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New Estimates for Div-Curl Products and Very Weak Solutions of PDEs

CARLO SBORDONE

In memory of my teacher Ennio De Giorgi

1. – Introduction

I am pleased to acknowledge the influence of Ennio De Giorgi on my mathematical career. His pioneering ideas on elliptic PDEs and Γ -convergence in the Calculus of Variations have shaped my own work from the very beginning.

It seems appropriate to begin this survey by introducing the general elliptic operator

$$(1.1) \quad L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

In particular, the theory of G-convergence, initiated in the late sixties by De Giorgi and Spagnolo [S], [DS] has enjoyed extensive developments.

Central to the theory of such operators is the energy functional

$$(1.2) \quad \mathcal{E}[u] = \int_{\Omega} \langle a(x) \nabla u, \nabla u \rangle dx$$

where $a(x) = (a_{ij}(x))$ denotes a measurable positive definite coefficient matrix.

A fruitful idea in studying such functionals is to view them as integrals of a product of two vector fields:

$$(1.3) \quad \mathcal{E}[u] = \int_{\Omega} \langle B(x), E(x) \rangle dx$$

where $E = \nabla u$ and $B = a \nabla u$. Thus

$$(1.4) \quad \begin{aligned} \operatorname{curl} E &= \left(\frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} \right)_{i,j=1,\dots,n} = 0 \\ \operatorname{div} B &= \sum_{i=1}^n \frac{\partial B_i}{\partial x_i} = 0 \end{aligned}$$

in the sense of distributions.

This is the view which we shall adopt and develop in this article. The question immediately arises as to whether the div-curl products enjoy a higher degree of regularity than the generic products of arbitrary vector fields. The first result in this direction can be traced back to the familiar div-curl lemma of F. Murat and L. Tartar [Mu], [T] and the subsequent theory of compensated compactness, with applications to G-convergence.

LEMMA 1.1 (Div-Curl). *Suppose the vector fields $B_k, E_k \in L^2(\Omega, \mathbb{R}^n)$ verify $\operatorname{div} B_k = \operatorname{curl} E_k = 0$ and converge weakly in $L^2(\Omega, \mathbb{R}^n)$ to B and E respectively, then*

$$\lim_{k \rightarrow \infty} \langle B_k, E_k \rangle = \langle B, E \rangle$$

in the sense of Schwartz distributions $\mathcal{D}'(\Omega)$.

In other words, the lack of compactness of the product of two general vector fields in $L^2(\Omega, \mathbb{R}^n)$ is compensated by the assumption that $\operatorname{div} B$ and $\operatorname{curl} E$ vanish.

This remarkable observation has many generalizations. The interested reader is referred to [CLMS]. Among them is the following celebrated result.

THEOREM 1.1 ([CLMS]). *The div-curl product $\langle B, E \rangle$ of vector fields $B \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ and $E \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{div} B = \operatorname{curl} E = 0$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$.*

In this connection we should also recognize the earlier work of S. Muller [M1] and H. Wente [W].

Unfortunately we will not be able to mention many other contributions here, though we include some of them in the references. The points we wish to make are concerned with integration of div-curl quantities under minimal degree of integrability of the vector fields B and E .

This idea, which has been developed in a series of joint papers with T. Iwaniec, not only extended earlier results about div-curl expressions [IS1] but also initiated the study of so-called very weak solutions of nonlinear elliptic PDEs [IS2].

It is our goal here to present these new techniques in some detail. However for the sake of brevity, we shall need to rephrase some results in lesser generality.

2. – Estimates below the natural exponent

We begin with a simple consequence of Hölder's inequality. For vector fields $B \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ and $E \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $1 < p, q < \infty$ and test function $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$(2.1) \quad \left| \int_{\mathbb{R}^n} \varphi \langle B, E \rangle dx \right| \leq \|\varphi\|_\infty \|B\|_q \|E\|_p.$$

In order to exploit certain cancellations in the above integral we now assume that

$$(2.2) \quad \operatorname{div} B = 0 \quad \text{and} \quad \operatorname{curl} E = 0$$

in the sense of distributions.

Unless otherwise stated, this assumption will remain valid throughout this article. The following basic estimates bring us quickly to the substance of our approach.

THEOREM 2.1. *Let $1 < p, q < \infty$ be a Hölder conjugate pair, $\frac{1}{p} + \frac{1}{q} = 1$, and let $1 < r, s < \infty$ be a Sobolev conjugate pair, $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{n}$. Then there exists a constant $c_n = c_n(p, s)$ such that for each test function $\varphi \in C_0^\infty(\Omega)$ we have*

$$(2.3) \quad \left| \int_{\Omega} \varphi \frac{\langle B, E \rangle}{|B|^{\epsilon} |E|^{\epsilon}} \right| \leq c_n \epsilon \|\varphi\|_{\infty} \|E\|_{p-\epsilon p}^{1-\epsilon} \|B\|_{q-\epsilon q}^{1-\epsilon} + c_n \|\nabla \varphi\|_{\infty} \|E\|_{s-\epsilon s}^{1-\epsilon} \|B\|_{r-\epsilon r}^{1-\epsilon}$$

whenever $0 \leq 2\epsilon \leq \min \left\{ \frac{p-1}{p}, \frac{q-1}{q}, \frac{r-1}{r}, \frac{s-1}{s} \right\}$ and $\operatorname{div} B = \operatorname{curl} E = 0$.

The key tool used in establishing this estimate is the stability of the Hodge decomposition theorem under nonlinear perturbations of the vector fields, first discovered by T. Iwaniec [I1] and developed in [IS2].

LEMMA 2.1. *Let X be a vector field of class $L^{p-\epsilon p}(\mathbb{R}^n, \mathbb{R}^n)$ with $1 < p < \infty$ and $-\infty < 2\epsilon < 1 - \frac{1}{p}$. Consider the Hodge decomposition of $|X|^{-\epsilon} X$*

$$(2.4) \quad |X|^{-\epsilon} X = B + \mathcal{E}$$

where $B \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{div} B = 0$ and $\mathcal{E} \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ with $\operatorname{curl} \mathcal{E} = 0$.

Then the following estimates hold:

$$(2.5) \quad \|B\|_p \leq c_p(n) |\epsilon| \|X\|_{p-\epsilon p}^{1-\epsilon} \quad \text{whenever} \quad \operatorname{curl} X = 0$$

$$(2.6) \quad \|\mathcal{E}\|_p \leq c_p(n) |\epsilon| \|X\|_{p-\epsilon p}^{1-\epsilon} \quad \text{whenever} \quad \operatorname{div} X = 0.$$

Of course, the L^p -theory of Hodge decomposition has been well known since the work of C. B. Morrey [Mo]. This approach yields the following estimate in either case

$$(2.7) \quad \|\mathcal{E}\|_p + \|B\|_p \leq c_p(n) |\epsilon| \|X\|_{p-\epsilon p}^{1-\epsilon}.$$

The innovation in Lemma 2.1 lies in the presence of the factor $|\epsilon|$ which can be as small as we want. The subject is linked with the rather abstract theory of nonlinear commutators and interpolation.

But we cannot discuss this theory here, see [I1], [I2], [IS2].

Having presented these preliminary results we can give a variant of (2.3) corresponding to $\Omega = \mathbb{R}^n$ and $\varphi \equiv 1$. This reads as

$$(2.8) \quad \int_{\mathbb{R}^n} \frac{\langle B, E \rangle}{|B|^\epsilon |E|^\epsilon} \leq c_n \epsilon \|E\|_{p-\epsilon p}^{1-\epsilon} \cdot \|B\|_{q-\epsilon q}^{1-\epsilon}$$

with $\operatorname{div} B = \operatorname{curl} E = 0$.

Let us decompose, according to Lemma 2.1, with $X = E$

$$(2.9) \quad \begin{cases} |E|^{-\epsilon} E = \mathcal{B}_1 + \mathcal{E}_1, & \operatorname{div} \mathcal{B}_1 = \operatorname{curl} \mathcal{E}_1 = 0 \\ \|\mathcal{B}_1\|_p \leq c_p(n) \epsilon \|E\|_{p-\epsilon p}^{1-\epsilon} \end{cases}$$

and then with $X = B$

$$(2.10) \quad \begin{cases} |B|^{-\epsilon} B = \mathcal{B}_2 + \mathcal{E}_2, & \operatorname{div} \mathcal{B}_2 = \operatorname{curl} \mathcal{E}_2 = 0 \\ \|\mathcal{E}_2\|_q \leq c_q(n) \epsilon \|B\|_{q-\epsilon q}^{1-\epsilon}. \end{cases}$$

Since divergence free vector fields are orthogonal to the curl free vector fields, the integral in question reduces to:

$$\int_{\mathbb{R}^n} \langle |B|^{-\epsilon} B, |E|^{-\epsilon} E \rangle = \int_{\mathbb{R}^n} \langle \mathcal{B}_2 + \mathcal{E}_2, \mathcal{B}_1 + \mathcal{E}_1 \rangle = \int_{\mathbb{R}^n} \langle \mathcal{B}_2, \mathcal{B}_1 \rangle + \int_{\mathbb{R}^n} \langle \mathcal{E}_2, \mathcal{E}_1 \rangle.$$

Using Hölder inequality and (2.9), (2.10) we may estimate the latter integrals.

For example

$$\int_{\mathbb{R}^n} \langle \mathcal{B}_2, \mathcal{B}_1 \rangle \leq \|\mathcal{B}_2\|_q \|\mathcal{B}_1\|_p \leq c'_p(n) \epsilon \|E\|_{p-\epsilon p}^{1-\epsilon} \cdot \|B\|_{q-\epsilon q}^{1-\epsilon}.$$

Here we applied (2.7) to estimate \mathcal{B}_2 and (2.9) to estimate \mathcal{B}_1 .

Similarly we handle the second integral, completing the proof of (2.8).

The general inequality (2.3) follows by combining (2.8) with Sobolev imbedding inequalities. The routine but lengthy computations are presented in [IS1], [I2].

One interesting inference from inequality (2.3) arises when $\epsilon = 0$:

$$(2.11) \quad \left| \int_{\Omega} \varphi \langle B, E \rangle \right| \leq c_n \|\nabla \varphi\|_{\infty} \|E\|_s \|B\|_r$$

where $1 < s, r < \infty$ and $\frac{1}{s} + \frac{1}{r} = 1 + \frac{1}{n}$.

It is important to realize that, in spite of the absence of the p, q -norms, inequality (2.11) still requires that $\|E\|_p < \infty$ and $\|B\|_q < \infty$ for some Hölder conjugate pair (p, q) .

Nevertheless, applying (2.11) we can give a meaning to $\langle B, E \rangle$ as a Schwartz distribution for arbitrary vector fields $B \in L^r(\Omega)$ and $E \in L^s(\Omega)$ with $\operatorname{div} B = 0$ and $\operatorname{curl} E = 0$.

Indeed we may evaluate the distribution $\langle B, E \rangle$ at the test function $\varphi \in C_0^\infty(\Omega)$ by the rule

$$\langle B, E \rangle(\varphi) = \lim_{k \rightarrow \infty} \int_{\Omega} \varphi \langle B_k, E_k \rangle$$

where B_k, E_k are arbitrary smooth vector fields ($\operatorname{div} B_k = \operatorname{curl} E_k = 0$) converging to B and E respectively in L^r_{loc} and L^s_{loc} .

To see that the limit exists, just decompose:

$$\langle B_k, E_k \rangle - \langle B, E \rangle = \langle B_k - B, E_k \rangle + \langle B, E_k - E \rangle$$

and apply (2.11) to each of the terms.

The distribution $\langle B, E \rangle \in \mathcal{D}'(\Omega)$ is referred to as a *weak div-curl product*. It is natural to ask whether the distribution $\langle B, E \rangle$ is regular. Clearly, if the distribution $\langle B, E \rangle$ is nonnegative, that is $\langle B, E \rangle(\varphi) \geq 0$ for nonnegative test functions, then it is of order one, and, therefore is represented by a Borel measure. S. Müller [M2] showed that, if this measure has no singular part with respect to Lebesgue measure, then it coincides with the pointwise product $\langle B(x), E(x) \rangle$, see the remark at the end of Section 3 for a converse.

We close this section with one more interesting result.

Let $L \log L(K)$ denote the Zygmund space over a set $K \subset \mathbb{R}^n$, $0 < |K| < \infty$, equipped with the norm

$$\|f\|_{L \log L(K)} = \int_K |f| \log \left(e + \frac{|f|}{|f|_K} \right).$$

PROPOSITION 2.1. *Suppose $E_j \rightarrow E$ in $L^p(\Omega, \mathbb{R}^n)$ and $B_j \rightarrow B$ in $L^q(\Omega, \mathbb{R}^n)$, where, as usual, $\operatorname{div} B_j = \operatorname{curl} E_j = 0$.*

If, in addition, $\langle E_j, B_j \rangle \geq 0$ a.e. for $j = 1, 2, \dots$, then

$$(2.13) \quad \lim_{j \rightarrow \infty} \|\langle E_j, B_j \rangle - \langle E, B \rangle\|_{L \log L(K)} = 0$$

for every compact $K \subset \Omega$.

It follows from [CLMS], see also Theorem 1.1, that $\langle B_j, E_j \rangle \rightarrow \langle B, E \rangle$ in $\mathcal{H}^1_{\text{loc}}(\Omega)$. Then the convergence in $L \log L(K)$ is immediate from a general fact concerning sequences of nonnegative functions in the Hardy space $\mathcal{H}^1_{\text{loc}}(\Omega)$, see [IV].

3. – The grand L^q -spaces

Another consequence of inequality (2.3) arises by taking the supremum with respect to the parameter ϵ in a suitable interval of positive numbers. New function spaces emerge [IS1], [G].

Let Ω be an open cube in \mathbb{R}^n .

DEFINITION 3.1. For $1 < p < \infty$ the *grand* L^p -space, denoted by $L^p(\Omega)$, consists of functions $f \in \bigcap_{1 \leq s < p} L^s(\Omega)$ such that

$$(3.1) \quad \|f\|_p = \sup_{0 < \epsilon \leq p-1} \left(\int_{\Omega} |f|^{p-\epsilon} \right)^{\frac{1}{p-\epsilon}} < \infty$$

where $f_{\Omega} |h|$, also denoted at times by $|h|_{\Omega}$, is the integral mean of $|h|$ over the cube Ω .

Note that $\|\cdot\|_p$ is a norm and $L^p(\Omega)$ is a Banach space containing $L^p(\Omega)$. The $L^p(\Omega)$ space is not dense in $L^p(\Omega)$. Its closure, denoted by $\mathcal{L}^p(\Omega)$, consists of functions $f \in \bigcap_{1 \leq s < p} L^s(\Omega)$ such that

$$(3.2) \quad \lim_{\epsilon \rightarrow 0} \left(\int_{\Omega} |f|^{p-\epsilon} \right)^{\frac{1}{p-\epsilon}} = 0.$$

Two well known function spaces are contained in $L^p(\Omega)$. First the Marcinkiewicz space $L^{p,\infty}(\Omega)$ consisting of functions f such that

$$(3.3) \quad |\{x : |f(x)| > t\}| \leq \left(\frac{[f]_p}{t} \right)^p$$

for all $t > 0$ where $[f]_p$ is a constant depending on f and p . We have

$$(3.4) \quad \|f\|_p \leq A_p [f]_p.$$

Second is the Zygmund space $L^p \log^{-1} L$ of functions f for which the nonlinear functional

$$(3.5) \quad \|f\|_{L^p \log^{-1} L} = \left[\int_{\Omega} |f|^p \log^{-1} \left(e + \frac{|f|}{|f|_{\Omega}} \right) \right]^{\frac{1}{p}}$$

is finite. This is a subspace of $\mathcal{L}^p(\Omega)$ for which we also have the uniform bound

$$(3.6) \quad \|f\|_p \leq c_p \|f\|_{L^p \log^{-1} L}.$$

With the above notation we can now formulate few corollaries of Theorem 2.1.

Assume that $E \in L^p(\Omega, \mathbb{R}^n)$, $B \in L^q(\Omega, \mathbb{R}^n)$ with $1 < p, q < \infty$ Hölder conjugate. Then

$$(3.7) \quad \left| \int_{\Omega} \varphi \frac{\langle B, E \rangle}{|B|^{\epsilon} |E|^{\epsilon}} \right| \leq c_n[\varphi] \|E\|_p^{1-\epsilon} \|B\|_q^{1-\epsilon}$$

where $c_n[\varphi] = c_n(p, s) \left(\|\varphi\|_{\infty} + |\Omega|^{\frac{1}{n}} \|\nabla \varphi\|_{\infty} \right)$ and $0 < 2\epsilon < \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}$. Thus, in particular, the integrals in the left hand side of (3.7) stay bounded as $\epsilon \rightarrow 0$, though the product $\langle B, E \rangle$ need not be integrable.

COROLLARY 3.1. *Let $E \in L^p \log^{-1} L(\Omega, \mathbb{R}^n)$ and $B \in L^q \log^{-1} L(\Omega, \mathbb{R}^n)$ with $1 < p, q < \infty$ Hölder conjugate exponents. Then, for $0 < 2\epsilon \leq \min\{\frac{1}{p}, \frac{1}{q}\}$ we have*

$$(3.8) \quad \left| \int_{\Omega} \varphi \frac{\langle B, E \rangle}{|B|^\epsilon |E|^\epsilon} \right| \leq c_n[\varphi] \|B\|_{L^q \log^{-1} L}^{1-\epsilon} \|E\|_{L^p \log^{-1} L}^{1-\epsilon}$$

for each test function $\varphi \in C_0^\infty(\Omega)$.

The Monotone Convergence Theorem yields

COROLLARY 3.2. *Under the above assumptions, if the product $\langle B, E \rangle$ is non-negative a.e., then it is locally integrable and*

$$(3.9) \quad \int_{\Omega} \langle B, E \rangle \leq c_n[\varphi] \|B\|_{L^q \log^{-1} L} \|E\|_{L^p \log^{-1} L}$$

for each non negative test function $\varphi \in C_0^\infty(\Omega)$.

As a matter of fact since $\mathcal{L}^p(\Omega, \mathbb{R}^n)$ contains $L^p \log^{-1} L(\Omega, \mathbb{R}^n)$ we can also handle div-curl products which change sign (see [I2]).

COROLLARY 3.3. *For $B \in L^q \log^{-1} L(\Omega, \mathbb{R}^n)$ and $E \in L^p \log^{-1} L(\Omega, \mathbb{R}^n)$, the weak product $\langle B, E \rangle$ can be defined equivalently without referring to any approximation by smooth vector fields as follows*

$$(3.10) \quad \langle B, E \rangle(\varphi) = \lim_{\epsilon \searrow 0} \int_{\Omega} \varphi \frac{\langle B, E \rangle}{|B|^\epsilon |E|^\epsilon}, \quad \varphi \in C_0^\infty.$$

The importance of this formula is attested to by its applications to the degree theory of mappings with nonintegrable Jacobian [GISS], [I].

One more point of emphasis is that if $\langle B, E \rangle$ happens to be non negative, it is locally integrable and as such represents a distribution which turns out to be the weak product $\langle B, E \rangle$.

This observation, also noted by L. Greco [G], provides a converse to a result of S. Müller [M2].

4. – The grand Sobolev spaces

For $q > 1$ and Ω a bounded open set in \mathbb{R}^n , the grand Sobolev space $W_0^{1,q}$ consists of all functions $u \in \bigcap_{0 < \epsilon \leq q-1} W_0^{1,q-\epsilon}(\Omega)$ such that

$$\|u\|_{W_0^{1,q}(\Omega)} = \sup_{0 < \epsilon \leq q-1} \left[\epsilon \int_{\Omega} |\nabla u|^{q-\epsilon} dx \right]^{\frac{1}{q-\epsilon}} < \infty.$$

This space, slightly larger than $W_0^{1,q}$ was introduced in [IS1] in connection with the regularity properties of the Jacobian.

In the case $q = n$, an imbedding theorem of Sobolev-Trudinger type was established in [FLS].

To state this result we need a few definitions.

The Orlicz space EXP_α , $\alpha > 0$ is defined according to the norm

$$(4.1) \quad \|f\|_{EXP_\alpha} = \inf \left\{ \lambda > 0 : \int_{\Omega} \exp \left| \frac{f}{\lambda} \right|^\alpha \leq 2 \right\}.$$

It is well known that L^∞ is not dense in EXP_α . In [CS] the following formulas for the distance

$$\text{dist}_{EXP_\alpha}(f, L^\infty) = \inf \left\{ \lambda > 0 : \int_{\Omega} \exp \left(\frac{|f|}{\lambda} \right)^\alpha < \infty \right\} = e \limsup_{q \rightarrow \infty} \frac{1}{q} \left(\int_{\Omega} |f|^{|\alpha q|} \right)^{\frac{1}{q}}$$

were established. We will denote by exp_α the closure of L^∞ in EXP_α . In the same paper the following formula for the distance to L^∞ from L^q was also proven (see also [G])

$$\text{dist}_{L^q}(f, L^\infty) = \limsup_{\epsilon \rightarrow 0} \left(\epsilon \int_{\Omega} |f|^{q-\epsilon} \right)^{\frac{1}{q-\epsilon}}.$$

We have the following imbedding results ([FLS])

THEOREM 4.1. *Let Ω be a bounded open set in \mathbb{R}^n . Then there exist $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that for $u \in W_0^{1,n}(\Omega)$ the estimate*

$$\int_{\Omega} \exp \left(\frac{|u|}{c_1 |\Omega|^{\frac{1}{n}} \|u\|_{W_0^{1,n}}} \right) dx \leq c_2$$

holds.

THEOREM 4.2. *If $u \in W_0^{1,1}(\Omega)$ satisfies for some $\sigma > 0$ the condition*

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^\sigma \int_{\Omega} |\nabla u|^{n-\epsilon} = 0$$

then $u \in \text{exp}_\alpha(\Omega)$, with $\alpha = \frac{n}{n-1+\sigma}$.

REMARK 4.1. If $g \in L^1_{loc}(\Omega)$ satisfies for a $\sigma > 0$

$$\int_{\Omega} |g|^n \log^{-\sigma}(e + |g|) < \infty$$

then, has already mentioned in the case $\sigma = 1$ in Section 3 (see [BFS], Lemma 3, for a proof)

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^\sigma \int_{\Omega} |g|^{n-\epsilon} = 0.$$

From this and Theorem 4.2 we deduce that if $u \in W_0^{1,1}(\Omega)$ and

$$|\nabla u|^n \log^{-\sigma}(e + |\nabla u|) \in L^1(\Omega)$$

then $u \in \text{exp}_\alpha$, $\alpha = \frac{n}{n-1+\sigma}$

REMARK 4.2. The exponent α cannot be improved. In fact, if

$$u(x) = \frac{\log |x|}{|\log |\log |x||^\theta}$$

for $|x|$ small, $\theta > \frac{1}{n}$, then $|\nabla u|^n \log^{-1}(e + |\nabla u|) \in L^1$ while for any $c > 0$, $\delta > 0$

$$\int_{\Omega} \exp\left(\frac{|u|^{1+\delta}}{c}\right) = \infty.$$

REMARK 4.3. Let us point out that functions $u \in W_0^{1,1}(\Omega)$ with $\nabla u \in L^n \log^{-\sigma} L$ need not belong to *BMO* or *VMO* when $\sigma > 0$. Recall the definition of the *BMO* class of John and Nirenberg: a function $h \in L^1_{loc}(\Omega)$ belongs to *BMO* if

$$\sup_B \int_B |h - h_B| = \|h\|_{BMO} < \infty$$

where the supremum is taken over all balls $B \subset \Omega$.

Recall also the definition of the *VMO* class of Sarason: a function $u \in BMO$ belongs to *VMO* if

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |u - u_{B_r}| dy = 0$$

uniformly with respect to x , where $B_r = B_r(x)$ is the ball, with radius r , centered at x .

EXAMPLE 4.1. Let us consider the function $h = h(x)$ with support in the unit ball B of \mathbb{R}^n

$$h(x) = \begin{cases} \log 2 & \text{if } |x| \leq \frac{1}{2} \\ -\log |x| & \text{if } \frac{1}{2} \leq |x| \leq 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

Let x_j be a sequence of points in an open set $\Omega \subset \mathbb{R}^n$ such that the balls $B_j = B_{r_j}(x_j) \subset \Omega$, $(r_j = 2^{-j^2})$ are disjoint for j sufficiently large. Next we define

$$h_j(x) = a_j h\left(\frac{x - x_j}{r_j}\right)$$

where $1 \leq a_j \rightarrow \infty$ and $\sum_j a_j^n \log(1 + j^{-2}) < \infty$. Moreover we set $f = \sum_j h_j$, so that $f(x) = h_j(x)$ if $|x - x_j| < r_j$. We have

$$a_j \int_B |h - h_B| = \int_{B_j} |h_j - (h_j)_{B_j}| \leq \|f\|_{BMO}$$

so that $f \notin BMO$. On the other hand

$$|\nabla h_j| \leq \begin{cases} \frac{a_j}{|x - x_j|} & \text{if } \frac{r_j}{2} \leq |x - x_j| \leq r_j \\ 0 & \text{if } |x - x_j| \leq \frac{r_j}{2}. \end{cases}$$

Using $\frac{r_j}{2} = r_j^{k_j}$, $(k_j = 1 + j^{-2})$ we obtain

$$\begin{aligned} \int_{|x-x_j| \leq r_j} \frac{|\nabla h_j|^n}{\log(e + |\nabla h_j|)} &\leq a_j^n \int_{r_j^{k_j} \leq |x-x_j| \leq r_j} \frac{|x - x_j|^{-n}}{-\log |x - x_j|} \\ &= n\omega_n a_j^n \int_{r_j^{k_j}}^{r_j} \frac{dt}{-t \log t} = n\omega_n a_j^n \log k_j \end{aligned}$$

which implies $|\nabla f| \in L^n \log^{-1} L$.

5. – Very weak solutions of PDEs

Consider the Leray-Lions operator

$$(5.1) \quad Lu = \operatorname{div} A(x, \nabla u)$$

in a regular domain $\Omega \subset \mathbb{R}^n$, where the mapping

$$(5.2) \quad \mathcal{A} = \mathcal{A}(x, \xi) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

verifies ($p > 1, m \geq 1$) the conditions

$$(5.3) \quad |\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leq m |\xi - \eta| (|\xi| + |\eta|)^{p-2}$$

$$(5.4) \quad m^{-1} |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} \leq \langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle$$

$$(5.5) \quad \mathcal{A}(x, 0) = 0 \quad \text{for a. e. } x \in \Omega; \quad \forall \xi, \eta \in \mathbb{R}^n.$$

The p -harmonic type operator (5.1), with

$$\mathcal{A}(x, \xi) = a(x) |\xi|^{p-2} \xi$$

and $a, \frac{1}{a} \in L^\infty(\Omega)$, verifies the above assumptions.

Consider the Dirichlet problem ($\frac{1}{p} + \frac{1}{q} = 1$)

$$(5.6) \quad \begin{cases} Lu = \operatorname{div} F & F \in L^q(\Omega, \mathbb{R}^n) \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

This is the natural setting of the p -harmonic problems: for any such F there is a unique solution u to (5.6). A nonlinear operator

$$(5.7) \quad \mathcal{H} : L^q(\Omega, \mathbb{R}^n) \rightarrow L^p(\Omega, \mathbb{R}^n)$$

is defined, which carries F into ∇u .

The uniform estimate

$$(5.8) \quad \|\mathcal{H}F - \mathcal{H}G\|_p^{p-1} \leq c_p \|F - G\|_q^\alpha (\|F\|_q + \|G\|_q)^{1-\alpha}$$

with $\alpha = \min\{1, p - 1\}$ holds. In the particular case $p = 2$ it expresses the Lipschitz continuity of \mathcal{H} .

The L^q space is the most natural domain of \mathcal{H} , but having in mind to study equations of the type

$$Lu = \mu$$

μ a Radon measure, it is also of interest to find other spaces to which \mathcal{H} eventually extends as a continuous operator.

In a joint paper with T. Iwaniec ([IS2]) the p -harmonic type equation

$$Lu = \operatorname{div} F$$

was examined for $F \in L^{q(1 \pm \epsilon)}$ (ϵ small).

Estimates for *very weak solutions*, which are a-priori in $W_0^{1,p(1 \pm \epsilon)}$ were established:

$$\int_\Omega |\nabla u|^{p(1 \pm \epsilon)} \leq c_p(n) \int_\Omega |F|^{q(1 \pm \epsilon)}.$$

Unfortunately, existence and uniqueness of such solutions remain unclear (unless $p = 2$).

The case $p = 2$ was examined in [FS] where the following result was proved

THEOREM 5.1. *There exists $\epsilon_0 = \epsilon_0(m, n)$ such that if $0 < \epsilon < \epsilon_0$, if $F, G \in L^{2-\epsilon}(\Omega, \mathbb{R}^n)$, then each of the equations*

$$\begin{aligned}\operatorname{div} \mathcal{A}(x, \nabla u) &= \operatorname{div} F \\ \operatorname{div} \mathcal{A}(x, \nabla v) &= \operatorname{div} G\end{aligned}$$

has unique solution in $W_0^{1,2-\epsilon}$ and

$$\int_{\Omega} |\nabla u - \nabla v|^{2-\epsilon} \leq c(m, n) \int_{\Omega} |F - G|^{2-\epsilon}.$$

In the linear case $\mathcal{A}(x, \xi) = a(x)\xi$, it reduces to the classical Meyers' theorem.

The proof in [FS] relies on the stability of Hodge decomposition.

In the general case $p > 1$, we were able in [GIS] to prove the following theorem dealing with the grand Sobolev space.

THEOREM 5.2. *For each $F \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$ the p -harmonic type equations*

$$\begin{aligned}\operatorname{div} \mathcal{A}(x, \nabla u) &= \operatorname{div} F \\ \operatorname{div} \mathcal{A}(x, \nabla v) &= \operatorname{div} G\end{aligned}$$

have exactly one solution in $W_0^{1,p}$ and

$$\|\nabla u - \nabla v\|_p^{p-1} \leq c_p \|F - G\|_q^\alpha (\|F\|_q + \|G\|_q)^{1-\alpha}$$

where $\alpha = \alpha(p) \in (0, 1)$.

As far as we are aware this is the only existence and uniqueness result for p -harmonic type equations below the natural spaces. It seems a good motivation for studying grand Sobolev spaces $W_0^{1,p}$.

Let us conclude this section with two results for elliptic equations with *measures* in the right hand side:

$$Lu = \operatorname{div} \left(a(x) |\nabla u|^{p-2} \nabla u \right) = \mu.$$

The linear case ($p=2$) was extensively treated by G. Stampacchia [St] who gave appropriate definition of a solution by duality in term of the Green operator. It turns out that there exists unique solution

$$u \in W_0^{1,s}(\Omega) \quad \forall 1 \leq s < \frac{n}{n-1}.$$

In dimensions $n > 2$ there are, however, weaker solutions than those introduced by Stampacchia, whose uniqueness is not guaranteed.

In [IS3] we proved that if $a \in VMO(\mathbb{R}^n)$ and $\mu \in \mathcal{M}_0(\mathbb{R}^n)$, that is $\int_{\mathbb{R}^n} d\mu = 0$ then the Dirichlet problem

$$\begin{cases} \operatorname{div}(a\nabla u) = \mu & \text{in } \mathbb{R}^n \\ u \in W_0^{1, \frac{n}{n-1}}(\mathbb{R}^n) \end{cases}$$

admits a unique solution.

In the general case $2 - \frac{1}{n} < p \leq n$ the existence of a solution by approximation is due to Boccardo-Gallouet [BG] and it satisfies

$$u \in W_0^{1,s}(\Omega) \quad \forall 1 \leq s < \frac{n(p-1)}{n-1}.$$

Uniqueness was proved when $p < n$ under extra conditions on u . ([Mu2])

Here we report on existence and uniqueness results in the case $p = n$. ([GIS], [FS])

THEOREM 5.3. *For each Radon measure μ with finite mass on the regular open set Ω in \mathbb{R}^n , there exists exactly one solution $u \in W_0^{1,n}(\Omega)$ of the equation*

$$Lu = \operatorname{div} \left(|\nabla u|^{n-2} \nabla u \right) = \mu$$

and

$$\|u\|_{W_0^{1,n}(\Omega)} \leq c \int_{\Omega} |d\mu|.$$

This result shows that the proper space for studying n -harmonic equation with a measure on the right hand side is the grand Sobolev space $W_0^{1,n}(\Omega)$.

PROOF (of Theorem 5.3). Given $\mu \in \mathcal{M}(\Omega)$. First solve the auxiliary equation

$$\operatorname{div} F = \mu.$$

F is found from the formula

$$F(x) = \frac{1}{nw_n} \int_{\Omega} \frac{x-y}{|x-y|^n} d\mu(y).$$

For $1 \leq s < \frac{n}{n-1}$, by Minkowski inequality

$$\begin{aligned} \|F\|_{L^s} &\leq \frac{1}{nw_n} \int_{\Omega} \left\| \frac{1}{|x-y|^{n-1}} \right\|_{L^s} d\mu(y) \\ &\leq \frac{1}{nw_n} \sup_{y \in \Omega} \left\| \frac{1}{|x-y|^{n-1}} \right\|_{L^s} \int_{\Omega} |d\mu|. \end{aligned}$$

Now:

$$\left\| \frac{1}{|x-y|^{n-1}} \right\|_{L^s} \sim \left[\int_0^1 \frac{dr}{r^{(n-1)(s-1)}} \right]^{\frac{1}{s}}$$

so

$$\|F\|_{L^s} \leq \frac{c(n)}{[s - n(s-1)]^{\frac{1}{s}}} \int_{\Omega} |\mu|.$$

Taking the supremum for $s < \frac{n}{n-1}$ we have established for solution F to

$$\operatorname{div} F = \mu$$

the estimate

$$\|F\|_{L^{\frac{n}{n-1}}} \leq c \int_{\Omega} |d\mu| = c\|\mu\|.$$

Now we can apply Theorem 5.2 to the equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} F,$$

write the estimate

$$\|\nabla u\|_{L^n}^{n-1} = \|\mathcal{H}F\|_{L^n}^{n-1} \leq c_n \|F\|_{L^{\frac{n}{n-1}}}$$

and get the conclusion in the case μ a general Radon measure with finite mass.

Moreover $Lu_i = \mu_i \Rightarrow$

$$\|\nabla u_i - \nabla u_j\|_n^{n-1} \leq c_n \|\mu_i - \mu_j\|^{\frac{2}{n}} [\|\mu_i\| + \|\mu_j\|]^{1-\frac{2}{n}}.$$

To prove that in the special case

$$\mu = f(x)dx, \quad f \in L^1$$

we actually have

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \int_{\Omega} |\nabla u|^{n-\epsilon} = 0$$

that is $\nabla u \in$ closure of L^n with respect to the topology of L^n , it is sufficient to use the following simple lemma.

LEMMA. Let $T : L^1 \rightarrow W$ be a continuous operator such that $T(L^\infty) \subset L^\infty$. Then

$$\operatorname{dist}_W(Tf, L^\infty) = 0 \quad \forall f \in L^1(\Omega).$$

6. – Nonuniformly elliptic operators

We shall discuss briefly how estimates for div-curl expressions can be applied to the equation

$$(6.1) \quad \operatorname{div}(a(x)\nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n$$

where a is a measurable function on Ω with values in the space of symmetric positive definite matrices. Precisely we assume that

$$\alpha(x) |\xi|^2 \leq \langle a(x)\xi, \xi \rangle$$

and

$$(6.2) \quad |a(x)\xi| \leq \beta(x) |\xi|$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, where $0 < \alpha(x) \leq \beta(x) < \infty$. By a very weak solution to equation (6.1) we mean a function $u \in W_{\text{loc}}^{1,1}(\Omega)$ with $\beta(x) |\nabla u| \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$ such that

$$(6.3) \quad \int_{\Omega} \langle a(x)\nabla u, \nabla \varphi \rangle = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. In order to take the theory of such equation off the ground one must first give a meaning to the energy integrals

$$(6.4) \quad \int_{\Omega} \langle a(x)\nabla u, \nabla u \rangle \varphi(x) dx = 0 \quad \varphi \in C_0^\infty(\Omega)$$

for every weak solution u . Denote by $E = \nabla u \in L^1_{\text{loc}}$ and $B = a\nabla u \in L^1_{\text{loc}}$. Thus

$$\operatorname{div} B = \operatorname{curl} E = 0.$$

Let $1 < r, s < \infty$ be any Sobolev conjugate pair, that is $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{n}$. Assuming that

$$(6.5) \quad \int_{\Omega} |\nabla u|^s + [\beta(x) |\nabla u|]^r < \infty$$

we may define the right hand side of (6.4) as $\langle E, B \rangle(\varphi)$. However, we can do better by exploiting the fact that $\langle E, B \rangle \geq 0$ pointwise. For example if

$$(6.6) \quad \int_{\Omega} \frac{[1 + (\beta(x))^2] |\nabla u(x)|^2}{\log(e + |\nabla u(x)|)} dx < \infty$$

then $E, B \in L^2 \log^{-1} L(\Omega, \mathbb{R}^n)$ and therefore

$$(6.7) \quad \int_G \langle a(x)\nabla u, \nabla u \rangle dx < \infty$$

for each compact $G \subset \Omega$.

We can deduce also the following result which implies a slightly higher degree of regularity of the energy for local solutions of nonuniformly elliptic equations

THEOREM 6.1. *Let $\beta \in L^\infty(\Omega)$ and $\frac{1}{\alpha} \in EXP(\Omega)$. Suppose $u \in W_{loc}^{1,1}(\Omega)$ is a very weak solution to equation (6.1) with $\int_\Omega \langle a(x)\nabla u, \nabla u \rangle < \infty$. Then*

$$(6.8) \quad \langle a(x)\nabla u, \nabla u \rangle \in L \log \log L(G)$$

for any $G \subset\subset \Omega$.

PROOF. First we note that under our assumptions there exists $\lambda_0 > 0$ such that

$$(6.9) \quad \lambda_0 \int_\Omega \frac{|\nabla u|^2}{\log \left(e + \frac{|\nabla u|}{\frac{1}{|\Omega|} \int_\Omega |\nabla u|} \right)} \leq \int_\Omega \langle a(x)\nabla u, \nabla u \rangle$$

for any $u \in W^{1,1}(\Omega)$. This follows from generalized Hölder inequality in Orlicz spaces (see [GIM]).

Assume now that u is a very weak solution to (6.1) with finite energy on Ω . By (6.9) we deduce (6.6), since β is bounded. Then by previous remark we get $B, E \in L^2 \log^{-1}(\Omega, \mathbb{R}^n)$.

Since it can be proved that nonnegative div-curl expressions $\langle B, E \rangle$ with $B, E \in L^2 \log^{-1}(\Omega, \mathbb{R}^n)$ satisfy the following estimate

$$\int_G \langle B, E \rangle \log \log \left(e + \frac{\langle B, E \rangle}{\frac{1}{|G|} \int_G \langle B, E \rangle} \right) \leq c(n) \|B\|_{L^2 \log^{-1} L(\Omega, \mathbb{R}^n)} \|E\|_{L^2 \log^{-1} L(\Omega, \mathbb{R}^n)}$$

for $G \subset\subset \Omega$ (see [GIM], [Mo]), we deduce (6.8).

There is also an interesting analogy with the theory of quasiconformal mappings, see [IS3]. We shall take a moment to put equation (6.1) in this context.

A pair $\Phi = (E, B)$ of vector fields $E, B \in L^2(\Omega, \mathbb{R}^n)$ with $\text{curl} E = \text{div} B = 0$ is said to be of finite distortion if there is a measurable function $K : \Omega \rightarrow [1, \infty)$ such that

$$(6.10) \quad |\Phi(x)|^2 \leq 2K(x)J(x, \Phi) \quad \text{a.e.}$$

where $|\Phi|^2 = |E|^2 + |B|^2$ and $J(x, \varphi) = \langle E(x), B(x) \rangle$. If K is bounded, say $K(x) \leq K_0 < \infty$, then there exists $p = p(n, K_0) > 2$ such that $\Phi \in L_{loc}^p(\Omega, \mathbb{R}^n \times \mathbb{R}^n)$. There is no higher integrability theory for Φ with distortion functions K in the space $EXP(\Omega)$. An unexpected twist is that if $K \in EXP_\gamma(\Omega)$ for some $\gamma > 1$, that is

$$\|K\|_{EXP_\gamma(\Omega)} = \inf \left\{ k : \int_\Omega e^{\frac{|f|^\gamma}{k}} \leq 2 \right\} < \infty$$

then $\Phi \in L^2 \log^\alpha L(G)$ for all $\alpha \geq 0$ and any compact $G \subset \Omega$ (see [MM]). For the solution $u \in W_{\text{loc}}^{1,1}$ of equation (6.1) the pair $\Phi = (E, B)$ has finite distortion

$$K(x) = \max_{|\xi|=1} \frac{|\xi|^2 + |a(x)\xi|^2}{\langle a(x)\xi, \xi \rangle} \leq \frac{1 + \beta(x)^2}{\alpha(x)}.$$

Assuming that $K \in EXP_\gamma(\Omega)$, we then conclude with the inequality

$$\int_G |\nabla u|^2 \log^\alpha(e + |\nabla u|) < \infty.$$

REFERENCES

- [BG] L. BOCCARDO – T. GALLOUET, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989), 149-169.
- [BFS] H. BREZIS – N. FUSCO – C. SBORDONE, *Integrability for the Jacobian of orientation preserving mappings*, J. Functional Analysis **115** (1993), 425-431.
- [CS] M. CAROZZA – C. SBORDONE, *The Distance to L^∞ in some Function Spaces and Applications*, Diff. Integr. Equations **10**, 4 (1997), 599-607.
- [CLMS] R. R. COIFMAN – P. L. LIONS – Y. MEYER – S. SEMMES, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. (1993), 247-286.
- [DS] E. DE GIORGI – S. SPAGNOLO, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, Boll. Un. Mat. Ital. (4) **8** (1973), 391-411.
- [DV] T. DEL VECCHIO, *Nonlinear elliptic equations with measure data*, Potential Analysis **4** (1995), 185-203.
- [FS] A. FIORENZA – C. SBORDONE, *Existence and Uniqueness Results for Nonlinear Equations with right hand side in L^1* , Studia Math. (to appear).
- [FLS] N. FUSCO – P. L. LIONS – C. SBORDONE, *Sobolev imbedding theorems in borderline cases*, Proc. Amer. Math. Soc. **124** 2 (1996), 561-565.
- [G] L. GRECO, *A remark on the equality $\det Df = \text{Det} Df$* , Diff. Int. Eq. **6** 5 (1993), 1089-1100.
- [GIS] L. GRECO – T. IWANIEC – C. SBORDONE, *Inverting the p -harmonic operator*, Manuscripta math. **92** (1997), 249-258.
- [GISS] L. GRECO – T. IWANIEC – C. SBORDONE – B. STROFFOLINI, *Degree formulas for maps with nonintegrable Jacobian*, Topological Methods in Nonlinear Analysis **6** (1) (1995), 81-95.
- [I1] T. IWANIEC, *p -harmonic tensors and quasiregular mappings*, Ann. Math. **136** (1992), 589-624.
- [I2] T. IWANIEC, "Integrability Theory of the Jacobians", Lipschitz Lectures, 1995.
- [IS1] T. IWANIEC – C. SBORDONE, *On the integrability of the Jacobian under minimal hypothesis*, Arch. Rat. Mech. Anal. **119** (1992), 129-143.
- [IS2] T. IWANIEC – C. SBORDONE, *Weak minima of variational integrals*, J. Reine Angew. Math. **454** (1994), 143-161.

- [IS3] T. IWANIEC – C. SBORDONE, *Riesz Transforms and Elliptic pde s with VMO Coefficients*, preprint n.9, Dipart. Mat. Appl. Univ. Napoli, 1997.
- [IV] T. IWANIEC – A. VERDE, *Note on the Operator $\mathcal{L}(f) = f \log |f|$* , preprint 102, Dip. Mat. Appl. R. Caccioppoli Univ. Napoli, 1996.
- [MM] L. MIGLIACCIO – G. MOSCARIELLO, *Mappings with Unbounded Dilatation*, preprint, 3, Univ. di Salerno, 1997.
- [Mo] C. B. MORREY, “Multiple Integrals in Calculus of Variations”, Springer 1966 (Ch.7).
- [M1] S. MÜLLER, *Higher integrability of Determinants and weak convergence in L^1* , J. Reine Angew. Math. **412** (1990), 20-34.
- [M2] S. MÜLLER, *Det $\nabla u = \det \nabla u$* , Comptes Rendus Acad. Sci. Paris **311** (1990), 13-17.
- [Mos] G. MOSCARIELLO, *On the integrability of the Jacobian in Orlicz spaces*, Math. Japonica **40** (2) (1994), 323-329.
- [Mu1] F. MURAT, *Compacité per compensation*, Ann. Scuola Normale Sup. Pisa (4) **5** (1978), 489-507.
- [Mu2] F. MURAT, *Soluciones renormalizadas de EDP elipticas no lineair*, Publications du Labo. Anal. Numer. Paris (1993).
- [S] S. SPAGNOLO, *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 577-597.
- [St] G. STAMPACCHIA, *Le problème de Dirichlet pour les équations elliptiques du second ordre á coefficients discontinus*, Ann. Inst. Fourier, Grenoble **15** (1965).
- [T] L. TARTAR, *Convergence d'opérateurs différentiels*, in: “Analisi convessa e applicazioni” (Roma), 101-104, Quaderno di Ricerca del CNR (1974).
- [W] H. WENTE, *An existence theorem for surfaces of constant mean curvature*, J. Math. Analysis Appl. **26** (1969) 318-344.

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