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Some Theoretical Results Concerning Non Newtonian Fluids of the Oldroyd Kind

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1. – Introduction

In this paper we consider incompressible visco-elastic fluids satisfying the Oldroyd constitutive law ([1], [15], [20]):

$$\sigma + \lambda_1 \frac{D_a \sigma}{Dt} = 2\eta \left(D(u) + \lambda_2 \frac{D_a D(u)}{Dt} \right) . \tag{1.1}$$

Here σ is the tensor of tangential stresses, $D(u) = \frac{1}{2}(\nabla u + {}^t\nabla u)$ is the deformation tensor, u is the velocity field, λ_1 and λ_2 are respectively the relaxation and retardation times ($0 \leq \lambda_2 \leq \lambda_1$) and η is the fluid viscosity. In (1.1), D_a/Dt denotes the following "objective derivative" ([1], [16], [20]):

$$\frac{D_a \sigma}{Dt} = \sigma' + (u \cdot \nabla)\sigma + g_a(\sigma, \nabla u) ,$$

σ' is the time derivative,

$$g_a(\sigma, \nabla u) = \sigma W(u) - W(u)\sigma - a(D(u)\sigma + \sigma D(u))$$

with $a \in [-1, 1]$ and, finally, $W(u) = \frac{1}{2}(\nabla u - {}^t\nabla u)$ is the vorticity tensor.

The case $\lambda_1 = \lambda_2 = 0$ (respectively $\lambda_1 > 0$, $\lambda_2 = 0$) corresponds to a purely viscous or Newtonian (respectively a purely elastic) fluid. In this paper, we will assume $0 < \lambda_2 < \lambda_1$; for λ_1 and λ_2 in these conditions, (1.1) can be used to describe the behaviour of a large variety of polymers (for instance, see [25]).

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The constitutive law (1.1) must accompany the motion and continuity equations for homogeneous incompressible fluids

$$\rho(u' + (u \cdot \nabla)u) = \nabla \cdot \Sigma + f, \qquad \nabla \cdot u = 0\,, \tag{1.2}$$

where u' is again a time derivative, $\Sigma = -pId + \sigma$ is the stress tensor (p is the pressure), f is a given field of exterior forces and ρ is the fluid density (a given constant). By putting $\sigma = \tau_N + \tau$ with

$$\tau_N = 2\eta \frac{\lambda_2}{\lambda_1} D(u)\,,$$

we see that (1.1)-(1.2) can also be written as follows:

$$\rho(u' + (u \cdot \nabla)u) - \eta(1-\alpha)\Delta u + \nabla p = \nabla \cdot \tau + f, \qquad \nabla \cdot u = 0\,, \tag{1.3}$$

$$\tau + \lambda_1(\tau' + (u \cdot \nabla)\tau + g_a(\tau, \nabla u)) = 2\eta\alpha D(u)\,. \tag{1.4}$$

Here, we have introduced the parameter $\alpha = 1 - \lambda_2/\lambda_1$ (notice that $0 < \alpha < 1$). The previous system can be adimensionalized in the usual way with characteristic values U and L for the velocity and the length. For the dimensionless variables, one finds:

$$\mathrm{Re}(u' + (u \cdot \nabla)u) - (1-\alpha)\Delta u + \nabla p = \nabla \cdot \tau + f, \qquad \nabla \cdot u = 0\,, \tag{1.5}$$

$$\mathrm{We}(\tau' + (u \cdot \nabla)\tau + g_a(\tau, \nabla u)) + \tau = 2\alpha\, D(u)\,, \tag{1.6}$$

where $\mathrm{Re} = \rho UL/\eta$ is the Reynolds number (the ratio between inertial and viscous forces acting on the fluid) and $\mathrm{We} = \lambda_1 U/L$ is the Weissenberg number (a measure of the elasticity of the fluid). We will assume that this system is satisfied in $\Omega \times (0, T)$, with $\Omega \subset \mathbb{R}^3$ being a bounded connected open set whose boundary $\partial\Omega$ is smooth and $(0, T)$ is a time interval. We will complete (1.5)-(1.6) with the following boundary and initial conditions:

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T)\,, \tag{1.7}$$

$$u\,|_{t=0} = u_0, \qquad \tau\,|_{t=0} = \tau_0 \quad \text{in} \quad \Omega\,. \tag{1.8}$$

The unknowns are u, p and τ; the data of the problem are the functions u_0, τ_0 and f and the constants Re, We, α and a.

The main goal of this paper is to deduce existence, uniqueness and stability results for (1.5)-(1.8). Several results have already been established, up to now all in a Hilbert framework. In [9], C. Guillopé and J.-C. Saut proved that there exists a unique strong local in time solution using the techniques in [23], [24] (roughly speaking, rewriting (1.5)-(1.8) as a fixed point equation and applying Schauder's theorem). They also proved that the solution is globally defined if the data are small. The existence and uniqueness of a global solution for large data have been demonstrated by the same authors in [10] in the particular cases of two-dimensional Couette and Poiseuille flows. Unfortunately, there is no general result concerning global in time existence for arbitrarily large data.

On the other hand, the main result for the stationary problem associated to (1.5)-(1.7) is due to M. Renardy ([19]). It states that, when f is sufficiently small and $0 < \alpha \leq 1$, there exists exactly one small strong solution. Some numerical questions for (1.5)-(1.8) and other related problems have been analyzed in [2], [3] and [21].

This paper is organized as follows. In Section 2, we prove the existence and uniqueness of a local $L^s - L^r$ solution to (1.5)-(1.8) (again global in time if the data are small). We use the techniques in [23] together with some recent results due to Y. Giga and H. Sohr [8] on the $L^s - L^r$ regularity of the solutions of the Stokes problem. In Section 3, uniqueness and stability results for (1.5)-(1.8) are given (the former can be viewed as an analog of the well known uniqueness result for regular solutions of the Navier-Stokes equations). Section 4 deals with the existence and uniqueness of a small L^r solution to the stationary problem associated to (1.5)-(1.7). In Section 5, we deduce global in time existence and uniqueness for Poiseuille flow in cylindrical domains (the flow of a fluid between two concentrical cylinders). Section 6 deals with some final remarks. Some of these results have already been announced in [5].

2. – The evolution problem (I): existence and uniqueness of a strong solution

In the sequel, one will have r, $s \in (1, \infty)$. Unless it is explicitly specified, $\Omega \subset \mathbb{R}^3$ is a bounded connected open set such that $\partial\Omega$ is Lipschitz-continuous. We will use the following notation (see [8]):

a) $L^r = L^r(\Omega)$, $H^1(\Omega)$, etc ... $H_r = \{v \in L^r(\Omega)^3; \nabla \cdot u = 0, v \cdot n = 0 \text{ on } \partial\Omega\}$. Here, $n = n(x)$ is a unitary vector, normal to $\partial\Omega$ at x and oriented towards the exterior of Ω; H_r endowed with the norm of $L^r(\Omega)^3$ is a reflexive Banach space. When $r = 2$, we will put H instead of H_2. The norm and the scalar product in L^2 will be denoted by $|\cdot|$ and $(\cdot, \cdot)$, respectively.

b) $V = H \cap H_0^1(\Omega)^3$, a Hilbert space for the usual norm in $H_0^1(\Omega)^3$. V' is its dual space ($\langle \cdot, \cdot \rangle$ is the corresponding duality pairing). The norm in H_0^1 will be denoted by $\|\cdot\|$.

c) $P_r : L^r(\Omega)^3 \to H_r$ is the Helmholtz projector in L^r. It is a bounded linear operator characterized by the equality $P_r v = v_0$, where v_0 is given by the so called Helmholtz decomposition

$$v = v_0 + \nabla q, \quad \text{with} \quad v_0 \in H_r \quad \text{and} \quad q \in W^{1,r}.$$

d) $A_r = P_r(-\Delta)$: $D(A_r) \to H_r$ is the Stokes operator in H_r. Here, $D(A_r) = W^{2,r}(\Omega)^3 \cap W_0^{1,r}(\Omega)^3 \cap H_r$ is a Banach space for the norm $\|A_r v\|_{L^r} + \|v\|_{L^r}$.

e) $D_r^s = \{v \in H_r;\ \|v\|_{D_r^s} \equiv \|v\|_{L^r} + (\int_0^\infty \|A_r e^{-tA_r} v\|_{L^r}^s dt)^{1/s} < \infty\}$.

We will frequently use functions with values in $\mathbb{R}^3$ or in the space $\mathcal{L}(\mathbb{R}^3)$ of real 3×3 matrices. In all cases, the notation will be abreged. For instance, $u \in H_0^1(\Omega)$ or simply $u \in H_0^1$ means that each component of u belongs to $H_0^1(\Omega)$. Whenever $X(\Omega)$ is a Banach space formed by functions defined in Ω, $L^p(X)$ stands for $L^p(0, T; X(\Omega))$ and $C(X)$ for $C([0, T]; X(\Omega))$.

THEOREM 2.1. *Assume* $\partial\Omega \in C^{2,\mu}(\mu > 0)$, $1 < s < +\infty$, $3 < r < +\infty$ *and* $T > 0$. *If*

$$u_0 \in D_r^s, \quad \tau_0 \in W^{1,r}, \quad f \in L^s(L^r),$$

then there exist $T_* \in (0, T]$ *and a unique strong solution* $\{u, p, \tau\}$ *to* (1.5)-(1.8) *in* $[0, T_*]$, *with*

$$u \in L^s(0, T_*; D(A_r)), \qquad u' \in L^s(0, T_*; H_r),$$

$$\tau \in C([0, T_*]; W^{1,r}), \qquad \tau' \in L^s(0, T_*; L^r).$$

THEOREM 2.2. *Assume* $\partial\Omega \in C^{2,\mu}(\mu > 0)$, $1 < s < +\infty$. *Then, for each* $T > 0$, *there exists* $\alpha_0(T) \in (0, 1)$ *such that, when* $0 < \alpha \le \alpha_0(T)$ *and the data*

$$u_0 \in D_r^s, \quad \tau_0 \in W^{1,r}, \quad f \in L^s(L^r)$$

have sufficiently small norms in their respective spaces, problem (1.5)-(1.8) *possesses exactly one strong solution* $\{u, p, \tau\}$ *in* $[0, T]$, *with*

$$u \in L^s(D(A_r)), \qquad u' \in L^s(H_r),$$

$$\tau \in C(W^{1,r}), \qquad \tau' \in L^s(L^r).$$

Theorem 2.1 slightly improves Theorem 1 in [5]. When $\Omega \subset \mathbb{R}^N$, results similar to Theorems 2.1 and 2.2 hold again for all finite $r > N$.

Theorems 2.1 and 2.2 can be compared with the results in [9]. In [9] (Theorems 2.4 and 3.3), C. Guillopé and J.-C. Saut impose stronger regularity hypotheses on the data (in particular, $f \in L^2(H^1)$ and $f' \in L^2(H^{-1})$ in Theorem 2.4 and $f \in L^\infty(H^1)$ and $f' \in L^\infty(H^{-1})$ in Theorem 3.3). They also show that if the data are small enough there exists a strong solution in $[0, \infty)$; contrarily, in Theorem 2.2, T is arbitrarily large but finite (and $\alpha_0(T) \to 0$ as $T \to +\infty$).

For the proofs of Theorem 2.1 and 2.2, we need the following three lemmas:

LEMMA A. *Assume* $\partial\Omega \in C^{2,\mu}$ $(\mu > 0)$, $1 < r, s < +\infty$ *and* $T > 0$. *If*

$$u_0 \in D_r^s, \quad F \in L^s(H_r),$$

then there exists a unique function u such that

$$u \in L^s(D(A_r)), \quad u' \in L^s(H_r)$$

and

$$\text{(A.1)} \qquad \mathrm{Re}\, u' + (1-\alpha)A_r u = F \quad a.e.\ in \quad (0,T), \qquad u\,|_{t=0} = u_0\,.$$

Furthermore,

$$\text{(A.2)} \qquad \|u\|^s_{L^s(D(A_r))} + \|u'\|^s_{L^s(H_r)} \le \left(\frac{C_1}{1-\alpha}\right)^s \left(\|u_0\|^s_{D^s_r} + \|F\|^s_{L^s(H_r)}\right),$$

where $C_1 = C_1(r, s, \mathrm{Re}, \Omega)$ *does not depend on* F, u_0 *and* T.

The fact that C_1 is independent from T is very important and will be used below.

LEMMA B. *Assume* $\partial\Omega \in C^1$, $3 < r < +\infty$, $1 < s < +\infty$ *and* $T > 0$. *If*

$$\bar{u} \in L^s(D(A_r)), \qquad \tau_0 \in W^{1,r},$$

then there exists a unique function τ *such that*

$$\tau \in C(W^{1,r}), \qquad \tau' \in L^s(L^r)$$

and

$$\text{(B.1)} \qquad \begin{cases} \mathrm{We}(\tau' + (\bar{u}\cdot\nabla)\tau + g_a(\tau, \nabla\bar{u})) + \tau = 2\alpha D(\bar{u})\ a.e.\ in\ \Omega\times(0,T)\,, \\ \tau\,|_{t=0} = \tau_0\,. \end{cases}$$

Furthermore, one has the following estimates, where $C_2 = C_2(r, a, \Omega)$:

$$\text{(B.2)}\ \|\tau\|_{L^\infty(W^{1,r})} + \frac{4\alpha}{C_2\mathrm{We}} \le \left(\|\tau_0\|_{W^{1,r}} + \frac{4\alpha}{C_2\mathrm{We}}\right)\exp\left(C_2\|\bar{u}\|_{L^1(W^{2,r})}\right) \equiv \Lambda\,,$$

$$\text{(B.3)}\ \|\tau'\|_{L^s(L^r)} \le 2^{1-\frac{1}{s}} C_2 \Lambda \left(\|\bar{u}\|_{L^s(W^{1,r})} + \frac{T^{1/s}}{C_2\mathrm{We}}\right).$$

LEMMA C. *The solution to* (A.1) *furnished by Lemma* A *satisfies:*

$$u \in C(L^r) \cap L^{2s}(W^{1,r}) \cap L^{2rs/3}(L^\infty)\,.$$

Furthermore, one has:

$$\text{(C.1)} \qquad \|u\|_{L^{2s}(W^{1,r})} \le C_3 \|u\|^{1/2}_{L^\infty(L^r)} \|u\|^{1/2}_{L^s(W^{2,r})}$$

and

$$\text{(C.2)} \qquad \|u\|_{L^{2sr/3}(L^\infty)} \le C_4 \|u\|^{1-\frac{3}{r}}_{L^\infty(L^r)} \|u\|^{\frac{3}{r}}_{L^{2s}(W^{1,r})}$$

where $C_3 = C_3(r, \Omega)$ *and* $C_4 = C_4(r, \Omega)$.

For the Proof of Lemma A, see [8]. Lemmas B and C are demonstrated in the Appendix.

PROOF OF THEOREM 2.1. We are going to rewrite (1.5)-(1.8) as a fixed point equation. Then, Lemmas A and B will be applied. This, together with Lemma C, will serve to check that all hypotheses of Schauder's theorem are satisfied.

For arbitrary $T > 0$, $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$, let us introduce the set

$$\begin{aligned} Y(T) = \{(\bar{u}, \bar{\tau});\ & \bar{u} \in L^s(D(A_r)), \bar{u}' \in L^s(H_r), \bar{u}(0) = u_0\,, \\ & \bar{\tau} \in L^\infty(W^{1,r}), \bar{\tau}' \in L^s(L^r), \bar{\tau}(0) = \tau_0\,, \\ & \|\bar{u}\|^s_{L^s(D(A_r))} + \|\bar{u}'\|^s_{L^s(H_r)} \le R_1^s\,, \\ & \|\bar{\tau}\|_{L^\infty(W^{1,r})} \le R_2, \|\bar{\tau}'\|_{L^s(L^r)} \le R_3\}\,. \end{aligned} \tag{2.1}$$

Let us see that, if R_1 and R_2 are sufficiently large, then $Y(T) \neq \emptyset$ for all $T > 0$ (and for all $R_3 > 0$). Indeed, let u_* be the unique solution to the Stokes problem

$$\mathrm{Re}\, u_*' + (1-\alpha) A_r u_* = 0 \text{ a.e. in } (0, T), \quad u_* |_{t=0} = u_0\,. \tag{2.2}$$

For Lemma A, we know there exists, $C_1 > 0$ such that

$$\|u_*\|^s_{L^s(D(A_r))} + \|u_*'\|^s_{L^s(H_r)} \le \left(\frac{C_1}{1-\alpha}\right)^s \|u_0\|^s_{D_r^s}$$

(C_1 does not depend on T). If one chooses

$$R_1 \ge \frac{C_1}{1-\alpha}\|u_0\|_{D_r^s}, \qquad R_2 \ge \|\tau_0\|_{W^{1,r}} \tag{2.3}$$

then (u_*, τ_0) belongs to $Y(T)$ for all $T > 0$ (and also for all $R_3 > 0$).

In the sequel, R_1 and R_2 will be assumed to satisfy (2.3). Let us introduce the Banach space $X_T = L^s(W_0^{1,r}) \times C(L^r)$ and the mapping Φ: $Y(T) \to X_T$, given by $\Phi(\bar{u}, \bar{\tau}) = (u, \tau)$, where u is the unique solution to (A.1) with

$$F = P_r(-\mathrm{Re}(\bar{u} \cdot \nabla)\bar{u} + \nabla \cdot \bar{\tau} + f)$$

and τ is the solution to (B.1). Obviously, a fixed point of Φ solves (1.5)-(1.8). Let us see that, for some $T_* \in (0, T]$, one has $\Phi(Y(T_*)) \subset Y(T_*)$. Indeed, if $(\bar{u}, \bar{\tau}) \in Y(T)$ then

$$\|F\|^s_{L^s(L^r)} \le C_5 \left(\mathrm{Re}^s \sum_{i,j} \int_0^T \left(\int_\Omega \left| \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \right|^r \right)^{\frac{s}{r}} + T \cdot \|\bar{\tau}\|^s_{L^\infty(W^{1,r})} + \|f\|^s_{L^s(L^r)} \right).$$

We deduce that

$$\|F\|^s_{L^s(L^r)} \le C_5' \left(T^{\frac{r-3}{2r}} \cdot \|\bar{u}\|^s_{L^{2sr/3}(L^\infty)} \|\nabla \bar{u}\|^s_{L^{2s}(L^r)} + T \cdot \|\bar{\tau}\|^s_{L^\infty(W^{1,r})} + \|f\|^s_{L^s(L^r)} \right) .$$

Now, using (C.2), (C.1), the definition of $Y(T)$ and the inequality

$$\|\bar{u}\|_{L^\infty(L^r)} \le \|u_0\|_{H^r} + \|\bar{u}'\|_{L^1(L^r)} \le \|u_0\|_{H_r} + T^{1-\frac{1}{s}} \|\bar{u}'\|_{L^s(L^r)} , \tag{2.4}$$

one obtains:

$$\|F\|^s_{L^s(L^r)} \le C_6 \left(\|u_0\|_{H_r}^{\frac{3s(r-1)}{2r}} R_1^{\frac{s(r+3)}{2r}} \cdot T^{\frac{r-3}{2r}} + R_1^{2s} \cdot T^{\frac{3s(r-1)}{2r}-1} + R_2^s \cdot T + \|f\|^s_{L^s(L^r)} \right) ,$$

where $C_6 = C_6(r, s, \mathrm{Re}, \Omega)$. From (A.2), we see that

$$\|u\|^s_{L^s(D(A_r))} + \|u'\|^s_{L^s(H_r)} \le \left(\frac{C_1}{1-\alpha} \right)^s \left(\|u_0\|^s_{D_r^s} \right.$$
$$\left. + C_6 \left(\|u_0\|_{H_r}^{\frac{3s(r-1)}{2r}} R_1^{\frac{s(r+3)}{2r}} \cdot T^{\frac{r-3}{2r}} + R_1^{2s} \cdot T^{\frac{3s(r-1)}{2r}-1} + R_2^s \cdot T + \|f\|^s_{L^s(L^r)} \right) \right) .$$

On the other hand, from (B.2) and (B.3), one has:

$$\|\tau\|_{L^\infty(W^{1,r})} + \frac{4\alpha}{C_2 \mathrm{We}} \le \left(\|\tau_0\|_{W^{1,r}} + \frac{4\alpha}{C_2 \mathrm{We}} \right) \exp\left(C_2 R_1 \cdot T^{1-\frac{1}{s}} \right) \equiv \Lambda' ,$$

$$\|\tau'\|_{L^s(L^r)} \le 2^{1-\frac{1}{s}} C_2 \Lambda' \left(R_1 + \frac{T^{1/s}}{C_2 \mathrm{We}} \right) .$$

Consequently, if T_*, R_1, R_2 and R_3 are chosen in such a way that the right sides of the three last inequalities (with T replaced by T_*) are bounded by R_1^s, R_2 and R_3 respectively, then $\Phi(Y(T_*))$ will be a subset of $Y(T_*)$. For example, it suffices to take

$$R_1^s = \left(\frac{C_1}{1-\alpha} \right)^s \left(\|u_0\|^s_{D_r^s} + C_6 \left(2 + \|u_0\|_{H_r}^{\frac{3s(r-1)}{2r}} + \|f\|^s_{L^s(L^r)} \right) \right) ,$$

$$R_2 = \left(\|\tau_0\|_{W^{1,r}} + \frac{4\alpha}{C_2 \mathrm{We}} \right) \exp C_2 ,$$

$$R_3 = 2^{1-\frac{1}{s}} C_2 R_2 \left(R_1 + \frac{1}{C_2 \mathrm{We} R_2} \right)$$

and

$$T_* = \min \left\{ R_1^{-\frac{s(r+3)}{r-3}} , R_1^{-\frac{4sr}{3s(r-1)-2r}} , R_1^{-\frac{s}{s-1}} , R_2^{-s} , T \right\} ,$$

Thus, we have found three constants R_1, R_2 and R_3 (depending on the data) and a time $T_* \in (0, T]$ (depending on R_1, R_2 and the data) such that $\Phi(Y(T_*)) \subset Y(T_*)$. Since $Y(T_*)$ is a convex compact subset of X_{T_*} and Φ is continuous, Schauder's theorem can be applied. This proves the existence of a strong solution to (1.5)-(1.8) in $\Omega \times (0, T_*)$. That this solution is unique stems on Theorem 3.1 (see Section 3). □

PROOF OF THEOREM 2.2. Let us consider again the set $Y(T)$, given by (2.1). Notice that once α, R_1 and R_2 are fixed, if u_0 and τ_0 satisfy (2.3), then the couple $(u_*, \tau_0) \in Y(T)$; here, u_* is the solution to (2.2). Hence, $Y(T) \neq \emptyset$ (and this true for all $R_3 > 0$).

As in the proof of Theorem 2.1, we introduce the space X_T and the mapping Φ. Let us see that there exists $\alpha_0(T) \in (0,1)$ such that $\Phi(Y(T)) \subset Y(T)$ for $\alpha \in (0, \alpha_0(T)]$. If $(\bar{u}, \bar{\tau}) \in Y(T)$ and we set $F = P_r(-\mathrm{Re}(\bar{u} \cdot \nabla)\bar{u} + \nabla \cdot \bar{\tau} + f)$, then

$$\|F\|^s_{L^s(L^r)} \leq C_7' \left(\mathrm{Re}^s \sum_{i,j} \int_0^T \left(\|\bar{u}_j\|_{L^r} \left\| \frac{\partial \bar{u}_i}{\partial x_j} \right\|_{L^\infty} \right)^s + T \cdot \|\bar{\tau}\|^s_{L^\infty(W^{1,r})} + \|f\|^s_{L^s(L^r)} \right) .$$

Using the fact that

$$\|\bar{u}\|_{L^\infty(L^r)} \leq C(T) \Big(\|\bar{u}\|_{L^s(L^r)} + \|\bar{u}'\|_{L^s(L^r)} \Big) ,$$

one finds

$$\|F\|^s_{L^s(L^r)} \leq C_7 \Big(\|\bar{u}\|^s_{L^s(L^r)} \|\bar{u}\|^s_{L^s(W^{2,r})} + \|\bar{u}'\|^s_{L^s(L^r)} \|\bar{u}\|^s_{L^s(W^{2,r})} + T \cdot \|\bar{\tau}\|^s_{L^\infty(W^{1,r})} + \|f\|^s_{L^s(L^r)} \Big) ,$$

where $C_7 = C_7(r, s, \mathrm{Re}, \Omega, T)$. Using the definition of $Y(T)$, one also has

$$\|F\|^s_{L^s(L^r)} \leq C_7 \left(2R_1^{2s} + R_2^s \cdot T + \|f\|^s_{L^s(L^r)} \right)$$

and this, together with (A.2), gives the following inequality:

$$\|u\|^s_{L^s(D(A_r))} + \|u'\|^s_{L^s(H_r)} \leq \left(\frac{C_1}{1-\alpha} \right)^s \left(\|u_0\|^s_{D^s_r} + C_7 \left(2R_1^{2s} + R_2^s \cdot T + \|f\|^s_{L^s(L^r)} \right) \right) .$$

On the other hand, from (B.2) and (B.3), one deduces:

$$\|\tau\|_{L^\infty(W^{1,r})} + \frac{4\alpha}{C_2 \mathrm{We}} \leq \left(\|\tau_0\|_{W^{1,r}} + \frac{4\alpha}{C_2 \mathrm{We}} \right) \exp\left(C_2 R_1 \cdot T^{1-\frac{1}{s}} \right) \equiv \Lambda' ,$$

$$\|\tau'\|_{L^s(L^r)} \leq 2^{1-\frac{1}{s}} C_2 \Lambda' \left(R_1 + \frac{T^{1/s}}{C_2 \mathrm{We}} \right) .$$

Consequently, if α, R_1, R_2, R_3, and the norms of u_0, τ_0 and f are such that the right sides of these three inequalities are bounded by R_1^s, $R_2 + \frac{4\alpha}{C_2 \mathrm{We}}$ and R_3 respectively, then $\Phi(Y(T)) \subset Y(T)$. First, we choose α_1, R_1 and R_2 such that

$$\left(\frac{C_1}{1-\alpha_1}\right)^s C_7 (2R_1^{2s} + R_2^s \cdot T) < R_1^s .$$

Then, we choose $\alpha_0 \in (0, \alpha_1]$ such that

$$\frac{4\alpha_0}{C_2 \mathrm{We}} \cdot \exp\left(C_2 R_1 \cdot T^{1-\frac{1}{s}}\right) < R_2 .$$

Finally, R_3 is fixed, such that

$$2^{1-\frac{1}{s}} C_2 \left(R_1 + \frac{T^{1/s}}{C_2 \mathrm{We}}\right) \cdot \frac{4\alpha_0}{C_2 \mathrm{We}} \cdot \exp\left(C_2 R_1 \cdot T^{1-\frac{1}{s}}\right) < R_3 .$$

When $0 < \alpha \leq \alpha_0 = \alpha_0(T)$ and the norms of u_0, τ_0 and f are sufficiently small, one has:

$$\left(\frac{C_1}{1-\alpha}\right)^s \left(\|u_0\|_{D_r^s}^s + C_7 (2R_1^{2s} + R_2^s \cdot T + \|f\|_{L^s(L^r)}^s)\right) \leq R_1^s ,$$

$$\left(\|\tau_0\|_{W^{1,r}} + \frac{4\alpha}{C_2 \mathrm{We}}\right) \exp\left(C_2 R_1 \cdot T^{1-\frac{1}{s}}\right) \leq R_2 + \frac{4\alpha}{C_2 \mathrm{We}} ,$$

$$2^{1-\frac{1}{s}} C_2 \left(R_1 + \frac{T^{1/s}}{C_2 \mathrm{We}}\right) \left(\|\tau_0\|_{W^{1,r}} + \frac{4\alpha}{C_2 \mathrm{We}}\right) \exp\left(C_2 R_1 \cdot T^{1-\frac{1}{s}}\right) \leq R_3 .$$

As in the proof of Theorem 2.1, we can now deduce that Φ possesses a fixed point in $Y(T)$ and also that this is the unique solution to (1.5)-(1.8). □

3. – The evolution problem (II): uniqueness and stability

THEOREM 3.1. *If* $u_0 \in H$, $\tau_0 \in L^2$, $f \in L^1(V')$ *and* (1.5)-(1.8) *possesses two weak solutions* $\{u^i, p^i, \tau^i\}$ $(i = 1, 2)$ *in* $[0, T]$ *(in the usual sense), with*

$$u^i \in L^\infty(H) \cap L^2(V) \cap L^1(W^{1,\infty}), \quad \tau^i \in L^\infty(L^2) \cap L^2(L^\infty) \cap L^2(W^{1,3}) ,$$

then they coincide (p^1 *and* p^2 *coincide up to a function only depending on* t).

PROOF. Let us introduce $u = u^1 - u^2$, $p = p^1 - p^2$ and $\tau = \tau^1 - \tau^2$. Then

$$(3.1)\qquad \begin{cases} \mathrm{Re}\langle u' + (u^1\cdot\nabla)u + (u\cdot\nabla)u^2, v\rangle + (1-\alpha)(\nabla u, \nabla v) = \langle \nabla\cdot\tau, v\rangle\,, \\ \forall\, v \in V\,, \end{cases}$$

a.e. in $[0, T]$,

$$(3.2)\quad \mathrm{We}(\tau' + (u^2\cdot\nabla)\tau + (u\cdot\nabla)\tau^1 + g_a(\tau^1, \nabla u) + g_a(\tau, \nabla u^2)) + \tau = 2\alpha D(u)$$

a.e. in $\Omega \times (0, T)$ and

$$(3.3)\qquad u\,|_{t=0} = 0, \qquad \tau\,|_{t=0} = 0\,.$$

The regularity properties of u and τ lead to the conclusion that $u' \in L^2(V') + L^1(H)$ and $\tau' \in L^2(L^1) \cap L^1(L^2)$. Here, u' and τ' must be understood as the usual time derivatives in $\mathcal{D}'(0, T; V')$ and $\mathcal{D}'(0, T; L^1)$, respectively. We can compute the duality product $\langle (3.1), u\rangle$ (since $u \in L^2(V) \cap L^\infty(H)$) and also the L^2 scalar product $\left((3.2), \frac{1}{2\alpha}\tau\right)$. They give together:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\mathrm{Re}\,|\,u\,|^2 + \frac{\mathrm{We}}{2\alpha}\,|\,\tau\,|^2\right) + (1-\alpha)\|u\|^2 + \frac{1}{2\alpha}\,|\,\tau\,|^2 &= -\mathrm{Re}\langle (u\cdot\nabla)u^2, u\rangle \\ - \frac{\mathrm{We}}{2\alpha}((u\cdot\nabla)\tau^1, \tau) - \frac{\mathrm{We}}{2\alpha}(g_a(\tau^1, \nabla u), \tau) &- \frac{\mathrm{We}}{2\alpha}(g_a(\tau, \nabla u^2), \tau)\,. \end{aligned}$$

Hence

$$(3.4)\qquad \begin{aligned} \frac{d}{dt}\left(\mathrm{Re}\,|\,u\,|^2 + \frac{\mathrm{We}}{2\alpha}\,|\,\tau\,|^2\right) &\le C\Big(\|u^2\|_{W^{1,\infty}} + \|\tau^1\|^2_{L^\infty} \\ &+ \|\tau^1\|^2_{W^{1,3}}\Big)\left(\mathrm{Re}\,|\,u\,|^2 + \frac{\mathrm{We}}{2\alpha}\,|\,\tau\,|^2\right), \end{aligned}$$

with C being a constant. Now, from Gronwall's lemma and (3.3), one easily deduces that u and τ vanish. □

Theorem 3.1 slightly improves Theorem 3 in [5]. When $\Omega \subset \mathbb{R}^N$ $(N \ge 4)$, one can deduce, in a similar way, uniqueness in the class

$$u \in L^\infty(H) \cap L^2(V) \cap L^1(W^{1,\infty}), \quad \tau \in L^\infty(L^2) \cap L^2(L^\infty) \cap L^2(W^{1,N})\,.$$

On the other hand, if $\Omega \subset \mathbb{R}^2$, one obtains uniqueness in the class

$$u \in L^\infty(H) \cap L^2(V) \cap L^1(W^{1,\infty}), \quad \tau \in L^\infty(L^2) \cap L^2(W^{1,2+\delta})\ (\delta > 0)\,.$$

Notice that Theorem 3.1 plays the role of Theorem 6.9 in [14] (p. 84), which provides uniqueness for the strong solution to the usual Navier-Stokes problem.

THEOREM 3.2. *Assume* $\partial\Omega \in C^{2,\mu}$ $(\mu > 0)$. *Also, assume* $1 < s < +\infty$, $3 < r < +\infty$, $T > 0$ *and*

$$u_0 \in D_r^s, \quad \tau_0 \in W^{1,r}, \quad f \in L^s(L^r) \cap L^2(H^{-1}).$$

Let $\{u, p, \tau\}$ *be the strong solution to* (1.5)-(1.8) *corresponding to these data. Then for each* $\delta > 0$, *one has:*

(a) *There exists* $\overline{T} \in (0, T]$ *only depending on* u_0, τ_0, f *and* δ, *such that if the data* $\bar{u}_0$, $\bar{\tau}_0$ *and* $\bar{f}$ *satisfy*

$$(\bar{u}_0, \bar{\tau}_0, \bar{f}) \in \mathcal{B}_0^\delta = \Big\{(\bar{u}_0, \bar{\tau}_0, \bar{f}) \in D_r^s \times W^{1,r} \times L^s(L^r) \cap L^2(H^{-1}) \ ; \\ \|\bar{u}_0 - u_0\|^2_{D_r^s} + \|\bar{\tau}_0 - \tau_0\|^2_{W^{1,r}} + \|\bar{f} - f\|^2_{L^s(L^r)} \le \delta^2\Big\},$$

then the corresponding system (1.5)-(1.8) *possesses exactly one strong solution, defined in the whole interval* $[0, \overline{T}]$.

(b) *There exists* $C = C(u_0, \tau_0, f, \delta) > 0$ *such that, if* $(\bar{u}_0, \bar{\tau}_0, \bar{f})$, $(\bar{v}_0, \bar{\sigma}_0, \bar{g}) \in \mathcal{B}_0^\delta$ *and* $\{\bar{u}, \bar{p}, \bar{\tau}\}$ *and* $\{\bar{v}, \bar{q}, \bar{\sigma}\}$ *are the strong solutions corresponding to these data, one has:*

$$\begin{aligned} &\|\bar{u} - \bar{v}\|^2_{L^\infty(H)} + \|\bar{u} - \bar{v}\|^2_{L^2(V)} + \|\bar{\tau} - \bar{\sigma}\|^2_{L^\infty(L^2)} \\ &\quad \le C\left(|\,\bar{u}_0 - \bar{v}_0\,|^2 + |\,\bar{\tau}_0 - \bar{\sigma}_0\,|^2 + \|\bar{f} - \bar{g}\|^2_{L^2(H^{-1})}\right). \end{aligned} \tag{3.5}$$

PROOF. Arguing as in the proof of Theorem 2.1, we see that if (u_0, τ_0, f) is given and R_1 and R_2 satisfy

$$R_1^s = \left(\frac{C_1}{1-\alpha}\right)^s \left(\|u_0\|^s_{D_r^s} + C_6\left(2 + \|u_0\|_{H_r}^{\frac{3s(r-1)}{2r}} + \|f\|^s_{L^s(L^r)}\right)\right)$$

and

$$R_2 = \left(\|\tau_0\|_{W^{1,r}} + \frac{4\alpha}{C_2 \mathrm{We}}\right) \exp C_2,$$

then there exists a unique strong solution to (1.5)-(1.8), defined at least in $[0, T_*]$, where

$$T_* = \min\left\{R_1^{-\frac{s(r+3)}{r-3}}, R_1^{-\frac{4sr}{3s(r-1)-2r}}, R_1^{-\frac{s}{s-1}}, R_2^{-s}, T\right\}.$$

Let $\delta > 0$ and $(\bar{u}_0, \bar{\tau}_0, \bar{f}) \in \mathcal{B}_0^\delta$ be fixed. Let us denote by $\overline{R}_1$, $\overline{R}_2$ and $\overline{T}_*$ the corresponding values of R_1, R_2 and T_*. Then

$$\begin{aligned} \overline{R}_1^s &\le \left(\frac{C_1}{1-\alpha}\right)^s ((\|u_0\|_{D_r^s} + \delta)^s \\ &\quad + C_6(2 + (\|u_0\|_{H_r} + \delta)^{\frac{3s(r-1)}{2r}} + (\|f\|_{L^s(L^r)} + \delta)^s)) \equiv S_1^s(\delta), \\ \overline{R}_2 &\le \left(\|\tau_0\|_{W^{1,r}} + \delta + \frac{4\alpha}{C_2 \mathrm{We}}\right) \exp C_2 \equiv S_2(\delta) \end{aligned}$$

and, consequently,

$$\overline{T}_* \geq \min\left\{S_1(\delta)^{-\frac{s(r+3)}{r-3}}, S_1(\delta)^{-\frac{4sr}{3s(r-1)-2r}}, S_1(\delta)^{-\frac{s}{s-1}}, S_2(\delta)^{-s}, T\right\} \equiv \overline{T}.$$

Hence, part a) of this theorem holds with this $\overline{T}$.

Now, assume $(\bar{u}_0, \bar{\tau}_0, \bar{f})$, $(\bar{v}_0, \bar{\sigma}_0, \bar{g}) \in \mathcal{B}_0^\delta$ and let us denote by $\{\bar{u}, \bar{p}, \bar{\tau}\}$ and $\{\bar{v}, \bar{q}, \bar{\sigma}\}$ the corresponding strong solutions to (1.5)-(1.8) (both defined at least in $[0, \overline{T}]$). Arguing as in the proof of Theorem 3.1, one finds:

$$\begin{aligned}\frac{d}{dt}\left(\mathrm{Re}\mid \bar{u}-\bar{v}\mid^2 + \frac{\mathrm{We}}{2\alpha}\mid \bar{\tau}-\bar{\sigma}\mid^2\right) + (1-\alpha)\|\bar{u}-\bar{v}\|^2 &\leq C_8\|\bar{f}-\bar{g}\|^2_{H^{-1}} \\ &+ C_9 h\left(\mathrm{Re}\mid \bar{u}-\bar{v}\mid^2 + \frac{\mathrm{We}}{2\alpha}\mid \bar{\tau}-\bar{\sigma}\mid^2\right).\end{aligned} \tag{3.6}$$

Here, $h(t) = \|\bar{v}(t)\|_{W^{1,\infty}} + \|\bar{\tau}(t)\|^2_{L^\infty} + \|\bar{\tau}(t)\|^2_{W^{1,3}}$. Since $D(A_r) \subset W^{1,\infty}$, $W^{1,r} \subset L^\infty$, $W^{1,r} \subset W^{1,3}$, $\|\bar{v}\|_{L^s(D(A_r))} \leq S_1(\delta)$ and $\|\bar{\tau}\|_{L^\infty(W^{1,r})} \leq S_2(\delta)$, one has:

$$\|h\|_{L^1(0,\overline{T})} \leq C_{10}\left(S_1(\delta)\cdot \overline{T}^{1/s'} + S_2(\delta)^2 \cdot \overline{T}\right).$$

Integrating (3.6) with respect to time in $[0, \overline{T}]$ and using Gronwall's lemma, one easily obtains (3.5). This ends the proof. □

The previous is a stability (or continuous dependence) result for the strong solutions to (1.5)-(1.8) furnished by Theorem 2.1. With some obvious changes, one can also demonstrate similar stability results for global solutions corresponding to small data.

4. – The stationary problem

In this section, $\Omega \subset \mathbb{R}^N$ ($N = 2$ or $N = 3$) is a bounded connected open set and $\partial\Omega \in C^2$. We consider the stationary problem corresponding to (1.5)-(1.7), that is

$$\mathrm{Re}(u\cdot\nabla)u - (1-\alpha)\Delta u + \nabla p = \nabla\cdot\tau + f, \qquad \nabla\cdot u = 0 \text{ in } \Omega, \tag{4.1}$$

$$\mathrm{We}((u\cdot\nabla)\tau + g_a(\tau, \nabla u)) + \tau = 2\alpha D(u) \qquad \text{in } \Omega, \tag{4.2}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4.3}$$

THEOREM 4.1. (a) *If* $f \in L^r$ ($N < r < +\infty$) *has a sufficiently small* L^r *norm, then* (4.1)-(4.3) *possesses exactly one small strong solution* $\{u, p, \tau\}$ (p *is unique up to a constant*), *with*

$$u \in D(A_r), \qquad \tau \in W^{1,r}.$$

(b) *There exists a constant* $C = C(\alpha, r, a, \mathrm{Re}, \mathrm{We}, \Omega) > 0$ *such that, if* $\{u, p, \tau\}$ *and* $\{v, q, \sigma\}$ *are the small strong solutions corresponding to the data* f *and* g, *both with sufficiently small* L^r *norms, then:*

$$\|u - v\|^2 + | \tau - \sigma |^2 \leq C \|f - g\|^2_{H^{-1}} .$$

For the proof of Theorem 4.1, we need the following two lemmas:

LEMMA D. *If* $F \in L^r$ $(1 < r < +\infty)$, *then there exists exactly one* $\{u, q\}$ *(*q *is unique up to a constant), such that*

$$u \in D(A_r), \qquad q \in W^{1,r},$$

$$\text{(D.1)} \qquad -\Delta u + \nabla q = F \qquad a.e. \ in\ \Omega .$$

Furthermore, for some $C_1' = C_1'(r, \Omega)$, *one has:*

$$\text{(D.2)} \qquad \|u\|_{W^{2,r}} + \|\nabla q\|_{L^r} \leq C_1' \|F\|_{L^r} .$$

LEMMA E. *Assume* $N < r < \infty$ *and* $\bar{u} \in D(A_r)$. *One has the following:*
(a) *If* $G \in L^r$, *there exists exactly one* $\tau \in L^r$ *such that*

$$\text{(E.1)} \qquad \mathrm{We}(\bar{u} \cdot \nabla)\tau + \tau = G \quad a.e. \ in \quad \Omega .$$

Moreover,

$$\|\tau\|_{L^r} \leq \|G\|_{L^r} .$$

(b) *There exists* $C_2' = C_2'(r, \Omega)$ *such that, if* $G \in W^{1,r}$ *and*

$$\|\bar{u}\|_{W^{2,r}} \leq \frac{1}{2C_2'\mathrm{We}},$$

then $\tau \in W^{1,r}$ *and*

$$\text{(E.2)} \qquad \|\tau\|_{W^{1,r}} \leq 2\|G\|_{W^{1,r}} .$$

Lemma D is proved in [11] (Theorem 2, p. 67); on the other hand, Lemma E is demonstrated in [17].

PROOF OF THEOREM 4.1. We will only present the proof of part (a). The proof of part (b) can be achieved with arguments like those used for uniqueness in part (a).

1. *Existence* – For each $u \in D(A_r)$, let us introduce the bounded linear operators $\mathcal{R}(u)$ and $\mathcal{L}(u)$. First, we put

$$\text{(4.4)} \qquad \mathcal{R}(u) : L^r(\Omega) \to L^r(\Omega), \qquad \mathcal{R}(u)v = w ,$$

with w being the unique solution to

$$\mathrm{We}(u \cdot \nabla)w + w = v\,.$$

From Lemma E, one has

$$\|\mathcal{R}(u)v\|_{L^r} \leq \|v\|_{L^r}\,.$$

Furthermore, if $\|u\|_{W^{2,r}}$ is small, $\mathcal{R}(u)$ maps $W^{1,r}(\Omega)$ into itself continuously. Secondly, we set

$$\mathcal{L}(u) : L^r(\Omega) \to L^r(\Omega), \quad \mathcal{L}(u)v = (1-\alpha)v + \alpha\mathcal{R}(u)v\,. \tag{4.5}$$

It is easy to check that

$$\|\mathcal{L}(u)v\|_{L^r} \leq \|v\|_{L^r} \qquad \forall\, v \in L^r$$

(recall that $\alpha \in (0,1)$). The operator $\mathcal{L}(u)$ has an inverse

$$\mathcal{L}(u)^{-1} : L^r(\Omega) \to L^r(\Omega)\,, \tag{4.6}$$

with

$$\|\mathcal{L}(u)^{-1}v\|_{L^r} \leq \left(\frac{1+\alpha}{1-\alpha}\right)\|v\|_{L^r}$$

(see [17]). At present, we are going to use formal calculus. This will lead to a reformulation of (4.1)-(4.3) as a fixed point equation. By computing the divergence of both sides of (4.2), we obtain:

$$\mathrm{We}(u \cdot \nabla)(\nabla \cdot \tau) + \nabla \cdot \tau = \alpha\Delta u - \mathrm{We}(\nabla \cdot g_a(\tau, \nabla u) + \partial u : \partial\tau)\,.$$

Here, $\partial u : \partial\tau$ is the vector whose j-th component is equal to

$$(\partial u : \partial\tau)_j = \sum_{i,k} \frac{\partial u_k}{\partial x_i}\,\frac{\partial \tau_{ij}}{\partial x_k}\,.$$

Hence,

$$\nabla \cdot \tau = \mathcal{R}(u)(\alpha\Delta u - \mathrm{We}(\nabla \cdot g_a(\tau, \nabla u) + \partial u : \partial\tau))\,. \tag{4.7}$$

From (4.7) and (4.1), we see that

$$-(1-\alpha)\Delta u + \nabla p = -\mathrm{Re}(u \cdot \nabla)u + \mathcal{R}(u)(\alpha\Delta u - \mathrm{We}(\nabla \cdot g_a(\tau, \nabla u) + \partial u : \partial\tau)) + f\,.$$

This leads to the identities

$$-\mathcal{L}(u)\Delta u + \nabla p = -\mathrm{Re}(u \cdot \nabla)u - \mathrm{We}\mathcal{R}(u)(\nabla \cdot g_a(\tau, \nabla u) + \partial u : \partial\tau) + f$$

and

$$-\Delta u + \nabla q = -\mathrm{Re}\,\mathcal{L}(u)^{-1}((u\cdot\nabla)u) - \mathrm{We}\,\mathcal{L}(u)^{-1}\mathcal{R}(u)(\nabla\cdot g_a(\tau,\nabla u) + \partial u : \partial\tau)$$
$$+\mathcal{L}(u)^{-1}f + \alpha\mathrm{We}\,\mathcal{L}(u)^{-1}\mathcal{R}(u)({}^t\nabla u\cdot\nabla(\mathcal{R}(u)q)),$$

with $q = \mathcal{L}(u)^{-1}p$. Here, we have used the fact that

$$\mathcal{L}(u)^{-1}(\nabla p) = \nabla q - \alpha\mathrm{We}\mathcal{L}(u)^{-1}\mathcal{R}(u)({}^t\nabla u\cdot\nabla(\mathcal{R}(u)q))$$

(see [17]).

We will introduce a Banach space X, a convex compact set $Y(\varepsilon)$ and a continuous mapping $\Phi : Y(\varepsilon) \to Y(\varepsilon)$ in such a way that (4.1)-(4.3) is equivalent to a fixed point equation for Φ in $Y(\varepsilon)$. More precisely, let us set $X = W_0^{1,r}\times L^r\times L^r$ and

$$Y(\varepsilon) = \Big\{(\bar u,\bar q,\bar\tau);\ \bar u\in D(A_r),\ \bar q\in W^{1,r},\ \bar\tau\in W^{1,r},$$
$$\|\bar u\|_{W^{2,r}} + \|\nabla\bar q\|_{L^r} \le \frac{\varepsilon}{2C_2'\mathrm{We}},\ \|\bar\tau\|_{W^{1,r}} \le \frac{2\varepsilon}{C_2'\mathrm{We}}\Big\}$$

for each $\varepsilon > 0$. Obviously, $Y(\varepsilon)\neq\emptyset$; now, let $\Phi : Y(\varepsilon)\to X$ be given as follows: $\Phi(\bar u,\bar q,\bar\tau) = (u,q,\tau)$, with $\{u,q\}$ being the solution to (D.1) with

$$F = -\mathrm{Re}\,\mathcal{L}(\bar u)^{-1}((\bar u\cdot\nabla)\bar u) - \mathrm{We}\mathcal{L}(\bar u)^{-1}\mathcal{R}(\bar u)(\nabla\cdot g_a(\bar\tau,\nabla\bar u) + \partial\bar u:\partial\bar\tau)$$
$$+\mathcal{L}(\bar u)^{-1}f + \alpha\mathrm{We}\,\mathcal{L}(\bar u)^{-1}\mathcal{R}(\bar u)({}^t\nabla\bar u\cdot\nabla(\mathcal{R}(\bar u)\bar q)), \tag{4.8}$$

$$\int_\Omega q\,dx = 0$$

and with τ being the solution to (E.1) with

$$G = 2\alpha D(\bar u) - \mathrm{We}g_a(\bar\tau,\nabla\bar u).$$

If (u,q,τ) is a fixed point for Φ, then $\{u,p,\tau\}$ solves (4.1)-(4.3) with $p = \mathcal{L}(u)q$. Let us see that if $\varepsilon > 0$ and $\|f\|_{L^r}$ are small enough then $\Phi(Y(\varepsilon))\subset Y(\varepsilon)$. First, if $(\bar u,\bar q,\bar\tau)\in Y(\varepsilon)$, then for some constants $C_3' = C_3'(a,r,\Omega)$ and $C_4' = C_4'(a,r,\Omega)$, one has:

$$\|F\|_{L^r} \le C_3'\left(\frac{1+\alpha}{1-\alpha}\right)\Big(\mathrm{Re}\|\bar u\|_{W^{1,r}}^2 + \mathrm{We}\|\bar\tau\|_{W^{1,r}}\|\bar u\|_{W^{2,r}}$$
$$+\|f\|_{L^r} + \alpha\mathrm{We}\|\bar u\|_{W^{2,r}}\|\nabla\bar q\|_{L^r}\Big)$$

and

$$\|G\|_{W^{1,r}} \le 2\alpha\|\bar u\|_{W^{2,r}} + C_4'\mathrm{We}\|\bar\tau\|_{W^{1,r}}\|\bar u\|_{W^{2,r}}.$$

Using (D.2), (E.2) and the definition of $Y(\varepsilon)$, we obtain:

$$\|u\|_{W^{2,r}}+\|\nabla q\|_{L^r} \leq C_1' C_3' \left(\frac{1+\alpha}{1-\alpha}\right)\left(\left(\frac{\mathrm{Re}+\alpha\mathrm{We}}{(2C_2'\mathrm{We})^2}+\frac{\mathrm{We}}{(C_2'\mathrm{We})^2}\right)\varepsilon^2+\|f\|_{L^r}\right),$$

$$\|\tau\|_{W^{1,r}} \leq \left(\frac{2\alpha}{C_2'\mathrm{We}}\right)\varepsilon+\left(\frac{2C_4'\mathrm{We}}{(C_2'\mathrm{We})^2}\right)\varepsilon^2 .$$

Hence, one will have $\Phi(Y(\varepsilon)) \subset Y(\varepsilon)$ provided $\varepsilon > 0$ and $\|f\|_{L^r}$ are such that the right sides in the previous inequalities are bounded respectively by

$$\frac{\varepsilon}{2C_2'\mathrm{We}} \quad \text{and} \quad \frac{2\varepsilon}{C_2'\mathrm{We}} .$$

It is not difficult to find $\varepsilon > 0$ and f satisfying these conditions. The conclusion is similar to the one in the proof of Theorem 2.1, since Φ is continuous (see [17]).

2. *Uniqueness* – Proceeding as in the proof of Theorem 3.1, one obtains:

$$(1-\alpha)\|u\|^2+\frac{1}{2\alpha}\mid \tau \mid^2 = -\mathrm{Re}((u\cdot\nabla)u^2,u)-\frac{\mathrm{We}}{2\alpha}((u\cdot\nabla)\tau^1,\tau)$$
$$-\frac{\mathrm{We}}{2\alpha}(g_a(\tau^1,\nabla u),\tau)-\frac{\mathrm{We}}{2\alpha}(g_a(\tau,\nabla u^2),\tau) .$$

Thus,

$$(1-\alpha)\|u\|^2+\frac{1}{2\alpha}\mid \tau \mid^2 \leq \mathrm{Re}C_1\|u^2\|_{W^{2,r}}\|u\|^2+\frac{\mathrm{We}}{2\alpha}C_2\|\tau^1\|_{W^{1,r}}\mid \tau \mid\cdot\|u\|$$
$$+\frac{\mathrm{We}}{2\alpha}C_3\|u^2\|_{W^{2,r}}\mid \tau \mid^2 ,$$

where $C_i = C_i(r,\Omega)$, $i = 1,2,3$. Hence,

$$\|u\|^2+\mid \tau \mid^2 \leq K(\|u^2\|_{W^{2,r}}+\|\tau^1\|_{W^{1,r}})(\|u\|^2+\mid \tau \mid^2) ,$$

where K is a new constant. By assumption,

$$\|u^2\|_{W^{2,r}} \leq \delta \quad \text{and} \quad \|\tau^1\|_{W^{1,r}} \leq \delta .$$

Thus, from the above estimates, we deduce that $u \equiv 0$ and $\tau \equiv 0$ whenever $\delta > 0$ is sufficiently small. □

The use we have made of (4.8) in the right side of the Stokes problem (D.1) is suggested by the work of M. Renardy ([19]). If, for example, $f \in H^1$, one can use the contractive mapping principle as in [19], which leads to an iterative algorithm for the computation of the solution. In [9] (Corollary 4.3), C. Guillopé and J.-C. Saut present an existence, uniqueness and stability result for (4.1)-(4.3) when $0 < \alpha \leq \alpha_0 < 1$ and $f \in H^1$ is sufficiently small.

5. – Global existence for Poiseuille flows in cylindrical domains

In this section, we introduce cylindrical coordinates. We will consider the region

$$\Omega_{R_1,R_2} = \{(r,\varphi,z);\, 0 < R_1 < r < R_2,\, 0 \le \varphi < 2\pi,\, z \in \mathbb{R}\}$$

and we will study Poiseuille flows for an Oldroyd-like fluid in $\Omega_{R_1,R_2} \times (0,T)$. More precisely, with data of the form

$$f(0,0,f^z(r,t)),\;\; u_0 = (0,0,v_0(r)),\;\; \tau_0 = \begin{bmatrix} \tau_0^1(r) & 0 & \tau_0^3(r) \\ 0 & \tau_0^4(r) & 0 \\ \tau_0^3(r) & 0 & \tau_0^6(r) \end{bmatrix},$$

we will search for a solution to (1.5)-(1.6) of the form

$$u = (0,0,v(r,t)), \qquad \tau = \begin{bmatrix} \tau^1(r,t) & 0 & \tau^3(r,t) \\ 0 & \tau^4(r,t) & 0 \\ \tau^3(r,t) & 0 & \tau^6(r,t) \end{bmatrix}.$$

One must have:

$$p_r = \tau_r^1 + \frac{1}{r}(\tau^1 - \tau^4), \quad p_\varphi = 0,$$

$$\mathrm{Re}v_t - (1-\alpha)v_{rr} - (1-\alpha)\frac{1}{r}v_r = \tau_r^3 + \frac{1}{r}\tau^3 + f^z - p_z$$

and also the following constitutive equations:

$$\tau_t^1 + \frac{\tau^1}{\mathrm{We}} = -(1-a)\tau^3 v_r\,,$$

$$\tau_t^3 + \frac{\tau^3}{\mathrm{We}} = \frac{\alpha}{\mathrm{We}}v_r + \left(\frac{1+a}{2}\right)\tau^1 v_r - \left(\frac{1-a}{2}\right)\tau^6 v_r\,,$$

$$\tau_t^4 + \frac{\tau^4}{\mathrm{We}} = 0,$$

$$\tau_t^6 + \frac{\tau^6}{\mathrm{We}} = (1+a)\tau^3 v_r\,.$$

Here, τ_t^i stands for the time derivative of τ^i, etc. Assuming (as usual) that the "pressure gradient" $p_z = p_0(t)$ is prescribed, we are led to the following system (see [17] for the details):

$$\mathrm{Re}v_t - (1-\alpha)v_{rr} - (1-\alpha)\frac{1}{r}v_r = \tau_r^3 + \frac{1}{r}\tau^3 + f\,,$$

$$\tau_t^1 + \frac{\tau^1}{\mathrm{We}} = -(1-a)\tau^3 v_r\,,$$

$$\tau_t^3 + \frac{\tau^3}{\mathrm{We}} = \frac{\alpha}{\mathrm{We}}v_r + \left(\frac{1+a}{2}\right)\tau^1 v_r - \left(\frac{1-a}{2}\right)\tau^6 v_r\,,$$

$$\tau_t^6 + \frac{\tau^6}{\mathrm{We}} = (1+a)\tau^3 v_r\,.$$

Here $f = f^z - p_z$ is given. Introducing the new variables

$$\sigma^1 = -\left(\frac{1+a}{2}\right)\tau^1 + \left(\frac{1-a}{2}\right)\tau^6,\ \sigma^2 = \tau^3,\ \sigma^3 = (1+a)\tau^1 + (1-a)\tau^6,$$

this can be written as follows:

$$\mathrm{Re} v_t - (1-\alpha)v_{rr} - (1-\alpha)\frac{1}{r}v_r = \sigma^2 + \frac{1}{r}\sigma^2 + f\,, \tag{5.1}$$

$$\sigma_t^1 + \frac{\sigma^1}{\mathrm{We}} = (1-a^2)\sigma^2 v_r\,, \tag{5.2}$$

$$\sigma_t^2 + \frac{\sigma^2}{\mathrm{We}} = \frac{\alpha}{\mathrm{We}}v_r - \sigma^1 v_r\,, \tag{5.3}$$

$$\sigma_t^3 + \frac{\sigma^3}{\mathrm{We}} = 0\,.$$

The main task in this section is to solve (5.1)-(5.3) in $(R_1, R_2) \times (0, T)$, together with appropriate boundary and initial conditions. For simplicity, we assume

$$v(R_1, t) = v(R_2, t) = 0 \qquad \text{for} \quad t \in (0, T)\,, \tag{5.4}$$

$$v\mid_{t=0} = v_0, \quad \sigma^i \mid_{t=0} = \sigma_0^i \ \ (i = 1, 2) \quad \text{in} \quad (R_1, R_2)\,. \tag{5.5}$$

THEOREM 5.1. *Assume* $0 < R_1 < R_2$, $T > 0$, $\mid a \mid < 1$.

(a) *If*

$$v_0 \in L^2, \quad \sigma_0^i \in L^\infty \quad (i = 1, 2), \quad f \in L^2(H^{-1})\,,$$

then (5.1)-(5.5) *possesses exactly one "semi-strong" solution* $\{v, \sigma^1, \sigma^2\}$ *in* $[0, T]$, *i.e.*

$$v \in C(L^2) \cap L^2(H_0^1), \quad \sigma^i \in L^\infty(L^\infty) \cap C(L^2) \qquad (i = 1, 2)\,,$$

the equation (5.1) *is satisfied in the following weak sense*

$$\begin{aligned} \mathrm{Re}\langle v_t, w\rangle + (1-\alpha)(v_r, w_r) - (1-\alpha)\left(\frac{1}{r}v_r, w\right) &= \langle \sigma_r^2, w\rangle \\ + \left(\frac{1}{r}\sigma^2, w\right) + \langle f, w\rangle \quad \forall w \in H_0^1 \quad &\textit{a.e. in} \quad (0, T)\,, \end{aligned} \tag{5.1$'$}$$

and the equations (5.2)-(5.3) *are satisfied a.e. in* $(R_1, R_2) \times (0, T)$.

(b)

$$\textit{If} \quad v_0 \in H_0^1, \quad \sigma_0^i \in H^1 \qquad (i = 1, 2), \quad f \in L^2(L^2)\,,$$

then (5.1)-(5.5) *possesses exactly one strong solution* $\{v, \sigma^1, \sigma^2\}$ *in* $[0, T]$, *i.e.*

$$v \in C(H_0^1) \cap L^2(H^2), \qquad \sigma^i \in C(H^1) \qquad (i = 1, 2)$$

and (5.1)-(5.3) *are satisfied a.e. in* $(R_1, R_2) \times (0, T)$.

THEOREM 5.2. *Assume* $0 < R_1 < R_2$, $T > 0$, $|\, a \,| < 1$ *and*

$$v_0 \in L^2, \quad \sigma_0^i \in L^\infty \qquad (i = 1, 2), \quad f \in L^2(H^{-1}).$$

Then, for each $\delta > 0$, *there exists* $C = C(\|\sigma_0^1\|_{L^\infty}, \|\sigma_0^2\|_{L^\infty}, \delta) > 0$ *such that, if* $\{\bar v, \bar\sigma^1, \bar\sigma^2\}$ *and* $\{\bar w, \bar\tau^1, \bar\tau^2\}$ *solve* (5.1)-(5.5) *with data* $(\bar v_0, \bar\sigma_0^1, \bar\sigma_0^2, \bar f)$ *and* $(\bar w_0, \bar\tau_0^1, \bar\tau_0^2, \bar g)$ *respectively and these belong to*

$$\mathcal{B}_0^\delta = \left\{ (\bar v_0, \bar\sigma_0^1, \bar\sigma_0^2, \bar f) \in L^2 \times L^\infty \times L^\infty \times L^2(H^{-1}); \right.$$

$$\left. |\, \bar v_0 - v_0 \,|^2 + \sum_{i=1,2} \|\bar\sigma_0^i - \sigma_0^i\|_{L^\infty}^2 + \|\bar f - f\|_{L^2(H^{-1})}^2 \le \delta^2 \right\},$$

one has:

$$\|\bar v - \bar w\|_{L^\infty(L^2)}^2 + \|\bar v - \bar w\|_{L^2(H_0^1)}^2 + \sum_{i=1,2} \|\bar\sigma^i - \bar\tau^i\|_{L^\infty(L^2)}^2$$

$$\le C \left(|\, \bar v_0 - \bar w_0 \,|^2 + \sum_{i=1,2} |\, \sigma_0^i - \bar\tau_0^i \,|^2 + \|\bar f - \bar g\|_{L^2(H^{-1})}^2 \right).$$

Part (a) in Theorem 5.1 holds for two-dimensional Couette and Poiseuille flows, which have been studied by C. Guillopé and J.-C. Saut in [10]. On the other hand, the argument we use for the proof of part (b) is similar to the one in [10] (the main difference in the new term $\frac{1}{r}((1-\alpha)v_r + \sigma^2)$). One can also deduce a stability result for the strong solutions furnished by Theorem 5.1, part (b) (see [17]).

For simplicity, we will only present the part of the proof of Theorem 5.1 concerning existence. For the details and other proofs, see [17].

PROOF OF THEOREM 5.1 (EXISTENCE). Let us first prove part (b).

There exists $T_* \in (0, T]$ and a unique strong solution $\{v, \sigma^1, \sigma^2\}$ in $[0, T_*]$, with

$$v \in C([0, T_*]; H_0^1) \cap L^2(0, T_*; H^2), \quad \sigma^i \in C([0, T_*]; H^1) \quad (i = 1, 2).$$

This can be deduced as in the proof of Theorem 2.1. Notice that, if $T_* < T$, it can be assumed that

$$T_* = \frac{1}{\chi(\|v_0\|, \|\sigma_0^1\|_{H^1}, \|\sigma_0^2\|_{H^1}, \|f\|_{L^2(L^2)})}, \tag{5.6}$$

where χ is increasing in each argument; this will be an important fact in the passage from a local to a global solution. We can find estimates for $\{v, \sigma^1, \sigma^2\}$ not depending on T_*. It suffices to compute

$$\left((5.2) - \frac{\alpha}{\mathrm{We}^2} \right) \cdot \left(\sigma^1 - \frac{\alpha}{\mathrm{We}} \right) + (5.3) \cdot ((1 - a^2)\sigma^2).$$

Indeed, we find

$$\frac{d}{dt}\psi(r,t) + \frac{2}{\mathrm{We}}\psi(r,t) = -\frac{2\alpha}{\mathrm{We}^2}\left(\sigma^1 - \frac{\alpha}{\mathrm{We}}\right) \le \frac{\alpha^2}{\mathrm{We}^3} + \frac{1}{\mathrm{We}}\psi(r,t)\,,$$

with

$$\psi(r,t) = \left(\sigma^1 - \frac{\alpha}{\mathrm{We}}\right)^2 + (1+a^2)(\sigma^2)^2\,.$$

Hence,

$$\psi(r,t) \le \psi(r,0)e^{-t/\mathrm{We}} + \frac{\alpha^2}{\mathrm{We}^2}\left(1 - e^{-t/\mathrm{We}}\right)$$

and

$$\|\sigma^i\|_{L^\infty(0,T_*;L^\infty)} \le C_1(a, \alpha, \mathrm{We}, \|\sigma_0^1\|_{L^\infty}, \|\sigma_0^2\|_{L^\infty}) \quad (i = 1, 2)\,. \tag{5.7}$$

On the other hand, multiplying (5.1) by v in $L^2(R_1, R_2)$, one also finds:

$$\frac{1}{2}\frac{d}{dt}(\mathrm{Re} \mid v \mid^2) + (1-\alpha) \mid v_r \mid^2 = (1-\alpha)\left(\frac{1}{r}v_r, v\right) - (\sigma^2, v_r) + \left(\frac{1}{r}\sigma^2, v\right) + (f, v)\,.$$

This, together with (5.7), leads to the following:

$$\|v\|_{L^\infty(0,T_*;L^2)} + \|v\|_{L^2(0,T_*;H_0^1)} \le C_2(\alpha, \mathrm{Re}, R_1, R_2, C_1, \mid v_0 \mid, \|f\|_{L^2(L^2)})\,. \tag{5.8}$$

If we denote by $(5.2)_r$ (respectively $(5.3)_r$) the r-derivative of (5.2) (respectively (5.3)), by computing

$$\left((5.2)_r, \sigma_r^1\right) + \left((5.3)_r, (1-a^2)\sigma_r^2\right) + \left((5.1), -\frac{(1-a^2)\alpha}{\mathrm{We}}v_{rr}\right)\,,$$

we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\frac{(1-a^2)\alpha\mathrm{Re}}{\mathrm{We}} \mid v_r \mid^2 + \mid \sigma_r^1 \mid^2 + (1-a^2) \mid \sigma_r^2 \mid^2\right) + \frac{(1-a^2)\alpha(1-\alpha)}{\mathrm{We}} \mid v_{rr} \mid^2 \\
&\quad + \frac{1}{\mathrm{We}}(\mid \sigma_r^1 \mid^2 + (1-a^2) \mid \sigma_r^2 \mid^2) = -\frac{(1-a^2)\alpha(1-\alpha)}{\mathrm{We}}\left(\frac{1}{r}v_r, v_{rr}\right) \\
&\quad - \frac{(1-a^2)\alpha}{\mathrm{We}}\left(\frac{1}{r}\sigma^2, v_{rr}\right) - \frac{(1-a^2)\alpha}{\mathrm{We}}(f, v_{rr}) + (1-a^2)(\sigma^2 v_{rr}, \sigma_r^1) \\
&\quad - (1-a^2)(\sigma^1 v_{rr}, \sigma_r^2)\,.
\end{aligned}$$

Using (5.7) and (5.8) to estimate the terms in the right side, one finds

$$(5.9)\quad \|v\|_{L^\infty(0,T_*;H_0^1)} + \|v\|_{L^2(0,T_*;H^2)} \le C_3, \quad \|\sigma^i\|_{L^\infty(0,T_*;H^1)} \le C_3 \quad (i = 1, 2),$$

where $C_3 = C_3(a, \alpha, \mathrm{Re}, \mathrm{We}, R_1, R_2, T, \|v_0\|, \|\sigma_0^1\|_{H^1}, \|\sigma_0^2\|_{H^1}, \|f\|_{L^2(L^2)})$. Let us introduce

$$T_{\max} = \sup\{T_* > 0;\ \text{the unique solution to (5.1)-(5.5) is defined in } [0, T_*]\}.$$

From the estimates (5.9), we see that

$$(5.10)\quad \limsup_{t\to T_{\max}^-} \left(\|v\|_{C([0,t];H_0^1)} + \|v\|_{L^2(0,t;H^2)} + \sum_{i=1,2} \|\sigma^i\|_{C([0,t];H^1)} \right) < +\infty .$$

Now, (5.6), (5.10) and a standard argument show that the solution is defined in the whole interval $[0, T]$.

The proof of part (a) is divided in four steps.

First Step: The existence of approximate solutions. Let us choose sequences $\{v_0^m\}$, $\{\sigma_0^{im}\}$ $(i = 1, 2)$ and $\{f^m\}$ such that $v_0^m \in H_0^1$, $\sigma_0^{im} \in H^1$, $f^m \in L^2(L^2)$ for all m,

$$\begin{aligned} &v_0^m \to v_0 \quad \text{in} \quad L^2, \\ &\sigma_0^{im} \to \sigma_0^i \quad \text{weakly}^* \text{ in} \quad L^\infty \quad \text{and a.e.}, \\ &f^m \to f \quad \text{in} \quad L^2(H^{-1}). \end{aligned}$$

From part (b), which has already been demonstrated, we know that for each m there exist

$$v^m \in C(H_0^1) \cap L^2(H^2), \qquad \sigma^{im} \in C(H^1),$$

such that

$$(5.1)_m\quad \mathrm{Re}\langle v_t^m, w\rangle + (1-\alpha)(v_r^m, w_r) - (1-\alpha)\left(\frac{1}{r}v_r^m, w\right) = \langle \sigma_r^{2m}, w\rangle + \left(\frac{1}{r}\sigma^{2m}, w\right) + \langle f^m, w\rangle \qquad \forall\, w \in H_0^1,$$

$$(5.2)_m\quad \sigma_t^{1m} + \frac{\sigma^{1m}}{\mathrm{We}} = (1 - a^2)\sigma^{2m} v_r^m \qquad \text{a.e. in } (R_1, R_2)\times(0, T),$$

$$(5.3)_m\quad \sigma_t^{2m} + \frac{\sigma^{2m}}{\mathrm{We}} = \frac{\alpha}{\mathrm{We}} v_r^m - \sigma^{1m} v_r^m \qquad \text{a.e. in } (R_1, R_2)\times(0, T),$$

$$(5.4)_m\quad v^m|_{t=0} = v_0^m, \quad \sigma^{im}|_{t=0} = \sigma_0^{im}\ (i = 1, 2) \qquad \text{a.e. in } (R_1, R_2).$$

SECOND STEP: UNIFORM ESTIMATES FOR THE APPROXIMATE SOLUTIONS. By computing

$$\left((5.2)_m - \frac{\alpha}{\mathrm{We}^2}\right)\cdot\left(\sigma^{1m} - \frac{\alpha}{\mathrm{We}}\right) + (5.3)_m \cdot \left((1-a^2)\sigma^{2m}\right),$$

we easily obtain

$$\frac{d}{dt}\psi^m(r,t) + \frac{2}{\mathrm{We}}\psi^m(r,t) = -\frac{2\alpha}{\mathrm{We}^2}\left(\sigma^{1m} - \frac{\alpha}{\mathrm{We}}\right) \le \frac{\alpha^2}{\mathrm{We}^3} + \frac{1}{\mathrm{We}}\psi^m(r,t),$$

where

$$\psi^m(r,t) = \left(\sigma^{1m,} - \frac{\alpha}{\mathrm{We}}\right)^2 + (1-a^2)(\sigma^{2m})^2 .$$

Consequently,

$$\psi^m(r,t) \le \psi^m(r,0)e^{-t/\mathrm{We}} + \frac{\alpha^2}{\mathrm{We}^2}\left(1 - e^{-t/\mathrm{We}}\right)$$

and

(5.11) $$\|\sigma^{im}\|_{L^\infty(L^\infty)} \le C_1(a, \alpha, \mathrm{We}, \|\sigma_0^1\|_{L^\infty}, \|\sigma_0^2\|_{L^\infty}) \quad (i = 1, 2).$$

Taking $w = v^m$ in $(5.1)_m$, we see that

$$\frac{1}{2}\frac{d}{dt}(\mathrm{Re} \mid v^m \mid^2) + (1-\alpha) \mid v_r^m \mid^2 = (1-\alpha)\left(\frac{1}{r}v_r^m, v^m\right) - (\sigma^{2m}, v_r^m) + \left(\frac{1}{r}\sigma^{2m}, v^m\right) + \langle f^m, v^m\rangle .$$

From this and (5.11), one finds:

(5.12) $$\|v^m\|_{L^\infty(L^2)} + \|v^m\|_{L^2(H_0^1)} \le C_2(\alpha, \mathrm{Re}, R_1, R_2, C_1, \mid v_0 \mid, \|f\|_{L^2(H^{-1})}).$$

Using (5.11) and (5.12), one also has:

(5.13) $$\|v_t^m\|_{L^2(H^{-1})} \le C_3(C_1, C_2), \qquad \|\sigma_t^{im}\|_{L^2(L^2)} \le C_4(C_1, C_2).$$

Now, (5.11)-(5.13) can be used to prove the existence of subsequences (again indexed with m) and functions v and σ^i such that:

$$\begin{aligned}
&v^m \to v && \text{weakly} && \text{in} \quad L^2(H_0^1) \quad \text{weakly* in} \quad L^\infty(L^2),\\
&v_t^m \to v_t && \text{weakly} && \text{in} \quad L^2(H^{-1}),\\
&\sigma^{im} \to \sigma^i && \text{weakly*} && \text{in} \quad L^\infty(L^\infty),\\
&\sigma_t^{im} \to \sigma_t^i && \text{weakly} && \text{in} \quad L^2(L^2).
\end{aligned}$$

From well known compactness results, one also has:

$$v^m \to v \quad \text{strongly in} \quad L^2(L^2)\,.$$

This suffices to take limits in $(5.1)_m$, which gives $(5.1)'$. However, these convergence properties are not strong enough to pass to the limit in $(5.2)_m$-$(5.3)_m$ because of the terms $\sigma^{im} \cdot v_r^m$ (recall that σ^{im} and v_r^m only converge weakly).

THIRD STEP: $\{v^m, \sigma^{1m}, \sigma^{2m}\}$ IS A CAUCHY SEQUENCE. Notice that

$$\begin{aligned}(5.14)\quad &\operatorname{Re}\langle v_t^m - v_t^n, w\rangle + (1-\alpha)(v_r^m - v_r^n, w_r) - (1-\alpha)\left(\frac{1}{r}(v_r^m - v_r^n), w\right)\\ &= -(\sigma^{2m} - \sigma^{2n}, w_r) + \left(\frac{1}{r}(\sigma^{2m} - \sigma^{2n}), w\right) + \langle f^m - f^n, w\rangle\end{aligned}$$

for all $w \in H_0^1$,

$$(5.15)\quad \sigma_t^{1m} - \sigma_t^{1n} + \frac{\sigma^{1m} - \sigma^{1n}}{\mathrm{We}} = (1-a^2)(\sigma^{2m} - \sigma^{2n})v_r^m + (1-a^2)\sigma^{2n}(v_r^m - v_r^n)$$

and

$$(5.16)\quad \sigma_t^{2m} - \sigma_t^{2n} + \frac{\sigma^{2m} - \sigma^{2n}}{\mathrm{We}} = \frac{\alpha}{\mathrm{We}}(v_r^m - v_r^n) - (\sigma^{1m} - \sigma^{1n})v_r^m - \sigma^{1n}(v_r^m - v_r^n)\,.$$

Setting $w = \frac{\alpha}{\mathrm{We}}(v^m - v^n)$ in (5.14) and computing the scalar products in L^2 of (5.15) and (5.16) respectively by $\frac{1}{1-a^2}(\sigma^{1m} - \sigma^{1n})$ and $(\sigma^{2m} - \sigma^{2n})$, one finds:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{\alpha\mathrm{Re}}{\mathrm{We}} \mid v^m - v^n \mid^2 + \frac{1}{1-a^2} \mid \sigma^{1m} - \sigma^{1n} \mid^2 + \mid \sigma^{2m} - \sigma^{2n} \mid^2\right)\\ &\qquad + \frac{\alpha(1-\alpha)}{\mathrm{We}} \mid v_r^m - v_r^n \mid^2 + \frac{1}{\mathrm{We}}\left(\frac{1}{1-a^2} \mid \sigma^{1m} - \sigma^{1n} \mid^2 + \mid \sigma^{2m} - \sigma^{2n} \mid^2\right)\\ &\qquad = \frac{\alpha(1-\alpha)}{\mathrm{We}}\left(\frac{1}{r}(v_r^m - v_r^n), v^m - v^n\right)\\ &\qquad + \frac{\alpha}{\mathrm{We}}\left(\left(\frac{1}{r}(\sigma^{2m} - \sigma^{2n}), v^m - v^n\right) + \langle f^m - f^n, v^m - v^n\rangle\right)\\ &\qquad + (\sigma^{2n}(v_r^m - v_r^n), \sigma^{1m} - \sigma^{1n}) - (\sigma^{1n}(v_r^m - v_r^n), \sigma^{2m} - \sigma^{2n})\,.\end{aligned}$$

The right side is bounded by

$$\begin{aligned}&\frac{\alpha(1-\alpha)}{\mathrm{We}R_1} \mid v_r^m - v_r^n \mid \cdot \mid v^m - v^n \mid + \frac{\alpha}{\mathrm{We}R_1} \mid \sigma^{2m} - \sigma^{2n} \mid \cdot \mid v^m - v^n \mid\\ &\qquad + \frac{\alpha}{\mathrm{We}} \| f^m - f^n \|_{H^{-1}} \| v^m - v^n \|\\ &\qquad + C_1 \mid v_r^m - v_r^n \mid \cdot \mid \sigma^{1m} - \sigma^{1n} \mid + C_1 \mid v_r^m - v_r^n \mid \cdot \mid \sigma^{2m} - \sigma^{2n} \mid\,.\end{aligned}$$

It must be emphasized that, in the previous manipulation, those terms "difficult to handle", i.e. the terms

$$(1-a^2)(\sigma^{2m}-\sigma^{2n})v_r^m \quad \text{in (5.15) and} \quad -(\sigma^{1m}-\sigma^{1n})v_r^m \quad \text{in (5.16)},$$

have disappeared. We find at once:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\alpha\mathrm{Re}}{\mathrm{We}}\mid v^m-v^n\mid^2+\frac{1}{1-a^2}\mid\sigma^{1m}-\sigma^{1n}\mid^2+\mid\sigma^{2m}-\sigma^{2n}\mid^2\right)\\ &\quad+\frac{\alpha(1-\alpha)}{\mathrm{We}}\mid v_r^m-v_r^n\mid^2\\ &\leq C_5\left(\frac{\alpha\mathrm{Re}}{\mathrm{We}}\mid v^m-v^n\mid^2+\frac{1}{1-a^2}\mid\sigma^{1m}-\sigma^{1n}\mid^2+\mid\sigma^{2m}-\sigma^{2n}\mid^2\right)\\ &\quad+C_6\|f^m-f^n\|_{H^{-1}}^2 \end{aligned} \tag{5.17}$$

where the constants C_5 and C_6 only depend on a, α, Re, We, R_1 and C_1. From Gronwall's lemma, we obtain:

$$\begin{aligned} &\|v^m-v^n\|_{L^\infty(L^2)}^2+\|\sigma^{1m}-\sigma^{1n}\|_{L^\infty(L^2)}^2+\|\sigma^{2m}-\sigma^{2n}\|_{L^\infty(L^2)}^2\\ &\leq C_7\left(\|v_0^m-v_0^n\|_{L^2}^2+\|\sigma_0^{1m}-\sigma_0^{1n}\|_{L^2}^2+\|\sigma_0^{2m}-\sigma_0^{2n}\|_{L^2}^2+\|f^m-f^n\|_{L^2(H^{-1})}^2\right). \end{aligned}$$

This tells to us that $\{v^m\}$ and $\{\sigma^{im}\}$ $(i=1,2)$ are Cauchy sequences in $L^\infty(L^2)$. Coming back to (5.17), we also obtain that $\{v_r^m\}$ is a Cauchy sequence in $L^2(L^2)$.

FOURTH STEP: PASSAGE TO THE LIMIT. The strong convergence properties deduced from the previous step suffice to take limits in $(5.2)_m$ and $(5.3)_m$. One is led to (5.2)-(5.3). It is also easy to check that (5.5) is satisfied. □

When $a=+1$ or $a=-1$, results like these can be proved more easily. One can also prove results of this nature for cylinders moving with velocities $K_1(t)$ and $K_2(t)$. In this case, the boundary conditions are

$$v(R_1,t)=K_1(t), \qquad v(R_2,t)=K_2(t).$$

6. – Some complementary questions

In order to describe the behaviour of visco-elastic fluids, several constitutive laws have been extensively used. Many of them are of the differential kind and have the form:

$$\sigma+\lambda_1\frac{D_a\sigma}{Dt}+\beta(\sigma,D(u))=2\eta\left(D(u)+\lambda_2\frac{D_aD(u)}{Dt}\right), \tag{6.1}$$

where β is a (possibly nonlinear) function of σ and $D(u)$ (here, the notation is as in Section 1). Let us recall some "classical" modes for which the results in Sections 2, 3 and 4 hold:

The Oldroyd's 8-*constant model.* In this case,

$$\beta(\sigma, D(u)) = \mu_0 Tr(\sigma)D(u) + \mu_1 Tr(\sigma D(u))Id + \mu_2 D(u)^2 \\ + \mu_3 Tr(D(u)^2)Id + \mu_4(D(u)\sigma + \sigma D(u)),$$

where all μ_i are constants (see [16]). When $\mu_i = 0$ for all i, we find (1.1) again.

The Larson model. This corresponds to $a = 1$ in (6.1) and

$$\beta(\sigma, D(u)) = 2Tr(\sigma D(u)) \cdot \gamma(Tr(\sigma)) \cdot (\sigma + Id),$$

with γ being a scalar function of $Tr(\sigma)$ (see [12]).

The Phan Thien and Tanner model. Now, we have again $a = 1$, but

$$\beta(\sigma, D(u)) = \gamma \cdot Tr(\sigma) \cdot \sigma,$$

where γ is a constant ([18]).

The Giesekus model. As before, $a = 1$. In this model,

$$\beta(\sigma, D(u)) = \gamma\sigma^2,$$

with γ being a constant ([7]).

One can also consider models with several different relaxation times. For instance, in the framework of Oldroyd models, one can introduce $\tau_1, \dots \tau_k$ as follows:

$$\sigma = \tau_N + \sum_{i=1}^{k} \tau_i,$$

with $\tau_N = 2\eta_0 D(u)$ ($\eta_0 > 0$) and

$$\tau_i + \lambda_i \frac{D_a \tau_i}{Dt} = 2\eta_i D(u), \qquad \eta_i, \lambda_i > 0 \quad \text{for} \quad i = 1, \dots, k.$$

The analog to (1.3)-(1.4) is

$$\rho(u' + (u \cdot \nabla)u) - \eta_0 \Delta u + \nabla p = \sum_{i=1}^{k} \nabla \cdot \tau_i + f, \quad \nabla \cdot u = 0, \tag{6.2}$$

$$\tau_i + \lambda_i(\tau_i' + (u \cdot \nabla)\tau_i + g_a(\tau_i, \nabla u)) = 2\eta_i D(u), \quad i = 1, \dots, k. \tag{6.3}$$

Once these equations are adimensionalized, we find:

$$\text{(6.4)}\ \mathrm{Re}(u' + (u\cdot\nabla)u) - (1-\alpha)\Delta u + \nabla p = \sum_{i=1}^{k} \nabla\cdot\tau_i + f \quad \nabla\cdot u = 0\,,$$

$$\text{(6.5)}\ \mathrm{W}_i(\tau_i' + (u\cdot\nabla)\tau_i + g_a(\tau_i,\nabla u)) + \tau_i = 2\alpha_i D(u), \qquad i = 1,\dots,k\,,$$

where

$$\alpha = \frac{\sum_{i=1}^{k}\eta_i}{\eta}, \quad \alpha_i = \frac{\eta_i}{\eta}, \quad \eta = \eta_0 + \sum_{i=1}^{k}\eta_i, \quad \mathrm{Re} = \frac{\rho U L}{\eta}, \quad \mathrm{W}_i = \frac{\lambda_i U}{L}\,.$$

Again, the results in Sections 2, 3, 4 and 5 hold for (6.4)-(6.5).

Contrarily to what happens in the case of a Newtonian fluid (see e.g. [13], [14], [22]), the existence of a global weak solution to (1.5)-(1.8) for arbitrary data in unknown. Some difficulties arise in order to obtain global estimates for u and τ. In the particular case $a = 0$, one finds uniform estimates for u in $L^\infty(H)\cap L^2(V)$ and for τ in $L^\infty(L^2)$. However, this is not enough to pass to the limit in the term $g_a(\tau,\nabla u)$ (see [17] for more details).

Appendix: Proofs of Lemma B and Lemma C

PROOF OF LEMMA B. Let $\{\bar u^m\}$ be a sequence in $C^1([0,T];C^3(\overline\Omega))$ such that $\bar u^m \in L^s(0,T;D(A_r))$ for all m, $\bar u^m \to \bar u$ in $L^s(0,T;W^{2,r})$. Also, let $\{\tau_0^m\}$ be a sequence in $C^2(\overline\Omega)$ such that $\tau_0^m \to \tau_0$ in $W^{1,r}$. For each $m \ge 1$, there exists a unique solution to

$$\tau^m \in C^1([0,T];C^2(\overline\Omega))\,,$$

$$\text{(B.1)}_m \qquad \begin{cases} \mathrm{We}((\tau^m)' + (\bar u^m\cdot\nabla)\tau^m + g_a(\tau^m,\nabla\bar u^m)) + \tau^m = 2\alpha D(\bar u^m) \\ \tau^m(0) = \tau_0^m\,. \end{cases}$$

This can be easily seen introducing the characteristic curves associated to $\bar u^m$. In order to estimate τ^m and $(\tau^m)'$, we first compute the L^2 scalar product $((\mathrm{B}.1)_m, |\,\tau^m\,|^{r-2}\,\tau^m)$. Then, we take gradients in $(\mathrm{B}.1)_m$ and compute the scalar product of both sides of the resulting equation with $|\,\nabla\tau^m\,|^{r-2}\,\nabla\tau^m$. By addition, one finds:

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}(\mathrm{We}\|\tau^m\|^r_{W^{1,r}}) + \|\tau^m\|^r_{W^{1,r}} &= 2\alpha(D(\bar u^m), |\,\tau^m\,|^{r-2}\,\tau^m) \\ &\quad + 2\alpha(\nabla D(\bar u^m), |\,\nabla\tau^m\,|^{r-2}\,\nabla\tau^m) \\ &\quad - \mathrm{We}(g_a(\tau^m,\nabla\bar u^m), |\,\tau^m\,|^{r-2}\,\tau^m) \\ &\quad - \mathrm{We}(\nabla g_a(\tau^m,\nabla\bar u^m), |\,\nabla\tau^m\,|^{r-2}\,\nabla\tau^m) \\ &\quad - \mathrm{We}\Big(({}^t\nabla\bar u^m\cdot\nabla)\tau^m, |\,\nabla\tau^m\,|^{r-2}\,\nabla\tau^m\Big)\,,\end{aligned}$$

since $((\bar{u}^m \cdot \nabla)\tau^m, |\tau^m|^{r-2}\tau^m) = 0$ and

$$(\nabla((\bar{u}^m \cdot \nabla)\tau^m), |\nabla\tau^m|^{r-2}\nabla\tau^m) = (({}^t\nabla\bar{u}^m \cdot \nabla)\tau^m, |\nabla\tau^m|^{r-2}\nabla\tau^m).$$

Using the fact that $r > N$, we see that

$$\frac{1}{r}\frac{d}{dt}(\mathrm{We}\|\tau^m\|^r_{W^{1,r}}) + \|\tau^m\|^r_{W^{1,r}} \le 4\alpha\|\bar{u}^m\|_{W^{2,r}}\|\tau^m\|^{r-1}_{W^{1,r}} + C_2\mathrm{We}\|\bar{u}^m\|_{W^{2,r}}\|\tau^m\|^r_{W^{1,r}}.$$

Here, $C_2 = C_2(r, a, \Omega)$. Dividing by $\mathrm{We}\|\tau^m\|^{r-1}_{W^{1,r}}$, one obtains

$$\frac{d}{dt}(\|\tau^m\|_{W^{1,r}}) + \frac{1}{\mathrm{We}}\|\tau^m\|_{W^{1,r}} \le \frac{4\alpha}{\mathrm{We}}\|\bar{u}^m\|_{W^{2,r}} + C_2\|\bar{u}^m\|_{W^{2,r}}\|\tau^m\|_{W^{1,r}}$$

and, consequently,

$$(\mathrm{B}.2)_m \quad \|\tau^m\|_{L^\infty(W^{1,r})} + \frac{4\alpha}{C_2\mathrm{We}} \le \left(\|\tau_0^m\|_{W^{1,r}} + \frac{4\alpha}{C_2\mathrm{We}}\right)\exp(C_2\|\bar{u}^m\|_{L^1(W^{2,r})}) \equiv \Lambda_m.$$

On the other hand, from $(\mathrm{B}.1)_m$, the following is found:

$$(\tau^m)' = \frac{1}{\mathrm{We}}(2\alpha D(\bar{u}^m) - \tau^m) - (\bar{u}^m \cdot \nabla)\tau^m - g_a(\tau^m, \nabla\bar{u}^m)),$$

$$\|(\tau^m)'\|_{L^r} \le \frac{1}{\mathrm{We}}(2\alpha\|\bar{u}^m\|_{W^{1,r}} + \|\tau^m\|_{L^r}) + C_2\|\bar{u}^m\|_{W^{1,r}}\|\tau^m\|_{W^{1,r}}$$

(it can be assumed that C_2 is the same constant),

$$\|(\tau^m)'\|_{L^r} \le C_2\left(\|\bar{u}^m\|_{W^{1,r}} + \frac{1}{C_2\mathrm{We}}\right)\left(\|\tau^m\|_{W^{1,r}} + \frac{2\alpha}{C_2\mathrm{We}}\right),$$

$$\|(\tau^m)'\|^s_{L^s(L^r)} \le 2^{s-1}C_2^s\left(\|\bar{u}^m\|^s_{L^s(W^{1,r})} + \frac{T}{(C_2\mathrm{We})^s}\right)\left(\|\tau^m\|_{L^\infty(W^{1,r})} + \frac{2\alpha}{C_2\mathrm{We}}\right)^s.$$

Finally, using $(\mathrm{B}.2)_m$ we see that

$$(\mathrm{B}.3)_m \qquad \|(\tau^m)'\|_{L^s(L^r)} \le 2^{1-\frac{1}{s}}C_2\Lambda_m\left(\|\bar{u}^m\|_{L^s(W^{1,r})} + \frac{T^{1/s}}{C_2\mathrm{We}}\right).$$

From $(\mathrm{B}.2)_m$ and $(\mathrm{B}.3)_m$, it is clear that τ^m (respectively $(\tau^m)'$) remains uniformly bounded in $L^\infty(W^{1,r})$ (respectively $L^s(L^r)$). Accordingly, it can be assumed that a function τ exists with

$$\tau^m \to \tau \quad \text{weakly* in } L^\infty(W^{1,r}) \text{ and strongly in } C(L^r),$$

$$(\tau^m)' \to \tau' \quad \text{weakly in } L^s(L^r).$$

This suffices to take limits in $(B.1)_m$. Thus, (B.1) is obtained. Also, from $(B.2)_m$ and $(B.3)_m$, taking into account the lower semicontinuity of the norm with respect to the weak and weak* convergence, one obtains (B.2) and (B.3). From the results in [4] concerning the transport equation, it is not difficult to prove that $\nabla\tau \in C(L^r)$. The uniqueness assertion stems from the fact that (B.1) is linear. □

PROOF OF LEMMA C. Since $u \in L^s(W^{2,r}) \cap C(L^r)$, it is easy to obtain (C.1) from the inequality

$$\|\nabla u\|_{L^r} \leq K\|u\|_{W^{2,r}}^{1/2}\|u\|_{L^r}^{1/2}$$

(here, K is a constant; see [6], p. 27). In order to demonstrate (C.2), we will use the following result, which can also be found in [6]: If $1 \leq q \leq p \leq +\infty$ and $r > 3$, then

$$\|u\|_{L^p} \leq C(p,q,r)\|u\|_{L^q}^{1-a}\|u\|_{W^{1,r}}^a \quad \forall u \in W^{1,r}, \quad \text{where } a = \frac{\frac{1}{q}-\frac{1}{p}}{\frac{1}{q}+\frac{1}{3}-\frac{1}{r}}.$$

For $q = r$ and $p = +\infty$, one has:

$$\|u\|_{L^\infty} \leq C_r\|u\|_{L^r}^{1-a}\|u\|_{W^{1,r}}^a \quad \text{with} \quad a = \frac{3}{r};$$

hence,

$$\int_0^T \|u\|_{L^\infty}^{2sr/3}dt \leq C_r^{2sr/3}\|u\|_{L^\infty(L^r)}^{2s(r-3)/3}\int_0^T \|u\|_{W^{1,r}}^{2s}dt$$

and (C.2) holds. □

REFERENCES

[1] G. ASTARITA – G. MARRUCCI, "Principles of Non-Newtonian Fluid Mechanics", McGraw Hill, New York, 1974.

[2] J. BARANGER – D. SANDRI, *Finite element approximation of viscoelastic fluid flow: Existence of approximate solutions and error bounds*, Numer. Math. **63** (1992), 13-27.

[3] M. J. CROCHET – A. R. DAVIES – K. WALTERS, "Numerical Simulation of Non-Newtonian Flow", Elsevier, Amsterdam, 1985.

[4] R. DIPERNA – P.-L. LIONS, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), 511-547.

[5] E. FERNÁNDEZ-CARA – F. GUILLÉN – R. R. ORTEGA, *Existence et unicité de solution forte locale en temps pour des fluides non newtoniens de type Oldroyd (version $L^s - L^r$)*, C. R. Acad. Sci. Paris. Sér. I Math. **319** (1994), 411-416.

[6] A. FRIEDMAN, "Partial Differential Equations", Holt-Rinehart-Winston, New York, 1976.

[7] H. GIESEKUS, *A unified approach to a variety of constitutive models for polymer fluids based on the concept of configuration dependent molecular mobility*, Rheol. Acta **21** (1982), 366-375.

[8] Y. GIGA – H. SOHR, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal. **102** (1991), 72-94.

[9] C. GUILLOPÉ – J.-C. SAUT, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonlinear Anal. Vol. 15, No. 9, (1990), 849-869.

[10] C. GUILLOPÉ – J.-C. SAUT, *Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type*, Math. Mod. Numer. Anal. Vol. 24, No. 3, (1990), 369-401.

[11] O. A. LADYZHENSKAYA, "The Mathematical Theory of Viscous Incompressible Flow", Gordon and Breach, New York, 1969.

[12] R. G. LARSON, *A critical comparison of constitutive equations for polymer melts*, J. Non-Newtonian Fluid Mech. **23** (1987), 249-269.

[13] J. LERAY, *Sur le mouvement d'une liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193-248.

[14] J.L. LIONS, "Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires", Dunod, Gauthier-Villars, Paris, 1969.

[15] J. G. OLDROYD, *On the formulation of rheological equations of state*, Proc. Roy. Soc. London Ser. A **200** (1950), 523-541.

[16] J. G. OLDROYD, *Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids*, Proc. Roy. Soc. London Ser. A **245** (1958), 278-297.

[17] R. R. ORTEGA, Thesis, University of Seville (Spain), 1995.

[18] N. PHAN THIEN – R. I. TANNER, *A new constitutive equation derived from network theory*, J. Non-Newtonian Fluid Mech. **2** (1977), 353-365.

[19] M. RENARDY, *Existence of slow flows of viscoelastic fluids with differential constitutive equations*, Z. Angew. Math. Mech. **65** (1985), 449-451.

[20] M. RENARDY – W. J. HRUSA – J. A. NOHEL, "Mathematical Problems in Viscoelasticity", Longman, London, 1987.

[21] D. SANDRI, *Approximation par éléments finis d'écoulements de fluides viscoélastiques: Existence de solutions approchées et majoration d'erreur. II. Contraintes continues*, C. R. Acad. Paris Sér. I Math. **313** (1991), 111-114.

[22] R. TÉMAM, "Navier-Stokes Equations, Theory and Numerical Analysis", North-Holland, Amsterdam, 1977.

[23] A. VALLI, *Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10** (1983), 607-647.

[24] A. VALLI, *Navier-Stokes equations for compressible fluids: global estimates and periodic solutions*, Proc. Sympos Pure Math. **45** (1986), 467-478.

[25] K. WALTERS (ed.), "Rheometry: Industrial Applications", J. Wiley and Sons, 1980.

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