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A Class of Nonlinear Conservative Elliptic Equations in Cylinders

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Abstract. Let (M, g) be a compact n-dimensional manifold without boundary and Δ_g the Laplace-Beltrami operator on M. This paper studies the asymptotic properties of the following conservative system $(S) u_{tt} + \Delta_g u + u^q - \lambda u = 0$ on $\mathbb{R}^+ \times M$ and their links with the homogeneous solutions of (S).

1. - Introduction

The study of asymptotics of the following class of conformally invariant Emden-Fowler equations in $\mathbb{R}^N - \{0\}$

$$-\Delta u + \left(c/|x|^2\right) u = u^{(N+2)/(N-2)} \tag{1.1}$$

gives rise to the following nonlinear equation

$$v_{tt} + \Delta_{S^{N-1}} v - \left((N-2)^2/4 + c\right) v + v^{(N+2)/(N-2)} = 0 \tag{1.2}$$

in $(-\infty, \infty) \times S^{N-1}$, where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{N-1} of $\mathbb{R}^N$, via the following classical change of variable

$$v(t, \sigma) = r^{(N-2)/2} u(r, \sigma), (r, \sigma) \in (0, \infty) \times S^{N-1}, t = \operatorname{Ln}(r). \tag{1.3}$$

One of the main feature of this equation is the conservation of energy (equivalent to Pohozaev's identity):

$$\frac{d}{dt} \int_{S^{N-1}} \left(|\nabla v|^2 - v_t^2 + \left(\frac{(N-2)^2}{4} + c \right) v^2 - \frac{(N-2)}{N} v^{2N/(N-2)} \right) d\sigma = 0. \tag{1.4}$$

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As a result of the works of Obata [0b] and Caffarelli-Gidas-Spruck [CGS], the asymptotic behaviour of the solutions of (1.2) as well as the global solutions are now well understood in the case when $c = 0$, but it is important to notice that this understanding mainly comes from the equation (1.1) itself and not from the study of (1.2): the main point is that the solutions behave asymptoticaly like the solutions of the associated O.D.E. It appears that when c is not 0, nothing is known except in the radial case where the relation (1.4) plays a crucial role: in particular there may exist solutions of (1.2) under the form

$$v(t,\sigma) = \omega\left(e^{tA}(\sigma)\right) \tag{1.5}$$

where A is a skew symmetric matrix.

The purpose of this paper is to extend this type of problem to a more general setting by considering the following equation

$$u_{tt} + \Delta_g u - \lambda u + |u|^{q-1}u = 0 \tag{1.6}$$

in $[0,\infty) \times M$ where (M, g) be a n-dimensional compact Riemannian manifold without boundary, Δ_g the Laplacian on M and q and λ are constant, $q > 1$. Let us first study the stationnary equation associated to (1.6), that is

$$\Delta_g u - \lambda u + |u|^{q-1}u = 0 \tag{1.7}$$

and in particular look under what conditions all the (positive) solutions of (1.7) are constant (by a solution we always mean a $C^2(M)$-function). Let λ_1 denote the first nonzero eigenvalue of -Δ_g, then two types of results are obtained in that direction. The first one points out the role of the curvature tensor and in particular its trace, the Ricci tensor:

THEOREM 2.1. *Assume that the Ricci tensor* Ricc_g *of* g *satisfies*

$$\mathrm{Ricc}_g \geq Rg \tag{1.8}$$

for some nonnegative R, *that* λ *is nonnegative and*

$$1 < q \leq (n+2)/(n-2) \tag{1.9}$$

with

$$(q-1)\lambda \leq \lambda_1 + \frac{qn(n-1)}{q+n(n+2)}\left(R - \frac{n-1}{n}\lambda_1\right). \tag{1.10}$$

Assume also that one of the two inequlities (1.9)-(1.10) *is strict if* (M, g) *is conformally diffeomorphic to* (S^n, g_0), *that is* $g = kg_0$ *for some positive* C^∞ *function* k, *then any nonnegative solution* u *of* (1.7) *is a constant.*

In the above result (S^n, g_0) is the unit sphere of $\mathbb{R}^{n+1}$ with the standard metric g_0 induced by the Euclidean structure of $\mathbb{R}^{n+1}$. Moreover this result is optimal on (S^n, g_0). In the second result it is proved that small enough solutions (not necessarily positive) are constant:

THEOREM 2.2. *Assume $\lambda \geq 0$, $q > 1$ and that u is a solution of* (1.7) *which satisfies*

$$q\|u\|_{L^\infty}^{q-1} \leq \lambda + \lambda_1, \tag{1.11}$$

then u is a constant.

Furthermore this result is extendable to a product manifold $(M, g)\times(N, h) = (M \times N, g \otimes h)$. The estimate (1.11) is not easy to obtain, however, in the subcritical case, the following a priori estimate is proved:

THEOREM 2.3. *Assume that*

$$1 < q < (n+2)/(n-2), \tag{1.12}$$

then there exists a positive constant $C = C(M, g)$ such that for any $\lambda \geq 0$ any nonnegative solution u of (1.7) *satisfies*

$$\|u\|_{L^\infty} \leq C\lambda^{1/(q-1)}. \tag{1.13}$$

For the time dependent equation (1.6), the following form of the conservation of energy is derived:

$$\frac{d}{dt}E(u)(t) = \frac{d}{dt}(\text{vol}(M))^{-1}\int_M \left(-|\nabla u|^2 + u_t^2 - \lambda u^2 + \frac{2}{q+1}|u^{q+1}|\right) dv_g = 0. \tag{1.14}$$

Assuming that $\sigma \mapsto X(\sigma)$ is a Killing vector field on (M, g), that is a vector field on M which is the infinitesimal generator of a group of isometries $(e^{tX})_{t\in\mathbb{R}}$ and L_X the associated covariant derivative defined by $(L_X u)(\sigma) = \frac{d}{dt}u(e^{tX}(\sigma))\big|_{t=0}$, then some L_X "cinetic momentum" is conserved, namely

$$\frac{d}{dt}\int_M u_t L_X u \, dv_g = 0. \tag{1.15}$$

Therefore, there may exist a solution of (1.6) under the form

$$u(t, \sigma) = \omega\left(e^{tX}(\sigma)\right), \tag{1.16}$$

where ω solves some nonlinear elliptic equation on M. However, in many cases, the solution of (1.6) homogenizes when t tends to infinity. Let us consider the following ordinary differential equation whose solutions are homogeneous solutions of (1.6)

$$\varphi_{tt} - \lambda\varphi + |\varphi|^{q-1}\varphi = 0. \tag{1.17}$$

It is easy to check that all the orbits of (1.17) but two are closed; they are characterized by the value of the energy function E defined above (see [BVB]) and all the closed orbits correspond to periodic solutions of (1.17). The last two orbits are the two homoclinic orbits of the equilibrium (0,0). Calling γ_σ an orbit where $\sigma = E(u)(t)$ is the corresponding value of the energy function (for the two homoclinic orbits, $\sigma = 0$), the following will be proven:

THEOREM 3.1. *Assume u is a solution of* (1.6) *on* $[0, \infty) \times M$ *such that*

$$\sup_{t \geq T} \|u(t, .)\|_{L^\infty} \leq ((\lambda + \lambda_1)/q)^{1/(q-1)}, \tag{1.18}$$

for some $T > 0$, then

$$\lim_{t \to \infty} \operatorname{dist}_{C^2}(u(t, .), \gamma_\sigma) = 0. \tag{1.19}$$

If it is assumed moreover that (1.18) *is strict and that $\sigma \neq 0$, then there exists a solution φ in the orbit γ_σ such that*

$$\lim_{t \to \infty} \|u(t, .) - \varphi(.)\|_{C^2} = 0. \tag{1.20}$$

As for estimate (1.18), there is a cylindrical analogue of Theorem 2.3, namely, assuming that u is a bounded solution of (1.6) on $\mathbb{R} \times M$ and that

$$1 < q < (n + 3)/(n - 1), \tag{1.21}$$

then there exists a constant $C = C(M, g)$ such that

$$\|u\|_{L^\infty(R \times M)} \leq C\lambda^{1/(q-1)}. \tag{1.22}$$

For the existence of solution of (1.6) with a given initial data we have two types of results: existence from monotone operators theory and existence via perturbation methods. For example, it can be proven:

THEOREM 4.1. *For any $u_0 \in C(M)$ satisfying*

$$0 \leq u_0(\sigma) \leq \left(\lambda \frac{q+1}{2}\right)^{1/(q-1)} \tag{1.23}$$

in M, there exists a solution u of (1.6) *on* $[0, \infty) \times M$ *such that* $u \in C([0, \infty); L^\infty(M))$ *which satisfies $u(0, \sigma) = u_0(\sigma)$ and*

$$\lim_{t \to \infty} \|u(t, .)\|_{L^\infty} = 0. \tag{1.24}$$

As for homogeneous solutions, the application of Floquet's theory of differential equations with periodic coefficients yields the existence of solutions of (1.6) in the neighbourhood of a periodic solution y_0 of (1.17). More precisely it can be proven that there exists an infinite dimensional subspace F_2 of $C^{2,\alpha}(M)$ which is associated to the spectrum of the linearized form of (1.6) following y_0

$$\psi \mapsto \mathbf{L}_{y_0}(\psi) = \psi_{tt} + \Delta_g \psi + \left(q|y_0(t)|^{q-1} - \lambda\right)\psi, \tag{1.25}$$

with the following property:

There exists $\delta > 0$ *such that for any* $u_0 \in C^{2,\alpha}(M)$ *satisfying*

$$|u_0(\sigma) - y_0(0)| \leq \delta \tag{1.26}$$

and $u_0(x) - y_0(0) \in F_2$, *there exists a solution* u *of* (1.6) *on* $[0, \infty) \times M$ *such that* $u \in C([0, \infty); L^\infty(M))$, *which satisfies* $u(0, \sigma) = u_0(\sigma)$ *and*

$$\|u(t, .) - \varphi(t)\|_{L^\infty} \leq Ce^{-\mu t} \tag{1.27}$$

for any $t \geq 0$, *where* C *and* μ *are positive constants.*

The last section deals with some simple nonlocal versions of (1.6) in the particular case where $q = 3$. These are

$$u_{tt} + \Delta_g u - \lambda u + \bar{u}^3 = 0 \tag{1.28}$$

$$u_{tt} + \Delta_g u - \lambda u + u\bar{u}^2 = 0 \tag{1.29}$$

$$u_{tt} + \Delta_g u - \lambda u + u\overline{u^2} = 0 \tag{1.30}$$

where the general notation $\bar{g}$ means that the average of the function g on M is taken. It is proven that all the positive and bounded solutions of these equations are asymptotically homogeneous when t tends to infinity. Again one key tool for this study is the use of Floquet's theory.

This paper is organized as follows:

2. – Equations on compact manifolds

In this section it is assumed that (M, g) is a compact n-dimensional Riemannian manifold without boundary. Let Δ_g be the be the Laplacian on M and λ_1 the first nonzero eigenvalue of $-\Delta_g$ in $W^{1,2}(M)$. Considering the following equation on M

$$\Delta_g u - \lambda u + |u|^{q-1}u = 0, \tag{2.1}$$

where q is larger than 1, it is clear that the condition $\lambda > 0$ is a necessary condition in order to have positive solutions; it is also a sufficient condition as it implies, if it is fulfilled, the existence of a constant solution, namely

$$u_\lambda = \lambda^{1/(q-1)}. \tag{2.2}$$

If (2.1) is linearized at the value $u = u_\lambda$, the following operator is obtained

$$\mathbf{L} = \Delta_g + (q-1)\lambda I \tag{2.3}$$

and $\mathbf{L}$ is singular if $(q-1)\lambda = \lambda_1$. Therefore, this particular value of λ is generically a bifurcation value and for $\lambda > \lambda_1/(q-1)$ there exist nonconstant positive solutions of (2.1). Let $\mathrm{Ricc}_g = (R_{ij})$ be the Ricci 2-tensor of g, that is the contraction of the Riemann curvature 4-tensor $\mathrm{Riem}_g = (R^i_{jkl})$, then the following result shows how local and global properties of the metric g may interfere in order to prove uniqueness result for positive solutions of (2.1):

THEOREM 2.1. *Assume that*

$$\mathrm{Ricc}_g \geq Rg \tag{2.4}$$

for some nonnegative R, that $\lambda \geq 0$ and

$$1 < q \leq (n+2)/(n-2) \tag{2.5}$$

and that

$$(q-1)\lambda \leq \lambda_1 + \frac{qn(n-1)}{q+n(n+2)}\left(R - \frac{n-1}{n}\lambda_1\right). \tag{2.6}$$

Assume also that one of the two inequalities (2.5)-(2.6) *is strict if* (M, g) *is conformally diffeomorphic to* (S^n, g_0), *that is* $g = kg_0$ *for some positive* C^∞ *function* k, *then any nonnegative solution* u *of* (2-1) *is a constant.*

PROOF. It is essentially an algebraic computation based upon the classical Bochner-Weitzenböck formula which introduces naturally the Ricci tensor (see [BGM])

$$\frac{1}{2}\Delta_g \left|\nabla_g v\right|^2 = |\mathrm{Hess}\, v|^2 + \langle \nabla_g\left(\Delta_g v\right), \nabla_g v\rangle + \mathrm{Ricc}\left(\nabla_g v, \nabla_g v\right). \tag{2.7}$$

Setting $u = v^{-\beta}$ where $\beta \in \mathbb{R}^*$, then v satisfies

$$-\Delta_g v + (\beta+1)\frac{\left|\nabla_g v\right|^2}{v} + \frac{1}{\beta}\left(v^{1+\beta-\beta q} - \lambda v\right) = 0 \tag{2.8}$$

on M. The key-stone of the proof lies in the following identities:

PROPOSITION 2.1. *For any $\gamma \neq -2$ and $\beta \in \mathbb{R}^*$, the following identity is verified*

$$(2.9)\quad \begin{aligned} A\int_M v^{\gamma-2}\left|\nabla_g v\right|^4 dv_g &= \frac{\beta q}{\gamma}\int_M \left(v^\gamma J + v^\gamma \operatorname{Ricc}\left(\nabla_g v, \nabla_g v\right)\right) dv_g \\ &+ \frac{n+2}{2n}\lambda(q-1)\int_M v^\gamma \left|\nabla_g v\right|^2 dv_g - B\int_M \left(\Delta_g\left(v^{(\gamma+2)/2}\right)\right)^2 dv_g \end{aligned}$$

where

$$(2.10)\quad A = \frac{n+2}{2n}\left(\left(\beta+1+\frac{\gamma}{4}\right)(\beta q-\gamma)-(\beta+1)^2\right)+\frac{\beta q(\gamma-4)}{8},$$

$$(2.11)\quad B = \frac{2}{n(\gamma+2)^2}\left(n+2+2\frac{\beta q}{\gamma}(n-1)\right),$$

$$(2.12)\quad J = \left(|\operatorname{Hess}(v)|^2 - \frac{1}{n}\left(\Delta_g v\right)^2\right).$$

Moreover, in the case where $\gamma = -2$, the preceding relation becomes

$$(2.13)\quad \begin{aligned} A\int_M \left|\nabla_g(\ln v)\right|^4 dv_g &= -\frac{\beta q}{2}\int_M \left(v^{-2}J + v^{-2}\operatorname{Ricc}(\nabla_g v, \nabla_g v)\right) dv_g \\ &+ \frac{n+2}{2n}\lambda(q-1)\int_M \left|\nabla_g(\ln v)\right|^2 dv_g - B\int_M \left(\Delta_g(\ln v)\right)^2 dv_g \end{aligned}$$

where

$$(2.14)\quad A = \frac{n+2}{2n}\left(\left(\beta+\frac{1}{2}\right)(\beta q+2)-(\beta+1)^2\right)-\frac{3\beta q}{4},$$

$$(2.15)\quad B = \frac{1}{2n}(n+2-\beta q(n-1))$$

PROOF OF PROPOSITION 2.1. Multiplying (2.8) by $v^{\gamma-1}\left|\nabla_g v\right|^2$ and $v^\gamma \Delta_g v$ successively and integrating over M result in

$$(2.16)\quad \begin{aligned} \int_M v^{\gamma-1}\Delta v\left|\nabla_g v\right|^2 dv_g &= (\beta+1)\int_M v^{\gamma-2}\left|\nabla_g v\right|^4 dv_g \\ &+ \frac{1}{\beta}\int_M \left(v^{\beta-\beta q+\gamma} - \lambda v^\gamma\right)\left|\nabla_g v\right|^2 dv_g, \end{aligned}$$

$$(2.17)\quad \begin{aligned} \int_M v^\gamma \left(\Delta_g v\right)^2 dv_g &= (\beta+1)\int_M v^{\gamma-1}\Delta_g v\left|\nabla_g v\right|^2 dv_g \\ &- \frac{1}{\beta}\int_M (1+\beta-\beta q+\gamma)\left(v^{\beta-\beta q+\gamma} - \lambda(\gamma+1)v^\gamma\right)\left|\nabla_g v\right|^2 dv_g. \end{aligned}$$

By a linear combination between (2.16) and (2.17) the term $\int_M v^{\beta-\beta q+\gamma}|\nabla_g v|^2 dv_g$ can be eliminated and therefore

$$(2.18)\quad \begin{aligned}(\gamma-\beta q)\int_M v^{\gamma-1}\Delta_g v\,|\nabla_g v|^2\,dv_g+\int_M v^\gamma\left(\Delta_g v\right)^2 dv_g\\ +(\beta+1)(\beta q-\gamma-\beta-1)\int_M v^{\gamma-2}|\nabla_g v|^4\,dv_g\\ =\lambda(q-1)\int_M v^\gamma|\nabla_g v|^2\,dv_g.\end{aligned}$$

Multiplying (2.7) by v^γ, integrating on M and replacing the term $|\operatorname{Hess} v|^2$ by $J+\frac{1}{n}\left(\Delta_g v\right)^2$ (where J defined by (2.12) is nonnegative from the Schwarz inequality) imply the following identity:

$$(2.19)\quad \begin{aligned}\frac{3\gamma}{2}\int_M v^{\gamma-1}\Delta_g v\,|\nabla_g v|^2\,dv_g+\frac{1}{2}\gamma(\gamma-1)\int_M v^{\gamma-2}|\nabla_g v|^4\,dv_g\\ +\frac{n-1}{n}\int_M v^\gamma\left(\Delta_g v\right)^2 dv_g\\ =\int_M J v^\gamma dv_g+\int_M v^\gamma \operatorname{Ricc}(\nabla v,\nabla v)\,dv_g.\end{aligned}$$

If $\gamma\neq -2$, there holds

$$(2.20)\quad \begin{aligned}v^{\gamma-1}|\nabla_g v|^2\,\Delta_g v=\frac{4}{\gamma(\gamma+2)^2}\left(\Delta_g\left(v^{(\gamma+2)/2}\right)\right)^2\\ -\frac{\gamma}{4}v^{\gamma-2}|\nabla_g v|^4-\frac{1}{\gamma}v^\gamma\left(\Delta_g v\right)^2\end{aligned}$$

and if $\gamma=-2$, (2.20) reads

$$(2.21)\quad v^{-3}|\nabla_g v|^2\,\Delta_g v=-\frac{1}{2}\left(\Delta_g(\log v)\right)^2+\frac{1}{2}v^{-4}|\nabla_g v|+\frac{1}{2}v^{-2}\left(\Delta_g v\right)^2.$$

If, in (2.18)-(2.19), the term $\int_M v^{\gamma-1}\Delta_g v|\nabla_g v|^2 dv_g$ is replaced by the right-hand side of (2.20) or (2.21), this gives

$$(2.22)\quad \begin{aligned}\frac{\beta q}{\gamma}\int_M v^\gamma\left(\Delta_g v\right)^2 dv_g\\ +\left[\left(\beta+1+\frac{\gamma}{4}\right)(\beta q-\gamma)-(\beta+1)^2\right]\int_M|\nabla_g v|^4 v^{\gamma-2}dv_g\\ -\frac{4(\beta q-\gamma)}{\gamma(\gamma+2)^2}\int_M\left(\Delta_g v^{(\gamma+2)/2}\right)^2 dv_g=\lambda(q-1)\int_M v^\gamma|\nabla_g v|^2\,dv_g\end{aligned}$$

and

$$
(2.23)\quad
\begin{aligned}
&\frac{6}{(\gamma+2)^2}\int_M \left(\Delta_g v^{(\gamma+2)/2}\right)^2 dv_g + \frac{\gamma(\gamma-4)}{8}\int_M v^{\gamma-2}\left|\nabla_g v\right|^4 dv_g \\
&\quad - \frac{n+2}{2n}\int_M v^{\gamma}\left(\nabla_g v\right)^2 dv_g \\
&= \int_M v^{\gamma} J dv_g + \int_M v^{\gamma}\,\mathrm{Ricc}_g\left(\nabla_g v, \nabla_g v\right) dv_g
\end{aligned}
$$

if $\gamma \neq -2$, with an easy modification in the case $\gamma = -2$. In those two identities the terms $\int_M \left(\Delta_g(v)^{(\gamma+2)/2}\right)^2 dv_g$ and $\int_M v^{\gamma}(\Delta_g v)^2 dv_g$ are nonnegative but give no estimate; should one of them be eliminated between (2.22) and (2.23), for example, $\int_M v^{\gamma}\left(\Delta_g v\right)^2 dv_g$, the result is (2.9). Formula (2.13) is obtained in the same way.

End of the proof of Theorem 2.1. From the nonnegativity of J, Proposition 2.1 and the classical relation (from Fourier analysis)

$$
(2.24)\quad \int_M \left(\Delta_g v^{(\gamma+2)/2}\right)^2 dv_g \geq \frac{(\gamma+2)^2}{4}\lambda_1 \int_M v^{\gamma}\left|\nabla_g v\right|^2 dv_g,
$$

if $\gamma \neq -2$, with an immediate modification if $\gamma = -2$, it suffices to find a couple (β, γ) such that

$$
(2.25)\quad A \geq 0, B \geq 0 \text{ et } \frac{\beta}{\gamma} \leq 0.
$$

In fact, if such a couple exists, it can be deduced from the previous relations that

$$
(2.26)\quad
\begin{aligned}
&A\int_M v^{\gamma-2}\left|\nabla_g v\right|^4 dv_g \leq \frac{\beta q}{\gamma}\int_M v^{\gamma} J dv_g \\
&+ \left[\frac{n+2}{2n}(\lambda(q-1)-\lambda_1) + \frac{\beta q}{\gamma}\left(R - \lambda_1\frac{n-1}{n}\right)\right]\int_M v^{\gamma}\left|\nabla_g v\right|^2 dv_g.
\end{aligned}
$$

We set

$$
(2.27)\quad X = \frac{\beta}{\gamma}, \quad \delta = \frac{1}{\gamma} + \frac{1}{2} \text{ and } \tilde{A} = \frac{2n}{(n+2)\gamma^2}A
$$

and the problem is reduced to maximise X in $[-(n+2)/(2q(n-1)), 0]$ under the constraint

$$
\tilde{A} = -\delta^2 + 2\frac{q-(n+2)}{n+2}\delta X + (q-1)X^2 + \frac{q(n-1)}{2(n+2)}X \geq 0.
$$

Computing the derivative of $\tilde{A}$ with respect to δ results in:

$$\frac{d\tilde{A}}{d\delta} = -2\left[\delta - \frac{q-(n+2)}{n+2}\right].$$

Therefore the maximum of $\tilde{A}$ is achieved for $\delta = \delta_0 = \frac{q-(n+2)}{n+2}$, which gives

$$\tilde{A}(\delta_0, X) = X^2\left[q - 1 + \left(\frac{q-(n+2)}{n+2}\right)^2\right] + \frac{q(n-1)}{2(n+2)}X. \tag{2.30}$$

If X_0 is the negative root of the above polynomial in X, then

$$X_0 = -\frac{(n+2)(n-1)}{2(q+n(n+2))}, \tag{2.31}$$

and the condition

$$X_0 \geq -\frac{n+2}{2q(n-1)} \tag{2.32}$$

is equivalent to

$$q \leq (n+2)/(n-2). \tag{2.33}$$

For this specific value of $X = X_0$ there holds

$$\begin{aligned}&\left[\frac{n+2}{2n}(\lambda(q-1)-\lambda_1) + \frac{\beta q}{\gamma}\left(R - \lambda_1\frac{n-1}{n}\right)\right]\\ &= \frac{n+2}{2n}\left[\lambda(q-1) - \lambda_1 - \frac{qn(n-1)}{q+n(n+2)}\left(R - \lambda_1\frac{n-1}{n}\right)\right].\end{aligned} \tag{2.34}$$

Therefore, assuming that (2.33) is fulfilled and that

$$\lambda(q-1) \leq \lambda_1 + \frac{qn(n-1)}{q+n(n+2)}\left(R - \lambda_1\frac{n-1}{n}\right), \tag{2.35}$$

there are two possibilities:

i) either (M, g) is not conformally diffeomorphic to (S^n, g_0) and there exist no nonconstant positive solutions to the equation $J = 0$ (see [Ob], [OY]), or

ii) (M, g) is conformally diffeomorphic to (S^n, g_0) and, unless $v^{(\gamma+2)/2}$ is an eigenfunction of the Laplacian, the relation (2.24) is strict and B is positive if $q < (n+2)/(n-2)$. In that case v has also to be constant if (2.35) is fulfilled.

REMARK 2.1. It is interesting to notice that in estimate (2.6), the term $R - \lambda_1\frac{n-1}{n}$ is always nonpositive from Lichnerowicz well known result [Li].

Moreover, it vanishes if and only if (M, g) is isometric to (S^n, g_0), the standard n-sphere with radius 1 [Ob]. The formula (2.6) has to be compared with the previous one from [BVV] which only says that, if

$$(q-1)\lambda \leq \frac{n}{n-1} R \tag{2.36}$$

any positive solution of (2.1) is a constant, provided (2.5) is fulfilled, with a strict inequality when (M, g) is conformally diffeomorphic to (S^n, g_0). In the case where (M, g) is isometric to (S^n, g_0), the two results are the same. However, if (M, g) is flat $(R = 0)$, for example in the flat torus case $(M, g) = (T^n, g_0)$, the [BVV] result gave no real information, but formula (2.6) reads as

$$(q-1)\lambda \leq \lambda_1 \frac{n(n+2-q(n-2))}{q+n(n+2)}. \tag{2.37}$$

REMARK 2.2. There is numerical evidence that in the case where $(M, g) = (S^3, g_0)$ and $q > 5$, there exist positive solutions of (2.1) for any $\lambda > 0$; the smallest is λ, the highest is the maximum of the numerical solution.

As a consequence of this result new estimates are obtained for the infimum of the following quotient

$$Q_{\lambda,q}(u) = \frac{\int_M \left(\left|\nabla_g u\right|^2 + \lambda u^2\right) dv_g}{\left(\int_M |u|^{q+1}\, dv_g\right)^{2/(q+1)}}. \tag{2.38}$$

COROLLARY 2.1. *Suppose that the Ricci curvature of g satisfies* (2.4), *and that* (2.5) *and* (2.6) *hold, then*

$$S_{\lambda,q} = \inf\left\{Q_{\lambda,q}(u) : u \in W^{1,2}(M) - \{0\}\right\} = \lambda(\operatorname{vol} M)^{(q-1)/(q+1)}. \tag{2.39}$$

The proof is the same as the one of [BVV, Cor 6.2], by using directly the equation in the case, $1 < q < (n+2)/(n-2)$, and the left upper semi-continuity of $q \mapsto S_{\lambda,q}$ at $q = (n+2)/(n-2)$ as in Trudinger's article [Tr].

REMARK 2.3. As quoted in Remark 2.1, the result of Theorem 2.1 is optimal if $(M, g) = (S^n, g_0)$. It has been noticed by H. Hamza [Ha] that, if $q = (n+2)/(n-2)$, there exist non constant positive solutions of (2.1) on (M, g) whenever $\lambda = \lambda_1/(q-1) = (n-2)\lambda_1/4$ and

$$\lambda_1 > n\left(\frac{\operatorname{vol} S^n}{\operatorname{vol} M}\right)^{2/n}. \tag{2.40}$$

In fact, it is known from Aubin's results [Au], that

$$4\frac{n-1}{n-2} S_{\lambda,(n+2)/(n-2)} \leq n(n-1)\left(\operatorname{vol} S^n\right)^{2/n} \tag{2.41}$$

for any λ. If the only positive solutions of (2.1) were constant, it would imply that

$$\lambda_1(n-1)(\mathrm{vol}\, M)^{2/n} \le n(n-1)\left(\mathrm{vol}\, S^n\right)^{2/n} \tag{2.42}$$

and consequently

$$\lambda_1 \le n\left(\frac{\mathrm{vol}\, S^n}{\mathrm{vol}\, M}\right)^{2/n}. \tag{2.43}$$

Taking $(M, g) = (\mathbf{P}_n(\mathbb{R}), g_0)$, the n-dimensional real projective space, then (see [BGM])

$$\mathrm{vol}\, M = 1/2\, \mathrm{vol}\, S^n \text{ and } \lambda_1 = 2(n+1); \tag{2.44}$$

it is clear that (2.43) means $2(n+1) \le n2^{2/n}$, which is never true for $n > 1$.

More generally, if $q = (n+2)/(n-2)$, the fact that (2.1) admits only constant for positive solutions implies that

$$\lambda \le \frac{n(n-2)}{4}\left(\frac{\mathrm{vol}\, S^n}{\mathrm{vol}\, M}\right)^{2/n} \tag{2.45}$$

which in turn implies that there exists a positive non constant solution of (2.1) whenever $\lambda > \frac{n(n-2)}{4}\left(\frac{\mathrm{vol}\, S^n}{\mathrm{vol}\, M}\right) = \lambda(M)$. Moreover, from the upper semi-continuity of $(\lambda, q) \mapsto S_{\lambda,q}$ on the left at $q = 2^* - 1 = (n+2)/(n-2)$, it can be concluded that this result still holds in a neighborhood of $(\lambda(M), (n+2)/(n-2))$. In the particular case of the flat torus $(M, g) = (T^n, g_0)$, the condition reads as

$$\lambda(M) = \frac{n(n-2)}{\pi^2}\left(\mathrm{vol}\, S^n\right)^{2/n}. \tag{2.46}$$

Another interesting application deals with the uniqueness of Einstein metric with constant positive scalar curvature.

DEFINITION 2.1. A metric g on a n-dimensional differentiable manifold M is said to be Einstein if there exists a real number k such that

$$\mathrm{Ricc}_g = kg \tag{2.47}$$

Since the scalar curvature is the trace of the Ricci tensor, it satisfies

$$\mathrm{Scal}_g = nk \tag{2.48}$$

Let us recall some well known facts concerning the conformal change of metrics: if g is some metric on M, the metric g' is said to be conformal to g if $g' = v(x)g$ for some $C^\infty(M)$, positive function v. Writing $v = u^{4/(n-2)}$ $(n \ge 3)$ and $g_u = g' = u^{4/(n-2)}g$, then (see [LP]) u satisfies

$$-4\frac{n-1}{n-2}\Delta_g u + \mathrm{Scal}_g\, u - \mathrm{Scal}_{g_u}\, u^{(n+2)/(n-2)} = 0. \tag{2.49}$$

All the metrics g' on M which are conformal to g are said to belong to the conformal class of g. Another proof of Obata's uniqueness result [Ob] is given below.

COROLLARY 2.2. *Let (M, g) be a compact Einstein manifold different from (S^n, g_0). Then g is the unique metric in its conformal class to have constant scalar curvature and fixed volume.*

PROOF. If g is Einstein, (2.49) reads as

$$-4\frac{n-1}{n-2}\Delta_g u + nku - \mathrm{Scal}_{g_u} u^{(n+2)/(n-2)} = 0. \tag{2.50}$$

When, $\mathrm{Scal}_{g_u} < 0$, then $k < 0$ and the only positive solution of (2.50) is a constant from the maximum principle. If $\mathrm{Scal}_{g_u} = 0$, there exists non positive solution of (2.50), whatever is k. Therefore the remaining case is the one where $\mathrm{Scal}_{g_u} > 0$ and necessarily $k > 0$ by integrating (2.50) on M. Up to change u into θu, for some $\theta > 0$, (2.43) reduces to

$$\Delta_g u - \frac{n(n-2)k}{4(n-1)}u + u^{(n+2)/(n-2)} = 0. \tag{2.51}$$

Since $\mathrm{Ricc}_g \geq kg$, the Lichnerowicz theorem implies that $\lambda_1 \geq nk/(n-1)$, and the condition (2.6) reads as

$$\frac{4n(n-2)k}{4(n-2)(n-1)} \leq \frac{n}{n-1}k, \tag{2.52}$$

which is obviously satisfied with equality. Therefore u is constant.

The remaining part of this section is devoted to similar types of results under an a priori estimate on u.

THEOREM 2.2. *Assume $\lambda \geq 0$, $q > 1$ and that u is solution of* (2.1) *which satisfies*

$$q\|u\|_{L^\infty}^{q-1} \leq \lambda + \lambda_1, \tag{2.53}$$

then u is a constant.

PROOF. If u satisfies (2.1), let $\bar{u}$ be the average value of u on M; $\bar{u}$ satisfies

$$\Delta_g \bar{u} - \lambda \bar{u} + \overline{|u|^{q-1}u} = 0, \tag{2.54}$$

and from classical Fourier analysis,

$$-\int_M (u-\bar{u})\Delta_g(u-\bar{u})dv_g \geq \lambda_1 \int_M (u-\bar{u})^2 dv_g \tag{2.55}$$

with equality if and only if $u - \bar{u}$ belongs to the eigenspace associated to λ_1. By the mean value theorem there holds

$$\int_M (u-\bar{u})\left(u|u|^{q-1} - \overline{u|u|^{q-1}}\right) dv_g \leq q\|u\|_{L^\infty}^{q-1} \int_M (u-\bar{u})^2 dv_g \tag{2.56}$$

with equality only if u is a constant. Therefore

$$\left(\lambda + \lambda_1 - q\|u\|_{L^\infty}^{q-1}\right) \int_M (u-\bar{u})^2 dv_g \leq 0, \tag{2.57}$$

which implies that $u = \bar{u}$ if (2.53) holds.

This result can be extended to a finite product of compact manifolds without boundary. In the particular case of two elements where $(M, g) \times (N, h) = (M \times N, g \otimes h)$ the Laplacian on the product manifold is computed by the following formula

$$(2.58)\qquad \left(\Delta_{g\otimes h}\right)_{(\sigma,\tau)} f = \left(\Delta_g\right)_\sigma f + (\Delta_h)_\tau f, \quad \sigma \in (M, g), \quad \tau \in (N, h).$$

COROLLARY 2.3. *Assume* $\lambda \geq 0$, $q > 1$ *and that* u *is solution of*

$$(2.59)\qquad \Delta_{g\otimes h} u - \lambda u + |u|^{q-1} u = 0$$

on $M \times N$. *Let* $\lambda_{1,M}$ *(respectively* $\lambda_{1,N}$*) be the first nonzero eigenvalue of* $-\Delta_g$ *(respectively*$-\Delta_h$*) in* $W^{1,2}(M)$ *(respectively* $W^{1,2}(N)$*) and let* $\sigma \in M$, $\tau \in N$ *be the variables. Then*

(i) - *If* $q\|u\|_{L^\infty}^{q-1} \leq \lambda + \lambda_{1,M}$, *u is independent of* $\sigma \in M$,

(ii) - *If* $q\|u\|_{L^\infty}^{q-1} \leq \lambda + \lambda_{1,N}$, *u is independent of* $\tau \in N$,

(iii) - *If* $q\|u\|_{L^\infty}^{q-1} \leq \lambda + \min(\lambda_{1,M}, \lambda_{1,N})$, *u is constant.*

PROOF. Setting $\bar{u}^M$ (respectively $\bar{u}^N$) the average of u with respect to the M-variable (respectively the N-variable) then

$$(2.60)\qquad \Delta_h \bar{u}^M + \Delta_g \bar{u}^M - \lambda \bar{u}^M + \overline{|u|^{q-1}}^M = 0.$$

Substracting (2.60) to (2.59), multiplying the result by $u - \bar{u}^M$ and integrating over M and N yields

$$(2.61)\qquad \begin{aligned} &\int_N \int_M (u - \bar{u}^M)\Delta_g(u - \bar{u}^M) dv_g dv_h \\ &+ \int_M \int_N (u - \bar{u}^M)\Delta_h(u - \bar{u}^M) dv_h dv_g \\ &- \lambda \int_N \int_M (u - \bar{u}^M)^2 dv_g dv_h \\ &+ \int_N \int_M (u - \bar{u}^M)\left(u|u|^{q-1} - \overline{u|u|^{q-1}}^M\right) dv_g dv_h = 0. \end{aligned}$$

But

$$(2.62)\qquad -\int_M (u - \bar{u}^M)\Delta_g(u - \bar{u}^M) dv_g \geq \lambda_{1,M} \int_M (u - \bar{u}^M)^2 dv_g$$

and

$$(2.63)\qquad -\int_N (u - \bar{u}^M)\Delta_h(u - \bar{u}^M) dv_h = \int_N \left|\nabla_h(u - \bar{u}^M)\right|^2 dv_h.$$

Therefore it can be deduced from (2.56), as above,

$$(2.64)\qquad \left(\lambda + \lambda_{1,M} - q\|u\|_{L^\infty}^{q-1}\right) \int_N \int_M (u - \bar{u}^M)^2 dv_g dv_h \leq 0$$

which implies (i) or (ii) equivalently, as for (iii) it is a consequence of (i) and (ii).

The last result of this section is an a priori estimate for any positive solution of (2.1) in a subcritical case.

THEOREM 2.3. *Assume that*

$$1 < q < (n+2)/(n-2) \tag{2.65}$$

then there exists a positive constant $C = C(M, g)$ *such that for any* $\lambda \geq 0$ *any nonnegative solution* u *of* (1.1) *satisfies*

$$\|u\|_{L^\infty} \leq C\lambda^{1/(q-1)}. \tag{2.66}$$

PROOF. Let us suppose that (2.66) does not hold. Then there exist four sequences $\{\lambda_m\}$, $\{C_m\}$, and $\{\sigma_m\}$, such that $\lambda_m > 0$, u_m is a positive solution of

$$\Delta_g u_m - \lambda_m u_m + u_m^q = 0 \tag{2.67}$$

in M with the following properties

$$\lim_{m\to\infty} C_m = \infty, \tag{2.68}$$

$$u_m(\sigma_m) = \|u_m\|_{L^\infty} = C_m \lambda_m^{1/(q-1)} \tag{2.69}$$

$$\lim_{m\to\infty} \sigma_m = \sigma_0. \tag{2.70}$$

There are three possibilities:

CASE 1: $\|u_m\|_{L^\infty}$ tends to some nonzero limit c when m tends to infinity. From (2.68) (2.69), λ_m tends to 0. Setting $w_m = u_m/\|u_m\|_{L^\infty}$, then

$$\Delta_g w_m - \lambda_m w_m + \|u_m\|_{L^\infty}^{q-1} w_m^q = 0. \tag{2.71}$$

As $\|w_m\|_{L^\infty} = 1$, $\|u_m\|_{L^\infty}$ is bounded and λ_m tends to 0, it can be deduced from classical estimates in elliptic equations theory that w_m converges in the C^2-M topology to some w which solves

$$\Delta_g w + c^{q-1} w^q = 0 \tag{2.72}$$

on M and

$$w(\sigma_0) = 1. \tag{2.73}$$

which is impossible.

CASE 2: $\|u_m\|_{L^\infty}$ tends to zero when m tends to infinity. Then there exists some m_0 such that $\|u_m\|_{L^\infty} \leq \lambda_1^{1/(q-1)}$ for $m \geq m_0$. From Theorem 1.2, u_m is constant with obvious value $\lambda_m^{1/(q-1)}$, which contradicts (2.68)-(2.69).

CASE 3: $\|u_m\|_{L^\infty}$ tends to infinity when m tends to infinity. The formula (2.67) can be written in local coordinates (x^i) near σ_0

$$\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^j}\left(\sqrt{|g|}\, g^{ij} \frac{\partial u_m}{\partial x^i}\right) - \lambda_m u_m + u_m^q = 0 \tag{2.74}$$

where $g = (g_{ij})$ is the metric tensor and $|g| = \det(g_{ij})$. Without any loss of generality, it can be assumed that (2.74) holds in the n-ball of center x_0 and radius d. Let us introduce the following scaling

$$\tilde{x} = \frac{x - x_0}{\alpha_m}, \; v_m(\tilde{x}) = \alpha_m^{2/(q-1)} u_m(x) \tag{2.75}$$

where α_m is defined by

$$\alpha_m^{2/(q-1)} \|u_m\|_{L^\infty} = 1. \tag{2.76}$$

For m large enough, $v_m(\tilde{x})$ is defined in the ball B_{d/α_m} (0) of center 0 and radius d/α_n where it satisfies $\|v_m\|_{L^\infty} = v_m(0) = 1$ and

$$\frac{1}{\sqrt{|g_m|}} \sum_{i,j} \frac{\partial}{\partial x^j} \left(\sqrt{|g_m|} g_m^{ij} \frac{\partial v_m}{\partial x^i} \right) - \lambda_m \alpha_m^2 v_m + v_m^q = 0 \tag{2.77}$$

where $g_m = (g_{mij}(\tilde{x})) = (g_{ij}(\alpha_m \tilde{x} + x_0))$. As in [GS2] it can be noticed that the coefficients and the ellipticity constant of (2.77) remain bounded and bounded below respectively. From the Agmon-Douglis-Nirenberg estimates (see [GT]) for any R and any $p > 1$ there exist some integer m_R and a positive constant M_R such that

$$\|v_m\|_{W^{2,p}(B_R(0))} \leq M_R \tag{2.78}$$

for $m \geq m_R$. From Morrey imbedding theorem there exists $\tilde{M}_R$ such that

$$\|v_m\|_{C^{1,\beta}(B_R(0))} \leq \tilde{M}_R. \tag{2.79}$$

for some $\beta \in (0, 1)$. Therefore, since

$$\lim_{m \to \infty} (g_{mij}(\tilde{x})) = \lim_{m \to \infty} (g_{ij}(\alpha_m \tilde{x} + x_0)) = (g_{ij}(x_0)), \tag{2.80}$$

it can be deduced that there exists a subsequence $\{v_{m_k}\}$ and a nonnegative function v defined in whole $\mathbb{R}^n$ such that v_{m_k} converges to v in the $C^{1,\beta}_{\text{loc}}$-topology, and v solves

$$\sum_{i,j} \frac{\partial}{\partial x^j} \left(g^{ij}(x_0) \frac{\partial v}{\partial x^i} \right) + v^q = 0, \tag{2.81}$$

$$v(0) = 1, \tag{2.82}$$

which is impossible from [GS1] since $q < (n+2)/(n-2)$.

3. – Equations in cylinders

In this Section, (M, g) is still a compact n-dimensional Riemannian manifold without boundary and the following time-dependent equation is studied

$$u_{tt} + \Delta_g u - \lambda u + |u|^{q-1}u = 0, \tag{3.1}$$

where the variable (t, σ) belongs to $I \times M$, I being either $\mathbb{R}$ or $\mathbb{R}^+$. Since M is compact without boundary, an important class of solutions of (3.1) consists in the class of homogeneous solutions which are the solutions of the ordinary differential equation

$$\varphi_{tt} - \lambda\varphi + |\varphi|^{q-1}\varphi = 0. \tag{3.2}$$

The solutions of (3.2) are classified by the value of the energy

$$E(\varphi) = \frac{1}{2}\varphi_t^2 - \frac{\lambda}{2}\varphi^2 + \frac{1}{q+1}|\varphi|^{q+1} \tag{3.3}$$

which is independent of t. All the orbits of (3.2) are closed and correspond to periodic solutions with the exception of the two homoclinic orbits consisting of the solutions $\varphi_0^{\pm}$ which satisfy $E(\varphi_0^{\pm}) = 0$ and

$$\varphi_0^+ > 0, \ \lim_{t\to-\infty} \varphi_0^+(t) = 0^+, \ \lim_{t\to\infty} \varphi_0^+(t) = 0^+, \tag{3.4}$$

$$\varphi_0^- < 0, \ \lim_{t\to-\infty} \varphi_0^-(t) = 0^-, \ \lim_{t\to\infty} \varphi_0^-(t) = 0^-. \tag{3.5}$$

Concerning (3.1) the first observation is the conservative aspect as the following quantity is independent of t:

$$E(u) = (\mathrm{vol}(M))^{-1} \int_M \left[\frac{1}{2}u_t^2 - \frac{1}{2}\left|\nabla_g u\right|^2 + \frac{\lambda}{2}u^2 - \frac{1}{q+1}|u|^{q+1}\right] dv_g. \tag{3.6}$$

Other invariants for (3.1) can be defined if M admits a Killing vector field X, that is a vector field $\sigma \mapsto X(\sigma)$ such that the group of diffeomormisms associated $\tau \mapsto e^{\tau X}$ is a group of isometries of (M, g). To this vector field can be associated the Lie derivative L_X defined by

$$(L_X u)(\sigma) = \frac{d}{dt}\, u(e^{tX}(\sigma))\Big|_{t=0}. \tag{3.7}$$

Proposition 3.1. *For any solution of* (3.1), *there holds*

$$\int_M u_t L_X u dv_g = Cst. \tag{3.8}$$

PROOF. Multiplying (3.1) by $L_X u$ and integrating over M

$$\int_M u_{tt} L_X u dv_g + \int_M \Delta_g u L_X u dv_g + \int_M \left(-\lambda u + |u|^{q-1}u\right) L_X u dv_g = 0. \tag{3.9}$$

Since X is a Killing vector field, this gives

$$\int_M \Delta_g u L_X u dv_g = -\frac{1}{2}\int_M L_X\left(\left|\nabla_g u\right|^2\right) dv_g = 0, \tag{3.10}$$

and for any C^1 function ω defined on M, there holds $\int_M L_X \omega dv_g = 0$. In the same way

$$\int_M (-\lambda u + |u|^{q-1}u) L_X u\, dv_g = \int_M L_X\left(-\frac{\lambda}{2}u^2 + \frac{1}{q+1}|u|^{q+1}\right) dv_g = 0, \tag{3.11}$$

and for the remaining term

$$\int_M u_{tt} L_X u\, dv_g = \frac{d}{dt}\int_M u_t L_X u\, dv_g - \int_M u_t L_X u_t\, dv_g = \frac{d}{dt}\int_M u_t L_X u\, dv_g, \tag{3.12}$$

from the above observation, which implies (3.8).

The main homogenization result is the following:

THEOREM 3.1. *Assume u is a solution of* (3.1) *on* $[0, \infty) \times M$ *such that*

$$\sup_{t\geq T} \|u(t, .)\|_{L^\infty} \leq ((\lambda + \lambda_1)/q)^{1/(q-1)}, \tag{3.13}$$

for some $T > 0$ and let $\sigma = E(u)(t)$, then

$$\lim_{t\to\infty} \operatorname{dist}_{C^2}(u(t.,), \gamma_\sigma) = 0. \tag{3.14}$$

If assuming moreover that (3.13) *is strict and that $\sigma \neq 0$, then there exists a solution φ in the orbit γ_σ of* (3.2) *defined by $E(\varphi) = \sigma$ such that*

$$\lim_{t\to\infty} \|u(t, .) - \varphi(.)\|_{C^2} = 0. \tag{3.15}$$

PROOF. Recall that $\bar{u}$ is the average of u on M. Averaging (3.1) yields

$$(u-\bar{u})_{tt} + \Delta_g(u-\bar{u}) - \lambda(u-\bar{u}) + |u|^{q-1}u - \overline{|u|^{q-1}u} = 0. \tag{3.16}$$

Multiplying by $u - \bar{u}$ and integrating over M implies, as in (2.55)-(2.56),

$$\frac{1}{2}\frac{d^2}{dt^2}\int_M (u-\bar{u})^2 dv_g - \left(\lambda + \lambda_1 - q\|u\|^{q-1}_{L^\infty((T,\infty)\times M)}\right)\int_M (u-\bar{u})^2 dv_g \geq 0 \tag{3.17}$$

for $t \geq T$. Setting $(\lambda + \lambda_1 - q\|u\|^{q-1}_{L^\infty((T,\infty)\times M)}) = \beta \geq 0$, then (3.17) implies that the function $t \mapsto \|u(t,.) - \bar{u}(t)\|^2_{L^2}$ is convex and therefore there exists $\alpha \geq 0$ such that

$$\lim_{t\to\infty} \|(u - \bar{u})(t)\|_{L^2} = \alpha. \tag{3.18}$$

Let us prove first that $\alpha = 0$. If (3.13) is strict then $\beta > 0$; (3.17) and the maximum principle imply

$$\|u(t,.) - \bar{u}(t)\|_{L^2} \leq \|u(T,.) - \bar{u}(T)\|_{L^2} e^{-\sqrt{\beta}(t-T)} \tag{3.19}$$

for $t \geq T$ and $\alpha = 0$. Supposing that $\sup_{t\geq T} \|u(t,.)\|_{L^\infty} = ((\lambda + \lambda_1)/q)^{1/(q-1)}$ and that $\alpha > 0$ then there exists $\theta > 0$ such that $\operatorname{vol} A(t) \geq \theta$ where

$$A(t) = \{\sigma \in M : |u(\sigma, t) - \bar{u}(t)| \geq \alpha/2\}. \tag{3.20}$$

Therefore (3.16) yields

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{dt^2}\int_M (u - \bar{u})^2 dv_g \\ &\geq \int_M \left(q\|u\|^{q-1}_{L^\infty((T,\infty)\times M)} - \frac{u|u|^{q-1} - \bar{u}|\bar{u}|^{q-1}}{u - \bar{u}} \right)(u - \bar{u})^2 dv_g, \\ &\geq \frac{\alpha^2}{4}\int_{A(t)} \left(q\|u\|^{q-1}_{L^\infty((T,\infty)\times M)} - \frac{u|u|^{q-1} - \bar{u}|\bar{u}|^{q-1}}{u - \bar{u}} \right) dv_g. \end{aligned} \tag{3.21}$$

If Θ is defined by

$$\begin{aligned} \Theta = \min\Bigg\{ q\|u\|^{q-1}_{L^\infty} - \frac{|a|^{q-1}a - |b|^{q-1}b}{a - b} &: |a - b| \geq \alpha/2, \\ &\max(|a|, |b|) \leq \|u\|_{L^\infty} \Bigg\} \end{aligned} \tag{3.22}$$

then $\Theta > 0$ and

$$\frac{1}{2}\frac{d^2}{dt^2}\int_M (u - \bar{u})^2 dv_g \geq \frac{\alpha^2}{4}\theta\Theta, \tag{3.23}$$

for $t \geq T$, which is impossible. Therefore $\alpha = 0$. Consequently

$$\lim_{t\to\infty} \left\| \left(|u|^{q-1}u - \overline{|u|^{q-1}u} \right)(t) \right\|_{L^2} = 0. \tag{3.24}$$

From $W^{2,2}$-estimates in elliptic equations, it is deduced from (3.16)-(3.18) that

$$\lim_{t\to\infty} \|(u - \bar{u})(t)\|_{W^{2,2}(M)} = 0. \tag{3.25}$$

Using Sobolev and Morrey imbedding theorems and the classical elliptic equations regularity theory finally yields

$$\lim_{t\to\infty}\left(\|(u-\bar u)(t)\|_{C^2}+\|(u_t-\bar u_t)(t)\|_{C^1}\right)=0. \tag{3.26}$$

Moreover, u remains uniformly bounded in $C^{2,\gamma}([a-1,a+1]\times M)$ independently of $a\geq T+1$, for some $\gamma\in(0,1)$. Let $\{t_n\}$ be a sequence of real numbers tending to infinity and let us set $u_{\{t_n\}}(t,\sigma)=u(t+t_n,\sigma)$, then there exist a subsequence $\{t_{n_k}\}$ and a function φ such that $u_{\{t_{n_k}\}}$ converges to φ in the C^2_{loc} -topology of $\mathbb{R}\times M$. It is clear that φ is a solution of (3.1), independent of $\sigma\in M$ from (3.25), and therefore a solution of (3.2). Moreover, as $E(u)$ is constant with value η, it is clear that $E(\varphi)=\eta$. As the orbit γ_η is uniquely determined (double orbit in the case $\eta=0$), relation (3.14) follows.

If it is assumed that (3.13) is strict then (3.19) and the standard elliptic equations theory imply an exponential rate of homogeneisation, namely

$$\|(u-\bar u)(t)\|_{C^2}+\|(u_t-\bar u_t)(t)\|_{C^1}\leq Ce^{-\sqrt{\beta}(t-T)}. \tag{3.27}$$

Therefore, $\bar u$ satisfies

$$\bar u_{tt}-\lambda\bar u+|\bar u|^{q-1}\bar u=a(t)e^{-t\sqrt{\beta}}, \tag{3.28}$$

where a is a bounded function. From the assumption, it is assumed that the energy $E(u)=\eta$ is not zero and therefore there exist $P>0$ and a P-periodic solution φ of (3.2) such that γ_η is just generated by φ. As in [CGS], it can be be written

$$E(\bar u)(t)=\frac{1}{2}\bar u_t^2-\frac{\lambda}{2}\bar u^2+\frac{1}{q+1}|\bar u|^{q+1}=E(\varphi)+(\bar u^2+\bar u_t^2)O(e^{-t\sqrt{\beta}}), \tag{3.29}$$

which implies that $\lim_{t\to\infty}(\bar u(t+P)-\bar u(t))=0$, from the classical perturbation theory of periodic solutions of ordinary differential equations as in [CGS]. Therefore $\bar u(t)$, and then $u(t,\sigma)$, is asymptotic to a suitable translate of φ.

For the estimate (3.13), the following analogous of Theorem 2.3 holds:

THEOREM 3.2. *Assume that*

$$1<q<(n+3)/(n-1); \tag{3.30}$$

then there exists a positive constant $C=C(M,g)$ *such that for any* $\lambda\geq 0$ *any nonnegative bounded solution* u *of* (3.1) *in* $\mathbb{R}\times M$ *satisfies*

$$\|u\|_{L^\infty}\leq C\lambda^{1/(q-1)}. \tag{3.31}$$

PROOF. Let us assume that (3.31) does not hold. Then there exist five sequences $\{\lambda_m\}$, $\{u_m\}$, $\{\varepsilon_m\}$, $\{C_m\}$, and $\{(t_m, \sigma_m)\}$, such that $\lambda_m > 0$, u_m, is a positive solution of

$$\partial^2 u_m/\partial t^2 + \Delta_g u_m - \lambda_m u_m + u_m^q = 0 \tag{3.32}$$

in $\mathbb{R} \times M$ with the following properties

$$\lim_{m\to\infty} C_m = \infty, \ \lim_{m\to\infty} \varepsilon_m = 0, \tag{3.33}$$

$$u_m(t_m, \sigma_m) = \|u_m\|_{L^\infty} - \varepsilon_m = C_m \lambda_m^{1/(q-1)}, \tag{3.34}$$

$$\lim_{m\to\infty} \sigma_m = \sigma_0 \tag{3.35}$$

as for $\{t_m\}$ there are two possibilities: either

$$\lim_{m\to\infty} t_m = t_0, \tag{3.36}$$

or

$$\lim_{m\to\infty} t_m = \infty. \tag{3.37}$$

Three cases have to be considered.

CASE 1: $\|u_m\|_{L^\infty}$ tends to some nonzero limit c when m tends to infinity. From (3.33) (3.34), λ_m tends to 0. If we set $w_m = u_m/\|u_m\|_{L^\infty}$, then

$$\partial^2 w_m/\partial t^2 + \Delta_g w_m - \lambda_m w_m + \|u_m\|_{L^\infty}^{q-1} w_m^q = 0. \tag{3.38}$$

Since, $\|w_m\|_{L^\infty} = 1$, $\|u_m\|_{L^\infty}$ is bounded and λ_m tends to 0, it is deduced from classical estimates in elliptic equations theory that $\tilde{w}_m(t, \sigma) = w_m(t_m + t, \sigma)$ converges in the C^2_{loc}-$\mathbb{R} \times M$ topology to some w which solves

$$\partial^2 w/\partial t^2 + \Delta_g w + c^{q-1} w^q = 0 \tag{3.39}$$

on $\mathbb{R} \times M$ and

$$w(0, \sigma_0) = 1. \tag{3.40}$$

Let $\bar{w}$ be the average of w on M, then

$$\bar{w}_{tt} + c^{q-1} \bar{w}^q \le 0 \tag{3.41}$$

on $\mathbb{R}$, which is impossible.

CASE 2: $\|u_m\|_{L^\infty}$ tends to zero when m tends to infinity.

From (3.17) there holds

$$\frac{1}{2}\frac{d^2}{dt^2}\int_M (u_m-\bar{u}_m)^2 dv_g + \left(\lambda+\lambda_1-q\|u_m\|_{L^\infty}^{q-1}\right)\int_M (u_m-\bar{u}_m)^2 dv_g \geq 0 \tag{3.42}$$

where $\bar{u}_m$ is the average of u_m on M. Then there exists some integer m_0 such that $t \mapsto \int_M (u_m-\bar{u}_m)^2(t,\sigma)dv_g$ is a strictly convex, positive and bounded function defined on $\mathbb{R}$ for $m \geq m_0$. Therefore it is identically zero which implies that $u_m = \lambda_m^{1/(q-1)}$ which contradicts (3.33)-(3.34).

CASE 3: $\|u_m\|_{L^\infty}$ tends to infinity when m tends to infinity.
Writing (3.32) in local coordinates (x^i) near σ_0 gives

$$\frac{\partial^2 u_m}{\partial t^2} + \frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial}{\partial x^j}\left(\sqrt{|g|}g^{ij}\frac{\partial u_m}{\partial x^i}\right) - \lambda_m u_m + u_m^q = 0 \tag{3.43}$$

where $g = (g_{ij})$ is the metric tensor and $|g| = \det(g_{ij})$. Without any loss of generality, it can be assumed that (3.43) holds in $\mathbb{R}\times B_d(x_0)$ where $B_d(x_0)$ is the $(n-1)$-ball of center x_0 and radius $d > 0$. Let us introduce the following scaling

$$\tilde{t} = \frac{t-t_m}{\alpha_m},\ \tilde{x} = \frac{x-x_0}{\alpha_m},\ v_m(\tilde{t},\tilde{x}) = \alpha_m^{2/(q-1)}u_m(t,x) \tag{3.44}$$

where α_m is defined by

$$\alpha_m^{2/(q-1)}\|u_m\|_{L^\infty} = 1. \tag{3.45}$$

Therefore, proceeding as in the proof of Theorem 2.3, it follows that u_{m_k} converges in the C^2_{loc}-$\mathbb{R}\times\mathbb{R}^{n+1}$-topology to some nonzero, nonnegative v which satisfies

$$\frac{\partial^2 v}{\partial t^2} + \sum_{i,j}\frac{\partial}{\partial x^j}\left(g^{ij}(x_0)\frac{\partial v}{\partial x^i}\right) + v^q = 0 \tag{3.46}$$

in $\mathbb{R}^{n+1}$, which, again, is impossible from [GS1].

An immediate consequence of Theorem 3.2 is the following

COROLLARY 3.1. *Assume that* (3.30) *holds and that u is a positive and bounded solution of* (3.1) *on* $[0,\infty)\times M$. *Then*

$$\lim_{T\to\infty}\sup_{t\geq T}\|u(t,.)\|_{L^\infty} \leq C\lambda^{1/(q-1)} \tag{3.47}$$

where C is the constant appearing in Theorem 3.2.

Combining Theorem 3.1 and Corollary 3.1 yields

COROLLARY 3.2. *Let* (3.30) *and* $\lambda < \lambda_1(qC^{q-1}-1)^{-1}$ *hold and u be a positive and bounded solution of* (3.1) *on* $[0,\infty)\times M$. *Then* (3.14) *holds for some* $\sigma = E(u)(t)$. *Moreover if* $\sigma \neq 0$, *there exists a solution* φ *in the orbit* γ_σ *of* (3.2) *defined by* $E(\varphi)=\sigma$ *such that* (3.15) *is valid*

REMARK 3.1. The assumption on the boundedness of the nonnegative solutions of (3.1) is not easy to check. However, it has been proved by Bouhar and Veron [BV] that any such solution is bounded provided $1 < q < (n+1)/(n-1)$.

REMARK 3.2. It is clear that non constant solutions of (2.1) are non-homogeneous solutions of (3.1). Moreover, in the case where M admits a Killing vector field X there may exist soliton solutions of (3.1) under the following form

$$u(t,\sigma)=\omega(e^{tX}(\sigma)) \tag{3.48}$$

where ω solves

$$\Delta_g\omega + L_X L_X\omega - \lambda\omega + |\omega|^{q-1}\omega = 0. \tag{3.49}$$

Non trivial solutions of (3.49) can be obtained when $1 < q < (n+2)/(n-2)$ by studying the critical points of the following functional

$$\mathcal{E}(\varphi) = \int_M \left(\left|\nabla_g\varphi\right|^2 + (L_X\varphi)^2 + \lambda\varphi^2 - \frac{2}{q+1}|\varphi|^{q+1}\right) dv_g. \tag{3.50}$$

Other nontrivial solutions, without the restriction on q, can be obtained by bifurcation from the first nonzero eigenvalue of the linearized operator

$$\Delta_g + L_X L_X + (q-1)\lambda I \tag{3.51}$$

(see [BVV] for some particular cases).

4. – Existence of solutions

In this section the initial value problem, that is the question of the existence of solutions of

$$u_{tt} + \Delta_g u - \lambda u + |u|^{q-1}u = 0\,, \tag{4.1}$$

defined on $\mathbb{R}^+ \times M$ and such that $u(0,\sigma)=u_0(\sigma)$, is considered. The existence of solutions tending to 0 at infinity is taken as a start.

THEOREM 4.1. *For any continuous function u_0 defined on M and satisfying*

$$0 \leq u_0(\sigma) \leq \left(\lambda \frac{q+1}{2}\right)^{1/(q-1)}, \tag{4.2}$$

there exists a continuous nonnegative solution u of (4.1) *defined on $\mathbb{R}^+ \times M$ which tends to* 0 *at infinity and takes the value u_0 at $t = 0$.*

PROOF. First it can be noticed that the specific value $\left(\lambda \frac{q+1}{2}\right)^{1/(q-1)}$ is the maximal value that can take any positive solution of the associated ordinary differential equation (3.2) and that there exists a solution (the positive homoclinic orbit) φ_0^+ of (3.2) on $\mathbb{R}^+$ which satisfies

$$\varphi_0^+ \geq 0, \quad \varphi_0^+(0) = \left(\lambda \frac{q+1}{2}\right)^{1/(q-1)}, \quad \lim_{t\to\infty} \varphi_0^+(t) = 0^+. \tag{4.3}$$

If u_0 is positive, then $u_0 \geq \varphi_0^+(T)$ for T large enough and, from the classical result, there exists a solution u of (4.1) such that $u(0,\sigma) = u_0(\sigma)$ and

$$\varphi_0^+(t+T) \leq u(t,\sigma) \leq \varphi_0^+(t) \tag{4.4}$$

for $(t,\sigma) \in \mathbb{R}^+ \times M$.

In the general case the following iterating scheme is introduced

$$\begin{cases} y_0 = 0 \\ \partial^2 y_m/\partial t^2 + \Delta_g y_m - \lambda y_m = -y_{m-1}^q \\ y_m(0,\sigma) = u_0(\sigma). \end{cases} \tag{4.5}$$

STEP 1. The sequence $\{y_m\}$ is an increasing sequence of positive bounded functions which decay exponentially when t tends to infinity.

In fact, for y_1,

$$\|y_1(t,.)\|_{L^2} \leq e^{-t\sqrt{\lambda}} \|u_0(\,.\,)\|_{L^2}, \tag{4.6}$$

is obtained from explicit representation, which implies

$$\|y_1(t,.)\|_{L^\infty} \leq C e^{-t\sqrt{\lambda}} \|u_0(\,.\,)\|_{L^\infty} \tag{4.7}$$

for $C = C(M) > 0$. From the maximum principle

$$y_1(t,\sigma) \leq \varphi_0^+(t). \tag{4.8}$$

From the classical linearisation technique, for any $\gamma \in (0, \sqrt{\lambda})$ there exists $C_\gamma > 0$ such that

$$\varphi_0^+(t) \leq C_\gamma e^{-t\gamma} \tag{4.9}$$

on $\mathbb{R}^+$. As $y_1^q \in L^2((0,\infty)\times M)\cap L^1(0,\infty;L^2(M))\cap C^1((0,\infty)\times M)$, y_2 can be defined with the following formula (see [Ve] for details)

$$y_2(t) = S(t)u_0 + \int_0^t S(t-s)\int_s^\infty S(\tau - s)y_1^q(\tau)d\tau\, ds \tag{4.10}$$

where $S(t)$ is the continuous semigoup of contractions of $L^2(M)$ generated by $-(-\Delta_g + \lambda I)^{1/2}$. This semigroup satisfies

$$\|S(t)\psi\|_{L^\infty} \leq Ce^{-t\sqrt{\lambda}}\|\psi\|_{L^\infty}\,. \tag{4.11}$$

Therefore y_2 is a bounded strong solution and it satisfies

$$y_1(t,\sigma) \leq y_2(t,\sigma) \leq \varphi_0^+(t) \tag{4.12}$$

on $\mathbb{R}^+\times M$. Iterating this process with the above representation formula allows the construction of the sequence $\{y_m\}$ of continuous nontrivial solutions of (4.5) on $\mathbb{R}^+\times M$, with the order property

$$0 \leq y_{m-1}(t,\sigma) \leq y_m(t,\sigma) \leq \varphi_0^+(t) \leq C_\gamma e^{-t\gamma} \tag{4.13}$$

on $\mathbb{R}^+\times M$.

STEP 2. End of the proof. The sequence $\{y_m\}$ is increasing and converges to some continuous and positive solution u of (4.1) defined on $\mathbb{R}^+\times M$ wich takes the value u_0 at $t=0$ and satisfies

$$0 \leq u(t,\sigma) \leq \varphi_0^+(t)\,. \tag{4.14}$$

The next question that is considered is the existence of a global solution close to some homogeneous solution and asymptotic to this homogeneous solution at infinity. By an implicit function method a local theory is constructed for such a problem. Let $\{\lambda_k\}_{k\geq 0}$ be the sequence eigenvalues of $-\Delta_g$ in $W^{1,2}(M)$, with corresponding eigenspaces H^k with dimension d_k and orthonormal basis $\{\Theta_{j,k}\}$, $0\leq j\leq d_k$. If $y_0(t)$ is a T-periodic solution of (3.2) the linearization of (4.1) around y_0 yields the following linear equation

$$\psi \mapsto \mathbf{L}_{y_0}(\psi) = \psi_{tt} + \Delta_g\psi + (q|y_0(t)|^{q-1} - \lambda)\psi\,. \tag{4.15}$$

Let us write first the Fourier decomposition of any solution of $\mathbf{L}_{y_0}(\psi) = 0$ as

$$\psi(t,\sigma) = \sum_k \sum_{0\leq j\leq d_k} c_{j,k}(t)\Theta_{j,k}(\sigma)\,. \tag{4.16}$$

Then the $c_k = c_{j,k}$ satisfy

$$c_k'' + (q|y_0|^{q-1} - \lambda - \lambda_k)c_k = 0\,, \tag{4.17}$$

which is a linear differential equation with periodic coefficients for which it is necessary to recall some elements of Floquet's theory.

PROPOSITION 4.1. *Consider the following differential equation*

$$y'' + a_1(t)y' + a_2(t)y = 0 \tag{4.18}$$

where a_1 and a_2 are T-periodic; then there exist two linearly independent solutions of (4.18), *y_1 and y_2, such that*

(i) *either*

$$y_1(t) = e^{m_1 t} p_1(t), \quad y_2(t) = e^{m_2 t} p_2(t)\,, \tag{4.19}$$

where m_1 and m_2 are constants (real or complex) and p_1 and p_2 are T-periodic functions,

(ii) *or*

$$y_1(t) = e^{mt} p_1(t), \quad y_2(t) = e^{mt}(tp_1(t) + p_2(t)) \tag{4.20}$$

where m is a constant (real or complex) and p_1 and p_2 are T-periodic functions.

The constants m_j are the characteristic exponents of the equation; if $\rho_j = e^{m_j T}$, then the ρ_j are the solutions of

$$\rho^2 - D\rho + \exp\left(-\int_0^T a_1(t)dt\right) = 0 \tag{4.21}$$

where D is a constant called the discriminant of the equation. In the particular case of Hill's equation

$$y''(t) + (\eta + a(t))y(t) = 0 \tag{4.22}$$

where a is a T-periodic function and η a real number, let $D(\eta)$ be the corresponding discriminant. Then Floquet's theory reads as follows

PROPOSITION 4.2. *There exist two sequences of real numbers $\{\nu_k\}$, $\{\mu_k\}$ such that*

(i) *they appear in the following order*

$$\nu_0 < \mu_0 \leq \mu_1 < \nu_1 \leq \nu_2 < \mu_2 \leq \mu_3 < \nu_3 \leq \nu_4 < \dots \tag{4.23}$$

(ii) *on the intervals $[\nu_{2k}, \mu_{2k}]$, $D(\eta)$ decreases from 2 to -2,*

(iii) *on the intervals $[\mu_{2k+1}, \nu_{2k+1}]$, $D(\eta)$ increases from -2 to 2,*

(iv) *on the intervals (μ_{2k}, μ_{2k+1}), $D(\eta) < -2$,*

(vi) *on the intervals $(-\infty, \nu_0)$ and (ν_{2k+1}, ν_{2k+2}), $D(\eta) > 2$,*

moreover

(vii) *if η is one of the ν_j or μ_j then $|D(\eta)| = 2$,* (4.21) *possesses a double root and the solutions are given by* (4.20). *As for m it takes the values* 0 *or $i\pi/T$ according $D(\eta) = 2$ or $D(\eta) = -2$ and η belongs to a* **periodicity zone**.

(viii) *if $|D(\eta)| > 2$, then η belongs to an* **instability zone** *with the solutions given by* (4.19) *where m_1 and m_2 are opposite real numbers,*

(ix) *if $|D(\eta)| < 2$, then η belongs to a* **stability zone** *with the solutions given by* (4.19) *where m_1 and m_2 are conjugate imaginary numbers.*

We apply Floquet's theory to equation (4.17) with $a(t) = q|y_0(t)|^{q-1}$ and $\eta = \eta_k = -\lambda - \lambda_k$: there exist an integer k_0 and a positive real number θ_1 such that

$$\forall k > k_0, \quad \forall t > 0, \quad (q|y_0(t) - \lambda - \lambda_k|) < -\theta_1^2 . \tag{4.24}$$

For $k > k_0$, η_k belongs to the first instability zone in the sense of Proposition 4.2, that is $(-\infty, \nu_0)$ and the solutions of (4.17) are of two different exponential types. For $0 \le k \le k_0$ the general form of a solution of (4.17) is determined by the fact that η_k belongs or does not belong to an instability zone. If η_k belongs to an instability zone, set m_k^- and m_k^+ the corresponding characteristic exponents of the equation with $m_k^- < 0 < m_k^+$. Let θ be such that

$$0 < \theta < \min\{\theta_1, \min\{m_k^+ \mid k \le k_0 \text{ and } \eta_k \text{ instable}\}\} . \tag{4.25}$$

E_1 is defined as the subspace of $L^2(M)$ generated by the $\Theta_{j,k}$, $0 \le j \le d_k$, corresponding to the k such that η_k belongs to a zone of stability or periodicity in the sense of (vii) and (ix) and E_2 as the orthogonal complement of E_1 in $L^2(M)$; E_2 is the Hilbertian sum of the H^k for which η_k belongs to an instability zone in the sense of (ix). Let P_1 and P_2 be the orthogonal projectors of $L^2(M)$ onto E_1 and E_2 respectively. It is important to notice that E_1 is finite dimensional.

REMARK 4.1. There always holds $D(\eta_0) = D(-\lambda) = 2$ as y_0' is a T-periodic solution of (4.17) with $k = 0$. Moreover E_1 is never trivial as it contains the space of constant functions.

THEOREM 4.2. *There exists $\delta > 0$ such that if $u_0 = y_0(0) + z_0$ with $z_0 \in E_0$ and*

$$\|z_0\|_{C^{2,\alpha}} < \delta , \tag{4.26}$$

where $\alpha \in (0, 1)$, then there exists a continuous solution u of (4.1) *defined on $\mathbb{R}^+ \times M$ and such that $u(0, \sigma) = u_0(\sigma)$.*

Before proving this result it is necessary to define some functional spaces

$$E_\theta^{2,\alpha} = \{v \mid e^{\theta t} v \in W^{2,\infty}((0,\infty) \times M) \cap C^{2,\alpha}([0,\infty) \times M)\} , \tag{4.27}$$

$$E_\theta^{\alpha} = \{v \mid e^{\theta t} v \in L^{\infty}((0,\infty) \times M) \cap C^{\alpha}([0,\infty) \times M)\} , \tag{4.28}$$

with the natural corresponding norms defined on, which endow those spaces with a structure of real Banach spaces. Set $F_2 = E_2 \cap C^{2,\alpha}(M)$ and define $\mathbf{G}$ from $E_\theta^{2,\alpha}$ into $E_\theta^\alpha \times F_2$ by

$$\mathbf{G}(v) = \left(\mathbf{L}_{y_0}(v), P_2(v(0), .)\right) . \tag{4.29}$$

then the following holds,

PROPOSITION 4.3. **G** *is a Banach isomorphism between* $E_\theta^{2,\alpha}$ *and* $E_\theta^\alpha \times F_2$.

PROOF. It is clear that **G** is well defined and is a continuous linear mapping from $E_\theta^{2,\alpha}$ into $E_\theta^\alpha \times F_2$. If g belongs to E_θ^α, the following equation has to be considered

$$\psi_{tt} + \Delta_g \psi + (q|y_0(t)|^{q-1} - \lambda)\psi = g \tag{4.30}$$

in $\mathbb{R}^+ \times M$. Decomposing ψ and g as

$$\psi(t,\sigma) = \sum_{k\geq 0} \sum_{0\leq j\leq d_k} c_{j,k}(t)\Theta_{j,k}(\sigma), \quad g(t,\sigma) = \sum_{k\geq 0} \sum_{0\leq j\leq d_k} \gamma_{j,k}(t)\Theta_{j,k}(\sigma) \tag{4.31}$$

and setting $c_{j,k} = c_k$, $\gamma_{j,k} = \gamma_k$ results in

$$c_k''(q|y_0|^{q-1} - \lambda - \lambda_k)c_k = \gamma_k\,, \tag{4.32}$$

moreover there exists a constant N_g such that $|\gamma_k(t)| \leq N_g e^{-t\theta}$ for $k \geq 0$, $t \geq 0$. Three possibilities are encountered

CASE 1. η_k belongs to a zone of stability.
Then

$$c_k(t) = \frac{y_1(t)}{W} \int_t^\infty y_2(s)\gamma_k(s)ds - \frac{y_2(t)}{W} \int_t^\infty y_1(s)\gamma_k(s)ds \tag{4.33}$$

where y_1 and y_2 are two linearly independent (and bounded) solutions of the associated homogeneous equation and W is their Wronskian determinant, which is constant in that case as there exists no term in c_k'. An easy computation gives that

$$|c_k(t)| \leq Ce^{-t\theta} \tag{4.34}$$

and, from elliptic estimates, it can be deduced that

$$\|e^{t\theta}c_k(t)\|_{W^{2,\infty}\cap C^{2,\alpha}} \leq C\|e^{t\theta}\gamma_k(t)\|_{L^\infty \cap C^\alpha}\,. \tag{4.35}$$

CASE 2. η_k belongs to a zone of periodicity.
It is clear that (4.33)-(4.35) are still valid.

CASE 3. η_k belongs to a zone of instability.
In that case

$$c_k(t) = C_2 y_2(t) + \frac{y_1(t)}{W} \int_t^\infty y_2(s)\gamma_k(s)ds + \frac{y_2(t)}{W} \int_0^t y_1(s)\gamma_k(s)ds \tag{4.36}$$

with $y_1(t) = e^{m_k^+ t} p_1(t)$ and $y_2(t) = e^{m_k^- t} p_2(t)$ (it is important not to forget that $m_k^- < 0 < m_k^+$) where C_2 is determined by $c_k(0)$ which are the coefficients of

$P_2(\psi(0))$. It is easy to check that (4.34)-(4.35) still holds with a constant C independent of k. In order to complete the existence proof let us consider the projection of (4.30) onto E_2 by setting

$$\tilde{\psi} = P_2(\psi), \quad \tilde{g} = P_2(g). \tag{4.37}$$

This gives

$$\tilde{\psi}_{tt} + \Delta_g \tilde{\psi} + (q|y_0(t)|^{q-1} - \lambda)\tilde{\psi} = \tilde{g} \tag{4.38}$$

and as θ_1 satisfies (4.24) the result is

$$\frac{d^2}{dt^2}(\|\tilde{\psi}\|_{L^2}) - \theta_1^2 \|\tilde{\psi}\|_{L^2} \geq -\|\tilde{g}\|_{L^2},$$

which implies

$$\|\tilde{\psi}(t)\|_{L^2} \leq e^{-\theta_1 t}\|\tilde{\psi}(0)\|_{L^2} + \int_0^t e^{-\theta_1(t-s)} \int_s^\infty e^{-\theta_1(\tau - s)}\|\tilde{g}(\tau)\|_{L^2} d\tau \, ds. \tag{4.40}$$

But

$$\|\tilde{g}(t,.)\|_{L^2} \leq C e^{-\theta t}\|g\|_{E_\theta^\alpha}, \tag{4.41}$$

therefore

$$\|\tilde{\psi}(t)\|_{L^2} \leq e^{-\theta_1 t}\|\tilde{\psi}(0)\|_{L^2} + C' e^{-\theta t}\|g\|_{E_\theta^\alpha}. \tag{4.42}$$

Using elliptic equations estimates yields

$$\|\tilde{\psi}\|_{E_\theta^{2+\alpha}} \leq C'(\|\tilde{\psi}(0)\|_{C^{2,\alpha}} + e^{-\theta t}\|g\|_{E_\theta^\alpha}). \tag{4.43}$$

Therefore $\mathbf{G}$ is onto and the inverse mapping $\mathbf{G}^{-1}$ is continuous from $E_\theta^\alpha \times F_2$ into $E_\theta^{2,\alpha}$. In order to end the proof it is assumed that $\mathbf{G}(\psi) = 0$ for some ψ in $E_\theta^{2,\alpha}$, then $P_2(\psi(0,.)) = 0$ and, if the general solution of (4.17) under the form is

$$c_k(t) = a_1 y_1(t) + a_2 y_2(t), \tag{4.44}$$

then necessarily $a_1 = a_2 = 0$ if η_k belongs to a zone of stability or periodicity; if η_k belongs to a zone of instability $a_1 = 0$ as $P_2(\psi(0,.)) = 0$ and $a_2 = 0$ as y_2 is unbounded.

Proof of Theorem 4.2. We look for a solution u of (4.1) under the form

$$u(t,\sigma) = y_0(t) + w(t,\sigma) \tag{4.45}$$

and w satisfies

$$w_{tt} + \Delta_g w + (q|y_0|^{q-1} - \lambda)w + Q(w) = 0 \tag{4.46}$$

on $\mathbb{R}^+ \times M$ with

$$Q(w) = |y_0 + w|^{q-1}(y_0 + w) - |y_0|^{q-1}y_0 - q|y_0|^{q-1}w\,. \tag{4.47}$$

If the mapping Γ from $E_\theta^{2,\alpha}$ into $E_\theta^\alpha \times F_2$ is defined by

$$\Gamma(w) = \Big(w_{tt} + \Delta_g w + (q|y_0|^{q-1} - \lambda)w + Q(w),\, P_2(w(0,.))\Big)\,, \tag{4.48}$$

then $\Gamma(0) = (0,0)$ and $D\Gamma(0) = \mathbf{G}$ which is an isomorphism. By the local inversion theorem, there exists $\delta > 0$ such that for any $z_0 \in F_2$ satisfying $\|z_0\|_{C^{2,\alpha}} < \delta$, there exists a solution w of $\Gamma(w) = (0, z_0)$, that is a solution u of (4.1) defined on $\mathbb{R}^+ \times M$ and such that $u(0,\sigma) = u_0(\sigma)$, under the form (4.45) with $u_0(\sigma) = y_0(0) + z_0$.

5. – Partially homogenized equations

In this section a short view of some partially homogenized problems on $\mathbb{R}^+ \times M$ with the specific exponent $q = 3$ is given. The equations that are considered are the following

$$u_{tt} + \Delta_g u - \lambda u + \bar{u}^3 = 0\,, \tag{5.1}$$

$$u_{tt} + \Delta_g u - \lambda u + u\bar{u}^2 = 0\,, \tag{5.2}$$

$$u_{tt} + \Delta_g u - \lambda u + u\overline{u^2} = 0\,, \tag{5.3}$$

where the general notation $\bar{g}$ represents the average of g on M.

PROPOSITION 5.1. *The bounded solutions of* (5.1) *are asymptotically homogeneous when t tends to infinity.*

PROOF. The function $w = u - \bar{u}$ satisfies

$$w_{tt} + \Delta_g w - \lambda w = 0 \tag{5.4}$$

on $\mathbb{R}^+ \times M$ which implies that

$$\|w(t,.)\|_{L^\infty} \le Ce^{-t\sqrt{\lambda+\lambda_1}}\|w(0,.)\|_{L^\infty}\,. \tag{5.5}$$

REMARK 5.1. From (5.5) the equation (5.1) is just an exponential perturbation of the differential equation that is actually satisfied by $\bar{u}$

$$\varphi_{tt} - \lambda\varphi + \varphi^3 = 0\,. \tag{5.6}$$

Moreover the boundedness assumption can be replaced by a sub-exponential growth assumption like

$$\|u(t,.)\|_{L^\infty} = o(e^{t\sqrt{\lambda+\lambda_1}})\,. \tag{5.7}$$

PROPOSITION 5.2. *The bounded positive solutions of* (5.2) *are asymptotically homogeneous when t tends to infinity.*

PROOF. The function $w = u - \bar{u}$ satisfies

$$w_{tt} + \Delta_g w - \lambda w + \bar{u}^2 w = 0 \tag{5.8}$$

and

$$\frac{d^2}{dt^2}\|w(t,.)\|^2_{L^2} + 2(\bar{u}^2(t) - \lambda - \lambda_1)\|w(t,.)\|^2_{L^2} \geq 0\,. \tag{5.9}$$

The Fourier decomposition of u gives

$$u(\sigma,t) = \bar{u}(t) + \sum_{k>0}\ \sum_{0\leq j\leq d_k} c_{j,k}(t)\Theta_{j,k}(\sigma) \tag{5.10}$$

and the $c_{j,k} = c_k$ are solutions of

$$c_k'' - (\lambda + \lambda_k - \bar{u}^2)c_k = 0\,. \tag{5.11}$$

As for $\bar{u}$ it satisfies

$$\bar{u}_{tt} - \lambda\bar{u} + \bar{u}^3 = 0 \tag{5.12}$$

and either it is periodic or it tends exponentially to 0 when t tends to infinity. In the first case Floquet's theory can be applied to (5.11): as $\bar{u}$ is a solution of

$$y'' + (\bar{u}^2 - \lambda)y = 0\,, \tag{5.13}$$

this equation possesses a periodic solution and λ is at the limit of a zone of stability in the sense of Proposition 4.2. As $\bar{u}$ is positive, λ is on the boundary of the first stability zone and all the other equations (5.11) are in the instability domain. Therefore, for $k > 0$, there only exists a unique type on bounded solutions for these equations and these solutions are exponentially decaying. In the second case, when $\bar{u}$ is exponentially decaying, the classical exponential perturbation theory can be applied to (5.11) and conclude that all the bounded solutions of (5.11) are exponentially decaying. As a consequence w tends exponentially to 0 and Remark 5.1 still applies.

PROPOSITION 5.3. *The bounded positive solutions of* (5.3) *are asymptotically homogeneous when t tends to infinity.*

PROOF. The average $\bar{u}$ of u satisfies

$$\bar{u}'' - \lambda\bar{u} + \overline{u^2}\bar{u} = 0 \tag{5.14}$$

and the $c_{j,k} = c_k$ are solutions of

$$c_k'' - (\lambda + \lambda_k - \overline{u^2})c_k = 0\,. \tag{5.15}$$

STEP 1. Assume that

$$\lim_{t\to\infty} \bar{u}(t) = 0\,. \tag{5.16}$$

As u is positive and bounded, (5.16) implies that $u(t, .)$ tends to 0 in any $L^p(M)$ space for $p \in [1, \infty)$ when t tends to infinity. Therefore the nonlinear term is negligible in (5.3) and

$$\|u(t, .)\|_{L^\infty} \leq Ke^{-\sigma t} \tag{5.17}$$

for some K and σ, which is the homogeneity property.

STEP 2. Assume that

$$\limsup_{t\to\infty} \bar{u}(t) > 0\,. \tag{5.18}$$

Since u is given by (5.10), there holds

$$\overline{u^2}(t) = \bar{u}^2(t) + \sum_{k>0} \sum_{0\leq j\leq d_k} c_{j,k}^2(t)\,. \tag{5.19}$$

Replacing this value in (5.15) yields

$$\bar{u}'' - \left(\lambda - \bar{u}^2(t) + \sum_{k>0} \sum_{0\leq j\leq d_k} c_{j,k}^2(t)\right)\bar{u} = 0 \tag{5.20}$$

for $k = 0$ and

$$\bar{c}_k'' - \left(\lambda + \lambda_k - \bar{u}^2(t) + \sum_{k>0} \sum_{0\leq j\leq d_k} c_{j,k}^2(t)\right)\bar{c}_k = 0 \tag{5.21}$$

for $k \geq 1$. From the Sturm comparison theorem, between two zeros of c_k there exists one zero of $\bar{u}$. If c_k as two zeros, then $\bar{u}$ has at least one zero which contradicts the fact that u has constant sign. Therefore we can assume that c_k has a constant sign for t large enough and positive without any loss of generality. Multiplying (5.20) by c_k and (5.21) by $\bar{u}$ and substracting gives

$$c_k\bar{u}'' - \bar{u}c_k'' + \lambda_k\bar{u}c_k = 0\,. \tag{5.22}$$

If $A(t) = c_k'\bar{u} - c_k\bar{u}'$, then

$$A(t_0) - A(t) = \lambda_k \int_{t_0}^{t} c_k(\tau)\bar{u}(\tau)d\tau\,. \tag{5.23}$$

Let us consider $t_0 > 0$ such that $c_k(t) > 0$ on $[t_0, \infty)$, then $A(t)$ is decreasing on $[t_0, \infty)$. There are two possibilities:

CASE 1. There exists $t_1 > t_0$ such that $A(t) < 0$ on (t_1, ∞). In that case the function $c_k(t)/\bar{u}(t)$ is increasing and admits a finite or infinite but positive limit ℓ. If $\ell < \infty$; there exists $t_2 > t_1$ such that

$$\left(\frac{c_k(t)}{\bar{u}(t)}\right)' = \frac{\lambda_k \int_{t_1}^{t} c_k(\tau)\bar{u}(\tau)d\tau - A(t_2)}{\bar{u}^2(t)} \geq \lambda_k \frac{\int_{t_1}^{t} \ell\bar{u}^2(\tau)d\tau}{2\bar{u}^2(t)} \geq \delta \tag{5.24}$$

for some $\delta > 0$. Therefore

$$\lim_{t\to\infty} \frac{c_k(t)}{\bar{u}(t)} = \infty\,. \tag{5.25}$$

As $\limsup_{t\to\infty} \bar{u}(t) > 0$, this results in a contradiction. If $\ell = \infty$ the contradiction is the same and the other possibility is left.

CASE 2. for any $t > t_0$, $A(t) > 0$.
Then $c_k(t)/\bar{u}(t)$ is positive and decreasing and

$$A(t_0) > \lambda_k \int_{t_0}^{t} c_k(\tau)\bar{u}(\tau)d\tau\,. \tag{5.26}$$

From the definition of $A(t)$ there holds

$$c_k(t)\bar{u}(t) = \frac{\bar{u}(t)}{c_k(t)}c_k^2(t) > \frac{\bar{u}(t_0)}{c_k(t_0)}c_k^2(t)\,, \tag{5.27}$$

which gives

$$\int_{t_0}^{t} c_k^2(\tau)d\tau \leq \frac{c_k(t_0)}{\bar{u}(t_0)}A(t_0)\lambda_k^{-1} \tag{5.28}$$

for any $t > t_0$. Letting t tend to infinity and summing over k yields

$$\int_{t_0}^{\infty}\int_M (u(t,\sigma) - \bar{u}(t))^2 d\sigma\, dt = \sum_{k\geq 1}\sum_{0\leq j\leq d_k}\int_{t_0}^{\infty} c_{j,k}^2(t)dt < \sum_{k\geq 1} A(t_0)\frac{c_k(t_0)}{\bar{u}(t_0)}\frac{d_k}{\lambda_k}. \tag{5.29}$$

But

$$\frac{c_k(t_0)}{\bar{u}(t_0)}A(t_0) = c_k^2(t_0)\frac{\bar{u}'(t_0)}{\bar{u}(t_0)} - c_k'(t_0)c_k(t_0) \tag{5.30}$$

and this last quantity is bounded independently of k. Therefore

$$\int_{t_0}^{\infty}\int_M (u(t,\sigma) - \bar{u}(t))^2 d\sigma\, dt < \infty\,. \tag{5.31}$$

If $w = u - \bar{u}$, then

$$w_{tt} + \Delta_g w - \lambda w + \overline{u^2}w = 0 \tag{5.32}$$

and from L^p and Schauder estimates the result is

$$\int_0^{\infty}\int_M u_t^2(\tau,\sigma)d\sigma\, dt + \int_0^{\infty}\int_M u_{tt}^2(\tau,\sigma)d\sigma\, dt < \infty\,, \tag{5.33}$$

$$\|w\|_{C^{2,\alpha}((T-1,T+1)\times M)} < C\,, \tag{5.34}$$

independently of T. Therefore $w(t,.)$ tends to 0 in $C^2(M)$ when t tends to infinity which ends the proof.

REMARK 5.2. Using the same construction as the one of Section 4, it can be proved the existence of solutions u of (5.3) defined on $\mathbb{R}^+ \times M$ such that $u(0,\sigma) = u_0(\sigma)$is close enough to the initial data of a solution of the associated differential equation (5.6).

REMARK 5.3. When $M = S^1$ the method of [BV] can be adapted to prove that all the positive solutions of (5.3) on $\mathbb{R}^+ \times M$ are bounded.

REFERENCES

[Au] TH. AUBIN, Nonlinear Analysis on Manifolds, Monge-Ampère Equations, Grundlehren Math. Wiss., Springer, Berlin, 1982.

[AMR] R. ABRAHAM – J. E. MARSDEN – T. RATIU, Manifolds, Tensor Analysis and Applications, Appl. Math. Sci. Springer, Berlin, 1988.

[BGM] M. BERGER – P. GAUDUCHON – E. MAZET, Le spectre d'une variété riemannienne, Lect. Notes Math., **94**, Springer, Berlin, 1971.

[BVB] M. F. BIDAUT-VÉRON – M. BOUHAR, *On characterization of solutions of some nonlinear differential equations and applications*, SIAM J. Math. Anal. **25** (1994), 859-875.

[BVV] M. F. BIDAUT-VÉRON – L. VÉRON, *Nonlinear elliptic equations on compact riemannian manifolds and asymptotics of Emden equations*, Invent. Math. **106** (1991), 489-539.

[BV] M. BOUHAR – L. VÉRON, *Integral representation of solutions of semilinear elliptic equations and applications*, Nonlinear Anal. **23** (1994), 275-296.

[CGS] L. A. CAFFARELLI – B. GIDAS – J. SPRUCK, *Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271-297.

[CL] E. A. CODDINGTON – N. LEVINSON, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, 1955.

[GS1] B. GIDAS – J. SPUCK, *Global and local behaviour of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525-598.

[GS2] B. GIDAS – J. SPUCK, *A priori bound for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), 883-901.

[GT] D. GILBARG – N. S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, Grundlehren Math. Wiss., Springer, Berlin, 1983.

[Ha] H. HAMZA, personal communication.

[LP] J. M. LEE – T. H. PARKER, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37-91.

[Li] A. LICHNEROWICZ, Géométrie des groupes de transformations, Dunod, 1958.

[L1] J. R. LICOIS, *Asymptotiques d'un système dynamique dynamique associé à une équation elliptique non linéaire*, J. Analyse. Math. **66** (1995), 1-36.

[L2] J. R. LICOIS, Systèmes elliptiques non linéaires conservatifs dans des cylindres infinis, Thèse de Doctorat, Univ. F. Rabelais, Tours, 1994.

[LV] J. R. LICOIS – L. VÉRON, *Un théorème d'annulation pour des équations elliptiques non linéaires sur des variétés riemanniennes compactes*, C.R. Acad. Sci. Paris Sér. I Math. **320** (1995), 1337-1342.

[Ob] M. OBATA, *The conjecture on conformal transformations of Riemannian manifolds*, J. Differential Geom. **6** (1971), 247-258.

[OY] M. OBATA – K. YANO, *Sur le groupe de transformation conformes d'une variété de Riemann dont le scalaire de courbure est constant*, C.R. Acad. Sci. Paris Sér. I Math. **260** (1965), 2698-2700.

[Tr] N. S. TRUDINGER, *Remarks concerning the conformal deformation of Riemannian structure on compact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** (1968), 265-274.

[Ve] L. VÉRON, *Equations d'évolution semi-linéaires du second ordre dans L^1*, Rev. Roumaine Math. Pures Appl. **27** (1982), 95-123.

[WW] E. T. WHITTAKER – G. N. WATSON, A Course of Modern Analysis, Cambridge Univ. Press, 1927.

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