=1}$, where W_j is the polynomial provided by Proposition 2.1 with $d = 2^j$, so that the coordinate transformation $\varphi^{(j)} := \varphi_0 \circ \dots \circ \varphi_j$ takes the Hamiltonian H into H_s -normal form up to order $1 + 2^{j+1}$. Consequently, $\varphi := \lim_{j \rightarrow \infty} \varphi^{(j)}$ is a formal symplectic transformation such that $H \circ \varphi$ is in H_s -normal form.*

One verifies by a straightforward inductive argument that the following Lemma holds:

LEMMA 2.3. *Assume that H is in H_s -normal form up to order $2 + d$ and G is an integral of H , i.e. $\{H, G\} = 0$, then $G = G^\ell + G^{\ell+1} + \dots$ is in H_s -normal form up to order $\ell + d$.*

In view of Lemma 2.3 and Corollary 2.2, one concludes that, for a proof of Theorem 1.1, it remains to establish that the formal coordinate transformation φ of Corollary 2.2 is given by a convergent power series. For this purpose we need to estimate the function W obtained in Proposition 2.1. Recall from the introduction that $\rho^{(j)} \in \mathbb{Z}^n \setminus \{0\}$ ($1 \leq j \leq n$) is a basis of \mathbb{Z}^n with $\langle \rho^{(j)}, \mu \rangle = \delta_{jn}$.

LEMMA 2.4. *Let f be a convergent power series. Then the projection $\Pi_N f$ of f on its H_s -normal part is given by*

$$(2.8) \quad \Pi_N f(x, y) = \int_0^1 d\theta_1 \cdots \int_0^1 d\theta_{n-1} f(e^{2\pi i\theta} x, e^{-2\pi i\theta} y)$$

where $e^{2\pi i\theta} x$ is defined by

$$(2.9) \quad e^{2\pi i\theta} x = \left(e^{2\pi i \sum_{j=1}^{n-1} \theta_j \rho_1^{(j)}} x_1, \dots, e^{2\pi i \sum_{j=1}^{n-1} \theta_j \rho_n^{(j)}} x_n \right)$$

and where $e^{-2\pi i\theta} y$ is defined similarly.

PROOF. It suffices to consider the case where f is a monomial, $f(x, y) = x^\alpha y^\beta$. Then

$$f(e^{2\pi i\theta} x, e^{-2\pi i\theta} y) = x^\alpha y^\beta \exp \left(2\pi i \sum_{j=1}^{n-1} \theta_j \langle \rho^{(j)}, \alpha - \beta \rangle \right).$$

In order for such a term not to get averaged when integrated over $\theta_1, \dots, \theta_{n-1}$, it is necessary and sufficient that $\langle \rho^{(j)}, \alpha - \beta \rangle = 0$ for $1 \leq j \leq n - 1$, i.e. α, β in $\mathbb{Z}_{\geq 0}^n$ have to be of the form

$$\alpha = \gamma + \ell_1 \mu^+ + \ell_2 \mu^-, \quad \beta = \gamma + \ell_1 \mu^- + \ell_2 \mu^+$$

where $\gamma \in \mathbb{Z}_{\geq 0}^n$ and $\ell_1, \ell_2 \in \mathbb{Z}_{\geq 0}$. □

Consider the disc $\Omega_r := \{z \in \mathbb{C}^{2n} \mid |z_j| < \delta_j r \ (1 \leq j \leq 2n)\}$ where $r > 0$ and $\delta_j > 0$ ($1 \leq j \leq n$).

For a polynomial W introduce

$$(2.10) \quad |W|_r^{\text{sup}} := \sup_{z \in \Omega_r} |W(z)|.$$

LEMMA 2.5. *Let W be a polynomial with $\Pi_N W = 0$. Then*

$$(2.11) \quad |W|_r^{\text{sup}} \leq 2\pi \sum_{k=1}^{n-1} |\{\tau_k, W\}|_r^{\text{sup}}.$$

PROOF. Introduce, for $z = (x, y) \in \Omega_r$ fixed,

$$(2.12) \quad \widehat{W}(\theta) := W(e^{2\pi i\theta} x, e^{-2\pi i\theta} y)$$

where $e^{2\pi i\theta}x$ and $e^{-2\pi i\theta}y$ are defined as above with $\theta = (\theta_1, \dots, \theta_{n-1})$, $0 \leq \theta_j \leq 1$. Notice that $(\tilde{x}(\theta), \tilde{y}(\theta)) := (e^{2\pi i\theta}x, e^{-2\pi i\theta}y) \in \Omega_r$ and $\widehat{W}(\theta = 0) = W(z)$. By the mean value theorem,

$$|\widehat{W}(\theta) - \widehat{W}(0)| \leq \sum_{j=1}^{n-1} \max_{\theta \in [0,1]^{n-1}} \left| \frac{\partial \widehat{W}(\theta)}{\partial \theta_j} \right|$$

where

$$\frac{\partial \widehat{W}(\theta)}{\partial \theta_j} = \sum_{k=1}^n \frac{\partial W}{\partial x_k} \Big|_{\tilde{x}, \tilde{y}} 2\pi i \rho_k^{(j)} \tilde{x}_k - \sum_{k=1}^n \frac{\partial W}{\partial y_k} \Big|_{\tilde{x}, \tilde{y}} 2\pi i \rho_k^{(j)} \tilde{y}_k.$$

As $\tau_j = \sum_{\ell=1}^n \rho_\ell^{(j)} x_\ell y_\ell$ one sees that

$$\frac{\partial \tau_j}{\partial x_k} = \rho_k^{(j)} y_k \quad \text{and} \quad \frac{\partial \tau_j}{\partial y_k} = \rho_k^{(j)} x_k.$$

This leads to

$$\begin{aligned} \left| \frac{\partial W}{\partial \theta_j} \right|_r^{\sup} &\leq 2\pi \left| \sum_{k=1}^n \left(\frac{\partial W}{\partial x_k} \frac{\partial \tau_j}{\partial y_k} - \frac{\partial W}{\partial y_k} \frac{\partial \tau_j}{\partial x_k} \right) \right|_{\tilde{x}, \tilde{y}} \Big|_r^{\sup} \\ &\leq 2\pi |\{W, \tau_j\}|_r^{\sup} \end{aligned}$$

and, therefore,

$$|\widehat{W}(\theta) - \widehat{W}(0)|_r^{\sup} \leq 2\pi \sum_{j=1}^{n-1} |\{W, \tau_j\}|_r^{\sup}.$$

Using the assumption $\Pi_N W = 0$ one concludes from Lemma 2.4 that

$$\int_0^1 d\theta_1 \cdots \int_0^1 d\theta_{n-1} \widehat{W}(\theta) = 0.$$

Therefore, $W(z) = \int_0^1 d\theta_1 \cdots \int_0^1 d\theta_{n-1} (\widehat{W}(0) - \widehat{W}(\theta))$ and

$$|W|_r^{\sup} \leq 2\pi \sum_{j=1}^{n-1} \text{big} |\{W, \tau_j\}|_r^{\sup}.$$

According to Lemma 2.5, we obtain an estimate of W (as in Proposition 2.1) from an estimate of the Poisson brackets $\{\tau_k, W\}$ for $1 \leq k \leq n - 1$.

2.3. – Estimate for $\{\tau_k, W\}$

To estimate $\{\tau_k, W\}$ for $1 \leq k \leq n-1$ we make use of the assumption that G_1, \dots, G_n are integrals of H and that dG_1, \dots, dG_n are generically independent.

Assume that the lowest order part of G_j is of degree $s_j \geq 1$ so that G_j is of the form

$$(2.13) \quad G_j = G_j^{s_j} + G_j^{s_j+1} + G_j^{s_j+2} + \dots$$

where $G_j^{s_j+d}$ are polynomials homogeneous of degree $s_j + d$. As pointed out by Ito [It2, Lemma 4.4, p. 422] we may assume that $dG_1^{s_1}, \dots, dG_n^{s_n}$ are generically independent. Assume that H is in H_s -normal form up to order $1+d$ ($d \geq 1$). By Lemma 2.3, G_j is in H_s -normal form up to order $s_j + d - 1$. Write

$$(2.14) \quad G_j(z) = g_j(z) + \widehat{G}_j(z)$$

where g_j is in H_s -normal form, i.e. $g_j = \Pi_N g_j$ and $\widehat{G}_j(z) = O(|z|^{s_j+d})$. Notice that

$$(2.15) \quad \text{rank} \left(\frac{\partial(g_1^{s_1}, \dots, g_n^{s_n})}{\partial(\tau_1, \dots, \tau_{n+1})} \right) = n$$

on an open dense subset near $z = 0$, where we recall that we view $g_1^{s_1}, \dots, g_n^{s_n}$ as Laurent series in $\tau_1, \dots, \tau_{n+1}$.

Let $z := \varphi(\zeta) = X_W^{t=1}(\zeta)$ be the transformation described in Proposition 2.1. Then, again by Lemma 2.3,

$$(2.14) \quad G_j(\varphi(\zeta)) = g_j(\zeta) + \{g_j(\zeta), W(\zeta)\} + \widehat{G}_j(\zeta) + O(|\zeta|^{s_j+2d})$$

is in H_s -normal form up to order $s_j - 1 + 2d$. Therefore ($1 \leq j \leq n$)

$$(2.17) \quad \{g_j(\zeta), W(\zeta)\} + \Pi_R \widehat{G}_j(\zeta) = O(|\zeta|^{s_j+2d})$$

which can be written as ($1 \leq i \leq n$)

$$(2.18)_i \quad \sum_{j=1}^{n+1} \frac{\partial g_i}{\partial \tau_j} \cdot \{\tau_j, W\} = -\Pi_R \widehat{G}_i(\zeta) + O(|\zeta|^{s_i+2d}).$$

This is a linear system of n equations for $\{\tau_1, W\}, \dots, \{\tau_{n+1}, W\}$, from which we would like to derive estimates for $\{\tau_1, W\}, \dots, \{\tau_{n-1}, W\}$. Notice that $\{G_i, G_j\} = 0$ implies $\{g_i^{s_i}, g_j^{s_j}\} = 0$, which, by (2.6), implies ($1 \leq i, j \leq n$)

$$\frac{\partial g_i^{s_i}}{\partial \tau_n} \frac{\partial g_j^{s_j}}{\partial \tau_{n+1}} - \frac{\partial g_i^{s_i}}{\partial \tau_{n+1}} \frac{\partial g_j^{s_j}}{\partial \tau_n} = 0.$$

This means that the $n \times 2$ matrix $(\frac{\partial g^0}{\partial \tau_n}, \frac{\partial g^0}{\partial \tau_{n+1}})$ with $g^0 = (g_1^{s_1}, \dots, g_n^{s_n})$ has rank at most 1, i.e. $\frac{\partial g^0}{\partial \tau_n}$ and $\frac{\partial g^0}{\partial \tau_{n+1}}$ are linearly dependent. But due to (2.15) either $\frac{\partial g^0}{\partial \tau_n} \neq 0$ (case 1) or $\frac{\partial g^0}{\partial \tau_{n+1}} \neq 0$ (case 2). By reordering the G_j 's, if necessary, we have that either $\frac{\partial g_n^{s_n}}{\partial \tau_n} \neq 0$ (case 1) or $\frac{\partial g_n^{s_n}}{\partial \tau_{n+1}} \neq 0$ (case 2). Let us outline how one proceeds in case 1 to obtain estimates for $\{\tau_1, W\}, \dots, \{\tau_{n-1}, W\}$ from the system (2.18). One uses equation (2.18)_n to eliminate $\{\tau_n, W\}$ and obtains a system of $n - 1$ equations for $\{\tau_1, W\}, \dots, \{\tau_{n-1}, W\}$ and $\{\tau_{n+1}, W\}$. Using that $\{G_n, G_i\} = 0$, one concludes that the terms involving $\{\tau_{n+1}, W\}$ are of sufficiently high order and, therefore, can be included in the error term. In more detail, we eliminate $\{\tau_n, W\}$ from the system (2.18) by forming (2.18)_i $\frac{\partial g_n}{\partial \tau_n} -$ (2.18)_n $\frac{\partial g_i}{\partial \tau_n}$ to obtain ($1 \leq i \leq n - 1$)

$$(2.19)_i \quad \sum_{j=1}^{n-1} \alpha_{ij}(\zeta) \{\tau_j, W\} = -\alpha_{i,n+1}(\zeta) \{\tau_{n+1}, W\} - \frac{\partial g_n}{\partial \tau_n} \Pi_R \widehat{G}_i + \frac{\partial g_i}{\partial \tau_n} \Pi_R \widehat{G}_n + \frac{\partial g_n}{\partial \tau_n} O(|\zeta|^{s_i+2d}) - \frac{\partial g_i}{\partial \tau_n} O(|\zeta|^{s_n+2d})$$

where α_{ij} are defined as

$$(2.20) \quad \alpha_{ij}(\zeta) = \frac{\partial g_n}{\partial \tau_n} \frac{\partial g_i}{\partial \tau_j} - \frac{\partial g_n}{\partial \tau_j} \frac{\partial g_i}{\partial \tau_n}.$$

Both sides of (2.19)_i are meromorphic functions of ξ_j, η_j with a possible pole at points ξ, η with $\xi_j = \eta_j = 0$ for some $1 \leq j \leq n$. To remove the poles in (2.19)_i we multiply this equation by the polynomial $P(\zeta) := \tau_{n+1} \prod_{k=1}^n \xi_k \eta_k$. To see that P serves its purpose, write ($1 \leq j \leq n + 1$)

$$(2.21) \quad \frac{\partial}{\partial \tau_j} = D_{\tau_j} + \frac{\partial \tau_{n+2}}{\partial \tau_j} D_{\tau_{n+2}}$$

where D_{τ_j} denotes the partial derivative of a function $f = f(\tau_1, \dots, \tau_{n+1}, \tau_{n+2})$ (when not considering it as a function of $\tau_1, \dots, \tau_{n+1}$). Recall that $\tau_{n+1} \tau_{n+2} = \xi^{\mu^+ + \mu^-} \eta^{\mu^+ + \mu^-}$ and, therefore,

$$(2.22) \quad \frac{\partial \tau_{n+2}}{\partial \tau_{n+1}} = - \left(\frac{1}{\tau_{n+1}} \right)^2 \xi^{\mu^+ + \mu^-} \eta^{\mu^+ + \mu^-} = - \frac{\tau_{n+2}}{\tau_{n+1}}.$$

Further ($1 \leq j \leq n$)

$$\frac{\partial \tau_{n+2}}{\partial \tau_j} = \sum_{k=1}^n \frac{\partial \tau_{n+2}}{\partial (\xi_k \eta_k)} \frac{\partial (\xi_k \eta_k)}{\partial \tau_j}.$$

To compute this derivative introduce the inverse $B = (b_{kj})$ of A where A is the unimodular $n \times n$ matrix whose j 'th row is given by $\rho^{(j)}$. Then

$$\frac{\partial(\xi_k \eta_k)}{\partial \tau_j} = b_{kj}.$$

Moreover,

$$\frac{\partial \tau_{n+2}}{\partial(\xi_k \eta_k)} = \frac{1}{\tau_{n+1}} \frac{\partial(\xi^{\mu^+ + \mu^-} \eta^{\mu^+ + \mu^-})}{\partial(\xi_k \eta_k)} = \frac{|\mu_k|}{\tau_{n+1}} \frac{\xi^{\mu^+ + \mu^-} \eta^{\mu^+ + \mu^-}}{\xi_k \eta_k}.$$

Altogether, we obtain ($1 \leq j \leq n$)

$$(2.23) \quad \frac{\partial \tau_{n+2}}{\partial \tau_j} = \left(\sum_{k=1}^n b_{kj} \frac{|\mu_k|}{\xi_k \eta_k} \right) \tau_{n+2}.$$

From (2.22) and (2.23) we, therefore, conclude that

$$P(\zeta) \frac{\partial \tau_{n+2}}{\partial \tau_{n+1}} = - \left(\prod_{k=1}^n \xi_k \eta_k \right) \tau_{n+2}$$

and

$$P(\zeta) \frac{\partial \tau_{n+2}}{\partial \tau_j} = \tau_{n+1} \tau_{n+2} \sum_{k=1}^n b_{kj} |\mu_k| \left(\prod_{\substack{1 \leq j \leq n \\ j \neq k}} \xi_j \eta_j \right).$$

These computations are now used to estimate the term $-\alpha_{i,n+1}(\zeta)\{\tau_{n+1}, W\}$ on the right hand side of (2.19)_{*i*}. Using (2.6) we see that

$$(2.24) \quad \begin{aligned} P(\zeta) \alpha_{i,n+1}(\zeta) &= \left(\prod_1^n \xi_k \eta_k \right) \tau_{n+1} \left(\frac{\partial g_n}{\partial \tau_n} \frac{\partial g_i}{\partial \tau_{n+1}} - \frac{\partial g_n}{\partial \tau_{n+1}} \frac{\partial g_i}{\partial \tau_n} \right) \\ &= - \left(\prod_{k=1}^n \xi_k \eta_k \right) \{g_n, g_i\}. \end{aligned}$$

Now we make use of the assumption that $0 = \{G_n, G_i\}$ (integrability) to conclude that

$$(2.25) \quad \begin{aligned} \{g_n, g_i\} &= -\{g_n, \widehat{G}_i\} - \{\widehat{G}_n, g_i\} - \{\widehat{G}_n, \widehat{G}_i\} \\ &= O(|\zeta|^{s_n + s_i + d - 2}). \end{aligned}$$

Combining (2.24) and (2.25) we conclude that the order of $P(\zeta)\alpha_{i,n+1}(\zeta)$ is given by $2n + s_n + s_i + d - 2$, i.e.

$$(2.26) \quad -P(\zeta)\alpha_{i,n+1}(\zeta)\{\tau_{n+1}, W\} = O(|\zeta|^{m_i + 2d - 2})$$

where $m_i = 2n + s_n + s_i + |\mu| - 2$. This is the key point in this proof. It uses the integrability of the system to insure that $\{\tau_{n+1}, W\}$ can be treated as an error term in (2.19).

Further, one verifies that

$$P(\zeta) \left(\frac{\partial g_n}{\partial \tau_n} O(|\zeta|^{s_i+2d}) - \frac{\partial g_i}{\partial \tau_n} O(|\zeta|^{s_n+2d}) \right) = O(|\zeta|^{m_i+2d}).$$

The system of equations (2.19)_i is, therefore, reduced to $(1 \leq i \leq n - 1)$

$$(2.27)_i \quad \sum_{j=1}^n P(\zeta) \alpha_{ij}(\zeta) \{\tau_j, W\} = P(\zeta) \left(\frac{\partial g_i}{\partial \tau_n} \Pi_R \widehat{G}_n - \frac{\partial g_n}{\partial \tau_n} \Pi_R \widehat{G}_i \right) + O(|\zeta|^{m_i+2d}).$$

By comparing homogeneous parts of (2.27), we can obtain equations to be solved for W^{d+2}, \dots, W^{2d+1} inductively, which we describe below.

In case 2, we argue similarly to obtain $(1 \leq i \leq n - 1)$

$$(2.28)_i \quad \sum_{j=1}^{n-1} P(\zeta) \tilde{\alpha}_{ij}(\zeta) \{\tau_j, W\} = P(\zeta) \left(\frac{\partial g_i}{\partial \tau_{n+1}} \Pi_R \widehat{G}_n - \frac{\partial g_n}{\partial \tau_{n+1}} \Pi_R \widehat{G}_i \right) + O(|\zeta|^{2n+s_n+s_i+2d})$$

where $\tilde{\alpha}_{ij}(\zeta) := \frac{\partial g_n}{\partial \tau_{n+1}} \frac{\partial g_i}{\partial \tau_j} - \frac{\partial g_n}{\partial \tau_j} \frac{\partial g_i}{\partial \tau_{n+1}}$ ($1 \leq i, j \leq n$).

To treat case 1 and case 2 simultaneously, introduce the polynomials $a_{ij}(\zeta)$

$$(2.29) \quad a_{ij}(\zeta) = \begin{cases} P(\zeta) \left(\frac{\partial g_n}{\partial \tau_n} \frac{\partial g_i}{\partial \tau_j} - \frac{\partial g_n}{\partial \tau_j} \frac{\partial g_i}{\partial \tau_n} \right) & \text{(case 1)} \\ P(\zeta) \left(\frac{\partial g_n}{\partial \tau_{n+1}} \frac{\partial g_i}{\partial \tau_j} - \frac{\partial g_n}{\partial \tau_j} \frac{\partial g_i}{\partial \tau_{n+1}} \right) & \text{(case 2)}. \end{cases}$$

Denote by $a_{ij}^{m_i}(\zeta)$ the leading order part of $a_{ij}(\zeta)$,

$$a_{ij}^{m_i}(\zeta) = \begin{cases} P(\zeta) \left(\frac{\partial g_n^{s_n}}{\partial \tau_n} \frac{\partial g_i^{s_i}}{\partial \tau_j} - \frac{\partial g_n^{s_n}}{\partial \tau_j} \frac{\partial g_i^{s_i}}{\partial \tau_n} \right) & \text{(case 1)} \\ P(\zeta) \left(\frac{\partial g_n^{s_n}}{\partial \tau_{n+1}} \frac{\partial g_i^{s_i}}{\partial \tau_j} - \frac{\partial g_n^{s_n}}{\partial \tau_j} \frac{\partial g_i^{s_i}}{\partial \tau_{n+1}} \right) & \text{(case 2)} \end{cases}$$

with $m_i = s_n + s_i + 2n + |\mu| - 2$ in case 1 and $m_i = s_n + s_i + 2n$ in case 2.

We summarize the results obtained above in the following

LEMMA 2.6. Assume that there exists a polynomial $W = W^{d+2} + \dots + W^{2d+1}$ with $\Pi_N W = 0$ so that, with $z = \varphi(\zeta) = X_W^{t=1}(\zeta)$, $G_j \circ \varphi$ are in H_s -normal form up to order $s_j + 2d - 1$. Then each homogeneous polynomial $W^{\ell+2}$ ($d \leq \ell \leq 2d - 1$) satisfies the following system of $(n - 1)$ equations ($1 \leq i \leq n - 1$)

$$(2.30) \quad \sum_{j=1}^{n-1} a_{ij}^{m_i}(\zeta) \{\tau_j, W^{\ell+2}\} = F_i^{m_i+\ell+2}(\zeta)$$

where $F_i^{m_i+\ell+2}(\zeta)$ is given by

$$(2.31) \quad \left\{ \begin{array}{l} P(\zeta) \sum_{j=0}^{\ell-d} \left(\frac{\partial g_i^{s_i+j}}{\partial \tau_n} \cdot \Pi_R \widehat{G}_n^{s_n+\ell-j} - \frac{\partial g_n^{s_n+j}}{\partial \tau_n} \Pi_R \widehat{G}_i^{s_i+\ell-j} \right) \\ - \sum_{j=1}^{n-1} \sum_{k=1}^{\ell-d} a_{ij}^{m_i+k} \{\tau_j, W^{\ell+2-k}\} \end{array} \right. \quad (\text{case 1})$$

$$\left\{ \begin{array}{l} (\zeta) \sum_{j=0}^{\ell-d} \left(\frac{\partial g_i^{s_i+j}}{\partial \tau_{n+1}} \Pi_R \widehat{G}_n^{s_n+\ell-j} - \frac{\partial g_n^{s_n+j}}{\partial \tau_{n+1}} \Pi_R \widehat{G}_i^{s_i+\ell-j} \right) \\ - \sum_{j=1}^{n-1} \sum_{k=1}^{\ell-d} a_{ij}^{m_i+k} \{\tau_j, W^{\ell+2-k}\} \end{array} \right. \quad (\text{case 2})$$

Moreover,

$$(2.32) \quad p(\zeta) := \det[a_{ij}^{m_i}(\zeta)_{1 \leq i, j \leq n-1}] \neq 0.$$

The system (2.30) can be solved for $\{\tau_j, W^{\ell+2}\}$,

$$(2.33) \quad \{\tau_k, W^{\ell+2}\} = \frac{q_k^{s+\ell+2}(\zeta)}{p(\zeta)}$$

where $q_k^{s+\ell+2}(\zeta)$ is also a determinant and given by Cramer's rule.

We point out that, $\{\tau_k, W^{\ell+2}\}$ being a polynomial, formula (2.33) shows that the numerator $q_k^\ell(\zeta)$ is divisible by $p(\zeta)$.

2.4. – Estimate of W

We now combine Lemma 2.5 and Lemma 2.6 to obtain an estimate for W .

For convenience, we assume that the Hamiltonian H is normalized so that $\|\mu\| = \|\lambda\|$.

According to Ito [It2, Lemma 5.1], for a small but otherwise arbitrary positive number $r > 0$, there exist constants $0 < \delta_i < 1$ ($1 \leq i \leq 2n$) such that, for $\zeta \in \Delta_r := \{\zeta = (\zeta_1, \dots, \zeta_{2n}) \in \mathbb{C}^{2n} \mid |\zeta_j| = \delta_{jr} \ (1 \leq j \leq 2n)\}$,

$$(2.34) \quad |p(\zeta)| \geq c_1 r^s$$

where s is the degree of $p(\zeta) := \det \left((a_{ij}^{m_i}(\zeta))_{1 \leq i, j \leq n-1} \right)$ (cf. (2.32)) and $c_1 > 0$ is independent of r . Introduce the polydiscs $\Omega_r := \{\zeta \in \mathbb{C}^{2n} \mid |\zeta_j| < \delta_j r \ (1 \leq j \leq 2n)\}$ and denote by $A(\Omega_r)$ the space of power series in ζ which are absolutely convergent in $\overline{\Omega}_r$. Further, introduce the subspaces $A_m(\Omega_r) := \{f \in A(\Omega_r) \mid f = f^m + f^{m+1} + \dots\}$ and the following norms for $f \in A(\Omega_r)$,

$$\bullet \quad |f|_r^{\text{sup}} := \max_{\overline{\Omega}_r} |f(\zeta)|; \quad \|f\|_r := \sum_{j=0}^{\infty} |f^j|_r^{\text{sup}}$$

and, for $f \in A_m(\Omega_r)$,

$$\|f\|_{r,m} := \frac{\|f\|_r}{r^m}.$$

It follows from (2.32), (2.33) and the maximum principle that

$$(2.35) \quad |\{\tau_k, W^{\ell+2}\}|_r^{\text{sup}} \leq \frac{\max_{\zeta \in \Delta_r} |q_k^{s+\ell+2}(\zeta)|}{\min_{\zeta \in \Delta_r} |p(\zeta)|} \leq \frac{1}{c_1} \frac{|q_k^{s+\ell+2}|_r^{\text{sup}}}{r^s}$$

with $W = W^{d+2} + \dots + W^{2d+1}$. According to Cramer's rule, $q_k^{s+\ell+2}$ is given by

$$q_k^{s+\ell+2} = \det \begin{pmatrix} a_{11}^{m_1} & \dots & F_1^{m_1+\ell+2} & \dots & a_{1n-1}^{m_1} \\ \vdots & & \vdots & & \vdots \\ a_{n-11}^{m_{n-1}} & \dots & F_{n-1}^{m_{n-1}+\ell+2} & \dots & a_{n-1n-1}^{m_{n-1}} \end{pmatrix}$$

(cf. (2.31) for the definition of $F_i^{m_i+\ell+2}$).

In a straightforward way one then verifies that $(1 \leq k \leq n-1; d \leq \ell \leq 2d-1)$

$$(2.36) \quad \|\{\tau_k, W^{\ell+2}\}\|_r \leq c_2 \sum_{i=1}^{n-1} \frac{\|F_i^{m_i+\ell+2}\|_r}{r^{m_i}}$$

for some $c_2 > 0$.

We need to introduce some more notation: for a power series $f = \sum_{\alpha, \beta} c_{\alpha\beta} \xi^\alpha \eta^\beta$ denote by \tilde{f} the power series with coefficients $|c_{\alpha\beta}|$

$$\tilde{f} = \sum_{\alpha, \beta} |c_{\alpha\beta}| \xi^\alpha \eta^\beta.$$

Let $c_3 > 0$ be a constant such that

$$\sum_{1 \leq i, j \leq n} \frac{\|D_{\tau_j} \tilde{g}_i^{s_i}\|_r}{r^{s_i-2}} + \sum_{\substack{1 \leq i \leq n \\ k=n+1, n+2}} \frac{\|D_k \tilde{g}_i^{s_i}\|_r}{r^{s_i-|\mu|}} \leq c_3$$

and define $c_4 := \max(c_2, 1 + c_2 c_3)$. Using (2.36) one then proves (cf. [It2, Lemma 5.3]).

LEMMA 2.7. *If*

$$(2.37) \quad \sum_{i,j=1}^n \left\| \frac{\partial(\tilde{g}_i - \tilde{g}_i^{s_i})}{\partial \tau_j} \right\|_r \frac{1}{r^{s_i-2}} + \sum_{i=1}^n \sum_{j=n+1}^{n+2} \left\| \frac{\partial(\tilde{g}_i - \tilde{g}_i^{s_i})}{\partial \tau_j} \right\|_r \frac{1}{r^{s_i-|\mu|}} < \frac{1}{2c_4}$$

then the polynomials $\{\tau_k, W\}$ ($1 \leq k \leq n - 1$) satisfy

$$(2.38) \quad \|\{\tau_k, W\}\|_r \leq 2c_4 \sum_{i=1}^n \frac{\|\tilde{G}_i\|_r}{r^{s_i-2}}.$$

Set $\delta := \min_{1 \leq j \leq n} \delta_j$ and $c_5 := 4\pi c_4$.

LEMMA 2.8. *Let $0 < \rho < r$. Under the assumption of Lemma 2.7, the following estimates hold:*

- (i) $\|W\|_r \leq c_5 \sum_{i=1}^n \frac{\|\tilde{G}_i\|_r}{r^{s_i-2}}$
- (ii) $\left\| \frac{\partial W}{\partial \xi_i} \right\|_\rho \leq \frac{c_5}{\delta(r-\rho)} \sum_{k=1}^n \frac{\|\tilde{G}_k\|_r}{r^{s_k-2}} \quad (1 \leq i \leq 2n)$
- (iii) $\left\| \frac{\partial^2 W}{\partial \xi_i \partial \xi_j} \right\|_\rho \leq \frac{4c_5}{\delta^2(r-\rho)^2} \sum_{k=1}^n \frac{\|\tilde{G}_k\|_r}{r^{s_k-2}} \quad (1 \leq i, j \leq 2n).$

PROOF. (i) By Lemma 2.5,

$$|W^{\ell+2}|_r^{\text{sup}} \leq 2\pi \sum_{k=1}^{n-1} |\{\tau_k, W^{\ell+2}\}|_r^{\text{sup}}$$

which, combined with Lemma 2.7 implies (i). Estimate (ii) and (iii) follow from (i) by applying Cauchy’s integral formula. □

2.5. – Proof of Theorem 1.1

The proof of Theorem 1.1 is now completed as in Ito [It2]. One first provides, using Lemma 2.8, estimates at the k ’th iteration step for the flow $X_{W_k}^t$ and $G_i^{(k)} \cdot \varphi_k$ where $\varphi_k = X_{W_k}^{t=1}$ and then uses them to prove that the limit $\varphi = \lim_{k \rightarrow \infty} \varphi_0 \cdots \varphi_k$ (cf. Corollary 2.2) defines a holomorphic coordinate transformation in a neighborhood of the origin.

3. – On the level sets of an integrable system near a fixed point with a simple resonance

In this section, we consider real integrable systems of the type described in Theorem 1.1’. Such a system can be viewed as a complex Hamiltonian system by introducing the symplectic coordinates

$$(3.1) \quad (x, iy) = \left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}}, i \frac{\hat{x} - i\hat{y}}{\sqrt{2}} \right)$$

where $\hat{z} := (\hat{x}, \hat{y})$ are coordinates as provided by Theorem 1.1'. In order to keep the exposition simple, we will often use the complex notation x, y with the understanding that (3.1) always holds and use $z = (x, y)$ rather than (x, iy) as we are interested in the geometry of the the level sets rather than the dynamics of the system. Choose $G_j = \tau_j$ ($1 \leq j \leq n - 1$) and denote by

$$(3.2) \quad M_c := \{ \hat{z} := (\hat{x}, \hat{y}) \in (\mathbb{R}^{2n}, 0) \mid G_j(z) = c_j \ (1 \leq j \leq n) \}$$

a level set of the integrable system where $(\mathbb{R}^{2n}, 0)$ is a neighborhood of 0 in \mathbb{R}^{2n} which is invariant under the Hamiltonian flows corresponding the the Hamiltonians G_j , $1 \leq j \leq n - 1$, (cf. paragraph before Proposition 3.2) and by $M_{c,z}$ the connected component of M_c containing z .

3.1. – Generic level sets

Notice that for generic $\hat{z} := (\hat{x}, \hat{y}) \in (\mathbb{R}^{2n}, 0)$, $d_z G_1, \dots, d_z G_n$ are linearly independent. By Sard's theorem (cf. e.g. [Hi]) the following result holds:

PROPOSITION 3.1. *For generic c , $d_z G_1, \dots, d_z G_n$ are linearly independent for arbitrary \hat{z} in M_c .*

For the sequel assume that the connected component $M_{c,z}$ of M_c with $z \in M_{c,z}$ has the property that $d_{z'} G_1, \dots, d_{z'} G_n$ are linearly independent for arbitrary z' in $M_{c,z}$. Following a standard argument one uses the flows $\varphi_j^{t_j}(z)$ corresponding to the Hamiltonian vector fields X_{G_j} with initial condition $\varphi_j^0(z) = z$ ($1 \leq j \leq n$) to construct a local diffeomorphism

$$\begin{aligned} \Phi(\cdot, z) : \mathbb{R}^{n-1} \times I(z) &\rightarrow M_{c,z} \\ (t_1, \dots, t_n) &\mapsto \varphi_1^{t_1} \circ \dots \circ \varphi_n^{t_n}(z) \end{aligned}$$

where $I(z)$ denotes the maximal interval of existence of the flow $\varphi_n^{t_n}(z)$. As the flows corresponding to X_{G_j} ($1 \leq j \leq n - 1$) are periodic (the integrals G_j ($1 \leq j \leq n - 1$) are an incomplete set of action variables) one concludes that $M_{c,z}$ is either diffeomorphic to $(S^1)^n$ or $(S^1)^{n-1} \times I(z)$. (Flows can also be used to study the level sets of complex systems. In that case the flow variables t_j are complex and the orbits $\varphi_j^{t_j}(z)$ are Riemann surfaces whose properties however are not easily analyzed (cf. e.g. [Sb]).)

Our aim is to obtain additional information, depending on properties of μ and G_n , on M_c and the fibration they induce. Without loss of generality, we may assume that μ and $\varrho^{(j)}$ satisfy the following (normalization) conditions

$$(3.3) \quad \begin{aligned} \mu_j &= 0 \ (1 \leq j \leq \ell); & \mu_j &\neq 0 \ (\ell + 1 \leq j \leq n); & \mu_n &> 0; \\ \varrho_k^{(j)} &:= e_j \ (1 \leq j \leq \ell); & \varrho_k^{(j)} &= 0 \ (\ell + 1 \leq j \leq n; & 1 \leq k \leq l) \end{aligned}$$

where ℓ is some integer, $0 \leq \ell \leq n - 2$, and e_1, \dots, e_n is the standard basis in \mathbb{R}^n . Introduce

$$\Omega := \left\{ \omega = (\omega_1, \dots, \omega_n, \omega_{n+1}) \in \overline{\mathbb{R}}_+^n \times \mathbb{C} \mid \prod_{j=\ell+1}^n \omega_j^{|\mu_j|} = |\omega_{n+1}|^2 \right\}$$

where $\overline{\mathbb{R}}_+$ is the closure of $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$ and define

$$\psi : \{(x, y) \in \mathbb{C}^{2n} \mid x_j = \bar{y}_j (1 \leq j \leq n)\} \rightarrow \Omega$$

given by

$$\omega_k \equiv \psi_k(x, y) := x_k y_k \quad (1 \leq k \leq n); \quad \omega_{n+1} \equiv \psi_{n+1}(x, y) := x^{\mu^+} y^{\mu^-}.$$

For $(\Omega, 0)$ we use the standard neighborhood system $\{U_\epsilon\}_\epsilon$ with $U_\epsilon := \{\omega \in \Omega : \sum_{j=1}^{n+1} |\omega_j|^2 < \epsilon\}$. For $c = (c_1, \dots, c_n)$ with $|c| \ll \epsilon$, introduce

$$B_c := \{\omega \in U_\epsilon \mid G_j(\omega) = c_j \quad (1 \leq j \leq n)\}$$

where here we view the G_j 's as functions of $\omega_1, \dots, \omega_{n+1}$ which are real analytic in $\omega_1, \dots, \omega_n, \Re \omega_{n+1}, \Im \omega_{n+1}$. By a standard argument (cf. e.g. [Lo]), for generic c (i.e. for $c \notin \Delta$ for some proper analytic subspace $(\Delta, 0) \subset (\mathbb{R}^{n+1}, 0)$) and for $|c| \ll \epsilon$, B_c is a one dimensional real smooth C^∞ -manifold, and its C^∞ -type does not depend on the choice of ϵ and c . Since $\psi^{-1}(0) = 0$ and ψ is proper, $V_\epsilon = \psi^{-1}(U_\epsilon)$ defines a neighborhood system $\{V_\epsilon\}_\epsilon$ of $(\mathbb{R}^{2n}, 0)$. Define for $c = (c_1, \dots, c_n)$ with $|c| \ll \epsilon$

$$M_c := \{z \in \psi^{-1}(U_\epsilon) \mid G_j(z) = c_j \quad (1 \leq j \leq n)\}.$$

Notice that ψ induces a map $M_c \rightarrow B_c$ which we again denote by ψ .

The following result reduces the analysis of M_c to the one of B_c , at least for generic c .

PROPOSITION 3.2. *For generic $c \in (\mathbb{R}^n, 0)$ with $|c| \ll \epsilon$, the following statements hold:*

- (i) $\psi : M_c \rightarrow B_c$ is a fiber bundle with fiber $(S^1)^{n-1}$;
- (ii) $\psi : M_c \rightarrow B_c$ admits a real analytic trivialization Ψ ,

$$\begin{array}{ccc} B_c \times (S^1)^{n-1} & \xrightarrow{\Psi} & M_c \\ pr_1 \searrow & & \swarrow \psi \\ & & B_c \end{array}$$

where $pr_1 : B_c \times (S^1)^{n-1} \rightarrow B_c$ denotes the canonical projection.

REMARK 3.1. In Subsection 3.3, we present a different trivialization which is independent of c , obtained from using the Hamiltonian flows corresponding to τ_1, \dots, τ_n . The one constructed in the proof of Proposition 3.2 will be convenient in Subsection 3.2.

To prove Proposition 3.2 we need the following auxiliary result. Define $\Omega' := \{\omega \in \Omega \mid \omega_j \neq 0 \text{ for } 1 \leq j \leq n\}$.

LEMMA 3.3. For generic $c \in (\mathbb{R}^n, 0)$, $B_c \subset \Omega'$.

PROOF. For a generic $c \in (\mathbb{R}^n, 0)$ we may assume that $c_j \neq 0$ ($1 \leq j \leq n$). Thus, for any $\omega \in B_c$, $\omega_j = \tau_j = c_j \neq 0$ for $1 \leq j \leq \ell$. Assume that there exist $\omega \in B_c$ and j_0 , $\ell + 1 \leq j_0 \leq n$, with $\omega_{j_0} = 0$ or, equivalently, that $\omega_{n+1} = 0$. Then, the system of equations

$$(3.4) \quad \sum_{k \neq j_0} \rho_k^{(j)} \omega_k = c_j (1 \leq j \leq n - 1); \quad G_n(\omega_1, \dots, \omega_n, 0) = c_n$$

has a solution $\omega^0 := (\omega_1, \dots, \omega_{j_0-1}, 0, \omega_{j_0+1}, \dots, \omega_n, 0)$. Introduce

$$\Sigma := \{\omega \in (\Omega, 0) \mid \omega_{j_0} = 0\} (= (\mathbb{R}^{n-1}, 0)).$$

The closure of $G(\Sigma)$ is an analytic variety of dimension not greater than $n - 1$, and $c \in G(\Sigma)$. We conclude that c is not generic. This shows that for generic c , $\omega_{n+1} \neq 0$ for any $(\omega_1, \dots, \omega_{n+1}) \in B_c$. \square

PROOF OF PROPOSITION 3.2. Notice that the base space Ω' is a product $(\mathbb{R}_+)^{\ell} \times \tilde{\Omega}'$ where

$$\tilde{\Omega}' := \left\{ (\omega_{\ell+1}, \dots, \omega_{n+1}) \in (\mathbb{R}_+)^{n-\ell} \times \mathbb{C}^* \mid \prod_{j=\ell+1}^n \omega_j^{|\mu_j|} = |\omega_{n+1}|^2 \right\}$$

with $\mathbb{C}^* := \mathbb{C} \setminus 0$. Correspondingly, the map ψ can be written as $\psi_1 \times \psi_2$ where

$$\psi_1 : \{(x, y) \in \mathbb{C}^{2\ell} \mid x_j = \bar{y}_j \neq 0 (1 \leq j \leq \ell)\} \rightarrow (\mathbb{R}_+)^{\ell}$$

is given by $\psi_1(x, y) := (x_1 y_1, \dots, x_{\ell} y_{\ell})$ and

$$\psi_2 : \{(x, y) \in \mathbb{C}^{2n-2\ell} \mid x_j = \bar{y}_j \neq 0 (\ell + 1 \leq j \leq n)\} \rightarrow \tilde{\Omega}'$$

is given by $\psi_2(x, y) := (x_{\ell+1} y_{\ell+1}, \dots, x_n y_n, \prod_{j=\ell+1}^n x_j^{\mu_j^+} y_j^{\mu_j^-})$.

The map ψ_1 is a trivial fiber bundle

$$\begin{array}{ccc} (\mathbb{R}_+)^{\ell} \times \psi_1^{-1}(1, \dots, 1) & \xrightarrow{\psi_1} & \psi_1^{-1}((\mathbb{R}_+)^{\ell}) \\ \text{\scriptsize } pr_1 \searrow & & \swarrow \text{\scriptsize } \psi_1 \\ & & (\mathbb{R}_+)^{\ell} \end{array}$$

where Ψ_1 is given by

$$\Psi_1(\omega_1, \dots, \omega_\ell, x_1, \dots, x_\ell, \bar{x}_1, \dots, \bar{x}_\ell) := (\sqrt{\omega_j}x_j, \sqrt{\omega_j}\bar{x}_j)_{1 \leq j \leq \ell}$$

and the fiber $\psi_1^{-1}(1, \dots, 1)$ is

$$\psi_1^{-1}(1, \dots, 1) = \{(x_1, \dots, x_\ell, \bar{x}_1, \dots, \bar{x}_\ell) \mid x_j \bar{x}_j = 1 \ (1 \leq j \leq \ell)\} = (S^1)^\ell.$$

The map ψ_2 is also a trivial fiber bundle

$$\begin{array}{ccc} \tilde{\Omega}' \times \psi_2^{-1}(1, \dots, 1) & \xrightarrow{\Psi_2} & \psi_2^{-1}(\tilde{\Omega}') \\ \text{pr}_1 \searrow & & \swarrow \psi_2 \\ & \tilde{\Omega}' & \end{array}$$

where

$$\begin{aligned} & \Psi_2((\omega_j)_{\ell+1 \leq j \leq n+1}, (x_j, \bar{x}_j)_{\ell+1 \leq j \leq n}) : \\ & = \left(\sqrt{\omega_j} \left(\frac{\omega_{n+1}}{|\omega_{n+1}|} \right)^{\rho_j^{(n)}} x_j, \sqrt{\omega_j} \left(\frac{\omega_{n+1}}{|\omega_{n+1}|} \right)^{\rho_j^{(n)}} \bar{x}_j \right)_{\ell+1 \leq j \leq n}. \end{aligned}$$

Notice that $\psi_2^{-1}(1, \dots, 1) \simeq (S^1)^{n-l-1}$. Indeed, fix a point $(x, \bar{x}) \in \psi_2^{-1}(1, \dots, 1)$. Then $R : (S^1)^{n-l-1} \rightarrow \psi_2^{-1}(1, \dots, 1)$, defined by

$$R(z_{l+1}, \dots, z_{n-1}) = \left(\left(\prod_{j=l+1}^{n-1} z_j^{\rho_k^{(j)}} \right) x_k, \left(\prod_{j=l+1}^{n-1} \bar{z}_j^{\rho_k^{(j)}} \right) \bar{x}_k \right)$$

is an isomorphism where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. (Recall that by the normalization (3.3), $\rho_k^{(j)} = 0$ if $l+1 \leq j \leq n$ and $1 \leq k \leq l$.) □

The following two propositions relate the nature of the prime resonance vector μ with the topology of B_c .

PROPOSITION 3.4. *Assume that μ is oscillating, i.e. that there exists j_0 with $\ell + 1 \leq j_0 \leq n - 1$ such that $\mu_{j_0} < 0$. (Recall that $\mu_n > 0$.) Then, for $c \in \mathbb{R}^n$ sufficiently small, B_c is compact. If B_c is smooth (which is the case for generic c), then each connected component of B_c is diffeomorphic to S^1 .*

PROOF. It is to prove that for c sufficiently small, B_c is compact. Consider the linear system

$$\sum_{k=1}^n \rho_k^{(j)} \omega_k = c_j \quad (1 \leq j \leq n - 1)$$

which admits a solution $\omega_k = \frac{\mu_k}{\mu_n} \omega_n - b_k$ where

$$b_k \equiv b_k(c') := \sum_{j=1}^{n-1} \left(-b_{kj} + \frac{\mu_k}{\mu_n} b_{nj} \right) c_j \quad (1 \leq k \leq n-1); \quad b_n \equiv b_n(c') = 0$$

with $c' = (c_1, \dots, c_{n-1})$ and, as in Subsection 2.3, (b_{kj}) is the inverse of the unimodular $n \times n$ matrix whose rows are given by $\rho^{(1)}, \dots, \rho^{(n)}$. Introduce $J^\pm := \{j \mid \ell+1 \leq j \leq n; \pm\mu_j > 0\}$. For $j \in J^+$ we write $\omega_j = \frac{\mu_j}{\mu_n} (\omega_n - \frac{\mu_n}{\mu_j} b_j)$ to conclude from $\omega_j \geq 0$ that $\omega_n \geq \frac{\mu_n}{\mu_j} b_j$. Similarly, for $j \in J^-$, we write $\omega_j = (-\frac{\mu_j}{\mu_n})(\frac{\mu_n}{\mu_j} b_j - \omega_n)$ and conclude that $\frac{\mu_n}{\mu_j} b_j \geq \omega_n$. Therefore, if $\omega \in B_c$,

$$0 \leq M^+ \leq \omega_n \leq M_- < \infty$$

where $M^+ \equiv M^+(c') := \max\{\frac{\mu_n}{\mu_j} b_j \mid j \in J^+\}$ and $M_- \equiv M_-(c') := \min\{\frac{\mu_n}{\mu_j} b_j \mid j \in J^-\}$. Notice that $M_- < \infty$ as $J^- \neq \emptyset$ and $0 \leq M^+$ as $n \in J^+$ and $b_n = 0$. In particular, we observe that $B_c = \emptyset$ if $M_- < M^+$. For the case $M^+ \leq M_-$ recall that $\omega \in B_c$ implies that $\prod_{\ell+1}^n \omega_j^{|\mu_j|} = |\omega_{n+1}|^2$. To rewrite this equation in a convenient way let

$$f_+(t) := \prod_{j \in J^+} \left(t - \frac{\mu_n}{\mu_j} b_j \right)^{\mu_j}, \quad f_-(t) := \prod_{j \in J^-} \left(\frac{\mu_n b_j}{\mu_j} - t \right)^{|\mu_j|}$$

and

$$f(t) := C f_+(t) f_-(t)$$

where

$$C := \left(\prod_{j=\ell+1}^n \left| \frac{\mu_j}{\mu_n} \right|^{|\mu_j|} \right) > 0.$$

One verifies that, with $\omega_j = \frac{\mu_j}{\mu_n} \omega_n - b_j$, $|\omega_{n+1}|^2 = \prod_{j=\ell+1}^n \omega_j^{|\mu_j|} = f(\omega_n)$, that $f(t)$ is a polynomial in t with $f(t) > 0$ for $M^+ < t < M_-$ and $f(M^+) = f(M_-) = 0$. Moreover, if $M^+ < M_-$, there exists a unique element t_* , $M^+ < t_* < M_-$, so that $\text{Max}_f = f(t_*)$ where

$$\text{Max}_f \equiv \text{Max}_f(c') := \max \{ f(t) \mid M^+ < t < M_- \}.$$

Notice that $\lim_{c' \rightarrow 0} b_j(c') = 0$ and therefore

$$\lim_{c' \rightarrow 0} M^+(c') = \lim_{c' \rightarrow 0} M_-(c') = \lim_{c' \rightarrow 0} \text{Max}_f(c') = 0.$$

Further, if $\omega \in B_c$, then

$$0 \leq \omega_n \leq M_-(c'), \quad \text{and} \quad |\omega_{n+1}|^2 \leq \text{Max}_f(c').$$

We conclude that for any ε -disc D_ε in $\overline{\mathbb{R}}_+^n \times \mathbb{C}$ there exists $\delta \equiv \delta(\varepsilon) > 0$ so that if $|c| < \delta$, then $B_c \cap D_\varepsilon \subset D_{\varepsilon/2}$ i.e. for c sufficiently small, B_c is compact. If B_c is smooth, it then follows that it is a disjoint union of circles. \square

PROPOSITION 3.5. Assume that $\mu_j \geq 0$ ($1 \leq j \leq n$). Then for generic c in $(\mathbb{R}^n, 0)$ with $|c| \ll 1$ and generic G_n (generic in the sense that \mathcal{A} is a generic Poisson algebra with simple resonance) the number $\#_{\text{open}}$ of open connected components of B_c (i.e. components which are diffeomorphic to the unit interval $(0, 1)$) is given as follows:

- (i) if $|\mu| = 2$ or 3 , then $\#_{\text{open}} = 1$;
- (ii) if $|\mu| = 4$, then either $\#_{\text{open}} = 0$ or $\#_{\text{open}} = 1$
- (iii) if $|\mu| \geq 5$, then $\#_{\text{open}} = 0$.

PROOF. In a first step we show that for generic c , B_c can be identified with the Milnor fiber of a hypersurface, obtained from G_n . Using the same notation as in the proof of Proposition 3.4, $f(t) = \prod_{j=\ell+1}^n (\frac{\mu_j}{\mu_n})^{\mu_j} (t - \frac{\mu_n}{\mu_j} b_j)^{\mu_j}$ as $J^- = \emptyset$. Notice that

$$f : (M^+, \infty) \longrightarrow (0, \infty)$$

is a real analytic isomorphism and admits a square root, $f^{\frac{1}{2}} : (M^+, \infty) \longrightarrow (0, \infty)$, which is again a real analytic isomorphism. Denote by $g \equiv g_{c'}$ the inverse of $f^{\frac{1}{2}}$ where $c' = (c_1, \dots, c_{n-1})$. Substituting $\omega_j = \frac{\mu_j}{\mu_n} g(r) - b_j$ ($\ell+1 \leq j \leq n-1$), and taking into account that $f(\omega_n) = r^2$ and $\omega_{n+1} = r e^{i\theta}$, B_c can be identified with the Milnor fiber

$$\{(r, \theta) \mid r \in (\overline{\mathbb{R}}_+, 0); H(g(r), r, \theta) = c_n\}$$

where $H(\omega_n, r, \theta) \equiv H_{c'}(\omega_n, r, \theta) := G_n(\omega)$.

In a next step consider the space B_0 corresponding to $c_j = 0$ ($1 \leq j \leq n$), i.e.

$$B_0 := \{(t, r, \theta) \mid t \in (\mathbb{R}_+, 0); t^{|\mu|} = C^{|\mu|} r^2; H(t, r, \theta) = 0\}$$

where $C \equiv C(\mu) := \prod_{k=\ell+1}^{n-1} (\frac{\mu_k}{\mu_n})^{-\mu_k/|\mu|}$. H admits an expansion of the form

$$H(t, r, \theta) := \sum_{\ell, k \geq 0} t^\ell r^k \left(\sum_{j+j'=k} a_{k\ell j} (\cos \theta)^j (\sin \theta)^{j'} \right).$$

Substituting $t = C(\mu)r^{2/|\mu|}$, one obtains

$$H_1(r, \theta) \equiv H(C(\mu)r^{2/|\mu|}, r, \theta) = \sum_{q=k+2\ell/|\mu|} P_q(\theta)r^q$$

where $P_q(\theta) := \sum_{q=k+2\ell/|\mu|} C^\ell (\sum_{j+j'=k} a_{k\ell j} (\cos \theta)^j (\sin \theta)^{j'})$.

As $P_q(\theta)$ is a polynomial in $\cos \theta$ and $\sin \theta$ (and thus, in particular, periodic of period 2π) the number $\#_q$ of roots of $P_q(\theta)$ in the interval $[0, 2\pi)$, counted with multiplicity, is even. Let

$$q_0 := \min \left\{ q = k + \frac{2\ell}{|\mu|} \mid P_q \not\equiv 0; k, \ell \geq 0 \right\}.$$

If $P_{q_0}(\theta)$ has only simple roots, we claim that, for generic c in $(\mathbb{R}^n, 0)$, the number $\#_{\text{open}}$ of noncompact components of B_c is given by $\frac{1}{2}\#_{q_0}$. This can be proved as follows: consider a sufficiently small, fixed disc $D_\varepsilon := \{(r, \theta) \mid |r| \leq \varepsilon\}$. The equation defining B_c is a perturbation of $r^{q_0}P_{q_0}(\theta) = 0$. The solution set $\{(r, \theta) \mid P_{q_0}(\theta) = 0\}$ are rays which intersect ∂D_ε transversely (in fact, orthogonally). Since intersecting transversally is an open property, the number of noncompact components of B_c , for $|c|$ sufficiently small, is given by

$$\frac{1}{2}\#(B_c \cap \partial D_\varepsilon) = \frac{1}{2}\#\{\theta \in [0, 2\pi) \mid P_{q_0}(\theta) = 0\}$$

and the above claim follows.

It remains to analyze P_{q_0} . For that purpose, expand $G_n(\omega)$,

$$G_n(\omega) = \sum_1^n a_j \omega_j + \sum_{j,k} a_{jk} \omega_j \omega_k + \beta_1 \text{Re}(\omega_{n+1}) + \beta_2 \text{Im}(\omega_{n+1}) + \dots$$

where the dots stand for terms of higher order, and substitute, for $1 \leq j \leq n$, $\omega_j = \frac{\mu_j}{\mu_n} \omega_n = \frac{\mu_j}{\mu_n} C r^{\frac{2}{|\mu|}}$ to obtain, as $|\mu| \geq 2$,

$$H(Cr^{\frac{2}{|\mu|}}, r, \theta) = \left(\sum_1^n a_j \mu_j \right) \frac{C}{\mu_n} r^{\frac{2}{|\mu|}} + \alpha r^{\frac{4}{|\mu|}} + O(r^{\frac{6}{|\mu|}}) + \beta_1 r \cos \theta + \beta_2 r \sin \theta + O(r^{1+\frac{2}{|\mu|}})$$

where $\alpha := (\frac{C}{\mu_n})^2 \sum_{j,k} a_{jk} \mu_j \mu_k$.

As $G_n \in \mathcal{A}$ and μ is a prime resonant vector for \mathcal{A} , $\sum_{j=1}^n a_j \mu_j = 0$. Further, for G_n generic, $\alpha \neq 0$ as well as $(\beta_1, \beta_2) \neq (0, 0)$. Therefore, if $|\mu| \geq 5$, $q_0 = \frac{4}{|\mu|}$, $P_{q_0} \equiv \alpha$ and thus $\#_{q_0} = 0$. If $|\mu| = 2$ or 3 , then $q_0 = 1$, $P_{q_0}(\theta) = \beta_1 \cos \theta + \beta_2 \sin \theta$ has two simple roots in $[0, 2\pi)$ and therefore $\#_{q_0} = 2$. If $|\mu| = 4$, $P_{q_0}(\theta) = \alpha + \beta_1 \cos \theta + \beta_2 \sin \theta$, then $P_{q_0}(\theta)$ has generically only simple roots and, depending of the size of α, β_1, β_2 , either $\#_{q_0} = 0$ or $\#_{q_0} = 2$. □

To illustrate the above results, we present a number of examples.

EXAMPLE 1 (cf. Proposition 3.5). For a first set of examples, let $n = 2$, $\mu = (2, 1)$, choose $\rho^{(1)} = (-1, 2)$, $\rho^{(2)} = (1, -1)$, $G_1 := \tau_1 = -\omega_1 + 2\omega_2$ and let G_2 be a power series in $\omega_1, \omega_2, \Re \omega_3$ and $\Im \omega_3$ where $\omega_3 = x_1^2 x_2$. Then, with $c = (c_1, c_2)$,

$$B_c = \{(\omega_1, \omega_2, \omega_3) \in (\overline{\mathbb{R}}_+^2 \times \mathbb{C}, 0) \mid \omega_1 \omega_2 = |\omega_3|^2; G_1 = c_1; G_2 = c_2\}.$$

Writing $t := \omega_2, \omega_3 = r e^{i\theta}$ and $\omega_1 = 2t - c_1$, B_c can be described, with some abuse of notation,

$$B_c = \{(t, r, \theta) \mid (t, r) \in R_{c_1}; \theta \in \mathbb{R}; G_2(t, r, \theta) = c_2\}$$

where $R_{c_1} := \{(t, r) \in (\mathbb{R}^2, 0) \mid t \geq 0; r \geq 0; 2t - c_1 \geq 0; (2t - c_1)t = r^2\}$. The set R_{c_1} is the graph in the positive (t, r) -quadrant of a curve which is strictly increasing. We consider B_c as a fiber space above R_{c_1} , where $B_c \rightarrow R_{c_1}, (t, r, \theta) \mapsto (t, r)$ is the projection and the fiber above (t, r) is given by the solution set $F_{c_2}(t, r) := \{\theta \in \mathbb{R} \mid G_2(t, r, \theta) = c_2\}$.

Let $A = (a_{ij})$ be the unimodular 2×2 matrix whose rows are given by $\rho^{(1)}, \rho^{(2)}$. The rows of the transpose of the inverse, $(A^{-1})^T = (\beta_{ij})$, are then given by $\nu^{(1)} = (1, 1), \nu^{(2)} = (2, 1)$. Let $x_j = \sqrt{\omega_j} e^{i\theta_j}$ ($j = 1, 2$) and notice that (ω_j, θ_j) are symplectic polar coordinates, $\omega_j = (\hat{x}_j^2 + \hat{y}_j^2)/2, \theta_j = \arg(\hat{x}_j + i\hat{y}_j)$. The coordinate transformation $(\tau_1 := \sum a_{1j}\omega_j = -\omega_1 + 2\omega_2, s_1 := \sum \beta_{1j}\theta_j = \theta_1 + \theta_2; \tau_2 := \sum a_{2j}\omega_j = \omega_1 - \omega_2, s_2 := \sum \beta_{2j}\theta_j = 2\theta_1 + \theta_2)$ is canoncial. Note that $s_2 = \theta$ and the variable s_1 is a new time variable for the reduced Hamiltonian system under the reduction $G_1 = c_1$ (cf. [Ar2, p 259]). Instead of using the coordinates $t = \omega_2, r = \omega_1\sqrt{\omega_2}, \theta = s_2$ to describe the level sets B_c , we could equally well use the coordinates τ_1, τ_2, s_2 .

EX. 1.1. Let $G_2 = \Re\omega_3 = r \cos \theta$. Then $F_{c_2}(t, r) = \emptyset$ if $r < |c_2|$. The equation $r \cos \theta = c_2$ has one solution in $\theta \pmod{2\pi}$ if $r = |c_2|$ and two solutions if $r > |c_2|$. One concludes that for $|c_2| > 0, B_c$ is diffeomorphic to the unit interval $(0, 1)$.

As shown in [Ar2, p. 259], the intersection of $M_c \cap \{\theta_2 = \text{constant}\}$ with the (ω_1, θ_1) plane (Poincaré section) consists of two disconnected components of one dimension which are, generically, parametrized by ω_2 . These components are however connected when considered in the space Ω .

EX. 1.2. Let $G_2 = \Re\omega_3^k = r^k \cos k\theta$. By a similar argument, one concludes that for $c_2 \neq 0, B_c = \bigsqcup_k (0, 1)$, i.e. B_c is a disjoint union of k copies of the unit interval.

EX. 1.3. Let $G_2 = (\Re\omega_3)^4 + (\Im\omega_3)^4$. Notice that, for $c_2 < 0, B_c = \emptyset$ and for $c = 0, B_0$ consists of the origin only. Following the arguments of the proof of Proposition 3.5, we conclude that for $c_2 > 0, B_c$ is diffeomorphic to S^1 . \square

EXAMPLE 2 (cf. Proposition 3.4). For this second set of examples, let $n = 2, \mu = (1, -1)$ and $G_1 := \tau_1 = \omega_1 + \omega_2$. Then

$$B_c = \{(\omega_1, \omega_2, \omega_3) \in (\overline{\mathbb{R}}_+^2 \times \mathbb{C}, 0) \mid \omega_1\omega_2 = |\omega_3|^2; \omega_1 + \omega_2 = c_1; G_2 = c_2\}.$$

Writing again $t := \omega_2, \omega_3 = re^{i\theta}$, one has $\omega_1 = c_1 - t$ and obtains

$$B_c = \{(t, r, \theta) \mid (t, r) \in R'_{c_1}; \theta \in \mathbb{R}; G_2 = c_2\}$$

where $R'_{c_1} := \{(t, r) \in (\overline{\mathbb{R}}_+^2, 0) \mid c_1 - t \geq 0; (c_1 - t)t = r^2\}$. If $c_1 < 0$, then $B_c = \emptyset$ and if $c_1 = 0, B_c$ consists only of $(\omega_1, \omega_2, \omega_3) = (0, 0, 0)$.

EX. 2.1. Let $c_1 > 0$ and $G_2 := \Re\omega_3 = r \cos \theta$. Then $B_c = S^1$ if $|c_2| < \frac{c_1}{2}$ and $B_c = \emptyset$ for $|c_2| > \frac{c_1}{2}$.

Ex. 2.2. Let $c_1 > 0$ and $G_2 := \Re\omega_3^k = r^k \cos k\theta$. Then $B_c = \bigsqcup_k S^1$ (disjoint union of k copies of S^1) if $|c_2|^{\frac{1}{k}} < \frac{c_1}{2}$ and $B_c = \emptyset$ if $|c_2|^{\frac{1}{k}} > \frac{c_1}{2}$.

We remark that Ex. 2.1-2.2 fit into the following more general situation: According to the proof of Proposition 3.4 and in view of the equations $\omega_j = \frac{\mu_j}{\mu_n} \omega_n - b_j$ (notation as in the proof of Proposition 3.4), B_c is determined by the following equations ($t := \omega_n$; $\omega_{n+1} = r e^{i\theta}$)

- (i) $r^2 = f(t)$ where $M^+(c') < t < M_-(c')$; (ii) $G_n(\omega) \equiv H(t, r, \theta) = c_n$.

Assume that G_n depends only on ω_{n+1} . Then H is independent of t and, for generic c , the set of solutions $(r, \theta), r \in (\mathbb{R}_+, 0)$, satisfying (i) and (ii) is given by $F \bigsqcup_{\partial F} F$ (the disjoint union of two copies of F which are identified along the boundaries) where F is the closed fiber of H in the disc of radius $\sqrt{\text{Max}_f(c')}$, centered at 0,

$$F \equiv F_H(c) := \left\{ (r, \theta) : r \leq \sqrt{\text{Max}_f(c')}; H(r, \theta) = c_n \right\}.$$

To verify this, notice that for given $r > 0$, the equation $f(t) = r^2$ has no solution if $r^2 > \text{Max}_f(c')$, has exactly one solution if $r^2 = \text{Max}_f(c')$ and has two solutions if $r^2 < \text{Max}_f(c')$. In particular, if $\#_o$ and $\#_c$ are the number of open, respectively compact connected components of $F_H(c)$, then B_c has $\#_o + 2\#_c$ connected components, each analytically diffeomorphic to S^1 . (If $|c_n| \ll \text{Max}_f(c')$, then F is the Milnor fiber of G_n .) □

3.2. – Nongeneric level sets

Let us make a few remarks concerning the level sets M_c for nongeneric c . In this case, $B_c \setminus \Omega' \neq \emptyset$ and, for $\omega \in B_c \setminus \Omega'$, the fiber $\psi^{-1}(\omega)$ might be different from $(S^1)^{n-1}$ (cf. Proposition 3.2). To analyze M_c for nongeneric c , we stratify the set Ω . Taking into account the normalization conditions (3.3), one sees that Ω has a product decomposition, $\Omega = (\mathbb{R}_+^{-})^l \times \tilde{\Omega}$ where

$$\tilde{\Omega} := \left\{ (\omega_j)_{l+1 \leq j \leq n+1} \in (\overline{\mathbb{R}_+})^{n-l} \times \mathbb{C} \mid \prod_{j=l+1}^n \omega_j^{|\mu_j|} = |\omega_{n+1}|^2 \right\}.$$

For arbitrary subsets $I \subseteq \{1, \dots, l\}$ and $J \subseteq \{l+1, \dots, n\}$, define the stratum

$$\Omega_{I,J} := \{ \omega \in \Omega \mid \text{for } 1 \leq j \leq n : j \in I \cup J \text{ iff } \omega_j = 0 \}.$$

Notice that for $\omega \in \Omega_{I,J}$ with $J \neq \emptyset$, $\omega_{n+1} = 0$ and that $\Omega' = \Omega_{\emptyset, \emptyset}$ as well as $\Omega = \bigcup_{I,J} \Omega_{I,J}$. It turns out the topological type of the fiber $\psi^{-1}(\omega)$ depends only on the stratum $\Omega_{I,J}$ which contains ω . Indeed, arguing as in the proof of Proposition 3.2 we see that

$$\psi^{-1}(\omega) = \psi_1^{-1}(\omega_1, \dots, \omega_l) \times \psi_2^{-1}(\omega_{l+1}, \dots, \omega_{n+1})$$

where here $\psi_1(x, y) := (x_1y_1, \dots, x_ly_l)$ is defined on $\{(x, y) \in \mathbb{C}^{2l} \mid x_j = \bar{y}_j \ (1 \leq j \leq l)\}$ and $\psi_2(x, y) := (x_{l+1}y_{l+1}, \dots, x_ny_n, \prod_{j=l+1}^n x_j^{\mu_j^+} y_j^{\mu_j^-})$ is defined on $\{(x, y) \in \mathbb{C}^{2n-2l} \mid x_j = \bar{y}_j \ (l+1 \leq j \leq n)\}$. Identifying S^1 with $\{z \in \mathbb{C} \mid |z| = 1\}$, one has, for $\omega \in \Omega_{I,J}$, $\psi_1^{-1}(\omega_1, \dots, \omega_l) = (S^1)^{l-|J|}$ and, if $|J| \neq \emptyset$, $\psi_2^{-1}(\omega_{l+1}, \dots, \omega_n) = (S^1)^{n-l-|J|}$. By Proposition 3.2, for $J = \emptyset$, $\psi_2^{-1}(\omega_{l+1}, \dots, \omega_n) = (S^1)^{n-l-1}$. To study the degeneration of the fibers for $\omega \in \Omega_{I,J}$, the cases $J \neq \emptyset$ and $J = \emptyset$ are treated in the same way and thus we consider the case $J \neq \emptyset$ only. Choose a continuous path $\omega(t) \in \Omega$ ($0 \leq t \leq 1$) without self-intersections, so that $\omega(0) = \omega \in \Omega_{I,J}$ and $\omega(t) \in \Omega' \equiv \Omega_{\emptyset, \emptyset}$ for $0 < t \leq 1$. Choose a continuous lift $(x_k(t), \bar{x}_k(t))_{1 \leq k \leq n}$ of the path $\omega(t)$, i.e. $\psi((x_k(t), \bar{x}_k(t))_{1 \leq k \leq n}) = \omega(t)$. Introduce $Y := \psi^{-1}(\omega(0))$ and $X := \psi^{-1}(\{\omega(t) \mid 0 \leq t \leq 1\})$ and define (with $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$) $R_{I,J} : (S^1)^{n-1} \times [0, 1] \rightarrow X$ by

$$R_{I,J}(s_1, \dots, s_{n-1}, t) := \left(\left(\prod_{j=1}^{n-1} s_j^{\rho_k^{(j)}} \right) x_k(t), \left(\prod_{j=1}^{n-1} \bar{s}_j^{\rho_k^{(j)}} \right) \bar{x}_k(t) \right)_{1 \leq k \leq n}$$

Notice that $R_{I,J}$ is onto and $R_{I,J} \mid_{(S^1)^{n-1} \times (0,1]}$ is 1 – 1. Therefore, $R_{I,J}$ induces a strong deformation retract $r_{I,J} : X \rightarrow Y$ given by (with $s = (s_1, \dots, s_{n-1})$)

$$r_{I,J}(R_{I,J}(s, t)) := R_{I,J}((s, 0)).$$

Further, as $J \neq \emptyset$, $R_{I,J}((S^1)^{n-1} \times \{0\})$ can be identified with

$$\left\{ \left(\left(\prod_{j=1}^{n-1} s_j^{\rho_k^{(j)}} \right) x_k(0), \left(\prod_{j=1}^{n-1} \bar{s}_j^{\rho_k^{(j)}} \right) \bar{x}_k(0) \right)_{k \in \{1, \dots, n\} \setminus I \cup J} \mid (s_1, \dots, s_{n-1}) \in (S^1)^{n-1} \right\}$$

which is diffeomorphic to $(S^1)^{n-|I|-|J|}$ and $R_{I,J} : (S^1)^{n-1} \times \{0\} \rightarrow (S^1)^{n-|I|-|J|}$ can be identified to the projection $pr_{I,J} : (S^1)^{n-1} \rightarrow (S^1)^{n-|I|-|J|}$ on the corresponding factors.

The topological type of M_c for nongeneric c can now be determined as follows: To make the exposition simpler, assume that c is such that $B_{c,\omega} \setminus \Omega'$ consists of a finite set of points, $B_{c,\omega} \setminus \Omega' = \{\omega^{(1)}, \dots, \omega^{(d)}\}$ where $B_{c,\omega}$ denotes the connected component of B_c which contains ω . (If $B_{c,\omega} \setminus \Omega'$ is not finite, there exists $(I, J) \neq (\emptyset, \emptyset)$ so that $B_{c,\omega} \subseteq \Omega_{I,J}$. This case is treated similarly as the case $(I, J) = (\emptyset, \emptyset)$ and is left to the reader.)

For $1 \leq j \leq d$, there exists $(I_j, J_j) \neq (\emptyset, \emptyset)$, so that $\omega^{(j)} \in \Omega_{I_j, J_j}$. Let $z \in M_c$ be a lift of ω , i.e. $\psi(z) = \omega$. Then the connected component $M_{c,z}$ of M_c containing z can be identified with the quotient space $B_{c,\omega} \times (S^1)^{n-1} / \sim$ where for two elements $(\omega', z'), (\omega'', z'') \in B_{c,\omega} \times (S^1)^{n-1}$, $(\omega', z') \sim (\omega'', z'')$, if there exists $1 \leq j \leq d$, so that $\omega' = \omega'' = \omega^{(j)}$ and $pr_{I_j, J_j}(z') = pr_{I_j, J_j}(z'')$. The identification of $M_{c,z}$ with $B_{c,\omega} \times (S^1)^{n-1} / \sim$ can be obtained as follows: Since

all the fibers of ψ are connected, $B_{c,\omega}$ can be lifted to $M_{c,z}$ by a continuous, injective map $B_{c,\omega} \rightarrow M_{c,z}, \omega' \mapsto (x(\omega'), \bar{x}(\omega'))$. This lift is used to define a continuous map $\Lambda : B_{c,\omega} \times (S^1)^{n-1} \rightarrow M_{c,z}$ given by

$$\Lambda(\omega', s_1, \dots, s_{n-1}) := \left(\left(\prod_{j=1}^{n-1} s_j^{\rho_k^{(j)}} \right) x_k(\omega'), \left(\prod_{j=1}^{n-1} \bar{s}_j^{\rho_k^{(j)}} \right) \bar{x}_k(\omega') \right)_{1 \leq k \leq n}$$

3.3. – Fibration by the level sets

Consider a deformation $(M_c)_{c \in \mathcal{C}}$ of level sets, with \mathcal{C} denoting the parameter space of the deformation. We claim that $(M_c)_{c \in \mathcal{C}}$ can be obtained from the corresponding deformation $(B_c)_{c \in \mathcal{C}}$. In particular, if, for $I \subset \{1, \dots, l\}$ and $J \subset \{l + 1, \dots, n\}$ arbitrary, the deformations $(B_c \cap \Omega_{I,J})_{c \in \mathcal{C}}$ (cf. notation of Subsection 3.2) are topologically trivial, then $(M_c)_{c \in \mathcal{C}}$ is topologically trivial as well. First notice that this statement does not follow from the identification $M_{c,z} \simeq B_{c,\omega} \times (S^1)^{n-1} / \sim$ provided in Subsection 3.2, as the identification is constructed by choosing a section of ψ over $B_{c,\omega}$ and therefore depends on c . An identification which is independent of c can be obtained by using the Hamiltonian flows corresponding to τ_1, \dots, τ_n with initial conditions $x_j = \bar{x}_j := r_j \geq 0$ ($1 \leq j \leq n$). Define $\phi : (S^1)^n \times (\overline{\mathbb{R}_+})^n \rightarrow \mathbb{R}^{2n}$, given by $\phi(s_1, \dots, s_n, r_1, \dots, r_n) = (\phi_k, \overline{\phi_k})_{1 \leq k \leq n}$ where

$$\phi_k(s_1, \dots, s_n, r_1, \dots, r_n) := \left(\prod_{j=1}^n s_j^{\rho_k^{(j)}} \right) r_k.$$

Then ϕ induces an isomorphism between $(S^1)^n \times \mathbb{R}_+^n$ and $\{(x, y) \in \mathbb{C}^{2n} \mid x_k = \bar{y}_k; \prod_{k=1}^n x_k y_k \neq 0\}$. Further ϕ collapses some closed subgroups of $(S^1)^n$ over $\{(x, y) \in \mathbb{R}^{2n} \mid \prod_{k=1}^n x_k y_k = 0\}$. One verifies that the composition $\alpha := \psi \cdot \phi : (S^1)^n \times \overline{\mathbb{R}_+}^n \rightarrow \Omega$ is given by $\alpha(s_1, \dots, s_n, r_1, \dots, r_n) = (r_1^2, \dots, r_n^2, r^{\mu^+ + \mu^-} s_n)$ and is therefore universal, i.e. does not depend on the choice of $B_c \subset \Omega$ or c . Then M_c can be obtained from B_c by taking the inverse image $\alpha^{-1}(B_c)$ and collapsing this space by ϕ . Of course, for generic c , this construction gives an alternative proof of Proposition 3.2 and for arbitrary c , this construction leads to the same result as the one given in Subsection 3.2.

As an application we provide a basis for the fundamental group $\pi_1(\psi^{-1}(\omega))$ of $\psi^{-1}(\omega)$ where $\omega \in B_c$.

Let $\eta := (\sqrt{\omega_j})_{1 \leq j \leq n}$ and $\xi := (1, \dots, 1, s_n)$, where s_n is chosen to be 1 if $\omega_{n+1} = 0$, and define pathes $\alpha^{(1)}, \dots, \alpha^{(n-1)}$ in the fiber of $M_c \xrightarrow{\psi} B_c$ above ω by

$$\alpha^{(j)}(t_j) := \left(e^{-i\rho_k^{(j)} t_j} \phi_k(\xi, \eta), e^{i\rho_k^{(j)} t_j} \overline{\phi_k(\xi, \eta)} \right)_{1 \leq k \leq n}$$

where $0 \leq t_j \leq 2\pi$. Notice that on $\psi^{-1}(\omega), x_k dy_k$ is a closed one form ($1 \leq k \leq n$). Thus, by Stokes' theorem, $\int_{\alpha^{(j)}} x_k dy_k$ ($1 \leq k \leq n$) depends only on the

homotopy class $[\alpha^{(j)}]_{\pi_1(\psi^{-1}(\omega))}$ of $\alpha^{(j)}$ in the fundamental group $\pi_1(\psi^{-1}(\omega))$ of $\psi^{-1}(\omega)$.

It follows that $[\alpha^{(j)}]_{\pi_1(\psi^{-1}(\omega))}$ ($1 \leq j \leq n - 1$) spans $\pi_1(\psi^{-1}(\omega))$ and, for generic c , is a basis (over \mathbb{Z}) of $\pi_1(\psi^{-1}(\omega))$.

Appendix A: Decoupled resonances

The aim of this appendix is to indicate why the method used to prove Theorem 1.1 seems only to work in the case of a simple resonance.

We continue to use the notation introduced in Sections 1 and 2. In addition, we introduce the notion of decoupled resonances – they are the most elementary type of resonances one can think of. Let \mathcal{A} be an algebra as defined in (1.1) and denote by $\Lambda_{\mathcal{A}}$ its resonance lattice. This is a lattice in \mathbb{Z}^n such that $\Lambda_{\mathcal{A}} = \bigcap_{f \in \mathcal{A}} \Lambda_f$. We consider the case of multiple resonances, i.e., $R \equiv \dim \Lambda_{\mathcal{A}} \geq 2$.

DEFINITION. \mathcal{A} is said to be resonant with decoupled resonances at $z = 0$ if $\Lambda_{\mathcal{A}}$ admits a basis $\mu^{(1)}, \dots, \mu^{(R)}$ such that

- (i) $\text{supp} \mu^{(j)} \cap \text{supp} \mu^{(k)} = \emptyset$ ($j \neq k$)
- (ii) $|\mu^{(j)}| := \sum_{k=1}^n |\mu_k^{(j)}| \geq 2$.

Notice that $R \leq \frac{n}{2}$, if \mathcal{A} is resonant with decoupled resonances.

In the following, we want to indicate where the proof of Theorem 2.1 as presented in Section 2 seems to break down for an algebra \mathcal{A} , which is resonant with $R \geq 2$ decoupled resonances.

Assume that there is an element H in \mathcal{A} so that $H_{\text{nil}} = 0$ and $\Lambda_{H_s} = \Lambda_{\mathcal{A}}$. Let H_s be given by $H_s = \sum_{j=1}^n \lambda_j x_j y_j$ with $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

It is convenient to introduce the following notation slightly differing from the one used in the previous sections: Denote by $\rho^{(j)}$ ($R + 1 \leq j \leq n$), $\rho^{(j)} \in \mathbb{Z}^n \setminus \{0\}$, a basis (over \mathbb{Z}) of the sublattice of \mathbb{Z}^n , orthogonal to $\Lambda_{\mathcal{A}}$. Further choose $\rho^{(j)}$ ($1 \leq j \leq R$) so that $\rho^{(j)}$ ($1 \leq j \leq n$) is a basis of \mathbb{Z}^n . Let

$$\tau_j := \sum_{k=1}^n \rho_k^{(j)} x_k y_k \quad (1 \leq j \leq n)$$

and

$$\sigma_j^+ := x^{\mu^{(j)+}} y^{\mu^{(j)-}}, \quad \sigma_j^- := x^{\mu^{(j)-}} y^{\mu^{(j)+}} \quad (1 \leq j \leq R)$$

where $\mu_k^{(j)+} := \mu_k^{(j)}$ (if $\mu_k^{(j)} \geq 0$), $\mu_k^{(j)+} := 0$ (if $\mu_k^{(j)} < 0$) and $\mu^{(j)-} := \mu^{(j)+} - \mu^{(j)}$.

Let $f = f(z) \in \mathcal{P}$ be a formal power series in H_s -normal form considered as a power series in τ_j, σ_k^{\pm} ($1 \leq j \leq n; 1 \leq k \leq R$). Note that

$$(A.1) \quad \sigma_k^+ \sigma_k^- = \prod_{j=1}^n (x_j y_j)^{\mu_j^{(k)}}.$$

Therefore, f can be represented uniquely as

$$(A.2) \quad f(z) = \sum_{\text{sign}} f_{\text{sign}}(\tau, \sigma_{\text{sign}})$$

where $\tau := (\tau_1, \dots, \tau_n)$, $\text{sign} := (\gamma_1, \dots, \gamma_R)$ with $\gamma_i \in \{\pm\}$ and $\sigma_{\text{sign}} = (\sigma_1^{\gamma_1}, \dots, \sigma_R^{\gamma_R})$. Here $f_{\text{sign}}(\tau, \sigma_{\text{sign}})$ are power series in $\tau_1, \dots, \tau_n, \sigma_1^{\gamma_1}, \dots, \sigma_R^{\gamma_R}$.

It is useful to consider functions in H_s -normal form as Laurent series in $n + R$ variables $\tau_1, \dots, \tau_n, \sigma_1^+, \dots, \sigma_R^+$ by eliminating σ_j^- using (A.1). For an element $f \in \mathcal{P}$ in H_s -normal form we denote by $\frac{\partial f}{\partial \tau_j}$ ($1 \leq j \leq n$) and $\frac{\partial f}{\partial \sigma_j^+}$ ($1 \leq j \leq R$) the derivatives of f with respect to τ_j, σ_j^+ when f is considered as a (formal) Laurent series in $\tau_1, \dots, \tau_n, \sigma_1^+, \dots, \sigma_R^+$.

Using that

$$(A.3) \quad \begin{aligned} \{\tau_j, \tau_k\} &= 0 \quad (1 \leq j, k \leq n), \\ \{\sigma_j^+, \tau_k\} &= 0 \quad (1 \leq j \leq R, 1 \leq k \leq n; j \neq k), \\ \{\sigma_j^+, \tau_j\} &= \sigma_j^+; \quad \{\sigma_j^+, \sigma_k^+\} = 0 \quad (1 \leq j, k \leq R), \end{aligned}$$

one verifies that for $f, g \in \mathcal{P}$ in H_s -normal form

$$(A.4) \quad \{f, g\} = - \sum_{j=1}^R \left(\frac{\partial f}{\partial \tau_j} \frac{\partial g}{\partial \sigma_j^+} - \frac{\partial f}{\partial \sigma_j^+} \frac{\partial g}{\partial \tau_j} \right) \sigma_j^+.$$

Recall that we have denoted by $\Pi_N f$ the projection of a power series f onto its H_s -normal form part. $\Pi_N f$ can be computed by an averaging procedure

$$(A.5) \quad \Pi_N f(x, y) = \int_0^1 d\theta_{R+1} \cdots \int_0^1 d\theta_n f(e^{2\pi i \theta} x, e^{-2\pi i \theta} y)$$

where $e^{2\pi i \theta} x$ is defined by

$$(A.6) \quad e^{2\pi i \theta} x = \left(e^{2\pi i \sum_{R+1}^n \theta_j \rho_1^{(j)}} x_1, \dots, e^{2\pi i \sum_{R+1}^n \theta_j \rho_n^{(j)}} x_n \right)$$

and $e^{-2\pi i \theta} y$ is defined similarly. Formula (A.5) implies that, given a polynomial W with $\Pi_N W = 0$, the supremum norm $|W|^{\text{sup}}$ on a neighborhood of the origin can be computed by

$$(A.7) \quad |W|^{\text{sup}} \leq 2\pi \sum_{k=R+1}^n |\{\tau_k, W\}|^{\text{sup}}.$$

One of the main points of the proof of Theorem 1.1, as presented in Section 2, is to estimate $|W|^{\text{sup}}$. Let $z = \varphi(\zeta) \equiv X_W^{\zeta=1}(\zeta)$ be the transformation

described in Proposition 2.1 with W being a polynomial, $W = W^{d+2} + \dots + W^{2d+1}$. Then, by Lemma 2.3, with $G_j = g_j + \widehat{G}_j$ and $\widehat{G}_j(\zeta) = O(\zeta^{s_j+d})$,

$$(A.8) \quad G_j \circ \varphi(\zeta) = g_j(\zeta) + \{g_j(\zeta), W\} + \widehat{G}_j(\zeta) + O(|\zeta|^{s_j+2d})$$

is in H_s -normal form up to order $s_j - 1 + 2d$. Therefore, $(1 \leq j \leq n)$

$$(A.9) \quad \{g_j(\zeta), W(\zeta)\} + \Pi_R \widehat{G}_j(\zeta) = O(|\zeta|^{s_j+2d})$$

which can be written as $(1 \leq i \leq n)$

$$(A.10)_i \quad \sum_{j=1}^n \frac{\partial g_i}{\partial \tau_j} \{\tau_j, W\} + \sum_{j=1}^R \frac{\partial g_i}{\partial \sigma_j^+} \{\sigma_j^+, W\} = -\Pi_R \widehat{G}_i(\zeta) + O(|\zeta|^{s_i+2d}).$$

Notice that (A.10) is a linear system of n equations for $n + R$ quantities $\{\tau_1, W\}, \dots, \{\tau_n, W\}$ and $\{\sigma_1^+, W\}, \dots, \{\sigma_R^+, W\}$.

We then try to eliminate the $2R$ variables $\{\tau_j, W\}, \{\sigma_j^+, W\}$ $(1 \leq j \leq R)$ from (A.10) up to terms of higher order, and reduce these n equations to $n - R$ equations for $\{\tau_j, W\}$ with $j = R + 1, \dots, n$. To illustrate this procedure, we recall the case of a simple resonance $R = 1$ (cf. Section 2): we first write (A.10) in matrix form,

$$(A.11) \quad \begin{pmatrix} \frac{\partial g_1}{\partial \tau_1} & \frac{\partial g_1}{\partial \sigma_1^+} \\ \frac{\partial g_i}{\partial \tau_1} & \frac{\partial g_i}{\partial \sigma_1^+} \end{pmatrix} \begin{pmatrix} \{\tau_1, W\} \\ \{\sigma_1^+, W\} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_i \end{pmatrix} \quad (2 \leq i \leq n)$$

where F_1 and F_i include the variables $\{\tau_j, W\}$ with $j = 2, \dots, n$. Let us consider, for example, the case where $(\frac{\partial g_i}{\partial \sigma_1^+}, -\frac{\partial g_1}{\partial \sigma_1^+}) \neq (0, 0)$ (cf. case 2 in Section 2). Then eliminating $\{\sigma_1^+, W\}$ from (A.11) (Cramer's rule), we obtain, after multiplication with $\tau_1 \sigma_1^+$,

$$(A.12) \quad \tau_1 \sigma_1^+ \det \begin{pmatrix} \frac{\partial g_1}{\partial \tau_1} & \frac{\partial g_1}{\partial \sigma_1^+} \\ \frac{\partial g_i}{\partial \tau_1} & \frac{\partial g_i}{\partial \sigma_1^+} \end{pmatrix} \{\tau_1, W\} = \tau_1 \sigma_1^+ \frac{\partial g_i}{\partial \sigma_1^+} F_1 - \tau_1 \sigma_1^+ \frac{\partial g_1}{\partial \sigma_1^+} F_i.$$

Notice that the determinant in (A.12) is related to the Poisson bracket $\{g_1, g_i\}$ (cf. (A.4)). Using $\{G_1, G_i\} = 0$ (integrability), we see that, for $2 \leq i \leq n$, $\{g_1, g_i\} = O(\zeta^{s_1+s_i+d-2})$. Therefore, the left hand side of (A.12) is $O(\zeta^{s_1+s_i+d+d+2})$, so that we obtain from (A.10), up to error terms $O(\zeta^{s_1+s_i+2d})$, a linear system of $n - 1$ equations for $\{\tau_i, W\}$ $(2 \leq i \leq n)$.

Now we consider the case $R = 2$. The equation corresponding to (A.11) is given by ($3 \leq i, j \leq n$)

$$A(1, 2; i, j) \begin{pmatrix} \{\tau_1, W\} \\ \{\sigma_1^+, W\} \\ \{\tau_2, W\} \\ \{\sigma_2^+, W\} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_i \\ F_j \end{pmatrix}$$

where F_1, F_2, F_i and F_j include the variables $\{\tau_k, W\}$ with $3 \leq k \leq n$, and $A(1, 2; i, j)$ is a 4×4 matrix given by

$$(A.13) \quad A(1, 2; i, j) := \begin{pmatrix} \frac{\partial g_1}{\partial \tau_1} & \frac{\partial g_1}{\partial \sigma_1^+} & \frac{\partial g_1}{\partial \tau_2} & \frac{\partial g_1}{\partial \sigma_2^+} \\ \frac{\partial g_2}{\partial \tau_1} & \frac{\partial g_2}{\partial \sigma_1^+} & \frac{\partial g_2}{\partial \tau_2} & \frac{\partial g_2}{\partial \sigma_2^+} \\ \frac{\partial g_i}{\partial \tau_1} & \frac{\partial g_i}{\partial \sigma_1^+} & \frac{\partial g_i}{\partial \tau_2} & \frac{\partial g_i}{\partial \sigma_2^+} \\ \frac{\partial g_j}{\partial \tau_1} & \frac{\partial g_j}{\partial \sigma_1^+} & \frac{\partial g_j}{\partial \tau_2} & \frac{\partial g_j}{\partial \sigma_2^+} \end{pmatrix}$$

for $3 \leq i, j \leq n$ and $i \neq j$. Analogous to the case $R = 1$, we would like to show that (A.10) leads to a linear system (similar to (A.12)), where the terms involving the four variables $\{\tau_1, W\}, \dots, \{\sigma_2^+, W\}$ are of the order of error terms. A necessary condition is that, for some $3 \leq i < j \leq n$, $\tau_1 \tau_2 \sigma_1^+ \sigma_2^+ \det A(1, 2; i, j) = O(\zeta^{s_1+s_2+s_i+s_j+d})$. However, this is not true in general as can be seen as follows: By the integrability $\{G_i, G_j\} = 0$, we obtain by (A.4) relations among the functions, $\frac{\partial g_i}{\partial \tau_k}, \frac{\partial g_j}{\partial \sigma_\ell^+}$ ($1 \leq i, j, k \leq n, 1 \leq \ell \leq 2$),

$$(A.14)_{ij} \quad \sum_{k=1}^2 \left(\frac{\partial g_i}{\partial \tau_k} \frac{\partial g_j}{\partial \sigma_k^+} - \frac{\partial g_i}{\partial \sigma_k^+} \frac{\partial g_j}{\partial \tau_k} \right) \tau_k \sigma_k^+ = O(\zeta^{s_i+s_j+d}).$$

It is convenient to introduce the following notations,

$$a_k^i := \tau_k \frac{\partial g_i}{\partial \tau_k}, \quad b_k^i := \sigma_k^+ \frac{\partial g_i}{\partial \sigma_k^+}$$

and to denote the coefficients $\tau_k \sigma_k^+ \left(\frac{\partial g_i}{\partial \tau_k} \frac{\partial g_j}{\partial \sigma_k^+} - \frac{\partial g_i}{\partial \sigma_k^+} \frac{\partial g_j}{\partial \tau_k} \right)$ in (A.14)_{ij} by $[i, j]_k$.

Then

$$[i, j]_k = \det \begin{pmatrix} a_k^i & b_k^i \\ a_k^j & b_k^j \end{pmatrix} = O(\zeta^{s_i+s_j}).$$

Equations (A.14)_{ij} and (A.14)_{kℓ} lead to the 2×2 system

$$\begin{pmatrix} [i, j]_1 & [i, j]_2 \\ [k, \ell]_1 & [k, \ell]_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} O(\zeta^{s_i+s_j+d}) \\ O(\zeta^{s_k+s_\ell+d}) \end{pmatrix}$$

and therefore, by Cramer’s rule, $(1 \leq i, j, k, \ell \leq n)$

$$(A.15) \quad [i, j]_1[k, \ell]_2 - [i, j]_2[k, \ell]_1 = O(\zeta^{s_i+s_j+s_k+s_l+d}).$$

The identities (A.15) give, in general, a complete set of bilinear relations for the expressions of the form $[i, j]_1[i', j']_2$ for $1 \leq i, j, i', j' \leq n$.

Analogous to the case $R = 1$, we would like to show that, at least for some $3 \leq i < j \leq n$, the term $\tau_1 \tau_2 \sigma_1^+ \sigma_2^+ \det A(1, 2; i, j)$ is factorized by the relation (A.15). Let us express such a term as a product of terms $[i, j]_k$. This can be done, by using the Laplace expansion for a determinant [Ma 1, p. 189], as follows

$$\begin{aligned} \tau_1 \tau_2 \sigma_1^+ \sigma_2^+ \det A(1, 2; i, j) &= \tau_1 \tau_2 \sigma_1^+ \sigma_2^+ \det (A(1, 2; i, j)^T) \\ &= \det \begin{pmatrix} a_1^1 & a_1^2 & a_1^i & a_1^j \\ b_1^1 & b_1^2 & b_1^i & b_1^j \\ a_2^1 & a_2^2 & a_2^i & a_2^j \\ b_2^1 & b_2^2 & b_2^i & b_2^j \end{pmatrix} = I - II + III \end{aligned}$$

where I, II and III are given by

$$\begin{aligned} I &= [1, 2]_1[i, j]_2 + [1, 2]_2[i, j]_1, \\ II &= [1, i]_1[2, j]_2 + [1, i]_2[2, j]_1, \\ III &= [1, j]_1[2, i]_2 + [1, j]_2[2, i]_1. \end{aligned}$$

Using (A.15) and $[i, j]_k = O(\zeta^{s_i+s_j})$, we obtain

$$\begin{aligned} \tau_1 \tau_2 \sigma_1^+ \sigma_2^+ \det A(1, 2; i, j) &= 2\{[1, 2]_1[i, j]_2 - [1, i]_1[2, j]_2 \\ &\quad + [1, j]_1[2, i]_2\} = O(\zeta^{s_1+s_2+s_i+s_j}). \end{aligned}$$

Notice that the terms in (A.16) are similar to the Plücker relations [cf. Ma 2, p 4], applied to $v_k := (a_k^1, a_k^2, a_k^i, a_k^j) (= \tau_k(\frac{\partial g_1}{\partial \tau_k}, \frac{\partial g_2}{\partial \tau_k}, \frac{\partial g_i}{\partial \tau_k}, \frac{\partial g_j}{\partial \tau_k}))$ and $w_k := (b_k^1, b_k^2, b_k^i, b_k^j)$ ($k = 1, 2$). The Plücker relations are obtained from $(v_k \wedge w_k) \wedge (v_k \wedge w_k) = 0$,

$$[1, 2]_k[i, j]_k - [1, i]_k[2, j]_k + [1, j]_k[2, i]_k = 0.$$

However these relations cannot be used to improve the estimate (A.16) further, as in (A.16) both subindices, 1 and 2, appear.

Since the relations (A.15) are the only bilinear relations among the terms $[k, l]_1 \cdot [k', l']_2$, we conclude that in a generic situation, with no further assumptions,

$$\tau_1 \tau_2 \sigma_1^+ \sigma_2^+ \det A(1, 2; i, j) = O(\zeta^{s_1+s_2+s_i+s_j})$$

(and not better) for $\{1, 2\} \cap \{i, j\} = \emptyset$ and $i \neq j$.

Appendix B: Generic level sets of complex systems

In this appendix, we want to present results corresponding to various results of Section 3 for complex systems.

Assume that $(x, y) \in (\mathbb{C}^{2n}, 0)$ are coordinates so that the conclusions of Theorem 1.1 hold. To analyze the level sets $M_c := \{z = (x, y) \in (\mathbb{C}^{2n}, 0) \mid G_j(z) = c_j \ (1 \leq j \leq n)\}$ we argue similarly as in Section 3. We again assume the normalization conditions (3.3) for μ and introduce the following subsets of \mathbb{C}^{n+2}

$$\begin{aligned} \Omega &:= \left\{ \omega \in \mathbb{C}^{n+2} \mid \prod_{j=\ell+1}^n \omega_j^{|\mu_j|} = \omega_{n+1} \omega_{n+2} \right\}, \\ \Omega' &:= \{ \omega \in \mathbb{C}^{n+2} \mid \omega_j \neq 0 (1 \leq j \leq \ell); \omega_{n+1} \neq 0 \}, \\ \Omega'' &:= \{ \omega \in \mathbb{C}^{n+2} \mid \omega_j \neq 0 (1 \leq j \leq \ell); \omega_{n+2} \neq 0 \} \end{aligned}$$

and set $\Omega^0 := \Omega' \cup \Omega''$.

Define $\psi : \mathbb{C}^{2n} \rightarrow \Omega$ by setting

$$\psi(x, y) := \left((x_j y_j)_{1 \leq j \leq n}, \prod_{j=\ell+1}^n x_j^{\mu_j^+} y_j^{\mu_j^-}, \prod_{j=\ell+1}^n x_j^{\mu_j^-} y_j^{\mu_j^+} \right)$$

and introduce for $c = (c_1, \dots, c_n)$

$$B_c := \{ \omega \in (\Omega, 0) \mid G_j(\omega) = c_j \ (1 \leq j \leq n) \}.$$

Notice that ψ induces a map $M_c \rightarrow B_c$ which we again denote by ψ . For the remainder of this subsection it is convenient to denote by M_c the inverse image $\psi^{-1}(B_c)$. The level set M_c is then semilocal as the generic fiber of ψ is equal to $(\mathbb{C}^*)^{n-1}$. (If we would consider $\psi_{local} : (\mathbb{C}^{2n}, 0) \rightarrow (\Omega, 0)$ and define M_c by $\psi_{local}^{-1}(B_c)$, the generic fiber $\psi_{local}^{-1}(\omega)$, with $\omega \in (\Omega, 0)$, is an annular domain $\{z = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1} \mid a_j < |z_j| < b_j, \ (1 \leq j \leq n-1)\}$ where $0 < a_j < b_j < \infty$ might depend on the choice of ω . Notice that for different choices of a_j 's and b_j 's, these domains are not bianalytically isomorphic.)

Analogous to Proposition 3.2 we have

PROPOSITION B.1. *For generic $c \in (\mathbb{C}^n, 0)$*

- (i) $\psi : M_c \rightarrow B_c$ is a fiber bundle with fiber $(\mathbb{C}^*)^{n-1}$;
- (ii) $\psi : M_c \rightarrow B_c$ admits an analytic trivialization

$$\begin{array}{ccc} B_c \times (\mathbb{C}^*)^{n-1} & \xrightarrow{\Psi} & M_c \\ & \searrow pr_1 & \swarrow \psi \\ & & B_c \end{array}$$

where $pr_1 : B_c \times (\mathbb{C}^*)^{n-1} \rightarrow B_c$ denotes the canonical projection.

To prove Proposition B.1 we need the following auxiliary result, which is proved in a similar way as Lemma 3.3.

LEMMA B.2. *For generic $c \in (\mathbb{C}^n, 0)$, $B_c \subset \Omega^0$.*

PROOF OF PROPOSITION B.1. Notice that the inverse image $\psi^{-1}(\Omega^0)$ is a product $E_1 \times_{\Omega^0} E_2$ where

$$E_1 := \{((x_k, y_k)_{1 \leq k \leq \ell}, \omega) \in (\mathbb{C}^*)^{2\ell} \times \Omega^0 \mid x_j y_j = \omega_j (1 \leq j \leq \ell)\}$$

and

$$E_2 := \left\{ ((x_k, y_k)_{\ell+1 \leq k \leq n}, \omega) \in (\mathbb{C}^*)^{2n-2\ell} \times \Omega^0 \mid x_j y_j = \omega_j (\ell + 1 \leq j \leq n); \right. \\ \left. \prod_{j=\ell+1}^n x_j^{\mu_j^+} y_j^{\mu_j^-} = \omega_{n+1}; \prod_{j=\ell+1}^n x_j^{\mu_j^-} y_j^{\mu_j^+} = \omega_{n+2} \right\}.$$

Let $\psi_j : E_j \rightarrow \Omega^0$ denote the projection ($j = 1, 2$) and observe that

$$\begin{array}{ccc} \psi^{-1}(\Omega^0) & \xrightarrow{\cong} & E_1 \times_{\Omega^0} E_2 \\ \psi \searrow & & \swarrow \psi_1 \times \psi_2 \\ & \Omega^0 & \end{array}$$

For $\psi_1 : E_1 \rightarrow \Omega^0$, a holomorphic trivialization $\Psi_1 : (\mathbb{C}^*)^\ell \times \Omega^0 \rightarrow E_1$ is given by

$$\Psi_1(t_1, \dots, t_\ell, \omega) = \left(\left(t_j \omega_j, \frac{1}{t_j} \right)_{1 \leq j \leq \ell}, \omega \right).$$

It remains to show that the bundle induced by $E_2 \rightarrow \Omega^0$ over B_c is holomorphically trivial for generic c . This is done in three steps:

(S1) To define local trivializations of $E_2 \rightarrow \Omega^0$ introduce the fiber

$$\mathbb{F} = \left\{ (z_{\ell+1}, \dots, z_n) \in (\mathbb{C}^*)^{n-\ell} \mid \prod_{j=\ell+1}^n z_j^{\mu_j} = 1 \right\}$$

and recall that $\rho^{(n)} = (0, \dots, 0, \rho_{\ell+1}^{(n)}, \dots, \rho_n^{(n)}) \in \mathbb{Z}^n$ satisfies $\langle \rho^{(n)}, \mu \rangle = 1$. Then $\Psi'_2 : \mathbb{F} \times \Omega' \rightarrow \psi_2^{-1}(\Omega')$ given by

$$\Psi'_2(z_{\ell+1}, \dots, z_n, \omega) := \left(\left(\left(\omega_k^{\frac{\mu_k^-}{|\mu_k|}} z_k \omega_{n+1}^{\rho_k^{(n)}}, \omega_k^{\frac{\mu_k^+}{|\mu_k|}} z_k^{-1} \omega_{n+1}^{-\rho_k^{(n)}} \right)_{\ell+1 \leq k \leq n} \right), \omega \right)$$

is a trivialization of $\psi_2 : E_2 \rightarrow \Omega^0$ above Ω' and $\Psi_2'' : \mathbb{F} \times \Omega'' \rightarrow \psi_2^{-1}(\Omega'')$ given by

$$\Psi_2''(z_{\ell+1}, \dots, z_n, \omega) := \left(\left(\omega_k^{\frac{\mu_k^+}{|\mu_k|}} z_k \omega_{n+2}^{-\rho_k^{(n)}}, \omega_k^{\frac{\mu_k^-}{|\mu_k|}} z_k^{-1} \omega_{n+2}^{+\rho_k^{(n)}} \right)_{\ell+1 \leq k \leq n}, \omega \right)$$

is a trivialization of $\psi_2 : E_2 \rightarrow \Omega^0$ above Ω'' . (Notice that Ψ_2' and Ψ_2'' are not symmetric.) The gluing map of the two trivializations, $G : \Omega' \cap \Omega'' \rightarrow \text{Aut}(\mathbb{F})$ is given by

$$(B.1) \quad G(\omega) := \text{diag} \left(\frac{(\omega_{n+1}\omega_{n+2})^{\rho_{\ell+1}^{(n)}}}{\omega_{\ell+1}}, \dots, \frac{(\omega_{n+1}\omega_{n+2})^{\rho_n^{(n)}}}{\omega_n} \right)$$

i.e. $G(\omega)$ is a diagonal matrix. Thus $\psi_2^{-1}(\Omega^0) = \mathbb{F} \times \Omega' \sqcup \mathbb{F} \times \Omega'' / \sim$, where $(z', \omega) \sim (z'', \omega)$ if $\omega \in \Omega' \cap \Omega''$, $(z', \omega) \in \Omega'$, $(z'', \omega) \in \mathbb{F} \times \Omega''$ and $z_j'' = (\omega_{n+1}\omega_{n+2})^{\rho_j^{(n)}} \omega_j^{-1} z_j'$ ($\ell + 1 \leq j \leq n$).

(S2) In view of Lemma 3.7, we may assume that $B_c \subset \Omega^0$. Consider the pullback $M_c^{(1)} \times_{B_c} M_c^{(2)} \rightarrow B$ of $E_1 \times_{\Omega^0} E_2 \rightarrow \Omega^0$. Then $M_c^{(1)} \rightarrow \psi_1 B_c$ is a trivial fibration whose fiber is $(\mathbb{C}^*)^\ell$. To investigate the fibration $M_c^{(2)} \rightarrow \psi_2 B_c$ we consider the line bundles induced by the diagonal elements of the gluing map G , introduced above. Denote by \mathcal{L}_j ($\ell + 1 \leq j \leq n$) the line bundle $(\mathbb{C} \times \Omega' \sqcup \mathbb{C} \times \Omega'') / \sim$ above Ω^0 where the equivalence relation is defined as follows: for $(u', \omega') \in \mathbb{C} \times \Omega'$ and $(u'', \omega'') \in \mathbb{C} \times \Omega''$, $(u', \omega') \sim (u'', \omega'')$ if $\omega' = \omega'' \in \Omega' \cap \Omega''$ and $u'' = (\omega_{n+1}\omega_{n+2})^{\rho_j^{(n)}} \omega_j^{-1} u'$. We claim that the pullback $\mathcal{L}_j|_{B_c} \rightarrow B_c$ of the line bundles $\mathcal{L}_j \rightarrow \Omega^0$ over B_c are all trivial. This follows from the following three observations:

(01) The exact sequence of sheaves of holomorphic functions

$$\mathcal{O} \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\text{exp}} \mathcal{O}^* \rightarrow \mathcal{O}$$

induces a long exact sequence in cohomology

$$\dots \rightarrow H^1(B_c, \mathcal{O}) \rightarrow H^1(B_c, \mathcal{O}^*) \rightarrow H^2(B_c, \mathbb{Z}) \rightarrow \dots$$

(Recall that \mathcal{O}^* denotes the sheaf of holomorphic functions on B_c which vanish nowhere on B_c .)

(02) Since B_c is a non compact analytic variety of dimension 1, B_c is a Stein space and, as a consequence, $H^1(B_c, \mathcal{O}) = 0$ (cf. e.g., [KK, p. 224]).

(03) Since B_c is a Stein space and of dimension 1, it is homotopy equivalent to a real one dimensional CW-complex, hence $H^2(B_c, \mathbb{Z}) = 0$ (cf. e.g., [GR, p. 156]).

Thus we conclude that $H^1(B_c, \mathcal{O}^*) = 0$, or, equivalently, that any line bundle over B_c is trivial. Denote by Θ_j a trivialization of $\mathcal{L}_j; B_c$

$$\begin{array}{ccc} \mathbb{C} \times B_c & \xrightarrow{\Theta_j} & \mathcal{L}_j; B_c \\ & \searrow pr_2 & \swarrow \\ & & B_c \end{array}$$

and introduce $B'_c := B_c \cap \Omega'$, $B''_c := B_c \cap \Omega''$ and the restrictions $\Theta'_j := \Theta_j \upharpoonright_{\mathbb{C} \times B'_c}$, $\Theta''_j := \Theta_j \upharpoonright_{\mathbb{C} \times B''_c}$. Recall that $\mathcal{L}_j = \mathbb{C} \times \Omega^0 \cap \mathbb{C} \times \Omega^0 / \sim$ and therefore there exist $u'_j \in \mathcal{O}^*(B'_c)$ and $u''_j \in \mathcal{O}^*(B''_c)$ so that

$$\Theta'_j(v, \omega) = (u'_j(\omega)v, \omega); \quad \Theta''_j(v, \omega) = (u''_j(\omega)v, \omega).$$

For $\omega \in B'_c \cap B''_c$,

$$(B.2) \quad u''_j(\omega) = (\omega_{n+1}\omega_{n+2})^{\rho_j^{(n)}} \omega_j^{-1} u'_j(\omega).$$

In the next step, the functions u'_j, u''_j ($\ell + 1 \leq j \leq n$) are used to construct a trivialization of $M_c \rightarrow B_c$.

(S3) The pull back $M_c = \psi_2^{-1}(B_c) \rightarrow B_c$ of $\psi_2 : E_2 \rightarrow \Omega^0$ over B_c has a presentation of the form

$$\psi_2^{-1}(B_c) = \mathbb{F} \times B'_c \sqcup \mathbb{F} \times B''_c / \sim$$

where $(z', \omega) \in \mathbb{F} \times B'_c$ and $(z'', \omega) \in \mathbb{F} \times B''_c$ are equivalent, $(z', \omega) \sim (z'', \omega)$, if $\omega \in B'_c \cap B''_c$ and $z''_j = (\omega_{n+1}\omega_{n+2})^{\rho_j^{(n)}} \omega_j^{-1} z'_j$ ($\ell + 1 \leq j \leq n$). Define $\Delta_2 : \psi_2^{-1}(B_c) \rightarrow (\mathbb{C}^*)^{n-1} \times B_c$ as follows: $\Delta'_2 := \Delta_2 \upharpoonright_{\mathbb{F} \times B'_c}$ and $\Delta''_2 := \Delta_2 \upharpoonright_{\mathbb{F} \times B''_c}$ are given by

$$\Delta'_2(z', \omega) := \left(\left(\frac{z'_j}{u'_j(\omega)} \right)_{\ell+1 \leq j \leq n}, \omega \right)$$

$$\Delta''_2(z'', \omega) := \left(\left(\frac{z''_j}{u''_j(\omega)} \right)_{\ell+1 \leq j \leq n}, \omega \right).$$

Notice that for $(z', \omega) \sim (z'', \omega)$, one has $\omega \in B'_c \cap B''_c$ and

$$z''_j = (\omega_{n+1}\omega_{n+2})^{\rho_j^{(n)}} \omega_j^{-1} z'_j$$

and therefore,

$$\frac{z''_j}{u''_j(\omega)} = \frac{(\omega_{n+1}\omega_{n+2})^{\rho_j^{(n)}} \omega_j^{-1} u'_j(\omega)}{u''_j(\omega)} \frac{z'_j}{u'_j(\omega)} = \frac{z'_j}{u'_j(\omega)}$$

where for the last equality we have used (B.2). This shows that Δ_2 is well defined. Notice that Δ_2 is an embedding and that the following diagram is commutative

$$\begin{array}{ccc} \psi_2^{-1}(B_c) & \xrightarrow{\Delta_2} & (\mathbb{C}^*)^{n-\ell} \times B_c \\ & \searrow & \swarrow pr_2 \\ & B_c & \end{array}$$

In order to identify the range of Δ_2 , range (Δ_2) , we argue over B'_c and B''_c separately. Let $(v_j)_{\ell+1 \leq j \leq n} \in (\mathbb{C}^*)^{n-\ell}$ so that there exists $((z'_j)_{\ell+1 \leq j \leq n}, \omega) \in \psi_2^{-1}(B'_c)$ with $\frac{z'_j}{u'_j(\omega)} = v_j$, or $z'_j = v_j u'_j(\omega)$ ($\ell + 1 \leq j \leq n$). Substituting these identities into $\prod_{\ell+1}^n (z'_j)^{\mu_j} = 1$ leads to

$$(B.3) \quad \left(\prod_{\ell+1}^n v_j^{\mu_j} \right) \left(\prod_{\ell+1}^n u'_j(\omega)^{\mu_j} \right) = 1.$$

The same argument can be used over B''_c to conclude that

$$(B.4) \quad \left(\prod_{\ell+1}^n v_j^{\mu_j} \right) \left(\prod_{\ell+1}^n u''_j(\omega)^{\mu_j} \right) = 1.$$

Further, it follows from (B.2) that

$$(B.5) \quad \prod_{\ell+1}^n u'_j(\omega)^{\mu_j} = \prod_{\ell+1}^n u''_j(\omega)^{\mu_j}.$$

and therefore we can define $\delta \in \mathcal{O}^*(B_c)$ by

$$\delta(\omega) = \begin{cases} \prod_{j=\ell+1}^n u'_j(\omega)^{\mu_j} & \text{if } \omega \in B'_c \\ \prod_{j=\ell+1}^n u''_j(\omega)^{\mu_j} & \text{if } \omega \in B''_c. \end{cases}$$

Conditions (B.3) and (B.4) can thus be expressed over all of B_c by

$$\delta(\omega) \prod_{\ell+1}^n v_j^{\mu_j} = 1$$

and $\Delta_2 : M_c = \psi_2^{-1}(B_c) \rightarrow \{((v_j)_{\ell+1 \leq j \leq n}, \omega) \in (\mathbb{C}^*)^{n-\ell} \times B_c \mid \delta(\omega) \prod_{\ell+1}^n v_j^{\mu_j} = 1\}$ is a bianalytic isomorphism.

(S4) To find a local trivialization of $\Delta_2(M_c)$, define

$$T_2 : \Delta_2(M_c) \rightarrow \left\{ ((\tilde{v}_j)_{\ell+1 \leq j \leq n}, \omega) \in (\mathbb{C}^*)^{n-\ell} \times B_c \mid \prod_{\ell+1}^n \tilde{v}_j^{\mu_j} = 1 \right\}$$

by setting $T_2((v_j)_{\ell+1 \leq j \leq n}, \omega) = ((\delta(\omega)^{-\rho_j^{(n)}} v_j)_{\ell+1 \leq j \leq n}, \omega)$. Then

$$\prod_{\ell+1}^n \tilde{v}_j^{\mu_j} = \left(\prod_{\ell+1}^n v_j^{\mu_j} \right) \delta(\omega)^{-\sum \mu_j \rho_j^{(n)}} = \delta(\omega) \frac{1}{\delta(\omega)} = 1.$$

We have thus shown that $T_2 \cdot \Delta_2 : M_c = \psi_2^{-1}(B_c) \rightarrow B_c \times \mathbb{F}$ is a bianalytic isomorphism. \square

REFERENCES

- [Ar1] V. I. ARNOLD, “Mathematical methods of classical mechanics”, 2nd ed., Springer, 1989.
- [Ar2] V. I. ARNOLD (Ed.), “Dynamical systems III, *Encycl. Math. Sci.*”, vol. 3, Springer, 1988.
- [CB] R. H. CUSHMAN – L. M. BATES, “Global Aspects of Classical Integrable Systems”, Birkhäuser, 1997.
- [Du] J. J. DUISTERMAAT, *On global action-angle coordinates*, *Comm. Pure Appl. Math.* **33** (1980), 687-706.
- [El] H. ELIASSON, *Normal form of Hamiltonian systems with Poisson commuting integrals – elliptic case*, *Comment. Math. Helv.* **65** (1990), 4-35.
- [Fo] A. T. FOMENKO (Ed.), “Topological Classification of Integrable Systems”, *Adv. in Soviet Math.* **6** AMS (1991).
- [GR] H. GRAUERT – R. REMMERT, “Theory of Stein Spaces”, *Grundle. Math. Wiss.* 236, Springer-Verlag, 1979.
- [Hi] M. HIRSCH, “Differential Topology”, *Graduate Text in Mathematics*, Springer-Verlag, 1976.
- [It1] H. ITO, *Convergence of Birkhoff normal forms for integrable systems*, *Comment. Math. Helv.* **64** (1989), 363-407.
- [It2] H. ITO, *Integrability of Hamiltonian systems and Birkhoff normal forms in the simple resonance case*, *Math. Ann.* **292** (1992), 411-444.
- [It3] H. ITO, *Action-angle coordinates at singularities for analytic integrable systems*, *Math. Z.* **206** (1991), 363-407.
- [KK] L. KAUP – B. KAUP, “Holomorphic functions of several variables”, *de Gruyter Studies in Mathematics* 3, Walter de Gruyter, Berlin-NewYork, 1983.
- [Lo] E. J. N. LOOIJENGA, “Isolated singular points on complete intersections”, *Cambridge University Press*, 1984.
- [Ma1] M. MARCUS, “Finite dimensional multilinear algebra”, *M. Dekker*, New York, 1973.
- [Ma2] M. MARCUS, “Finite dimensional multilinear algebra, Part 2”, *M. Dekker*, New York, 1975.

- [Mo] J. MOSER, "Lectures on Hamiltonian systems", Memoir AMS, vol. 81, 1968.
- [Sb] Y. SIBUYA, "Linear differential equations in the complex domain: problems of analytic continuation", vol. 82, Translations of mathematical monographs AMS, 1990.
- [Si] C. L. SIEGEL, *On the integrals of canonical systems*, Ann. of Math. **42** (1941), 806-822.
- [Ve] J. VEY, *Sur certain systèmes dynamiques séparables*, Amer. J. Math. **100** (1978), 591-614.

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