

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 26, n° 2 (1998), p. 207-232

http://www.numdam.org/item?id=ASNSP_1998_4_26_2_207_0

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Duality in the Spaces of Solutions of Elliptic Systems

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Introduction

The aim of this paper is to give representations of the strong dual of the space of solutions of a linear elliptic system $Pu = 0$ of partial differential equations on an open subset of \mathbb{R}^n . We consider both determined and overdetermined elliptic systems.

Let U be an open subset of the domain $X \subset \mathbb{R}^n$ where the operator P is defined. Denote by $\text{sol}(U, P)$ the vector space of all smooth solutions to the equation $Pu = 0$ on U , with the usual Fréchet-Schwartz topology. We will write it simply $\text{sol}(U)$ when no confusion can arise.

Denote by $\text{sol}(U)'$ the dual space of $\text{sol}(U)$, i.e., the space of all continuous linear functionals on $\text{sol}(U)$. We tacitly assume that this dual space $\text{sol}(U)'$ is endowed with the *strong topology*, i.e., the topology of uniform convergence on every bounded subset of $\text{sol}(U)$.

Any successful characterization of the dual space $\text{sol}(U)'$ results in the analysis of solutions to $Pu = 0$ (*Golubev series*, etc., see Havin [4], Tarkhanov [15]).

There are a few classical examples of representation of this dual space, such as *Grothendieck duality* and *Poincaré duality* (see for instance Tarkhanov [16, Ch. 5]). The Grothendieck duality is of analytical nature; it has been of particular interest in complex analysis. On the other hand, the Poincaré duality can be stated in an abstract framework.

For determined elliptic operators of the type P^*P we obtain in Section 3 an analogue of the duality result of Grothendieck [3] (cf. Mantovani and Spagnolo [7]). Note that the system $P^*Pu = 0$ is a straightforward generalization of the *Laplace equation*. In this way we obtain what we shall call generalized harmonic functions, or simply *harmonic functions* when no confusion can arise.

Our main result for general elliptic systems is concerned with the case where the coefficients of P are real analytic and U is a relatively compact subdomain of X with real analytic boundary. In this case we prove the following theorem.

THEOREM A. *Let the coefficients of the operator P be real analytic on X and $\mathcal{D} \Subset X$ be a domain with real analytic boundary. Suppose that, given any neighborhood U of $\overline{\mathcal{D}}$, there is a neighborhood $U' \subset U$ of $\overline{\mathcal{D}}$ such that $\text{sol}(U')$ is dense in $\text{sol}(\mathcal{D})$. Then*

$$\text{sol}(\mathcal{D})' \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}).$$

In fact, in Sections 6, 7 below, we will formulate and prove a stronger statement with weaker assumptions on analyticity. Moreover, in these sections we provide also an explicit formula for the pairing.

In fact, there is a transparent heuristic explanation of this duality. Given any solution $v \in \text{sol}(\overline{\mathcal{D}})$, the *Petrovskii theorem* shows that v is real analytic in a neighborhood of $\overline{\mathcal{D}}$. On the other hand, each $u \in \text{sol}(\mathcal{D})$ is real analytic in \mathcal{D} , and so u is a hyperfunction there. As the sheaf of hyperfunctions is *flabby*, u can be extended to a hyperfunction in X with support in the closure of \mathcal{D} . Thus, v can be paired with every $u \in \text{sol}(\mathcal{D})$.

By *Runge theorem*, the approximation assumption of Theorem A holds for every determined elliptic operator with real analytic coefficients or in the case where P is an elliptic operator with constant coefficients and \mathcal{D} is convex.

The approximation condition on the couple P and \mathcal{D} in this theorem is to some extent an analogue of the so-called *approximation property* introduced by Grothendieck [3]. In several complex variables a close concept is known as *Runge property* (cf. Hörmander [5]).

For the space of holomorphic functions in simply connected domains in \mathbb{C} and in (p, q) -circular domains in \mathbb{C}^2 a similar result was obtained by Aizenberg and Gindikin [1]. For the spaces of harmonic and holomorphic functions a similar result was recently obtained by Stout [13]. However they constructed isomorphisms different from ours. The advantage of our approach is the fact that it highlights the close connection between the duality of Theorem A and the Grothendieck duality (see Section 3).

Acknowledgment. The authors thank Prof. E. Vesentini for many useful discussions.

1. – Preliminaries

Assume that X is an open set in \mathbb{R}^n , and $E = X \times \mathbb{C}^k$, $F = X \times \mathbb{C}^l$ are (trivial) vector bundles over X . Sections of E and F of a class \mathfrak{C} on an open set $U \subset X$ can be interpreted as columns of complex valued functions from $\mathfrak{C}(U)$, that is, $\mathfrak{C}(E|_U) \cong [\mathfrak{C}(U)]^k$, and similarly for F .

Throughout the paper we will usually write the letters u, v for sections of E , and f, g for sections of F .

A differential operator P of order $p \geq 1$ and type $E \rightarrow F$ can be written in the form $P(x, D) = \sum_{|\alpha| \leq p} P_\alpha D^\alpha$, with suitable $(l \times k)$ -matrices $P_\alpha(x)$ of smooth functions on X .

The *principal symbol* $\sigma(P)$ of P is a function on the cotangent bundle of X with values in the space of bundle morphisms $E \rightarrow F$. Given any $(x, \xi) \in X \times \mathbb{R}^n$, we have $\sigma(P)(x, \xi) = \sum_{|\alpha|=p} P_\alpha(x) \xi^\alpha$.

We say that P is *elliptic* if the mapping $\sigma(P)(x, \xi) : \mathbb{C}^k \rightarrow \mathbb{C}^l$ is injective for every $x \in X$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Hence it follows that $l \geq k$; we say that P is *determined elliptic* if $l = k$, and *overdetermined elliptic* if $l > k$.

Every elliptic operator is hypoelliptic, i.e. all distribution sections satisfying $Pu = 0$ on an open set U of X are infinitely differentiable there. If U is an open subset of X , then we denote by $\text{sol}(U, P)$ the vector space of all C^∞ solutions of the equation $Pf = 0$ on U . We will write it simply $\text{sol}(U)$ when no confusion can arise.

We endow the space $\text{sol}(U)$ with the topology of uniform convergence on compact subsets of U . This topology is generated by the family of seminorms

$$\|u\|_{C(E|_K)} = \sup_{x \in K} |u(x)|,$$

where K runs over all compact subsets of U .

LEMMA 1.1. *If $U \subset X$ is open, then the topology in $\text{sol}(U)$ coincides with that induced by $C_{\text{loc}}^\infty(E|_U)$. In particular, $\text{sol}(U)$ is a Fréchet-Schwartz space.*

PROOF. By a priori estimates for solutions of elliptic equations, if K' and K'' are compact subsets of U and K' is a subset of the interior of K'' , then

$$(1.1) \quad \sup_{|\alpha| \leq j} \|D^\alpha u\|_{C(E|_{K'})} \leq c \|u\|_{C(E|_{K''})} \text{ for all } u \in \text{sol}(U),$$

with a constant c depending only on K' , K'' and j . Hence it follows that the original topology on $\text{sol}(U)$ coincides with that induced by $C_{\text{loc}}^\infty(E|_U)$. To finish the proof we use the fact that $C_{\text{loc}}^\infty(E|_U)$ is a Fréchet-Schwartz space. \square

Throughout this paper we assume that the operator P possesses the following Unique Continuation Property:

$$(U)_s \quad \text{given any domain } \mathcal{D} \subset X, \text{ if } u \in \text{sol}(\mathcal{D}) \text{ vanishes on a non-empty open subset of } \mathcal{D}, \text{ then } u \equiv 0 \text{ on } \mathcal{D}.$$

Here and in the sequel, by a domain is meant any open connected subset of \mathbb{R}^n . This property holds, for instance, if the coefficients of the operator P are real analytic.

It is natural to consider solutions of the system $Pu = 0$ on open sets. However, some problems require to consider solutions on sets $\sigma \subset X$ which are not open. Here we are interested not simply in restrictions of solutions to the given set, but also in the *local solutions* of the system $Pu = 0$ on σ , that is, solutions of the system in some (open) neighborhoods of σ .

If σ is a closed subset of X , then $\text{sol}(\sigma)$ stands for the space of (equivalence classes of) local solutions of $Pu = 0$ on σ . Two such solutions are *equivalent* if there is a neighborhood of σ where they are equal. In $\text{sol}(\sigma)$, a sequence $\{u_\nu\}$ is said to converge if there exists a neighborhood \mathcal{N} of σ such that all the solutions are defined at least in \mathcal{N} and converge uniformly on compact subsets of \mathcal{N} .

Alternatively $\text{sol}(\sigma)$ can be described as the inductive limit of the spaces $\text{sol}(U_\nu)$, where $\{U_\nu\}$ is any decreasing sequence of open sets containing σ such that each neighborhood of σ contains some U_ν and such that each connected component of each U_ν intersects σ . (This latter condition guarantees that the maps $\text{sol}(U_\nu) \rightarrow \text{sol}(\sigma)$ are injective. Then the space $\text{sol}(\sigma)$ is necessarily a Hausdorff space.)

LEMMA 1.2. *Let the operator P possess the Unique Continuation Property $(U)_s$. Then the space $\text{sol}(\sigma)$ is separated, a subset is bounded if and only if it is contained and bounded in some $\text{sol}(U_\nu)$, and each closed bounded set is compact.*

PROOF. This follows by the same method as in Köthe [6, p. 379]. \square

2. – Green's function

Denote by $E^* = X \times (\mathbb{C}^k)'$ the conjugate bundle of E , and similarly for F . For the operator P , we define the transpose P' as usual, so that P' is a differential operator of type $F^* \rightarrow E^*$ and order p on X .

Fix the standard Hermitian structure in the fibers $E_x = \mathbb{C}^k$ ($x \in X$) of E : $(u, v)_x = \sum_{j=1}^k u_j \bar{v}_j$ for $u, v \in \mathbb{C}^k$. This gives the conjugate linear bundle isomorphism $\star_E : E \rightarrow E^*$ by $\langle \star_E v, u \rangle_x = (u, v)_x$ for $u, v \in E_x$.

Using matrix operation conventions, we have $\langle \star_E v, u \rangle_x = v^* u$ for $u \in \mathbb{C}^k$ where v^* is the conjugate matrix: we have $\star_E v = v^*$ under this identification.

The operator \star_E also acts on sections of E via $(\star_E u)(x) = \star_E(u(x))$ for all $x \in X$. Thus, for a class \mathcal{C} of sections of E we have $\star_E : \mathcal{C}(E) \rightarrow \mathcal{C}(E^*)$.

The operator \star_E is similar to Hodge's *star operator* on differential forms. We write simply \star when no confusion can arise.

We are now in a position to endow the spaces $C_{\text{comp}}^\infty(E)$ and $C_{\text{comp}}^\infty(F)$, consisting of infinitely differentiable sections with compact supports of E and F respectively, with (L^2-) pre-Hilbert structures by $(u, v)_X = \int_X \langle \star v, u \rangle_x dx$.

Under these structures, the operator P has a formal adjoint operator which is denoted by P^* . This is the differential operator of type $F \rightarrow E$ and order p on X given by $P^*g(x) = \sum_{|\alpha| \leq p} D^\alpha (P_\alpha(x)^* g(x))$ for $g \in C_{\text{comp}}^\infty(F)$.

The relation between the transposed operator and its (formal) adjoint becomes clear by using the bundle isomorphism \star . Namely, $P^* = \star_E^{-1} P' \star_F$ (see Tarkhanov [15, 4.1.4] for more details).

The operator $\Delta = P^*P$ is usually referred to as the generalized *Laplacian* associated to P . It is easy to see that Δ is an elliptic differential operator of type $E \rightarrow E$ and order $2p$ on X .

Throughout the paper we shall assume that the operator Δ possesses the Unique Continuation Property $(U)_s$. Obviously, this implies that P does so.

If P is the gradient operator in \mathbb{R}^n , then $\Delta = P^*P$ is the usual Laplace operator up to a -1 factor. On the other hand, if P is the Cauchy-Riemann operator in \mathbb{C}^n , then $\Delta = P^*P$ coincides with the usual Laplace operator on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ up to a $-1/4$ factor.

In the general case, the solutions of the system $\Delta u = 0$ are said to be generalized *harmonic functions*.

Let $\mathcal{O} \Subset X$ be a domain with C^∞ boundary. Denote by $n(x)$ the unit outward normal vector to the boundary $\partial\mathcal{O}$ at a point x . The system of boundary operators $\{(\partial/\partial n)^j\}_{j=0,1,\dots,p-1}$ is known to be a Dirichlet system of order $p-1$ on $\partial\mathcal{O}$.

We formulate the Dirichlet problem for the generalized Laplacian Δ in the following way.

PROBLEM 2.1. Given a section f of E over \mathcal{O} , find a section u of E over \mathcal{O} such that $\Delta u = f$ in \mathcal{O} and $(\partial/\partial n)^j u = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \dots, p-1$.

As in the classical case, Problem 2.1 is verified to be an elliptic boundary value problem. Moreover, it is formally selfadjoint and possesses at most one solution in reasonable function spaces for u . So, this problem may be treated by standard tools in the scale $\{H^s(E|_{\mathcal{O}})\}_{s \in \mathbb{R}}$ of Sobolev spaces on \mathcal{O} (see Roitberg [11]).

From this treatment, we briefly sketch the relevant material on Green's function. For more details we refer the reader to Roitberg [11] and Tarkhanov [15, 9.3.8].

It turns out that the inverse of the operator corresponding to Problem 2.1 is integral. Namely, there exists a unique kernel $\mathcal{G}(x, y)$ on $\mathcal{O} \times \mathcal{O}$ such that, for each data $f \in H^{s-2p}(E|_{\mathcal{O}})$, the function

$$(2.1) \quad u(x) = \int_{\mathcal{O}} \mathcal{G}(x, y) f(y) dy \quad (x \in \mathcal{O})$$

belongs to $H^s(E|_{\mathcal{O}})$ and satisfies $\Delta u = f$ in \mathcal{O} and $(\partial/\partial n)^j u = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \dots, p-1$. Such a kernel $\mathcal{G}(x, y)$ is said to be the *Green's function* for Problem 2.1.

We will later give a precise meaning to the integrals in (2.1), specifying to which spaces the Green's function belongs.

The Green's function $\mathcal{G}(\cdot, y)$ is alternatively defined as the solution to the Dirichlet problem with the data $f = \delta_y$, the Dirac *delta-function* supported at $y \in \mathcal{O}$. This data is easily verified to belong to all Sobolev spaces $H^s(\mathcal{O})$ with $s < -\frac{n}{2}$.

THEOREM 2.2. *The kernel \mathcal{G} is a C^∞ section of the bundle $E \otimes E^*|_{\overline{\mathcal{O}} \times \overline{\mathcal{O}}}$ away from the diagonal of $\overline{\mathcal{O}} \times \overline{\mathcal{O}}$.*

PROOF. See Roitberg [11, 7.4]. □

A discussion of the singularity of $\mathcal{G}(x, y)$ at the diagonal $\{(x, x) : x \in \overline{\mathcal{O}}\}$ can be found in Roitberg [11, Th. 7.4.3]. For our purpose, it suffices to know that the mapping (2.1), when restricted to $f \in C_{\text{comp}}^{\infty}(E|_{\mathcal{O}})$, is a pseudodifferential operator of type $E|_{\mathcal{O}} \rightarrow E|_{\mathcal{O}}$ and order $-2p$. Thus, if f is sufficiently smooth, the integral in (2.1) is actually a usual Lebesgue integral.

Green's formula enables us to prove that the Green's function is a solution of the adjoint boundary value problem in the y variable. To explain this more accurately, denote by I_k the identity ($k \times k$)-matrix.

THEOREM 2.3. *Given any $x \in \mathcal{O}$, we have:*

$$(2.2) \quad \begin{cases} \Delta'(y, D)\mathcal{G}(x, y) = \delta_x(y)I_k & \text{for } y \in \mathcal{O}, \\ (\partial/\partial n(y))^j \mathcal{G}(x, y) = 0 & \text{for } y \in \partial\mathcal{O} \quad (j = 0, 1, \dots, p-1). \end{cases}$$

PROOF. See Tarkhanov [15, Th. 9.3.24]. □

We are now in a position to state the symmetry of Green's function in the variables x and y . This symmetry could be expected from the fact that the Dirichlet problem is (formally) selfadjoint.

COROLLARY 2.4. *The matrix $\mathcal{G}(x, y)$ is Hermitian, i.e., $\mathcal{G}(x, y)^* = \mathcal{G}(y, x)$ for all $x, y \in \overline{\mathcal{O}}$.*

PROOF. Indeed, since the solution of Problem 2.1 is unique, it follows from Theorem 2.3 that

$$\begin{aligned} \mathcal{G}(y, x) &= \star_x \mathcal{G}(x, y) \star_y^{-1} \\ &= \mathcal{G}(x, y)^*, \end{aligned}$$

as desired. □

3. – Grothendieck duality for harmonic functions

In the sequel, we shall denote by \mathcal{O} a fixed relatively compact domain in X with C^{∞} boundary $\partial\mathcal{O}$, as in Section 2.

Inspired by the work of Grothendieck [3] who used solutions to $\Delta v = 0$ at infinity, we shall consider the manifold with boundary $\widehat{\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O}$ as the compactification of \mathcal{O} .

We use $\widehat{\mathcal{O}}$ instead of $\overline{\mathcal{O}}$ to conceptually distinguish this manifold with boundary from the closed subset $\overline{\mathcal{O}}$ of X .

The topology of $\widehat{\mathcal{O}}$ is given by the following neighborhoods bases:

- If $x \in \mathcal{O}$, then we take the usual basis of neighborhoods of x (for example, the family $\{B \cap \mathcal{O}\}$, where B runs over all balls in X centered at x).

- If $x \in \partial\mathcal{O}$, then the basis of neighborhoods of x is defined to be the family $\{B \cap (\mathcal{O} \cup \partial\mathcal{O})\}$, where B runs over all balls in X centered at x .

We shall say that an open set U in $\widehat{\mathcal{O}}$ is a neighborhood of infinity if U contains the part at infinity $\partial\mathcal{O}$ of $\widehat{\mathcal{O}}$.

We shall also need the concept of a solution to $\Delta u = 0$ in a neighborhood $B \cap (\mathcal{O} \cup \partial\mathcal{O})$ of a point $x \in \partial\mathcal{O}$.

By this, we mean any solution to $\Delta u = 0$ on $B \cap \mathcal{O}$ (finite part) which is C^∞ up to $B \cap \partial\mathcal{O}$ (infinite part) and satisfies $(\partial/\partial n)^j u = 0$ on $B \cap \partial\mathcal{O}$ for $j = 0, 1, \dots, p - 1$.

Given an open set $U \subset \widehat{\mathcal{O}}$, denote by $\text{sol}(U, \Delta)$ the set of all solutions to $\Delta u = 0$ on U .

LEMMA 3.1. *Let U be a neighborhood of infinity in $\widehat{\mathcal{O}}$. Then $\text{sol}(U, \Delta)$ is a closed subspace of $\text{sol}(U \cap \mathcal{O}, \Delta)$.*

PROOF. Pick a sequence $\{u_\nu\}$ in $\text{sol}(U, \Delta)$ converging to a solution u_∞ in $\text{sol}(U \cap \mathcal{O}, \Delta)$. We shall establish the lemma by proving that u_∞ is C^∞ up to the boundary of \mathcal{O} and $(\partial/\partial n)^j u_\infty = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \dots, p - 1$.

To this end, let U' be a sufficiently thin open band close to the boundary in \mathcal{O} , so that $\partial\mathcal{O} \subset \partial U'$ and $U' \Subset U$. We can certainly assume that the boundary of U' is of class C^∞ .

By the above, the Dirichlet problem for the Laplacian in U' is coercive. Hence for any integer $s \geq p$ there is a constant c such that

$$(3.1) \quad \|u\|_{H^s(E|_{U'})} \leq c \left(\sum_{j=0}^{p-1} \|(\partial/\partial n)^j u\|_{H^{s-j-1/2}(E|_{\partial U'})}^2 \right)^{\frac{1}{2}}$$

whenever $u \in H^s(E|_{U'}) \cap \text{sol}(U', \Delta)$.

Let us apply this estimate to a solution $u \in \text{sol}(U, \Delta)$. Since the normal derivatives of u up to order $p - 1$ vanish on the part $\partial\mathcal{O}$ of the boundary of U' , we can assert that the norm of u in $H^s(E|_{U'})$ is dominated by Sobolev norms of the normal derivatives of u up to order $p - 1$ on the remaining part of the boundary of U' . What is especially important here is that this remaining part $\partial U' \setminus \partial\mathcal{O}$ is a subset in $U \cap \mathcal{O}$. Hence combining the *Sobolev embedding theorem* with the interior a priori estimates (1.1) yields

$$(3.2) \quad \sup_{|\alpha| \leq j} \|D^\alpha u\|_{C(E|_{\overline{U'}})} \leq c \|u\|_{C(E|_K)} \quad \text{for all } u \in \text{sol}(U, \Delta),$$

with K a compact subset of $U \cap \mathcal{O}$, whose interior contains $\partial U' \setminus \partial\mathcal{O}$, and c a constant depending only on U' , k and j .

We can now return to the sequence $\{u_\nu\}$. It follows from (3.2) that, given any multi-index α , the sequence of derivatives $\{D^\alpha u_\nu\}$ is a Cauchy sequence in $C(E|_{\overline{U'}})$. Therefore $\{u_\nu\}$ converges to a section $u \in C^\infty(E|_{\overline{U'}})$ uniformly on $\overline{U'}$ and together with all derivatives.

Obviously, $u_\infty = u$ in U' . This shows at once that u_∞ is C^∞ up to the boundary of \mathcal{O} and $(\partial/\partial n)^j u_\infty = 0$ on $\partial\mathcal{O}$ for $j = 0, 1, \dots, p - 1$, as desired. \square

In the case where U is an open subset of $\widehat{\mathcal{O}}$ containing $\partial\mathcal{O}$ we endow $\text{sol}(U, \Delta)$ with the topology induced by $\text{sol}(U \cap \mathcal{O}, \Delta)$. Then Lemmas 1.1 and 3.1 show that $\text{sol}(U, \Delta)$ is a Fréchet-Schwartz space. (For the moment we shall say nothing about the topology of $\text{sol}(U, \Delta)$ in the general case.)

We now invoke the construction of the *inductive limit* of a sequence of Fréchet spaces in order to define the space $\text{sol}(\sigma, \Delta)$ also for those closed sets σ in $\widehat{\mathcal{O}}$ which are “approximable” by open subsets of $\widehat{\mathcal{O}}$ containing $\partial\mathcal{O}$. These are nothing but the close subsets of $\widehat{\mathcal{O}}$ containing the “infinitely far” hypersurface $\partial\mathcal{O}$.

Next we fix a *Green operator* G_P for the differential operator P . By definition, G_P is a bidifferential operator of type $(F^*, E) \rightarrow \Lambda^{n-1}T^*(X)$ (where $\Lambda^{n-1}T^*(X)$ is the bundle of exterior forms of degree $n - 1$ on X) and order $p - 1$, such that $dG_P(\star g, u) = ((Pu, g)_x - (u, P^*g)_x)dx$ pointwise on X , for all smooth sections g of F and u of E .

We immediately obtain:

LEMMA 3.2. *A Green operator for the Laplacian Δ is given by*

$$(3.3) \quad G_\Delta(\star v, u) = G_P(\star Pv, u) - \overline{G_P(\star Pu, v)}.$$

Having disposed of these preliminary steps, we fix now an open subset U of \mathcal{O} and turn to describing the dual space for $\text{sol}(U, \Delta)$.

Given any solution $v \in \text{sol}(\mathcal{O} \setminus U, \Delta)$, we define a linear functional \mathcal{F}_v on $\text{sol}(U, \Delta)$ as follows.

There is an open set $\mathcal{N}_v \Subset U$ with piecewise smooth boundary such that v is still defined and satisfies $\Delta v = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}_v$. Put

$$(3.4) \quad \langle \mathcal{F}_v, u \rangle = \int_{\partial\mathcal{N}_v} G_\Delta(\star v, u) \quad u \in \text{sol}(U, \Delta).$$

It follows from Stokes’ formula that the value $\langle \mathcal{F}_v, u \rangle$ is independent of the particular choice of \mathcal{N}_v with the properties previously mentioned.

LEMMA 3.3. *The functional \mathcal{F}_v defined by (3.4) is a continuous linear functional on the space $\text{sol}(U, \Delta)$.*

PROOF. Use estimate (1.1) with $K' = \partial\mathcal{N}_v$ and $j = 2p - 1$. □

The following result is related to the work of Grothendieck [3] where the concept of solution to $\Delta v = 0$ regular at the point of infinity of the one-point compactification of \mathcal{O} was used.

THEOREM 3.4. *Let the operator P^*P possess the Unique Continuation Property $(U)_s$ on X . Then for each open set $U \subset \mathcal{O}$, the correspondence $v \mapsto \mathcal{F}_v$ induces a topological isomorphism*

$$\text{sol}(U, \Delta)' \stackrel{\text{top.}}{\cong} \text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta).$$

PROOF. Pick a continuous linear functional \mathcal{F} on $\text{sol}(U, \Delta)$. Since $\text{sol}(U, \Delta)$ is a subspace of $C_{\text{loc}}(E|_U)$, the space of continuous sections of E over U , this functional can be extended, by the Hahn-Banach theorem, to an E^* -valued measure m with compact support in U . Set $K = \text{supp } m$.

Let $\mathcal{N} \Subset U$ be any open set with piecewise smooth boundary such that $K \subset \mathcal{N}$. For each solution $u \in \text{sol}(U, \Delta)$, we have, by Green's formula,

$$u(x) = - \int_{\partial \mathcal{N}} G_{\Delta}(\mathcal{G}(x, y), u(y)) \quad (x \in \mathcal{N}).$$

(Here $\mathcal{G}(x, y)$ is the Green's function of the Dirichlet problem for the Laplacian in \mathcal{O} , as in Section 2.) Therefore

$$\langle \mathcal{F}, u \rangle = \int_U \langle dm, u \rangle_x = \int_{\partial \mathcal{N}} G_{\Delta}(\star v, u),$$

where $v(y) = - \star_y^{-1} \int_U \langle dm, \mathcal{G}(\cdot, y) \rangle_x$.

Now we look more closely at the properties of this function v called the "Fantappiè indicatrix" of \mathcal{F} . Since $\Delta'(y, D)\mathcal{G}(x, y) = \delta_x(y)I_k$, we deduce that $\Delta v = 0$ away from K .

Moreover, Theorems 2.2 and 2.3 show that v is C^∞ up to the boundary of \mathcal{O} and satisfies $(\partial/\partial n)^j v = 0$ on $\partial \mathcal{O}$ for $j = 0, 1, \dots, p - 1$.

From what has already been proved, it follows that $v \in \text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$ and $\mathcal{F} = \mathcal{F}_v$. Our next claim is that such v is unique.

To this end, we let $v \in \text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$ satisfy

$$(3.5) \quad \int_{\partial \mathcal{N}_v} G_{\Delta}(\star v, u) = 0 \quad \text{for all } u \in \text{sol}(U, \Delta),$$

where $\mathcal{N}_v \Subset U$ is an open set with piecewise smooth boundary, such that v is still defined and satisfies $\Delta v = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}_v$.

We represent v in the complement of \mathcal{N}_v by Green's formula. This is possible because of $(\partial/\partial n)^j v = 0$ on $\partial \mathcal{O}$ for $j = 0, 1, \dots, p - 1$. We get

$$v(y) = - \star_y^{-1} \int_{\partial \mathcal{N}_v} G_{\Delta}(\star v(x), \mathcal{G}(x, y)) \quad \text{for } y \in \mathcal{O} \setminus \overline{\mathcal{N}_v}.$$

For any fixed $y \in \mathcal{O} \setminus U$, we have $\mathcal{G}(\cdot, y) \in \text{sol}(U, \Delta)$, and so $v(y) = 0$ by condition (3.5). Since the operator P^*P possesses the Unique Continuation Property $(U)_s$, $v \equiv 0$ if $\overline{U} \subset \mathcal{O}$. To complete the proof in the case where \overline{U} is not contained in \mathcal{O} , we use the *Runge theorem* for solutions of the equation $\Delta u = 0$ (cf. Tarkhanov [15, 5.1.6]).

There exists an open set $\mathcal{N} \Subset U$ with the following properties:

- $\mathcal{N}_v \Subset \mathcal{N}$, and
- the complement of \mathcal{N} has no compact connected components in U .

(The second property can always be achieved by adding all compact connected components of $U \setminus \mathcal{N}$ to \mathcal{N} .)

Fix $y \in \mathcal{O} \setminus \mathcal{N}$. Then each column of the matrix $\mathcal{G}(\cdot, y)$ is in $\text{sol}(\mathcal{N}, \Delta)$. According to the *Runge theorem*, it can be approximated uniformly on compact subsets of \mathcal{O} by solutions in $\text{sol}(U, \Delta)$. Let $\{u_\nu\}$ be a resulting sequence for $\mathcal{G}(\cdot, y)$, so that the columns of u_ν belong to $\text{sol}(\mathcal{N}, \Delta)$ and $u_\nu \rightarrow \mathcal{G}(\cdot, y)$ uniformly on compact subsets of \mathcal{O} .

Applying (1.1) we can assert that the derivatives up to order $p - 1$ of u_ν also converge to the corresponding derivatives of $\mathcal{G}(\cdot, y)$ uniformly on compact subsets of \mathcal{N} . Therefore,

$$v(y) = - \lim_{\nu \rightarrow \infty} \int_{\partial \mathcal{N}_\nu} G_\Delta(\star v, u_\nu) = - \lim_{\nu \rightarrow \infty} 0 = 0.$$

Thus, $v = 0$ in $\mathcal{O} \setminus \mathcal{N}$, i.e., v is the zero element of $\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$.

We have proved that the correspondence $v \mapsto \mathcal{F}_v$ induces the isomorphism of vector spaces

$$\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta) \xrightarrow{\cong} \text{sol}(U, \Delta)'.$$

We are now going to invoke an operator-theoretic argument to conclude that this algebraic isomorphism is in fact a topological one.

To this end, we note that the spaces $\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$ and $\text{sol}(U, \Delta)'$ are both spaces of type *DFS*. (For $\text{sol}(\widehat{\mathcal{O}} \setminus U, \Delta)$, see the proof of Theorem 1.5.5 in Morimoto [8, p. 13]. For $\text{sol}(U, \Delta)'$, see Lemma 1,1 above.) As the *Closed Graph theorem* is correct for linear maps between spaces of type *DFS* (see Corollary A.6.4. in Morimoto [8, p. 254]), to see that $v \mapsto \mathcal{F}_v$ is a topological isomorphism, it suffices to show that it is continuous. This latter conclusion, however, is obvious from the way the inductive limit topology is defined, and the construction of \mathcal{F}_v . This completes the proof. \square

One may conjecture that Theorem 3.4 is still true for *arbitrary* open sets U in $\widehat{\mathcal{O}}$. But we have not been able to prove this.

4. – A corollary

In this section we derive the following consequence of Theorem 3.4.

COROLLARY 4.1. *Let $\mathcal{D} \in \mathcal{O}$ be a domain with real analytic boundary. Assume that the operator Δ satisfies the Unique Continuation Property $(U)_s$ on X and its coefficients are real analytic in a neighborhood of the boundary of \mathcal{D} . Then it follows that*

$$(4.1) \quad \text{sol}(\mathcal{D}, \Delta)' \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}, \Delta).$$

Before proving this corollary, we briefly discuss a result of Morrey and Nirenberg [9] to be used in the proof.

THEOREM 4.2. *Let Δ be a determined strongly elliptic differential operator of order $2p$ with real analytic coefficients on X . Assume that u is a solution to $\Delta u = 0$ in a domain $\mathcal{D} \subset X$. If u vanishes up to order $p - 1$ on an open real analytic portion S of the boundary of \mathcal{D} , then for each point $x_0 \in S$ there is a neighborhood $\mathcal{N}(x_0)$ on X depending only on the operator Δ and the domain near x_0 , such that u may be extended as a solution of $\Delta u = 0$ from $\mathcal{N}(x_0) \cap \mathcal{D}$ to the whole neighborhood $\mathcal{N}(x_0)$.*

PROOF. See Morrey and Nirenberg [9]. □

The important point to note is that the neighborhood $\mathcal{N}(x_0)$ in Theorem 4.2 is independent of the particular solution u .

In fact, Morrey and Nirenberg [9] proved the existence of $\mathcal{N}(x_0)$ by showing that there is a real $r > 0$ such that, for any $u \in \text{sol}(\mathcal{D}, \Delta)$ vanishing up to order $p - 1$ on S , the Taylor series of u at x_0 converges in the ball $B(x_0, r)$. Thus, the solution u holomorphically extends to a neighborhood $\tilde{\mathcal{N}}_{x_0}$ of x_0 in \mathbb{C}^n .

We are going to apply this corollary in the case where $\Delta = P^*P$ is the generalized Laplacian. To this end, we have to verify that the Laplacian is strongly elliptic (this notion becomes clear below).

LEMMA 4.3. *If P is an elliptic differential operator of order p , then the operator $\Delta = P^*P$ is strongly elliptic of order $2p$.*

PROOF. What has to be proved is that, given any non-zero vector $v \in \mathbb{C}^k$, we have

$$\text{Re } v^* \sigma(\Delta)(x, \xi) v \neq 0 \quad \text{for all } (x, \xi) \in X \times (\mathbb{R}^n \setminus \{0\}).$$

Suppose the lemma is false. Then there is a non-zero vector $v \in \mathbb{C}^k$ such that $\text{Re } v^* \sigma(\Delta)(x, \xi) v = 0$ for some $(x, \xi) \in X \times (\mathbb{R}^n \setminus \{0\})$. However,

$$\begin{aligned} \text{Re } v^* \sigma(\Delta)(x, \xi) v &= \text{Re}(\sigma(P)(x, \xi)v)^*(\sigma(P)(x, \xi)v) \\ &= |\sigma(P)(x, \xi)v|^2, \end{aligned}$$

and so $v = 0$ because $\sigma(P)(x, \xi) : \mathbb{C}^k \rightarrow \mathbb{C}^l$ is injective. This contradicts our assumption. □

We also need a slightly modified version of Theorem 4.2, a version which relates to *inhomogeneous* elliptic boundary value problems.

LEMMA 4.4. *We keep the assumptions of Theorem 4.2. Let $\{B_j\}_{j=0,1,\dots,p-1}$ be a Dirichlet system of order $p - 1$ with real analytic coefficients on S . If the Dirichlet data $u_j = B_j u|_S$ ($j = 0, 1, \dots, p - 1$) of a solution u to $\Delta u = 0$ in \mathcal{D} are real analytic on S , then for each point $x_0 \in S$ there exists a neighborhood $\mathcal{N}(x_0)$ on X depending only on Δ , the domain \mathcal{D} near x_0 and $\{u_j\}$, such that u may be extended to a solution of $\Delta u = 0$ on $\mathcal{N}(x_0)$.*

PROOF. For $j = p, p+1, \dots, 2p-1$, set $B_j = (\partial/\partial n)^j$, the j -th derivative along the unit outward normal vector to \mathcal{S} . This completes $\{B_j\}_{j=0,1,\dots,p-1}$ to a Dirichlet system of order $2p-1$ with real coefficients on \mathcal{S} .

By the *Cauchy-Kovalevskaya theorem*, there is a unique solution u' to the *Cauchy problem*

$$(4.2) \quad \begin{cases} \Delta u' = 0 & \text{in } \mathcal{N}, \\ B_j u' = u_j & \text{on } \mathcal{S} \quad (j = 0, 1, \dots, p-1), \\ B_j u' = 0 & \text{on } \mathcal{S} \quad (j = p, p+1, \dots, 2p-1), \end{cases}$$

defined on some neighborhood \mathcal{N} of \mathcal{S} in X . (We observe at once that u' is real analytic in \mathcal{N} .)

Let \mathcal{N}_{x_0} be the neighborhood of x_0 which is guaranteed by Theorem 4.2. We can certainly assume that u' is defined in \mathcal{N}_{x_0} , for if not, we replace \mathcal{N}_{x_0} by $\mathcal{N}_{x_0} \cap \mathcal{N}$.

By (4.2), the difference $u'' = u - u'$ satisfies the equation $\Delta u'' = 0$ in $\mathcal{D} \cap \mathcal{N}$ and vanishes up to order $p-1$ on \mathcal{S} .

Repeated application of Theorem 4.2 enables us to assert that there is a neighborhood of x_0 on X depending only on Δ and the domain $\mathcal{D} \cap \mathcal{N}$ near x_0 , such that u'' may be extended to a solution of $\Delta u'' = 0$ in this neighborhood. To shorten notation, we continue to write \mathcal{N}_{x_0} for this new neighborhood. Obviously, $u = u' + u''$ extends to \mathcal{N}_{x_0} , and the lemma follows. \square

We are now able to prove Corollary 4.1.

PROOF. By Theorem 3.4, we establish the corollary if we prove that

$$(4.3) \quad \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta) \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}, \Delta).$$

To this end, define a mapping $\mathcal{E} : \text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ in the following way (cf. Tarkhanov [15, 10.2.3]).

Given any $u \in \text{sol}(\overline{\mathcal{D}}, \Delta)$, there exists a unique solution v to the Dirichlet problem

$$(4.4) \quad \begin{cases} \Delta v = 0 & \text{in } \mathcal{O} \setminus \overline{\mathcal{D}}, \\ (\partial/\partial n)^j v = (\partial/\partial n)^j u & \text{on } \partial\mathcal{D} \quad (j = 0, 1, \dots, p-1), \\ (\partial/\partial n)^j v = 0 & \text{on } \partial\mathcal{O} \quad (j = 0, 1, \dots, p-1). \end{cases}$$

By the *regularity* of solutions to the Dirichlet problem, v is C^∞ up to the boundary of $\mathcal{O} \setminus \overline{\mathcal{D}}$ and so $v \in \text{sol}(\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}})$.

Let us denote by Ω the neighborhood of $\partial\mathcal{D}$ where the coefficients of P^*P are real analytic. By the *Petrovskii theorem* there is a neighborhood Ω' of $\partial\mathcal{D}$ where u is real analytic. Since the Dirichlet data $\{(\partial/\partial n)^j u\}_{j=0,1,\dots,p-1}$ are real analytic on the real analytic open portion $\partial\mathcal{D}$ of the boundary of $\Omega' \setminus \overline{\mathcal{D}}$ and $\partial\mathcal{D}$ is compact, Lemma 4.4 shows that there is a neighborhood \mathcal{N}_v of $\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}$

such that v extends as a solution of $\Delta v = 0$ to \mathcal{N}_v . Moreover, \mathcal{N}_v depends only on Δ , the domain $\Omega' \setminus \overline{\mathcal{D}}$ near $\partial\mathcal{D}$ and u .

For our case, we can derive a little bit more of information on \mathcal{N}_v than that given by Lemma 4.4. Namely, \mathcal{N}_v depends on the domain $\Omega' \cup \mathcal{D} \supset \mathcal{N}_u \supset \overline{\mathcal{D}}$ of u rather than on u . Indeed, the difference $v - u$ satisfies $\Delta(v - u) = 0$ in the open set $\mathcal{N}_u \setminus \overline{\mathcal{D}}$ and vanishes up to order $p - 1$ on the real analytic portion $\partial\mathcal{D}$ of its boundary. By Theorem 4.2, there is a neighborhood \mathcal{N} of $\mathcal{N}_u \setminus \overline{\mathcal{D}}$ depending only on Δ and $\mathcal{N}_v \setminus \overline{\mathcal{D}}$ near $\partial\mathcal{D}$, such that $v - u$ extends to a solution on \mathcal{N} . Then $v = u + (v - u)$ also extends to \mathcal{N} , and so we can add \mathcal{N} to \mathcal{N}_v .

It follows that $v \in \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$. We set $\mathcal{E}(u) = v$, thus obtaining the mapping $\mathcal{E}: \text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$.

Since the solution of the Dirichlet problem in \mathcal{D} is unique, the mapping \mathcal{E} is injective. On the other hand, since this problem is solvable for all Dirichlet data, the mapping \mathcal{E} is surjective. In other words, \mathcal{E} is an isomorphism of the vector spaces $\text{sol}(\overline{\mathcal{D}}, \Delta) \xrightarrow{\cong} \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$.

We now argue as at the end of the proof of Theorem 3.4 to conclude that this algebraic isomorphism is in fact a topological one. Since $\text{sol}(\overline{\mathcal{D}}, \Delta)$ and $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ are both spaces of type *DFS*, we are reduced to proving that \mathcal{E} is continuous.

To do this, pick a sequence $\{u_\nu\}$ in $\text{sol}(\overline{\mathcal{D}}, \Delta)$ converging to zero. By the definition of inductive limit topology, there is a neighborhood $\mathcal{N}_{\{u_\nu\}}$ of $\overline{\mathcal{D}}$ such that each u_ν is defined in $\mathcal{N}_{\{u_\nu\}}$ and $u_\nu \rightarrow 0$ uniformly on compact subsets of $\mathcal{N}_{\{u_\nu\}}$.

Set $v_\nu = \mathcal{E}(u_\nu)$. From what has already been proved it follows that there is a neighborhood $\mathcal{N}_{\{v_\nu\}}$ of $\widehat{\mathcal{O}} \setminus \mathcal{D}$ such that all the v_ν are defined in $\mathcal{N}_{\{v_\nu\}}$.

As the Dirichlet problem in $\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}$ is well-posed, we can assert that $v_\nu \rightarrow 0$ uniformly on $\widehat{\mathcal{O}} \setminus \mathcal{D}$. The same holds also for the derivatives of $\{v_\nu\}$. We have however to show that $v_\nu \rightarrow 0$ uniformly on some neighborhood of $\widehat{\mathcal{O}} \setminus \mathcal{D}$.

For this purpose, we find an $r > 0$ and a finite number of points x_1, \dots, x_J on $\partial\mathcal{D}$ such that

- the balls $\{B(x_j, r)\}_{j=1, \dots, J}$ cover $\partial\mathcal{D}$; and
- for any ν and j , the Taylor series of v_ν at x_j converges in the ball $(B(x_j, r))$.

(That such r and $\{x_j\}$ exists, follows from the comment to Theorem 4.2.) \square

Let $\mathcal{N} = (\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}) \cup (\bigcup_{j=1}^J B(x_j, r/2))$. This is a neighborhood of $\widehat{\mathcal{O}} \setminus \mathcal{D}$, and we have

$$(4.5) \quad \sup_{x \in \mathcal{N}} |v_\nu(x)| \leq \sup_{x \in \widehat{\mathcal{O}} \setminus \mathcal{D}} |v_\nu(x)| + \sum_{j=1}^J \sup_{x \in B(x_j, r/2)} |v_\nu(x)|.$$

As mentioned, $\sup_{x \in \widehat{\mathcal{O}} \setminus \mathcal{D}} |v_\nu(x)| \rightarrow 0$ when $\nu \rightarrow \infty$. It remains to estimate each term $\sup_{x \in B(x_j, r/2)} |v_\nu(x)|$.

Since the Taylor series of v_ν at x_j converges in the ball of radius r , we obtain by the *Cauchy-Hadamard formula*

$$\left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \leq \text{const}(\nu) \left(\frac{1}{r} \right)^{|\alpha|} \quad \text{for all } \alpha \in \mathbb{Z}_+.$$

Therefore

$$\begin{aligned} \sup_{x \in B(x_j, r/2)} |v_\nu(x)| &= \sup_{x \in B(x_j, r/2)} \left| \sum_{\alpha} \frac{D^\alpha v_\nu(x_j)}{\alpha!} (x - x_j)^\alpha \right| \\ &\leq \sum_{\alpha} \left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \left(\frac{r}{2} \right)^{|\alpha|} \\ &\leq \text{const}(\nu) \sum_{\alpha} \left(\frac{1}{2} \right)^{|\alpha|}. \end{aligned}$$

We may now invoke the *Theorem on Dominated Convergence* to conclude that

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \sup_{x \in B(x_j, r/2)} |v_\nu(x)| &\leq \sum_{\alpha} \left(\lim_{\nu \rightarrow \infty} \left| \frac{D^\alpha v_\nu(x_j)}{\alpha!} \right| \right) \left(\frac{r}{2} \right)^{|\alpha|} \\ &= 0, \end{aligned}$$

the last equality being a consequence of the fact that the derivatives of $\{v_\nu\}$ converge to zero uniformly on $\partial\mathcal{D}$.

Thus, (4.5) shows that the sequence $\{v_\nu\}$ converges to zero uniformly on \mathcal{N} . It follows that $\{v_\nu\}$ converges to zero in the topology of $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$, and so \mathcal{E} is continuous. This completes the proof. \square

An advantage in describing duality by (4.1) is the fact that it also provides an explicit formula for the pairing.

COROLLARY 4.5. *Under the hypothesis of Corollary 4.1, let \mathcal{F}_ν be defined by (3.4). Then the correspondence $\nu \mapsto \mathcal{F}_{\mathcal{E}(\nu)}$ induces the topological isomorphism (4.1).*

PROOF. This follows from Theorem 3.4 and the proof of Corollary 4.1. \square

5. – Miscellaneous

It turns out that the analyticity of the boundary of \mathcal{D} is essential for the validity of Corollary 4.5 (cf. Stout [13]).

EXAMPLE 5.1. If P is the Cauchy operator in $X = \mathbb{R}^2$, then P^*P is the usual Laplace operator Δ in \mathbb{R}^2 up to the factor $-\frac{1}{4}$. Assume that \mathcal{D} is a

bounded domain in \mathbb{R}^2 with connected boundary $\partial\mathcal{D}$ of class C^2 . According to the *Riemann theorem*, \mathcal{D} is holomorphically equivalent to the unit ball $B(0, 1)$ in \mathbb{R}^2 , i.e., there exists a conformal mapping $m : \mathcal{D} \rightarrow B(0, 1)$. Moreover, it is known that m is of class C^1 up to the boundary of \mathcal{D} and $m' \neq 0$ on $\overline{\mathcal{D}}$. We denote by x^0 the point of \mathcal{D} such that $m(x^0) = 0$. Let $\mathcal{O} = B(x^0, R)$, where R is a positive number, and $\mathcal{D} \subseteq B(x^0, R)$. For $u(x) = \log \left| \frac{x-x_0}{Rm(x)} \right|$, an easy verification shows that $\mathcal{E}(u)(x) = \log \frac{|x-x^0|}{R}$ belongs to $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$. Clearly, u is real analytic near the closure of \mathcal{D} if and only if $m(x)$ is. Thus, if the boundary of \mathcal{D} is not real analytic, then u can fail to be real analytic near the closure of \mathcal{D} . \square

However, Theorem A is still true for *certain* domains \mathcal{D} with non-analytic boundary.

EXAMPLE 5.2. Under the hypothesis of Example 5.1, the mapping $m : \mathcal{D} \rightarrow B(0, 1)$ induces a topological isomorphism of $\text{sol}(\mathcal{D}, \Delta) \xrightarrow{\cong} \text{sol}(B(0, 1), \Delta)$. Arguing in a similar way, we see that the complement of $\overline{\mathcal{D}}$ is holomorphically equivalent to the complement of the closed unit ball in \mathbb{R}^2 . And the corresponding conformal mapping induces a topological isomorphism of $\text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta) \xrightarrow{\cong} \text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{B(0, 1)}, \Delta)$. Using the Grothendieck duality and the reflexivity of the spaces $\text{sol}(B(0, 1), \Delta)$ and $\text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{B(0, 1)}, \Delta)$, we conclude that $\text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{B(0, 1)}, \Delta) \xrightarrow{\text{top.}} \text{sol}(B(0, 1), \Delta)$. Hence $\text{sol}(\mathcal{D}, \Delta) \xrightarrow{\text{top.}} \text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta)$. Finally, because of the Grothendieck duality, we have

$$\begin{aligned} \text{sol}(\mathcal{D}, \Delta)' &\xrightarrow{\text{top.}} \text{sol}(\widehat{\mathbb{R}^2} \setminus \overline{\mathcal{D}}, \Delta)' \\ &\xrightarrow{\text{top.}} \text{sol}(\overline{\mathcal{D}}, \Delta). \end{aligned}$$

What is still lacking is an explicit description of this duality (cf. Aizenberg and Gindikin [1]). \square

6. – Duality for solutions of $Pu = 0$

For a domain $\mathcal{D} \in \mathcal{O}$ with real analytic boundary, the pairing corresponding to the duality (5.1) is explicitly defined as follows.

Let $v \in \text{sol}(\overline{\mathcal{D}}, \Delta)$. Denote by $\mathcal{E}(v)$ the unique solution to the Dirichlet problem for the Laplacian in $\mathcal{O} \setminus \overline{\mathcal{D}}$, with Dirichlet data $\{(\partial/\partial n)^j v\}_{j=0,1,\dots,p-1}$ on $\partial\mathcal{D}$ and zero Dirichlet data on $\partial\mathcal{O}$ (cf. (4.4)). There exists an open set $\mathcal{N}_{\mathcal{E}(v)} \in \mathcal{D}$ with piecewise smooth boundary, such that $\mathcal{E}(v)$ still satisfies $\Delta\mathcal{E}(v) = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}_{\mathcal{E}(v)}$. Set

$$(6.1) \quad \langle \mathcal{F}_{\mathcal{E}(v)}, u \rangle = \int_{\partial\mathcal{N}_{\mathcal{E}(v)}} G_{\Delta}(\star\mathcal{E}(v), u) \quad (u \in \text{sol}(\mathcal{D}, \Delta)).$$

Then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ induces, by Corollary 4.5, the *topological* isomorphism $\text{sol}(\overline{\mathcal{D}}, \Delta) \xrightarrow{\cong} \text{sol}(\mathcal{D}, \Delta)'$

Since $\Delta = P^*P$, we have

$$\begin{aligned} \text{sol}(\mathcal{D}, P) &\hookrightarrow \text{sol}(\mathcal{D}, \Delta), \\ \text{sol}(\overline{\mathcal{D}}, P) &\hookrightarrow \text{sol}(\overline{\mathcal{D}}, \Delta) \end{aligned}$$

(and both subspaces are closed).

Moreover, equality (3.3) shows that the restriction of the functional (6.1) to the subspace $\text{sol}(\mathcal{D}, P)$ is given by

$$(6.2) \quad \langle \mathcal{F}_{\mathcal{E}(v)}, u \rangle = \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P \mathcal{E}(v), u) \quad (u \in \text{sol}(\mathcal{D}, P)).$$

Again it follows from *Stokes' formula* that the value $\langle \mathcal{F}_{\mathcal{E}(v)}, u \rangle$ is independent of the particular choice of \mathcal{N}_v with the properties previously mentioned. By the above, $\mathcal{F}_{\mathcal{E}(v)}$ is a continuous linear functional on the space $\text{sol}(\mathcal{D}, P)$.

Of course, it is no longer true that to different solutions v_1 and v_2 in $\text{sol}(\overline{\mathcal{D}}, \Delta)$ there correspond different functionals $\mathcal{F}_{\mathcal{E}(v_1)}$ and $\mathcal{F}_{\mathcal{E}(v_2)}$ on $\text{sol}(\mathcal{D}, P)$ by (6.2). However, this still holds if we vary v within $\text{sol}(\overline{\mathcal{D}}, P)$ only.

LEMMA 6.1. *If $v \in \text{sol}(\overline{\mathcal{D}}, P)$ satisfies*

$$(6.3) \quad \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P \mathcal{E}(v), u) = 0 \quad \text{for all } u \in \text{sol}(\mathcal{D}, P),$$

then $v = 0$.

PROOF. Take $u = v$ in (6.3). By Stokes' formula,

$$\begin{aligned} 0 &= \int_{\partial \mathcal{N}_{\mathcal{E}(v)}} G_P(\star P \mathcal{E}(v), v) \\ &= \int_{\partial \mathcal{D}} G_P(\star P \mathcal{E}(v), v) \\ &= \int_{\partial \mathcal{D}} G_P(\star P \mathcal{E}(v), \mathcal{E}(v)), \end{aligned}$$

the last equality being a consequence of the fact that $v = \mathcal{E}(v)$ up to order $p - 1$ on $\partial \mathcal{D}$. As $\mathcal{E}(v)$ vanishes up to order $p - 1$ on $\partial \mathcal{O}$ and $\Delta \mathcal{E}(v) = 0$ in $\mathcal{O} \setminus \overline{\mathcal{D}}$, we obtain by the definition of Green operators

$$\begin{aligned} 0 &= - \int_{\partial(\mathcal{O} \setminus \mathcal{D})} G_P(\star P \mathcal{E}(v), \mathcal{E}(v)) \\ &= - \int_{\mathcal{O} \setminus \mathcal{D}} |P \mathcal{E}(v)|^2 dx. \end{aligned}$$

Hence it follows that $P\mathcal{E}(v) = 0$ in $\mathcal{O} \setminus \overline{\mathcal{D}}$.

Consider the section

$$\tilde{v} = \begin{cases} v & \text{in } \mathcal{D}, \\ \mathcal{E}(v) & \text{in } \mathcal{O} \setminus \overline{\mathcal{D}}. \end{cases}$$

It is of class $C_{\text{loc}}^{p-1}(E|_{\mathcal{O}})$ and satisfies $P\tilde{v} = 0$ away from the hypersurface $\partial\mathcal{O}$. A familiar argument on removable singularities (see for instance Tarkhanov [14, Theorem 3.2]) shows that \tilde{v} is actually a solution to $P\tilde{v} = 0$ on the whole domain \mathcal{O} .

Since \tilde{v} vanishes up to order $p - 1$ on $\partial\mathcal{O}$, it follows that $\tilde{v} = 0$ in \mathcal{O} . Hence $v = 0$ in \mathcal{D} , as desired. \square

Thus, the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$, provides us with an injective mapping $\text{sol}(\overline{\mathcal{D}}, P) \rightarrow \text{sol}(\mathcal{D}, P)'$. One may ask whether this mapping is surjective. We prove that this is the case if and only if the domain \mathcal{D} possesses a convexity property with respect to the operator P .

THEOREM 6.2. *Let $\mathcal{D} \Subset \mathcal{O}$ be a domain with real analytic boundary. Assume that the operator P^*P possesses the Unique Continuation Property $(U)_s$ and has real analytic coefficients in a neighborhood of $\partial\mathcal{D}$. If, given any neighborhood U of $\overline{\mathcal{D}}$, there is a neighborhood $U' \subset U$ of $\overline{\mathcal{D}}$ such that $\text{sol}(U')$ is dense in $\text{sol}(\mathcal{D})$ then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$, when restricted to $v \in \text{sol}(\overline{\mathcal{D}}, P)$, induces the topological isomorphism*

$$\text{sol}(\mathcal{D}, P)' \stackrel{\text{top.}}{\cong} \text{sol}(\overline{\mathcal{D}}, P).$$

This result sharpens Theorem A in the Introduction.

7. – Proof of the main theorem

The main step in the proof consists of verifying the surjectivity of the mapping $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$.

Let \mathcal{F} be a continuous linear functional on $\text{sol}(\mathcal{D}, P)$. Since $\text{sol}(\mathcal{D}, P)$ is a subspace of $C_{\text{loc}}(E|_{\mathcal{D}})$, this functional can be extended, by the Hahn-Banach theorem, to an E^* -valued measure m with compact support in \mathcal{D} . We set $K = \text{supp } m$.

As in the previous section, we denote by Ω the neighborhood of $\partial\mathcal{D}$ where the coefficients of P^*P are real analytic. Fix an open set $\mathcal{N} \Subset \mathcal{D}$ with piecewise smooth boundary, such that $K \subset \mathcal{N}$ and $\partial\mathcal{N} \subset \Omega$. We first argue formally.

SKETCH OF THE PROOF OF SURJECTIVITY. For any $u \in \text{sol}(\mathcal{D}, P)$, we have by Green's formula

$$\begin{aligned} u(x) &= - \int_{\partial\mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u) \\ &= - \int_{\partial\mathcal{N}} G_P(\star P \mathcal{E}(\mathcal{E}^{-1}(\star^{-1} \mathcal{G}(x, \cdot))), u) \end{aligned}$$

whenever $x \in \mathcal{N}$.

Suppose that outside of a larger open set $\mathcal{N}' \Subset \mathcal{D}$ with piecewise smooth boundary $K(x, \cdot) = \mathcal{E}^{-1} \star^{-1} \mathcal{G}(x, \cdot)$ can be decomposed into the sum $K(x, \cdot) = K_1(x, \cdot) + K_2(x, \cdot)$, where $K_1(x, \cdot) \in \text{sol}(\overline{\mathcal{D}}, P)$ is sufficiently smooth in $x \in \mathcal{N}$, and $K_2(x, \cdot)$ is orthogonal to u under the pairing $\int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_2(x, \cdot)), u)$.

Then

$$u(x) = - \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_1(x, \cdot)), u), \quad x \in \mathcal{N},$$

and so

$$\langle \mathcal{F}, u \rangle = \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(v), u)$$

with $v(y) = -\langle dm, K_1(\cdot, y) \rangle_{\mathcal{N}}$.

Hence it follows that $v \in \text{sol}(\overline{\mathcal{D}}, P)$ and $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$, as desired. \square

We now proceed to give a rigorous proof. By Theorem 2.3, the columns of the Green's function $\star_y^{-1} \mathcal{G}(x, y)$ belongs to $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{N}, \Delta)$ in the variable y , for each fixed $x \in \mathcal{N}$. In the following we will apply different operators and notations to matrices, understanding that they hold for each of their columns.

Given any fixed $x \in \mathcal{N}$, let $K(x, \cdot) = \mathcal{E}^{-1}(\star^{-1} \mathcal{G}(x, \cdot))$, i.e., $K(x, y)$ be the unique solution to the following Dirichlet problem:

$$\begin{cases} \Delta(y, D)K(x, y) = 0 & \text{for } y \in \overline{\mathcal{D}}, \\ (\partial/\partial n(y))^j K(x, y) = (\partial/\partial n(y))^j (\star_y^{-1} \mathcal{G}(x, y)) & \text{for } y \in \partial \mathcal{D} \ (j=0, 1, \dots, p-1). \end{cases}$$

Since $\overline{\mathcal{N}} \subset \mathcal{D}$, it follows from Lemma 4.4 that there is a neighborhood $U \Subset \Omega \cup D$ of $\overline{\mathcal{D}}$ independent of $x \in \mathcal{N}$, such that $K(x, \cdot)$ belongs to $\text{sol}(U, \Delta)$. (We use here the fact that the Green's function is real analytic away from the diagonal in $\Omega \times \Omega$.)

Moreover, $K(x, \cdot)$ is real analytic in $x \in \mathcal{N} \cap \Omega$ because of the *Poisson formula* for solutions of the Dirichlet problem (cf. Tarkhanov [15, (9.3.12)]).

As mentioned, $\text{sol}(U, P)$ is a closed subspace of $\text{sol}(U, \Delta)$. Our next goal is to extract a summand from $K(x, \cdot)$ which corresponds to this subspace, so that the rest is orthogonal to $\text{sol}(\mathcal{D}, P)$ in a suitable sense. To this end, we invoke Hilbert space techniques.

We first recall a result of Nacinovich and Shlapunov [10].

LEMMA 7.1. *The Hermitian form*

$$(7.1) \quad h(u, v) = \int_{\mathcal{D}} (Pu, Pv)_x dx + \int_{\mathcal{O} \setminus \mathcal{D}} (P\mathcal{E}(u), P\mathcal{E}(v))_x dx \quad (u, v \in H^p(E|_{\mathcal{D}})),$$

defines a scalar product on $H^p(E|_{\mathcal{D}})$, and the topologies induced in $H^p(E|_{\mathcal{D}})$ by $h(\cdot, \cdot)$ and by the standard scalar product are equivalent.

PROOF. See *ibid* as well as in Tarkhanov [15, 10.2.3]. \square

An easy calculation shows that if moreover v is sufficiently smooth up to the boundary of \mathcal{D} (it suffices $v \in H^{2p}(E|_{\mathcal{D}})$) and u satisfies $Pu = 0$ in \mathcal{D} , then

$$(7.2) \quad h(u, v) = - \int_{\partial\mathcal{D}} G_P(\star P\mathcal{E}(v), \mathcal{E}(u)).$$

By assumption, there is a neighborhood $U' \Subset U$ of $\overline{\mathcal{D}}$ such that $\text{sol}(U', P)$ is dense in $\text{sol}(\mathcal{D}, P)$. We can certainly assume that $H^p(E|_{U'}) \cap \text{sol}(U', P)$ is dense in $\text{sol}(\mathcal{D}, P)$, for if not, we replace U' by a smaller neighborhood.

Denote by H_2 the closure of $H^p(E|_{U'}) \cap \text{sol}(U', P)$ in $H^p(E|_{\mathcal{D}})$; we endow H_2 with scalar product (7.1).

The following result is a particular case of a general theorem of Shlapunov and Tarkhanov [12] (see also [15, 12.1.2]).

LEMMA 7.2. *There exists an orthonormal basis $\{e_\nu\}$ in $H^p(E|_{U'}) \cap \text{sol}(U', P)$ such that the restriction of $\{e_\nu\}$ to \mathcal{D} is an orthogonal basis in H_2 .*

PROOF. Consider the mapping $R : H^p(E|_{U'}) \cap \text{sol}(U', P) \rightarrow H_2$ given by restricting sections over U' to \mathcal{D} . (It will cause no confusion if we use the same symbol for a section $u \in H^p(E|_{U'}) \cap \text{sol}(U', P)$ and its restriction Ru to \mathcal{D} .)

By the Unique Continuation Property $(U)_s$, R is injective. Moreover, by *Stiljtes-Vitali theorem* R is compact. It follows that R^*R is a compact selfadjoint operator of zero null-space in the Hilbert space $H^p(E|_{U'}) \cap \text{sol}(U', P)$. (Here R^* stands for the adjoint of R in the sense of Hilbert spaces.)

Let $\{e_\nu\}$ be a complete orthonormal system of eigenfunctions of the operator R^*R in $H^p(E|_{U'}) \cap \text{sol}(U', P)$ corresponding to eigenvalues $\{\lambda_\nu\}$. Since $H^p(E|_{U'}) \cap \text{sol}(U', P)$ is dense in H_2 , we can assert that

- $\{e_\nu\}$ is orthonormal basis in $H^p(E|_{U'}) \cap \text{sol}(U', P)$; and
- the system $\{Re_\nu\}$ is a basis in H_2 orthogonal with respect to the scalar product $h(\cdot, \cdot)$.

Thus, the system $\{e_\nu\}$ possesses the desired properties, and the lemma follows. □

Note that the Fourier coefficients of a section $u \in H^p(E|_{U'}) \cap \text{sol}(U', P)$ with respect to the system $\{e_\nu\}$ are given by

$$(7.3) \quad \begin{aligned} (u, e_\nu)_{H^p(E|_{U'})} &= \frac{1}{\lambda_\nu} (u, R^*Re_\nu)_{H^p(E|_{U'})} \\ &= \frac{1}{\lambda_\nu} h(Ru, Re_\nu) \\ &= \frac{1}{\lambda_\nu} h(u, e_\nu), \end{aligned}$$

where $\lambda_\nu = h(e_\nu, e_\nu)$.

Our next objective is to treat the “projection” of the kernel $K(x, \cdot)$ on the space $H^p(E|_{U'}) \cap \text{sol}(U', P)$. To do this, we need the following technical lemma.

LEMMA 7.3. *Let $\{e_\nu\}$ be an orthonormal system in a separable Hilbert space H , and $K(x)$ be a continuous function on a topological space T with values in H . Then the Fourier series $\sum_\nu (K(x), e_\nu)_H e_\nu$ converges in the norm of H uniformly in x on compact subset of T .*

PROOF. Denote by H_1 the closure of the linear span of $\{e_\nu\}$ in H . Pick a complete orthonormal system $\{b_\mu\}$ in the orthogonal complement of H_1 in H . Then $\{e_\nu\} \cup \{b_\mu\}$ is an orthonormal basis in H .

Given any $x \in T$, decompose $K(x)$ into the Fourier series with respect to this basis. Namely,

$$K(x) = \sum_{\nu=1}^{\infty} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{\infty} (K(x), b_\mu)_H b_\mu.$$

Hence it follows that

$$\begin{aligned} & \left\| K(x) - \left(\sum_{\nu=1}^N (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^N (K(x), b_\mu)_H b_\mu \right) \right\|_H^2 \\ &= \sum_{\nu=N+1}^{\infty} |(K(x), e_\nu)_H|^2 + \sum_{\mu=N+1}^{\infty} |(K(x), b_\mu)_H|^2, \end{aligned}$$

and so

$$(7.4) \quad \sum_{\nu=N+1}^{\infty} |(K(x), e_\nu)_H|^2 \leq \left\| K(x) - \left(\sum_{\nu=1}^N (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^N (K(x), b_\mu)_H b_\mu \right) \right\|_H^2$$

for all $\nu = 1, 2, \dots$

Since the Fourier series converges in the norm H , for every $x^0 \in T$ and $\varepsilon > 0$ there is a number N^0 depending on x^0 and ε , such that

$$\left\| K(x^0) - \left(\sum_{\nu=1}^{N^0} (K(x^0), e_\nu)_H e_\nu + \sum_{\mu=1}^{N^0} (K(x^0), b_\mu)_H b_\mu \right) \right\|_H < \varepsilon.$$

Moreover, from the continuity of $K(x)$ at x^0 we deduce that the set

$$\mathcal{N}(x^0) = \left\{ x \in T : \left\| K(x) - \left(\sum_{\nu=1}^{N^0} (K(x), e_\nu)_H e_\nu + \sum_{\mu=1}^{N^0} (K(x), b_\mu)_H b_\mu \right) \right\|_H < \varepsilon \right\}$$

is an open neighborhood of x^0 .

Applying (7.4) yields

$$\sum_{v=N+1}^{\infty} |(K(x), e_v)_H|^2 \leq \sum_{v=N^0+1}^{\infty} |(K(x), e_v)_H|^2 < \varepsilon^2,$$

for all $N \geq N^0$ and $x \in \mathcal{N}(x^0)$. Therefore, the series $\sum_v (K(x), e_v)_H e_v$ converges in the norm of H uniformly in $x \in \mathcal{N}(x^0)$.

As each compact subset of T can be covered by a finite number of such neighborhoods, the lemma follows. \square

By the above, $K(x, \cdot)$ is a continuous function of $x \in \mathcal{N}$ with values in the Hilbert space $H^p(E|_{U'}) \cap \text{sol}(U', \Delta)$. Lemma 7.3 thus shows that the series

$$(7.5) \quad K_1(x, \cdot) = \sum_{v=1}^{\infty} (k(x, \cdot), e_v)_{H^p(E|_{U'})} e_v$$

converges in $H^p(E|_{U'}) \cap \text{sol}(U', \Delta)$ uniformly in x on compact subsets of \mathcal{N} . As the same holds for the derivatives of $K(x, \cdot)$ with respect to the x variables, we conclude that $K_1(x, \cdot)$ is of class C^∞ in $x \in \mathcal{N}$.

We now apply the operator \mathcal{E} to both sides of equality (7.5). Since \mathcal{E} determines a topological isomorphism of $\text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$, there is an open set $\mathcal{N}' \Subset \mathcal{D}$ with piecewise smooth boundary, such that every $\mathcal{E}(e_v)$ extends to a solution of $\Delta v = 0$ in a neighborhood of $\mathcal{O} \setminus \mathcal{N}'$, and the series

$$\mathcal{E}(K_1(x, \cdot)) = \sum_{v=1}^{\infty} (K(x, \cdot), e_v)_{H^p(E|_{U'})} \mathcal{E}(e_v)$$

converges in $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{N}', \Delta)$. (By construction, \mathcal{N}' is larger than \mathcal{N} , since otherwise we obtain a gain in analyticity.)

LEMMA 7.4. For each $u \in \text{sol}(\mathcal{D}, P)$, it follows that

$$(7.6) \quad u(x) = - \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_1(x, \cdot)), u), \quad x \in \mathcal{N}.$$

PROOF. Pick a system $\{b_\mu\}$ in $H^p(E|_{\mathcal{D}}) \cap \text{sol}(\mathcal{D}, \Delta)$ such that $\{e_v\} \cup \{b_\mu\}$ is a basis in this space orthogonal with respect to scalar product (7.1).

For a fixed $x \in \mathcal{N}$, we decompose $K(x, \cdot)$ into the Fourier series with respect to this basis, i.e.,

$$\begin{aligned} (7.7) \quad K(x, \cdot) &= \sum_{v=1}^{\infty} \frac{h(K(x, \cdot), e_v)}{h(e_v, e_v)} e_v + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu \\ &= \sum_{v=1}^{\infty} (K(x, \cdot), e_v)_{H^p(E|_{U'})} e_v + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu \\ &= K_1(x, \cdot) + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_\mu)}{h(b_\mu, e_\mu)} b_\mu, \end{aligned}$$

the second equality being a consequence of (7.3). (Note that the series on the right-hand here converges in the norm of $H^p(E|_{\mathcal{D}})$.)

As \mathcal{E} is a topological isomorphism of

$$H^p(E|_{\mathcal{D}}) \cap \text{sol}(\mathcal{D}, \Delta) \rightarrow H^p(E|_{\mathcal{O} \setminus \overline{\mathcal{D}}}) \cap \text{sol}(\widehat{\mathcal{O}} \setminus \overline{\mathcal{D}}, \Delta),$$

we may apply \mathcal{E} to (7.7) termwise, thus obtaining

$$\star^{-1}\mathcal{G}(x, \cdot) = \mathcal{E}(K_1(x, \cdot)) + \sum_{\mu=1}^{\infty} \frac{h(K(x, \cdot), b_{\mu})}{h(b_{\mu}, e_{\mu})} \mathcal{E}(b_{\mu}), \quad x \in \mathcal{N},$$

the series converging in the norm of $H^p(E|_{\mathcal{O} \setminus \overline{\mathcal{D}}})$.

Having disposed of this preliminary step, we can now return to representation (7.6). Let $u \in \text{sol}(\mathcal{D}, P)$. By assumption, there exists a sequence $\{u_j\}$ in $H^p(E|_{U'}) \cap \text{sol}(U', P)$ converging to u together with all derivatives uniformly on compact subsets of \mathcal{D} . Given any $x \in \mathcal{N}$, we have therefore by Green's formula

$$\begin{aligned} u(x) &= - \int_{\partial \mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u) \\ &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{N}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u_j) \\ &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{D}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), u_j), \end{aligned}$$

the second equality being a consequence of Lemma 1.1, and the third equality being a consequence of Stokes' formula.

On the boundary of \mathcal{D} , we have $u_j = \mathcal{E}(u_j)$ up to order $p - 1$. Therefore

$$\begin{aligned} u(x) &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{D}} G_P(\star P \star^{-1} \mathcal{G}(x, \cdot), \mathcal{E}(u_j)) \\ &= \lim_{j \rightarrow \infty} h(u_j, K(x, \cdot)), \end{aligned}$$

which is due to (7.2).

On the other hand, since every u_j is in $H^p(E|_{U'}) \cap \text{sol}(U', P)$, we may write

$$u_j = \sum_{\nu=1}^{\infty} (u_j, e_{\nu})_{H^p(E|_{U'})} e_{\nu},$$

where the series converges in the norm of $H^p(E|_{U'})$. Combining this with (7.7) yields

$$h(u_j, K(x, \cdot)) = h(u_j, K_1(x, \cdot)),$$

for the systems of sections $\{e_{\nu}\}$ and $\{b_{\mu}\}$ are pairwise orthogonal with respect to $h(\cdot, \cdot)$.

Thus

$$\begin{aligned} u(x) &= \lim_{j \rightarrow \infty} h(u_j, K(x, \cdot)) \\ &= - \lim_{j \rightarrow \infty} \int_{\partial \mathcal{D}} G_P(\star P \mathcal{E}(K_1(x, \cdot)), \mathcal{E}(u_j)) \\ &= - \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(K_1(x, \cdot)), u), \end{aligned}$$

for $x \in \mathcal{N}$. This is precisely the assertion of the lemma. □

We are now in a position to finish the proof of Theorem 6.2.

PROOF OF THEOREM 6.2. From Lemma 7.4 it follows that

$$\langle \mathcal{F}, u \rangle = \int_{\partial \mathcal{N}'} G_P(\star P \mathcal{E}(v), u) \quad \text{for all } u \in \text{sol}(\mathcal{D}, P),$$

where $v(y) = -\langle dm, K_1(\cdot, y) \rangle_{\mathcal{N}}$.

One easily verifies that $Pv = 0$ in U' . Hence $v \in \text{sol}(\overline{\mathcal{D}}, P)$ and $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$, which proves the surjectivity of the mapping $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$.

When combined with Lemma 6.1, this shows that the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ induces the isomorphism of vector spaces

$$\text{sol}(\overline{\mathcal{D}}, P) \xrightarrow{\cong} \text{sol}(\mathcal{D}, P)'.$$

We now argue as at end of the proof of Theorem 3.4 to conclude that this algebraic isomorphism is in fact a topological one.

For this purpose, we note that the spaces $\text{sol}(\overline{\mathcal{D}}, P)$ and $\text{sol}(\mathcal{D}, P)'$ are both spaces of type *DFS*. (For $\text{sol}(\overline{\mathcal{D}}, P)$, see the proof of Theorem 1.5.5 in Morimoto [8, p. 13]. For $\text{sol}(\mathcal{D}, P)'$, see Lemma 1.1 above.) As the *Closed Graph theorem* is correct for linear maps between spaces of type *DFS* (see Corollary A.6.4 in Morimoto [8, p. 24]), to see that $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ is a topological isomorphism, it suffices to show that it is continuous.

The latter conclusion is however a consequence of the following two facts already proved:

- the mapping $v \mapsto \mathcal{F}_v$ of $\text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta) \rightarrow \text{sol}(\mathcal{D}, \Delta)$ is continuous (cf. Theorem 3.4); and
- the mapping $v \mapsto \mathcal{E}(v)$ of $\text{sol}(\overline{\mathcal{D}}, \Delta) \rightarrow \text{sol}(\widehat{\mathcal{O}} \setminus \mathcal{D}, \Delta)$ is continuous (cf. Corollary 4.1).

This completes the proof. □

Let us mention an important consequence of Theorem 6.2.

COROLLARY 7.5. *Under the hypotheses of Theorem 6.2, it follows that*

$$\text{sol}(\overline{\mathcal{D}}, P)' \overset{\text{top.}}{\cong} \text{sol}(\mathcal{D}, P).$$

PROOF. By Lemma 1.1, $\text{sol}(\mathcal{D}, P)$ is a Fréchet-Schwartz space. Therefore, it is a *Montel space*. That $\text{sol}(\mathcal{D}, P)$ is a Montel space implies that it is reflexive, i.e., under the natural pairing, we have

$$(\text{sol}(\mathcal{D}, P)')' \stackrel{\text{top.}}{\cong} \text{sol}(\mathcal{D}, P),$$

where both $\text{sol}(\mathcal{D}, P)'$ and $(\text{sol}(\mathcal{D}, P)')'$ are provided with the strong topology. Thus, the desired statement follows immediately from Theorem 6.2. \square

8. – The converse theorem

Assume that \mathcal{D} is a relatively compact subdomain of \mathcal{O} with real analytic boundary.

We have proved that if, for any neighborhood U of $\overline{\mathcal{D}}$, there exists a neighborhood $U' \subset U$ of $\overline{\mathcal{D}}$ such that $\text{sol}(U', P)$ is dense in $\text{sol}(\mathcal{D}, P)$, then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ induces the topological isomorphism of $\text{sol}(\overline{\mathcal{D}}, P)$ onto the dual space to $\text{sol}(\mathcal{D}, P)$.

We show that this condition is almost necessary.

THEOREM 8.1. *If the map $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$ of $\text{sol}(\overline{\mathcal{D}}, P) \rightarrow \text{sol}(\mathcal{D}, P)'$ is surjective, then $\text{sol}(\overline{\mathcal{D}}, P)$ is dense in $\text{sol}(\mathcal{D}, P)$.*

PROOF. Let \mathcal{F} be a continuous linear functional on $\text{sol}(\mathcal{D}, P)$ vanishing on $\text{sol}(\overline{\mathcal{D}}, P)$. By the Hahn-Banach theorem, our statement will be proved once we show that $\mathcal{F} \equiv 0$.

By assumption, there is a $v \in \text{sol}(\overline{\mathcal{D}}, P)$ such that $\mathcal{F}_{\mathcal{E}(v)} = \mathcal{F}$. It follows that

$$\begin{aligned} \langle \mathcal{F}_{\mathcal{E}(v)}, v \rangle &= \langle \mathcal{F}, v \rangle \\ &= 0, \end{aligned}$$

and so an argument similar to that in the proof of Lemma 6.1 shows that $v = 0$ in \mathcal{D} . Hence $\mathcal{F} \equiv 0$, as desired. \square

9. – Duality in complex analysis

Aizenberg and Gindikin [1] obtained Theorem A, formulated in the Introduction, in the case where P is the *Cauchy-Riemann operator* in \mathbb{C}^n , and $n = 1, 2$ (for simply connected domains with real analytic boundary in \mathbb{C} , and for the so-called (p, q) -circular domains in \mathbb{C}^2).

Stout [13] proved Theorem A for the Cauchy-Riemann operator in \mathbb{C}^n ($n \geq 1$) and for domains \mathcal{D} possessing the following property:

- the Szegő Kernel $\mathcal{K}(\cdot, \zeta)$ of \mathcal{D} has real analytic boundary values for each $\zeta \in \mathcal{D}$.

This condition is known to hold on some explicitly given domains. One supposes it to hold on *strictly pseudoconvex* domains with real analytic boundary. But, as far as Stout [13] has been able to determine, this result has not been written out anywhere.

However, the *approximation condition* in Theorem A holds true for strictly pseudoconvex domains in \mathbb{C}^n (cf. Hörmander [5]). Thus, our viewpoint sheds some new light on the result of Stout [13].

THEOREM 9.1. *Let $\mathcal{D} \Subset \mathbb{C}^n$ ($n \geq 2$) be a strictly pseudoconvex domain with real analytic boundary. Then the correspondence $v \mapsto \mathcal{F}_{\mathcal{E}(v)}$, when restricted to $v \in \text{hol}(\overline{\mathcal{D}})$, induces the topological isomorphism*

$$\text{hol}(\mathcal{D})' \stackrel{\text{top.}}{\cong} \text{hol}(\overline{\mathcal{D}}).$$

Here we use hol for the spaces of holomorphic functions.

PROOF. This follows immediately by combining Theorem 6.2 with the *Runge theorem* as stated in Hörmander [5]. \square

We note that, because the Cauchy-Riemann operator in \mathbb{C} is determined elliptic, Theorem 9.1 holds true for spaces of holomorphic functions in every bounded domain in \mathbb{C} with real analytic boundary.

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