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0. - Introduction

The purpose of this paper is to study systematically a version of the Legendre transformation which is relevant for the study of the Laplace transformation. The simplest results of the kind we have in mind are the Paley-Wiener-Schwartz theorem and the related results of Gelfand and Šilov [3], [4]. The latter paper led us more than 40 years ago to publish an announcement [5] of the statements in Section 2 and a part of Section 4 here. However, it is the later results on existence theorems with weighted bounds for the $\bar{\partial}$ operator which has made it natural to return to this topic. The main new result here is the invariance under a modified Legendre transformation of a class of functions in $\mathbb{C}^n$ that are concave in the real directions and (partially) plurisubharmonic.

Let us first recall the most classical definition of the Legendre transformation and its formal properties. Let φ be a real valued function in $C^2(\mathbb{R}^n)$. As is well known, it follows from the implicit function theorem that the equations

$$\partial\varphi(x)/\partial x_i = \xi_i, \ \ i = 1, \dots, n, \quad \tilde{\varphi}(\xi) = \langle x, \xi\rangle - \varphi(x); \qquad \langle x, \xi\rangle = \sum_1^n x_i\xi_i\,;$$

define a function $\tilde{\varphi}(\xi)$ in a neighborhood of $\varphi'(x)$ if $\det\varphi''(x) \neq 0$, and by differentiation one immediately obtains the equations $\partial\tilde{\varphi}(\xi)/\partial\xi_i = x_i$. Hence the relation between φ and $\tilde{\varphi}$ is expressed by the symmetric system of equations

$$\partial\varphi(x)/\partial x_i = \xi_i, \quad i = 1, \dots, n, \tag{0.1}$$

$$\partial\tilde{\varphi}(\xi)/\partial\xi_i = x_i, \quad i = 1, \dots, n, \tag{0.2}$$

$$\varphi(x) + \tilde{\varphi}(\xi) = \langle x, \xi\rangle\,. \tag{0.3}$$

The function $\tilde{\varphi}$ is called the Legendre transform of φ. Since the equations are symmetric, the Legendre transformation is an involution. Differentiation of (0.1) and (0.2) gives $\partial^2\varphi(x)/\partial x^2 = \partial\xi(x)/\partial x$ and $\partial^2\tilde{\varphi}(\xi)/\partial\xi^2 = \partial x(\xi)/\partial\xi$, which proves that

$$\partial^2\tilde{\varphi}(\xi)/\partial\xi^2 = (\partial^2\varphi(x)/\partial x^2)^{-1}\,. \tag{0.4}$$

To calculate the Legendre transform of a sum $\chi = \varphi + \psi$ we have the equations

$$\chi'(x) = \varphi'(x) + \psi'(x) = \xi, \quad \tilde{\chi}(\xi) = \langle x, \xi \rangle - \varphi(x) - \psi(x).$$

If we put $\varphi'(x) = \eta$ then $\psi'(x) = \xi - \eta$ and $\tilde{\varphi}(\eta) = \langle x, \eta \rangle - \varphi(x)$, $\tilde{\psi}(\xi - \eta) = \langle x, \xi - \eta \rangle - \psi(x)$, which gives

$$\tilde{\chi}(\xi) = \tilde{\varphi}(\eta) + \tilde{\psi}(\xi - \eta), \quad \partial(\tilde{\varphi}(\eta) + \tilde{\psi}(\xi - \eta))/\partial\eta = 0. \tag{0.5}$$

The second equation follows since $\tilde{\varphi}'(\eta) = x$ and $\tilde{\psi}'(\xi - \eta) = x$. Thus $\tilde{\chi}(\xi)$ is a critical value of $\eta \mapsto \tilde{\varphi}(\eta) + \tilde{\psi}(\xi - \eta)$.

Since the Legendre transformation is an involution, the Legendre transform of a function χ such that $\chi(x)$ is a critical value of $y \mapsto \varphi(y)+\psi(x-y)$ should be equal to $\tilde{\varphi}+\tilde{\psi}$. To define χ we have to solve the equation $\varphi'(y) = \psi'(x-y)$ for y and set $\chi(x) = \varphi(y)+\psi(x-y)$. Given x_0 and y_0 with $\varphi'(y_0) = \psi'(x_0-y_0)$, we can solve the equation $\varphi'(y) = \psi'(x-y)$ for y when x is in a neighborhood of x_0 so that $y(x_0) = y_0$, if $\varphi''(y_0) + \psi''(x_0 - y_0)$ is invertible. Then we have

$$\psi''(x - y) = (\varphi''(y) + \psi''(x - y))\partial y/\partial x,$$

and since $\chi'(x) = \psi'(x - y) = \varphi'(y)$ it follows that

$$\begin{aligned} \chi''(x) &= \varphi''(y)\partial y/\partial x \\ &= \varphi''(y)(\varphi''(y) + \psi''(x - y))^{-1}\psi''(x - y) \\ &= \varphi''(y) - \varphi''(y)(\varphi''(y) + \psi''(x - y))^{-1}\varphi''(y) \\ &= \psi''(x - y) - \psi''(x - y)(\varphi''(y) + \psi''(x - y))^{-1}\psi''(x - y), \end{aligned} \tag{0.6}$$

which is invertible if $\varphi''(y)$ and $\psi''(x - y)$ are invertible. Thus the Legendre transform of χ exists locally when χ, $\tilde{\varphi}$ and $\tilde{\psi}$ are defined locally, and then we have $\tilde{\chi} = \tilde{\varphi} + \tilde{\psi}$ since the Legendre transformation is an involution.

The equations defining $\tilde{\varphi}(\xi)$ can usually only be solved locally so the definition is not always valid globally. It is therefore desirable to find large classes of functions for which the solution is possible and unique in the large, and the definition can be expressed in a form that does not contain derivatives so that the differentiability assumptions on φ can be avoided. Now the equations (0.1) and (0.3) mean that $\tilde{\varphi}(y)$ is a stationary value of the function $x \mapsto \langle x, \xi \rangle - \varphi(x)$. If we require that this point shall be an absolute maximum then

$$\tilde{\varphi}(\xi) = \sup_x \left(\langle x, \xi \rangle - \varphi(x)\right). \tag{0.7}$$

It is clear that $\tilde{\varphi}$ is then a convex function. As has been proved by Mandelbrojt [8] and Fenchel [1], the transformation defined by (0.7) is involutive precisely for the functions φ that are convex and semi-continuous from below.

In Section 1 we shall recall this well-known result for any finite or infinite number of variables.

If instead we require that the stationary value shall be an absolute minimum, we are led to the definition

$$\tilde{\varphi}(\xi) = \inf_x \left(\langle x, \xi \rangle - \varphi(x)\right). \tag{0.8}$$

Since $(-\tilde{\varphi})(-\xi) = \sup_x \left(\langle x, \xi \rangle - (-\varphi)(x)\right)$, it is clear that this is involutive precisely in the class of concave and upper semi-continuous functions, so it gives nothing essentially new. However we can also single out the case of saddle points by taking the maximum over some variables and the minimum over the others, and this gives an interesting class of functions as the natural domain of definition of the transform. It will be studied in Section 2 with the minimum taken for x in a residue class with respect to a given subspace followed by the maximum over the residue classes. In Section 3 we introduce more restrictive conditions by studying functions in the direct sum of two spaces that are convex in the directions of one of the subspaces and concave in the directions of the other. These are essentially the "saddle functions" studied by Rockafellar [9], [10] with quite different motivations. The results of Section 3 are therefore not new. However, we need to emphasize the facts needed for the study in Section 6 of a class of functions in $\mathbb{C}^n$ which are further restricted by a plurisubharmonicity condition and occur naturally in the study of the Laplace transformation in Section 4. In that case the usual approach to the proof of the Paley-Wiener theorem leads to a modified Legendre transformation for functions in $\mathbb{C}^n$ where one takes first the supremum over $\mathbb{R}^n$ and then the infimum over $i\mathbb{R}^n$. (See Section 4.) With functions φ in the class $\mathcal{P}$ thus defined we associate a class of functions $\mathcal{S}_\varphi$ and prove that the Laplace transformation maps it isomorphically on the class defined by a modified Legendre transform of φ. This result includes the Paley-Wiener theorem and the lemmas on which the Schwartz definition of the Fourier transform is based, as well as the results of Gelfand and Šilov [3], [4] that were the original motivation for the announcement of some of the results in this paper given more than 40 years ago in [5]. After discussing the (modified) Legendre transform of quadratic polynomials at some length in Section 5 we prove in Section 6 that the class of functions $\mathcal{P}$ in $\mathbb{C}^n$ introduced in Section 4 is invariant under a modified Legendre transformation, as suggested by the results on the Laplace transformation. In Section 7 we discuss some lower bounds for functions in $\mathcal{P}$ and examine examples that in particular contain the results of [3], [4] when combined with existence theorems given in Section 8.

1. – The Legendre transform of convex or concave functions

Let E_1 and E_2 be two real vector spaces and suppose that there is defined a bilinear form $\langle x, \xi \rangle$ for $x \in E_1$ and $\xi \in E_2$. We introduce the weak topologies in E_1 and E_2 defined by the bilinear form. The topology in E_1 is separated if and only if $\langle x, \xi\rangle = 0$ for every $\xi \in E_2$ implies $x = 0$ and similarly for the topology in E_2. Although we do not assume these separation conditions every continuous linear form on E_1 (respectively on E_2) can be written $x \mapsto \langle x, \xi\rangle$ for some $\xi \in E_2$ (respectively $\xi \mapsto \langle x, \xi\rangle$ for some $x \in E_1$).

Let φ be a function defined in E_1 with values in $\mathbb{R}\cup\{\pm\infty\}$. In this section we define its Legendre transform $\tilde{\varphi}$ at first by

$$\tilde{\varphi}(\xi) = \sup_{x\in E_1} \big(\langle x, \xi\rangle - \varphi(x)\big), \quad \xi \in E_2 . \tag{1.1}$$

Our goal is to decide when the inversion formula $\tilde{\tilde{\varphi}} = \varphi$ is valid, that is,

$$\varphi(x) = \sup_{\xi\in E_2} (\langle x, \xi\rangle - \tilde{\varphi}(\xi)), \quad x \in E_1 . \tag{1.2}$$

It is clear that if $\tilde{\varphi} \not\equiv +\infty$ then (1.2) implies that φ is convex and semicontinuous from below with values in $\mathbb{R}\cup\{+\infty\}$, for an affine linear function and hence the supremum of a family of affine linear functions has these properties. Conversely, we have the following well-known result:

THEOREM 1.1. *If φ is a convex function in E_1 that is lower semicontinuous with values in $\mathbb{R}\cup\{+\infty\}$ and $\varphi \not\equiv +\infty$, then the Legendre transform $\tilde{\varphi}$ defined by* (1.1) *has the same properties in E_2 and the inversion formula* (1.2) *is valid.*

PROOF. As already pointed out it is obvious that $\tilde{\varphi}$ is convex and lower semicontinuous with values in $\mathbb{R}\cup +\infty$. From (1.1) it follows that $\tilde{\varphi}(\xi) \ge \langle x, \xi\rangle - \varphi(x)$, hence that $\varphi(x) \ge \langle x, \xi\rangle - \tilde{\varphi}(\xi)$, so

$$\varphi(x) \ge \sup_{\xi\in E_2} (\langle x, \xi\rangle - \tilde{\varphi}(\xi)) = \tilde{\tilde{\varphi}}(x) .$$

The proof will be achieved when we have proved the opposite inequality, which implies that $\tilde{\varphi} \not\equiv +\infty$. Thus we must prove that $t < \tilde{\tilde{\varphi}}(x)$ if $t < \varphi(x)$.

The epigraph U of φ defined by

$$U = \{(x, t) \in E_1 \oplus \mathbb{R};\, \varphi(x) \le t\}, \tag{1.3}$$

is convex, nonempty and closed by the hypotheses on φ. Take a fixed $x_0 \in E_1$ and $t_0 \in \mathbb{R}$ with $t_0 < \varphi(x_0)$. Then $(x_0, t_0) \notin U$, so it follows from the Hahn-Banach theorem that there is a separating hyperplane defined by an equation of the form $\langle x, \xi\rangle - ct = \alpha$ with $\xi \in E_2$ and $c, \alpha \in \mathbb{R}$. We may suppose that

$$\langle x_0, \xi\rangle - ct_0 > \alpha, \quad \langle x, \xi\rangle - ct \le \alpha \quad \text{when } (x, t) \in U . \tag{1.4}$$

From the second inequality it follows that $c \ge 0$. Suppose at first that $c > 0$; since we can divide by c we may as well assume that $c = 1$. Then the second inequality (1.4) means that $\tilde{\varphi}(\xi) \le \alpha$, and the first inequality gives that

$$\tilde{\tilde{\varphi}}(x_0) \ge \langle x_0, \xi\rangle - \tilde{\varphi}(\xi) > \alpha + t_0 - \alpha = t_0$$

as claimed. Now suppose that $c = 0$. Then (1.4) can be written

$$\langle x_0, \xi \rangle = \alpha + \varepsilon, \quad \langle x, \xi \rangle \le \alpha \quad \text{when } \varphi(x) < \infty, \tag{1.4$'$}$$

where $\varepsilon > 0$. This implies $\varphi(x_0) = +\infty$, and since $\varphi \not\equiv +\infty$ the first case, with $c > 0$, must occur for some x_0. Hence there exists some $\eta \in E_2$ with $\tilde{\varphi}(\eta) < \infty$. Returning to (1.4)$'$ we conclude from the second inequality that

$$\tilde{\varphi}(s\xi + \eta) = \sup_x (s\langle x, \xi \rangle + \langle x, \eta \rangle - \varphi(x)) \le s\alpha + \tilde{\varphi}(\eta), \quad \text{if } s > 0.$$

Using this inequality and the first part of (1.4)$'$ we get

$$\tilde{\tilde{\varphi}}(x_0) \ge \langle x_0, s\xi + \eta \rangle - \tilde{\varphi}(s\xi + \eta) \ge s(\alpha + \varepsilon) + \langle x_0, \eta \rangle - s\alpha - \tilde{\varphi}(\eta) = s\varepsilon + \langle x_0, \eta \rangle - \tilde{\varphi}(\eta),$$

and when $s \to +\infty$ it follows that $\tilde{\tilde{\varphi}}(x_0) = +\infty$. This completes the proof.

If $\varphi \equiv +\infty$ then $\tilde{\varphi} \equiv -\infty$ according to the definition (1.1). We shall therefore accept the function which is identically $-\infty$ as a convex function but apart from that convex functions will tacitly be assumed to have values in $(-\infty, +\infty]$.

EXAMPLE 1.1. If φ in addition to the hypotheses of Theorem 1.1 is positively homogeneous of degree one, that is, $\varphi(tx) = t\varphi(x)$ when $t > 0$, then

$$\tilde{\varphi}(\xi) = \sup_x(\langle x, \xi \rangle - \varphi(x)) = \sup_x(\langle tx, \xi \rangle - \varphi(tx)) = t \sup_x(\langle x, \xi \rangle - \varphi(x)) = t\tilde{\varphi}(\xi)$$

for every $t > 0$, which means that $\tilde{\varphi}(\xi) = 0$ or $\tilde{\varphi}(\xi) = +\infty$ everywhere. The set K where $\tilde{\varphi}(\xi) = 0$ is convex and closed, and $\varphi(x) = \sup_{\xi \in K} \langle x, \xi \rangle$ is the supporting function of K.

If $K = \{\xi \in \mathbb{R}^2; \xi_2^2 \le 4\xi_1\}$ is the closed interior of a parabola, then

$$\varphi(x) = \sup_{\xi \in K} \langle x, \xi \rangle = \begin{cases} -x_2^2/x_1, & \text{if } x_1 < 0, \\ 0, & \text{if } x = 0, \\ +\infty, & \text{if } x_1 \ge 0 \text{ and } x \ne 0. \end{cases}$$

Note that φ is not continuous at the origin with values in $(-\infty, +\infty]$ even when restricted to the half plane where $x_1 \le 0$, for on a parabola $ax_1 = x_2^2$ where $a < 0$ the limit at the origin is equal to $-a$. However, $\varphi(x) \to \varphi(0)$ if $x \to 0$ on a ray in the open left half plane.

For later reference we shall now give a detailed discussion of the semicontinuity condition in the finite dimensional case. (See e.g. Fenchel [1], [2] and Rockafellar [9], [10].) This will show that the observation made in the preceding example is valid quite generally. Recall that a convex set M in a finite dimensional vector space is contained in a minimal affine subspace $\mathrm{ah}(M)$, the affine hull of M, and that M has interior points as a subset of $\mathrm{ah}(M)$. The set M° of such points is a dense convex subset of M called the relative interior of M.

PROPOSITION 1.2. *Let φ be a convex function in a finite dimensional vector space E with values in $(-\infty, +\infty]$. Then $M = \{x \in E; \varphi(x) < \infty\}$ is a convex set, and φ is continuous in M°. If $x \in \overline{M} \setminus M^\circ$ and $x^\circ \in M^\circ$ then*

$$A(x) = \lim_{\lambda \to +0} \varphi((1-\lambda)x + \lambda x^\circ)$$

exists, $-\infty < A(x) \le \varphi(x)$, and $A(x) = \underline{\lim}_{y\to x}\varphi(y)$ is independent of x°. The largest convex lower semicontinuous minorant of φ is equal to $A(x)$ if $x \in \overline{M}\setminus M^\circ$ and equal to φ elsewhere. It is the only convex lower semicontinuous function which is equal to φ in the complement of $\overline{M}\setminus M^\circ$.

PROOF. That the limit $A(x)$ exists for a fixed $x^\circ \in M^\circ$ and that $-\infty < A(x) \le \varphi(x)$ is clear, for $\varphi((1-\lambda)x + \lambda x^\circ) + c\lambda$ is a decreasing function of $\lambda \in \left[0, \frac{1}{2}\right]$ if c is chosen so that a derivative at $\frac{1}{2}$ vanishes. If we prove that $\underline{\lim}_{y\to x}\varphi(y) = A(x)$ it will follow in particular that $A(x)$ is independent of x°. Since

$$\varphi((1-\lambda)y + \lambda x^\circ) \le (1-\lambda)\varphi(y) + \lambda\varphi(x^\circ), \quad 0 < \lambda < 1,$$

and $(1-\lambda)y + \lambda x^\circ \to (1-\lambda)x + \lambda x^\circ \in M^\circ$ when $y \to x$, for fixed $\lambda \in (0, 1)$, we obtain

$$\varphi((1-\lambda)x + \lambda x^\circ) \le (1-\lambda)\,\underline{\lim}_{y\to x}\,\varphi(y) + \lambda\varphi(x^\circ), \quad \text{if } 0 < \lambda < 1.$$

When $\lambda \to 0$ it follows that $A(x) \le \underline{\lim}_{y\to x}\varphi(y) \le A(x)$. The largest lower semicontinuous minorant ψ of φ is $\psi(x) = \underline{\lim}_{y\to x}\varphi(y)$; it is obviously convex and is equal to $A(x)$ when $x \in \overline{M}\setminus M^\circ$ and equal to φ elsewhere. Since $A(x)$ is determined by the restriction of φ to M°, the last statement follows.

REMARK. If $\psi \not\equiv \pm\infty$ is a lower semicontinuous convex function defined in a relatively open convex subset O of E, then a lower semicontinuous extension of ψ to E is given by

$$\varphi(x) = \begin{cases} \psi(x), & \text{if } x \in O, \\ \lim_{\lambda\to+0}\psi((1-\lambda)x + \lambda x^\circ), & \text{if } x \in \overline{O}\setminus O, \\ +\infty, & \text{if } x \in E\setminus\overline{O}, \end{cases}$$

where $x^\circ \in M^\circ$ with $M = \{x \in O; \varphi(x) < \infty\}$. In fact, if we first define $\psi(x) = +\infty$ in $E \setminus O$, then the hypotheses of Proposition 1.2 are fulfilled with φ replaced by ψ, so the limit in the definition above exists in $(-\infty, +\infty]$ and is independent of x°. If $x \in (O \cap \overline{M})\setminus M^\circ$ we have $\varphi(x) = \psi(x)$, for $\varphi(x) \le \psi(x)$ by Proposition 1.2, and $\psi(x) \le \varphi(x)$ by the semicontinuity assumed in O. By Proposition 1.2 φ is the only lower semicontinuous convex function which is equal to ψ in O and $+\infty$ in $E\setminus\overline{O}$. We have

$$\tilde{\varphi}(\xi) = \sup_{x\in O}(\langle x, \xi\rangle - \psi(x)),$$

for $\langle x, \xi \rangle - \varphi(x) \le \sup_{x' \in O}(\langle x', \xi \rangle - \psi(x'))$ when $x \in \overline{O} \setminus O$ by the definition above, and this is also trivially true when $x \in E \setminus \overline{O}$.

In the following results we no longer assume finite dimensionality. If ψ is a convex function then the largest lower semicontinuous convex minorant will be called the lower semicontinuous regularization of ψ.

PROPOSITION 1.3. *The limit φ of an increasing sequence of convex lower semicontinuous functions φ_j is convex and lower semicontinuous. If $\varphi \not\equiv +\infty$ then $\tilde{\varphi}_j$ is decreasing, and $\tilde{\varphi}$ is the lower semicontinuous regularization of* $\lim \tilde{\varphi}_j$.

PROOF. It is trivial that φ is convex and lower semicontinuous and that $\tilde{\varphi}_j$ is decreasing and bounded below by $\tilde{\varphi}$. If ψ is another lower semicontinuous convex minorant of $\lim \tilde{\varphi}_j$ then $\varphi_j \le \tilde{\psi}$ so $\varphi \le \tilde{\psi}$, hence $\psi \le \tilde{\varphi}$ as claimed.

If $\varphi \equiv +\infty$ it follows from the proof that there is no convex lower semicontinuous minorant of $\lim \tilde{\varphi}_j$, so it is natural to define that the lower semicontinuous regularization is identically $-\infty$ then. In the finite dimensional case it is then easy to see that $\tilde{\varphi}_j(x) \to -\infty$ in the relative interior of the convex set where $\tilde{\varphi}_j(x) < +\infty$ for some j, for a finite limit at one such point implies that the limit does not take the value $-\infty$.

PROPOSITION 1.4. *If φ_j is a decreasing sequence of lower semicontinuous convex functions then the Legendre transform of the lower semicontinuous regularization of* $\lim \varphi_j$ *is equal to* $\lim \tilde{\varphi}_j$.

PROOF. This is Proposition 1.3 applied to the sequence $\tilde{\varphi}_j$.

PROPOSITION 1.5. *If φ and ψ are convex lower semicontinuous functions in E_1 not identically $-\infty$ and $\chi = \varphi + \psi \not\equiv +\infty$, then $\tilde{\chi}$ is the lower semicontinuous regularization of*

$$E_2 \ni \xi \mapsto \inf_{\eta \in E_2} (\tilde{\varphi}(\eta) + \tilde{\psi}(\xi - \eta)) . \tag{1.5}$$

It is called the infimal convolution of $\tilde{\varphi}$ and $\tilde{\psi}$.

PROOF. Since

$$\begin{aligned}\tilde{\chi}(\xi) &= \sup_x (\langle x, \xi \rangle - \varphi(x) - \psi(x)) \\ &= \sup_x (\langle x, \eta \rangle - \varphi(x) + \langle x, \xi - \eta \rangle - \psi(x)) \le \tilde{\varphi}(\eta) + \tilde{\psi}(\xi - \eta) ,\end{aligned}$$

the lower semicontinuous regularization Γ of (1.5) is bounded below by $\tilde{\chi}$. Thus $\tilde{\Gamma} \le \tilde{\tilde{\chi}} = \chi$, and since

$$\tilde{\Gamma}(x) = \sup_\xi (\langle x, \xi \rangle - \Gamma(\xi)) \ge \sup_{\xi, \eta}(\langle x, \xi \rangle - \tilde{\varphi}(\eta) - \tilde{\psi}(\xi - \eta)) = \tilde{\tilde{\varphi}}(x) + \tilde{\tilde{\psi}}(x) = \chi(x),$$

it follows that $\tilde{\Gamma} = \chi$, hence $\Gamma = \tilde{\chi}$.

The properties of the Legendre transformation defined by

$$\tilde{\varphi}(\xi) = \inf_{x \in E_1} (\langle x, \xi \rangle - \varphi(x)), \tag{1.6}$$

are immediately reduced to those of (1.1) as pointed out in the introduction; we just have to interchange convexity and concavity, lower and upper semicontinuity and so on in the preceding statements. As an example we have the following analogue of Theorem 1.1:

THEOREM 1.6. *If φ is a concave function in E_1 which is upper semicontinuous with values in $\mathbb{R} \cup \{-\infty\}$ and $\varphi \not\equiv -\infty$, then the Legendre transform $\tilde{\varphi}$ defined by* (1.6) *has the same properties in E_2 and $\tilde{\tilde{\varphi}} = \varphi$.*

As in the case of convex functions we shall accept the function which is identically $+\infty$ as a concave function. It is the Legendre transform of the concave function which is identically $-\infty$, but all other concave functions take their values in $[-\infty, +\infty)$.

As a preparation for Section 3 we shall now give a slight extension of the preceding results. Let $E_1 \times E_2 \ni (x, \xi) \mapsto A(x, \xi) \in \mathbb{R}$ be affine linear in x for fixed ξ and in ξ for fixed x. We shall prove that $\langle x, \xi \rangle$ can be replaced by $A(x, \xi)$ in the preceding results. First we prove that there is a unique decomposition

$$A(x, \xi) = \langle x, \xi \rangle + L_1(x) + L_2(\xi) + C, \quad x \in E_1, \ \xi \in E_2,$$

where $\langle x, \xi \rangle$ is a bilinear form, L_1 and L_2 are linear forms, and C is a constant. In fact, suppose that we have such a decomposition. Then $A(0,0) = C$, $A(x, 0) = L_1(x) + C$ and $A(0, \xi) = L_2(\xi) + C$, so we must have

$$\langle x, \xi \rangle = A(x, \xi) - A(x, 0) - A(0, \xi) + A(0,0),$$
$$L_1(x) = A(x,0) - A(0,0), \ L_2(\xi) = A(0, \xi) - A(0,0), \ C = A(0,0).$$

It is immediately verified that the functions $\langle \cdot, \cdot \rangle$, L_1 and L_2 are respectively bilinear and linear forms.

Now we define as before the topologies in E_1 and in E_2 by means of the bilinear form $\langle x, \xi \rangle$. In doing so it is convenient to note that

$$\langle x - x_0, \xi - \xi_0 \rangle = A(x, \xi) - A(x, \xi_0) - A(x_0, \xi) + A(x_0, \xi_0).$$

We shall now determine when the Legendre transformation defined by

$$\tilde{\varphi}(\xi) = \sup_{x \in E_1} (A(x, \xi) - \varphi(x)), \quad \xi \in E_2, \tag{1.1$'$}$$

is involutive. (1.1)$'$ can be written in the form

$$\tilde{\varphi}(\xi) - L_2(\xi) - \tfrac{1}{2}C = \sup_{x \in E_1} \left(\langle x, \xi \rangle - \left(\varphi(x) - L_1(x) - \tfrac{1}{2}C\right)\right),$$

and
$$\tilde{\tilde{\varphi}}(x) - L_1(x) - \tfrac{1}{2}C = \sup_{\xi\in E_2}\left(\langle x,\xi\rangle - \left(\tilde{\varphi}(\xi) - L_2(\xi) - \tfrac{1}{2}C\right)\right).$$

Hence the Legendre transformation with respect to A defined in (1.1)$'$ is involutive if and only if the Legendre transformation (1.1) is involutive for $\varphi(x) - L_1(x) - \frac{1}{2}C$, that is, $\varphi(x) - L_1(x) - \frac{1}{2}C$ is convex and lower semicontinuous. The convexity is equivalent to convexity of φ, and the semi-continuity is equivalent to the semicontinuity of $x \mapsto \varphi(x) - A(x,\xi)$ for some (and hence all) $\xi \in E_2$, for $A(x,\xi) - L_1(x) - \frac{1}{2}C = \langle x,\xi\rangle + L_2(\xi) + \frac{1}{2}C$ is continuous with respect to $x \in E_1$. Hence we have:

THEOREM 1.1$'$. *If the Legendre transform is defined by* (1.1)$'$ *with a general affine bilinear* $A(x,\xi)$, *then* $\tilde{\tilde{\varphi}} = \varphi$ *if and only if* φ *is convex and* $x \mapsto \varphi(x) - A(x,\xi)$ *is lower semicontinuous with values in* $\mathbb{R}\cup\{+\infty\}$ *for some* (*and hence for all*) $\xi \in E_2$.

Note that $L_1(x)$ may not be continuous even in the finite dimensional case, for if $\langle x,\xi\rangle$ is singular, the topology is not separated. There is of course a similar extension of Theorem 1.6.

2. – A minimax definition of the Legendre transform

Let E_1 and E_2 be two real vector spaces and $\langle x,\xi\rangle$ be a bilinear form in $E_1 \times E_2$ defining separated weak topologies in E_1 and in E_2. Let F_1 be a closed subspace of E_1 and denote the annihilator in E_2 by F_2,

$$F_2 = \{\xi \in E_2;\ \langle x,\xi\rangle = 0 \text{ when } x \in F_1\},$$

which is automatically closed. Since F_1 is closed, the annihilator of F_2 is equal to F_1 by the Hahn-Banach theorem. The quotient spaces $R_1 = E_1/F_1$ and $R_2 = E_2/F_2$ are then separated and in duality with F_2 and F_1 respectively. The canonical map $E_1 \to R_1$ will be denoted $x \mapsto \dot{x}$. The constant value of $\langle x,\xi\rangle$ when ξ is fixed in F_2 and $\dot{x} = X$ is fixed in R_1 will also be denoted by $\langle X,\xi\rangle$; it is the bilinear form defining the duality between R_1 and F_2. Similarly we define $\langle x,\Xi\rangle$ when $x \in F_1$ and $\Xi \in R_2$.

In this section we define the Legendre transform as a mixed extreme value, the infimum over some variables and the supremum over the others. More precisely, for a function φ in E_1 we define

$$\tilde{\varphi}(\xi) = \sup_{X\in R_1}\left(\inf_{\dot{x}=X}(\langle x,\xi\rangle - \varphi(x))\right), \quad \xi \in E_2. \tag{2.1}$$

For a function ψ in E_2 we set

$$\tilde{\psi}(x) = \inf_{\Xi\in R_2}\left(\sup_{\dot{\xi}=\Xi}(\langle x,\xi\rangle - \psi(\xi))\right), \quad x \in E_1, \tag{2.2}$$

and for the intermediate steps in these transforms, with $\psi = \tilde{\varphi}$, we introduce the notation

$$\Phi_1(X,\xi) = \inf_{\dot{x}=X}(\langle x,\xi\rangle - \varphi(x)), \quad X \in R_1, \quad \xi \in E_2, \tag{2.3}$$

$$\Phi_2(x,\Xi) = \sup_{\dot{\xi}=\Xi}(\langle x,\xi\rangle - \tilde{\varphi}(\xi)), \quad x \in E_1, \quad \Xi \in R_2. \tag{2.4}$$

This means that

$$\tilde{\varphi}(\xi) = \sup_{X\in R_1} \Phi_1(X,\xi), \quad \xi \in E_2, \tag{2.5}$$

$$\tilde{\tilde{\varphi}}(x) = \inf_{\Xi\in R_2} \Phi_2(x,\Xi), \quad x \in E_1. \tag{2.6}$$

Lemma 2.1. *If* $\tilde{\tilde{\varphi}} = \varphi$ *then*

$$\Phi_1(\dot{x},\xi) + \Phi_2(x,\dot{\xi}) = \langle x,\xi\rangle, \quad x \in E_1,\ \xi \in E_2. \tag{2.7}$$

This should be understood as $\Phi_1(\dot{x},\xi) = \langle x,\xi\rangle - \Phi_2(x,\dot{\xi})$ *if the terms on the left are infinite.*

Proof. From the definition (2.3) of $\Phi_1(\dot{x},\xi)$ it follows that

$$\langle x,\xi\rangle - \Phi_1(\dot{x},\xi) = \sup_{\dot{y}=\dot{x}}(\langle x-y,\xi\rangle + \varphi(y)) = \sup_{y\in F_1}(\langle y,\dot{\xi}\rangle + \varphi(x-y)),$$

which is a function of x and $\dot{\xi}$. Similarly we find that $\langle x,\xi\rangle - \Phi_2(x,\dot{\xi})$ is a function of $\dot{x}$ and ξ. Now we get from (2.5) and (2.6) that

$$\tilde{\varphi}(\xi) \geq \Phi_1(\dot{x},\xi), \quad \varphi(x) = \tilde{\tilde{\varphi}}(x) \leq \Phi_2(x,\dot{\xi}), \quad x \in E_1,\ \xi \in E_2.$$

Hence (2.3) and (2.4) give

$$\Phi_1(\dot{x},\xi) = \inf_{\dot{y}=\dot{x}}(\langle y,\xi\rangle - \varphi(y)) \geq \inf_{\dot{y}=\dot{x}}(\langle y,\xi\rangle - \Phi_2(y,\dot{\xi})) = \langle x,\xi\rangle - \Phi_2(x,\dot{\xi}),$$

$$\Phi_2(x,\dot{\xi}) = \sup_{\dot{\eta}=\dot{\xi}}(\langle x,\eta\rangle - \tilde{\varphi}(\eta)) \leq \sup_{\dot{\eta}=\dot{\xi}}(\langle x,\eta\rangle - \Phi_1(\dot{x},\eta)) = \langle x,\xi\rangle - \Phi_1(\dot{x},\xi).$$

The last equality follows since the function whose infimum (supremum) is taken is in fact constant in the equivalence class. Combining these two inequalities we get (2.7).

Lemma 2.2. *If* $\tilde{\tilde{\varphi}} = \varphi$ *then*

(A) φ *is either* $\equiv +\infty$ *or everywhere* $< +\infty$, *concave and upper semicontinuous in every equivalence class modulo* F_1.

(B) $\Phi_1(X,\xi)$ *is concave and upper semicontinuous as a function of* $X \in R_1$.

PROOF. Using (2.6) and (2.7) we have

$$\varphi(x) = \tilde{\tilde{\varphi}}(x) = \inf_{\xi \in E_2} (\langle x, \xi \rangle - \Phi_1(\dot{x}, \xi)), \quad x \in E_1 .$$

In every equivalence class modulo F_1 it follows that φ is the infimum of a family of affine linear functions (which may be empty). This proves (A). Moreover, it follows from (2.4) that $\Phi_2(x, \Xi)$ is convex and semicontinuous from below as a function of x, which implies (B) by (2.7).

The converse is true:

LEMMA 2.3. *If φ satisfies* (A) *and* (B) *then* $\tilde{\tilde{\varphi}} = \varphi$.

PROOF. Since φ satisfies (A), it follows from the analogue of Theorem 1.1′ for concave functions, applied to the equivalence class of x and the space E_2, that

$$\varphi(x) = \inf_{\xi \in E_2} (\langle x, \xi \rangle - \Phi_1(\dot{x}, \xi)) . \tag{2.8}$$

We can write $\langle x, \xi \rangle - \Phi_1(\dot{x}, \xi) = \Psi(x, \dot{\xi})$, for the difference is constant in the equivalence classes modulo F_2 as proved at the beginning of the proof of Lemma 2.1. It follows from (B) that $\Psi(x, \dot{\xi}) - \langle x, \xi \rangle = -\Phi_1(\dot{x}, \xi)$ is convex and lower semicontinuous as a function of x with the topology induced by the bilinear form $\langle x, \eta \rangle$ considered only for $x \in E_1$ and η in the equivalence class of ξ, for this topology is simply the topology of $R_1 = E_1/F_1$. Now the (convex) Legendre transform of $E_1 \ni x \mapsto \Psi(x, \dot{\xi})$ at η is

$$\sup_{x \in E_1} (\langle x, \eta \rangle - \Psi(x, \dot{\xi})) = \sup_{x \in E_1} \Phi_1(\dot{x}, \eta) = \tilde{\varphi}(\eta), \quad \text{if } \dot{\eta} = \dot{\xi} ,$$

so it follows from Theorem 1.1′ that

$$\Psi(x, \dot{\xi}) = \sup_{\dot{\eta} = \dot{\xi}} (\langle x, \eta \rangle - \tilde{\varphi}(\eta)) . \tag{2.9}$$

Combining (2.8) and (2.9) we obtain $\tilde{\tilde{\varphi}}(x) = \varphi(x)$, which proves the lemma.

It is easy to see that the condition (B) can be split into the following two more familiar conditions:

(B_1) For every $\xi \in E_2$ the maximum principle is valid for $E_1 \ni x \mapsto \varphi(x) - \langle x, \xi \rangle$ in the following form: If $\varphi(x) - \langle x, \xi \rangle \le C$ for all $x \in E_1$ such that $\dot{x} = X_1$ or $\dot{x} = X_2$, where $X_1, X_2 \in R_1$, then the same inequality is valid if $\dot{x} = \lambda X_1 + (1 - \lambda) X_2$ and $0 \le \lambda \le 1$.

(B_2) For every $\xi \in E_2$ the function $E_1 \ni x \mapsto \varphi(x) - \langle x, \xi \rangle$ is lower semicontinuous with respect to R_1 in the following sense: Given $\varepsilon > 0$ and $x \in E_1$ with $\varphi(x) > -\infty$ there exists a neighborhood U of $\dot{x}$ in R such that in every class in U there is at least one y such that $\varphi(y) - \langle y, \xi \rangle > \varphi(x) - \langle x, \xi \rangle - \varepsilon$.

We leave the proof for the reader. (Similar arguments can also be found in Thorin [12].) Summing up, we have proved

THEOREM 2.4. *In order that $\tilde{\tilde{\varphi}} = \varphi$ with the definitions* (2.1) *and* (2.2) *of the Legendre transform it is necessary and sufficient that φ satisfies the conditions* (A) *and* (B) (*or equivalently* (A), (B_1) *and* (B_2)). *Then $-\tilde{\varphi}$ satisfies the analogous conditions with F_1 replaced by F_2.*

PROOF. That $\tilde{\varphi}$ satisfies these conditions follows since we have $\tilde{\tilde{\tilde{\varphi}}} = \tilde{\varphi}$.

EXAMPLE 2.1. If $F_1 = \{0\}$ then the conditions (A) and (B) mean that φ is convex and lower semicontinuous, so Theorem 2.4 contains Theorem 1.1.

EXAMPLE 2.2. If $F_1 = E_1$ then (A) and (B) mean that φ is concave and upper semicontinuous, so Theorem 2.4 contains Theorem 1.6 also.

EXAMPLE 2.3. Suppose that φ takes no other values than 0 and $-\infty$, let M be the set where $\varphi = 0$, and set $M(X) = M \cap \{x \in E_1; \dot{x} = X\}$ when $X \in R_1$. For φ to satisfy (A) and (B) it is necessary and sufficient that $M(X)$ is convex and closed for every $X \in R_1$, and that $M(X)$ is a lower semicontinuous convex family of convex sets. From the fact that $t\varphi(x) = \varphi(x)$ when $t > 0$ it follows that $\tilde{\varphi}(\xi)$ is positively homogeneous of degree 1. We can call $\tilde{\varphi}$ the supporting function of M. An explicit elementary example is

$$E_1 = E_2 = \mathbb{R}^2, \quad F_1 = \{x \in \mathbb{R}^2; x_1 = 0\}, \quad M = \{x \in \mathbb{R}^2; x_2^2 \le x_1^2 + 1\}.$$

Then $M(x_1)$ is an interval with length $2\sqrt{x_1^2+1}$,

$$\Phi_1(x_1, \xi) = \inf_{x_2^2 \le x_1^2+1} (x_1\xi_1 + x_2\xi_2) = x_1\xi_1 - \sqrt{x_1^2+1}\,|\xi_2|$$

is a concave function of x_1, and

$$\tilde{\varphi}(\xi) = \sup_{x_1} \left(x_1\xi_1 - \sqrt{x_1^2+1}\,|\xi_2| \right) = \begin{cases} -\sqrt{\xi_2^2 - \xi_1^2}, & \text{if } \xi_2^2 \ge \xi_1^2, \\ +\infty, & \text{if } \xi_2^2 < \xi_1^2. \end{cases}$$

Note the Lorentz invariance which shows that any other spacelike choice of F_1 would have given the same Legendre transform.

EXAMPLE 2.4. Let E_1 be finite dimensional and let $\varphi(x) = Q(x,x)$ where Q is a symmetric bilinear form in E_1. If the restriction of φ to F_1 is concave then $Q(x,x) \le 0$ for $x \in F_1$. Furthermore, if φ satisfies (B_1) we must have $Q(x,x) \ge 0$ if $Q(x, F_1) = 0$, for

$$\sup_{y \in F_1} Q(tx+y, tx+y) = t^2 Q(x,x)$$

must be a convex function of $t \in \mathbb{R}$. Conversely, if these conditions are fulfilled and Q is nonsingular, we can choose coordinates $x_1, \dots, x_n$ in E_1 such that F_1 is defined by $x'' = (x_{\nu+1}, \dots, x_n) = 0$ and for some $\mu \le n - \nu$, $\mu \le \nu$

$$\varphi(x) = -\sum_{\mu+1}^{\nu} x_j^2 + \sum_1^{\mu} x_j x_{\nu+j} + \sum_{\nu+\mu+1}^{n} x_j^2 .$$

Set $x' = (x_1, \dots, x_\nu)$. Then $\inf_{x'}(\langle x, \xi\rangle - \varphi(x)) = -\infty$ unless $\xi_j = x_{\nu+j}$ for $1 \le j \le \mu$; then it is equal to $\sum_{\nu+\mu+1}^n (x_j\xi_j - x_j^2) + \sum_1^\mu \xi_j\xi_{j+\mu} - \sum_{\mu+1}^\nu \xi_j^2/4$, which is concave in x''. Thus $\tilde\varphi$ is defined and

$$\tilde\varphi(\xi) = -\frac{1}{4}\sum_{\mu+1}^{\nu}\xi_j^2 + \sum_1^\mu \xi_j\xi_{j+\nu} + \frac{1}{4}\sum_{\nu+\mu+1}^{n}\xi_j^2,$$

which is precisely the transform defined by (0.1)-(0.3). This is not surprising, for if the sup inf in (2.1) is attained at a point x where φ is differentiable then $\varphi'(x) = \xi$. Note that $\tilde\varphi \ge 0$ in the orthogonal space F_2 defined by $\xi' = 0$, and that the transform does not depend on the choice of F_1 provided that it is chosen so that conditions (A) and (B) are fulfilled. If we choose F_1 with maximal dimension so that Q is negative definite there then $\tilde Q_2$ is positive definite in F_2. We shall study the Legendre transform of quadratic forms much more in Section 5.

Remark. The preceding example easily shows that the set of functions satisfying (A) and (B) is not closed under addition. If we take $E_1 = E_2 = \mathbb{R}^2$ and $F_1 = \mathbb{R}\times\{0\}$, $\varphi(x) = -x_1^2 + x_2^2$, $\psi(x) = ax_1^2 + 2bx_1x_2 + cx_2^2$, then φ and ψ satisfy (A) and (B) if $a < 0$ and $b^2 > ac$. However, $\varphi(x) + \psi(x) = (a-1)x_1^2 + 2bx_1x_2 + (c+1)x_2^2$ does not satisfy (B) unless $b^2 \ge (a-1)(c+1)$. If

$$a < 0, \quad ac < b^2 < (a-1)(c+1) = ac + a - c - 1\,,$$

then ψ satisfies (B) but $\varphi + \psi$ does not. If we choose $a < 0$ and $c = a - 2$ then the conditions on b can be fulfilled. By the inversion formula for Legendre transforms we conclude that the same problem occurs if one wants to define a "critical convolution" by (0.5). The reason for these flaws is of course that (B) is quite weak in the sense that it does not give any information on where the infimum in (2.3) is attained. In Section 3 we shall introduce more restrictive conditions which eliminate this problem.

Example 2.5. The hypotheses (A) and (B) are satisfied by some rather weird functions. For example, if $E_1 = E_2 = \mathbb{R}^2$ and $F_1 = \mathbb{R}\times\{0\}$ then

$$\varphi(x_1, x_2) = x_1x_2 + \psi(x_2)$$

obviously satisfies (A) for any ψ, and (B) is valid if ψ takes its values in $(-\infty, +\infty]$ since

$$\Phi_1(x_2, \xi) = \inf_{x_1}(x_1\xi_1 + x_2\xi_2 - \varphi(x_1, x_2)) = \begin{cases} -\infty, & \text{if } x_2 \ne \xi_1\,, \\ x_2\xi_2 - \psi(x_2), & \text{if } x_2 = \xi_1\,, \end{cases}$$

which is a concave function of x_2 when ξ is fixed. It is equal to $-\infty$ except at one point at most. The Legendre transform is

$$\tilde\varphi(\xi) = \sup_{x_2}\Phi_1(x_2, \xi) = \xi_1\xi_2 - \psi(\xi_1)\,.$$

The point of this example is that (B) only guarantees that Φ_1 is separately concave and separately upper semicontinuous in the variables X and ξ. The set where Φ_1 is finite may therefore be quite complicated. It would not make invariant sense to require $\Phi_1(X,\xi)$ to be a concave function, for

$$\Phi_1(X,\xi+\eta)=\Phi_1(X,\xi)+\langle X,\eta\rangle,\qquad X\in R_1,\ \xi\in E_2,\ \eta\in F_2\,.$$

If $\Phi_1(X,\xi)$ were concave as a function of (X,ξ) then the right-hand side would be concave too. Replacing η by $t\eta$ and letting $t\to+\infty$ after division by t we would conclude that $\langle X,\eta\rangle$ is a concave function in $R_1\times F_2$ which is not true. However, if a supplement of F_1 in E_1 is fixed, then the preceding objection is no longer valid. This is the situation that will be studied in Section 3. It occurs naturally in the applications to the Laplace transformation where $E_1=\mathbb{C}^n$, $F_1=\mathbb{R}^n$, which has the natural supplement $i\mathbb{R}^n$.

There is a lack of symmetry between (2.1) and (2.2) — the order of the supremum and infimum is reversed. This can be replaced by another asymmetry with a modified definition which is much better adapted to the application to the Fourier-Laplace transformation. Thus we define

$$\varphi^\dagger(\xi)=\inf_{X\in R_1}\Big(\sup_{\dot x=X}(\langle x,\xi\rangle+\varphi(x))\Big),\qquad \xi\in E_2\,. \tag{2.11}$$

This means that

$$-\varphi^\dagger(\xi)=\sup_{X\in R_1}\Big(\inf_{\dot x=X}(\langle x,\xi\rangle-\varphi(-x))\Big),\qquad \xi\in E_2\,,$$

so $-\varphi^\dagger$ is the Legendre transform of $\varphi(-\cdot)$ in the sense (2.1). If φ satisfies (A) and

(B)† $\Phi_1^\dagger(X,\xi)=\sup_{\dot x=X}(\langle x,\xi\rangle+\varphi(x))$ is convex and lower semicontinuous as a function of $X\in R_1$,

then it follows from Lemma 2.3 that

$$\varphi(-x)=\inf_{\Xi\in R_2}\Big(\sup_{\dot\xi=\Xi}(\langle x,\xi\rangle+\varphi^\dagger(\xi))\Big)=\varphi^{\dagger\dagger}(x)\,, \tag{2.12}$$

so the iterated transformation (2.11) behaves just as the iterated Fourier(-Laplace) transformation. We shall refer to the normalization (2.11) as the modified the Legendre transform whenever a confusion seems possible.

3. – The Legendre transform of concave-convex functions

For the sake of simplicity *we assume from now on that E_1 and E_2 are finite dimensional and that the bilinear form $E_1 \times E_2 \ni (x, \xi) \mapsto \langle x, \xi \rangle$ is non-degenerate.* Choose a supplement G_1 of F_1 in E_1, so that $E_1 = F_1 \oplus G_1$. If G_2 is the annihilator of G_1 in E_2 and F_2 as before is the annihilator of F_1 in E_2, it follows that $E_2 = F_2 \oplus G_2$. We shall denote the elements in E_1 by (x, y) where $x \in F_1$ and $y \in G_1$ and those in E_2 by (ξ, η) where $\xi \in F_2$ and $\eta \in G_2$. The bilinear form defining the duality of E_1 and E_2 can be written

$$E_1 \times E_2 \ni (x, y), (\xi, \eta) \mapsto \langle x, \eta \rangle_{F_1, G_2} + \langle y, \xi \rangle_{G_1, F_2}$$

where we shall usually omit the subscripts. We identify R_1 with G_1 and R_2 with G_2 now. If φ is a function in E_1 then (2.3) and (2.4) take the form

$$(3.1)\ \Phi_1(y, (\xi, \eta)) = \langle y, \xi \rangle + \Psi_1(y, \eta), \quad \Psi_1(y, \eta) = \inf_{x \in F_1} (\langle x, \eta \rangle - \varphi(x, y)),$$

$$(3.2)\ \Phi_2((x, y), \eta) = \langle x, \eta \rangle + \Psi_2(y, \eta), \quad \Psi_2(y, \eta) = \sup_{\xi \in F_2} (\langle y, \xi \rangle - \tilde{\varphi}(\xi, \eta)),$$

and (2.7) states that if $\tilde{\tilde{\varphi}} = \varphi$ then

$$\Psi_2(y, \eta) = -\Psi_1(y, \eta).$$

This is quite obvious, for

$$\tilde{\varphi}(\xi, \eta) \geq \langle y, \xi \rangle + \Psi_1(y, \eta), \quad \varphi(x, y) = \tilde{\tilde{\varphi}}(x, y) \leq \langle x, \eta \rangle + \Psi_2(y, \eta),$$

which implies $\Psi_1(y, \eta) \geq -\Psi_2(y, \eta)$ and $\Psi_2(y, \eta) \leq -\Psi_1(y, \eta)$. Thus we get again the necessity of condition (B) which can be stated

(B) $y \mapsto \Psi_1(y, \eta)$ is concave and upper semicontinuous,

for $y \mapsto \Psi_2(y, \eta)$ is obviously convex and lower semicontinuous. The sufficiency proved in Lemma 2.3 also follows right away, for

$$\varphi(x, y) = \inf_{\eta \in G_2} (\langle x, \eta \rangle - \Psi_1(y, \eta)) \tag{3.3}$$

by condition (A), and since $\tilde{\varphi}(\xi, \eta) = \sup_{y \in G_1} (\langle y, \xi \rangle + \Psi_1(y, \eta))$ it follows from (B) that

$$-\Psi_1(y, \eta) = \sup_{\xi \in F_2} (\langle y, \xi \rangle - \tilde{\varphi}(\xi, \eta)),$$

and this gives the inversion formula

$$\varphi(x, y) = \inf_{\eta \in G_2} \sup_{\xi \in F_2} (\langle x, \eta \rangle + \langle y, \xi \rangle - \tilde{\varphi}(\xi, \eta)).$$

Thus our present hypotheses give a slightly simpler proof of the results in Section 2. However, the main point is that we can now introduce stronger hypotheses which will be natural in the application to the Laplace transformation in Section 4.

By condition (B) $\Psi_1(y, \eta)$ is concave and upper semicontinuous as a function of $y \in G_1$, and the definition shows that it has these properties also as a function of $\eta \in G_2$. Conversely, every function Ψ_1 in $G_1 \oplus G_2$ with these properties defines by (3.3) a function $\varphi(x, y)$ for which the conditions (A) and (B), hence the inversion formula, are valid. In the rest of this section we shall strengthen these properties to

(C) The function $\Psi_1(y, \eta)$ defined in (3.1) is concave and upper semicontinuous.

This condition has been studied before by Rockafellar [10] where the relation to the saddle functions introduced in [9] was established. The results in this section could therefore be extracted from [9], [10] but we shall give a selfcontained exposition emphasizing the facts we need in Section 6.

Condition (C) is much stronger separate concavity and separate semicontinuity. Our next goal is to express it (C) in terms of the corresponding function φ in E_1 defined by (3.3). However, before doing so we shall switch to the modified definition of the Legendre transform in (2.11), so we set

$$\varphi^{\dagger}(\xi, \eta) = \inf_{y\in G_1} \sup_{x\in F_1} (\langle x, \eta\rangle + \langle y, \xi\rangle + \varphi(x, y))\,. \tag{3.4}$$

This means that $\varphi^{\dagger}(\xi, \eta) = -\tilde{\varphi}(-\xi, -\eta)$ with our earlier notation, and since $-\Psi_1(y, -\eta) = \Phi(y, \eta)$ where

$$\Phi(y, \eta) = \sup_{x\in F_1} (\langle x, \eta\rangle + \varphi(x, y)), \quad y \in G_1, \ \ \eta \in G_2\,, \tag{3.5}$$

the condition (C) becomes

(C)† The function Φ defined by (3.5) is convex and lower semicontinuous in $G_1 \oplus G_2$ with values in $(-\infty, +\infty]$.

The condition (A) means that

$$\varphi(x, y) = \inf_{\eta\in G_2} (\Phi(y, \eta) - \langle x, \eta\rangle)\,. \tag{3.6}$$

The infimum is a convex function of y, for if $\Phi(y_j, \eta_j) - \langle x, \eta_j\rangle \le C_j$, $j = 0, 1$, and $0 < \lambda < 1$, then

$$\Phi((1-\lambda)y_0 + \lambda y_1, (1-\lambda)\eta_0 + \lambda\eta_1) - \langle x, (1-\lambda)\eta_0 + \lambda\eta_1\rangle \le (1-\lambda)C_0 + \lambda C_1\,,$$

which means that

$$\varphi(x, (1-\lambda)y_0 + \lambda y_1) \le (1-\lambda)\varphi(x, y_0) + \lambda\varphi(x, y_1), \quad 0 < \lambda < 1\,, \tag{3.7}$$

where the right-hand side should be interpreted as $+\infty$ if one of the terms is $+\infty$. Set

$$M = \{(y, \eta) \in G_1 \oplus G_2;\ \Phi(y, \eta) < +\infty\}, \tag{3.8}$$

and let

$$\begin{aligned} Y_1 &= \{y \in G_1;\ \Phi(y, \eta) < +\infty \text{ for some } \eta \in G_2\}, \\ Y_2 &= \{\eta \in G_2;\ \Phi(y, \eta) < +\infty \text{ for some } y \in G_1\}, \end{aligned} \tag{3.9}$$

be the projections of M in G_1 and G_2 respectively. If $y \notin Y_1$ then $\varphi(x, y) = +\infty$ for every $x \in F_1$ but if $y \in Y_1$ then $F_1 \ni x \mapsto \varphi(x, y)$ is a concave function with values in $[-\infty, +\infty)$.

The relative interior Y_j° and the affine hull $\mathrm{ah}(Y_j)$ of Y_j are the projections in G_j of M° and $\mathrm{ah}(M)$. In fact, a simplex $S \subset M$ with $\mathrm{ah}(S) = \mathrm{ah}(M)$ is projected to a convex polyhedron $S_j \subset Y_j$ in G_j with $\mathrm{ah}(S_j) = \mathrm{ah}(Y_j)$, and the relative interior of S_j is projected to the relative interior of S_j.

PROPOSITION 3.1. *If φ satisfies* (A) *and* $(\mathrm{C})^\dagger$ *and Y_1 is defined by* (3.9), *then*

(i) *$\varphi(x, y) = +\infty$ if $x \in F_1$ and $y \in G_1 \setminus Y_1$.*
(ii) *$F_1 \ni x \mapsto \varphi(x, y)$ is a concave upper semicontinuous function $\not\equiv -\infty$ with values in $[-\infty, +\infty)$ if $y \in Y_1$.*
(iii) *$Y_1 \ni y \mapsto \varphi(x, y)$ is convex with values in $[-\infty, +\infty)$ for every $x \in F_1$.*
(iv) *If $y_0 \in Y_1 \setminus Y_1^\circ$ and $y_1 \in Y_1^\circ$, then $x \mapsto \varphi(x, y_0)$ is the upper semicontinuous regularization of $x \mapsto \lim_{\lambda\to+0} \varphi(x, (1-\lambda)y_0 + \lambda y_1)$. If $y_0 \in \overline{Y}_1 \setminus Y_1$ then the limit has no upper semicontinuous concave majorant except $+\infty$.*

Conversely, if φ is a function in E_1 with values in $[-\infty, +\infty]$ and Y_1 is a convex subset of G_1 such that the conditions (i)-(iv) *are fulfilled, then conditions* (A) *and* $(\mathrm{C})^\dagger$ *are fulfilled. We also have*

(v) *There is a convex subset X_1 of F_1 such that $\varphi(x, y) = -\infty$ in $(F_1 \setminus X_1) \times Y_1^\circ$ and $Y_1^\circ \ni y \mapsto \varphi(x, y)$ is a (continuous) convex function for every $x \in X_1$. If K is a compact subset of Y_1° and $y_0 \in K$, then the convex functions*

$$K \ni y \mapsto \varphi(x, y)/(1 + |x| + |\varphi(x, y_0)|)$$

are uniformly bounded and equicontinuous when $x \in X_1$. If $x \notin \overline{X}_1 \setminus X_1^\circ$ and $y_0 \in Y_1 \setminus Y_1^\circ$, $y_1 \in Y_1^\circ$, then $\varphi(x, y_0) = \lim_{\lambda\to+0} \varphi(x, (1-\lambda)y_0 + \lambda y_1)$.

PROOF. (i) and (ii) follow from (A), for if $\varphi(\cdot, y) \equiv -\infty$ then $\Phi(y, \cdot) \equiv -\infty$. Since $\varphi(x, y) < +\infty$ when $y \in Y_1$ we have already proved (iii) in (3.7).

With y_0 and y_1 as in (iv) it follows from Proposition 1.2 and (3.7) that $\psi(x) = \lim_{\lambda\to+0} \varphi(x, (1-\lambda)y_0 + \lambda y_1)$ exists and that $\psi(x) \le \varphi(x, y_0)$; it is clear that $\psi(x)$ is concave. To prove that the upper semicontinuous regularization of ψ is equal to $\varphi(\cdot, y_0)$ assume that $\eta_0 \in G_2$ and that $\langle x, \eta_0\rangle + \psi(x) \le A < \infty$ when $x \in F_1$. Choose η_1 so that $(y_1, \eta_1) \in M^\circ$ and consider the function

$$\varphi(x, (1-\lambda)y_0 + \lambda y_1) + \langle x, (1-\lambda)\eta_0 + \lambda\eta_1\rangle, \quad 0 < \lambda \le 1.$$

It is convex in λ and the limit as $\lambda \to 0$ is $\le A$. When $\lambda = 1$ it has a bound B, so it is bounded by $(1-\lambda)A + \lambda B$ when $0 < \lambda \le 1$. Thus

$$\Phi((1-\lambda)y_0 + \lambda y_1, (1-\lambda)\eta_0 + \lambda\eta_1) \le (1-\lambda)A + \lambda B \to A \quad \text{when } \lambda \to 0.$$

This implies that $\Phi(y_0, \eta_0) \le A$, for Φ is lower semicontinuous by condition (C)†. Thus we have proved that $\Phi(y_0, \eta_0) \le \sup_x (\langle x, \eta_0\rangle + \psi(x))$, and the opposite inequality is valid since $\psi(x) \le \varphi(x, y_0)$. Since ψ is concave it follows that $\varphi(\cdot, y_0)$ is the upper semicontinuous regularization of ψ. If $y_0 \in \overline{Y}_1$ and the limit ψ has an upper semicontinuous concave majorant $\not\equiv +\infty$ then we can choose $\eta_0 \in G_2$ and A so that $\langle x, \eta_0\rangle + \psi(x) \le A < \infty$ and conclude as before that $\Phi(y_0, \eta_0) \le A$, hence that $y_0 \in Y_1$. This proves (iv).

Assume now instead that (i)-(iv) are fulfilled. Condition (A) follows from (i) and (ii). The function Φ defined by (3.5) is convex, for $\Phi(y, \eta) = +\infty$ if $y \notin Y_1$, and since

$$Y_1 \times G_2 \ni (y, \eta) \mapsto \langle x, \eta\rangle + \varphi(x, y)$$

is convex for every $x \in F_1$ by condition (iii), it follows that Φ is convex in $Y_1 \times G_2$ with values in $(-\infty, +\infty]$. To prove that Φ is lower semicontinuous assume that $(y_j, \eta_j) \to (y_0, \eta_0)$ and that $\Phi(y_j, \eta_j) \le A < \infty$. This means that

$$\langle x, \eta_j\rangle + \varphi(x, y_j) \le A, \quad x \in F_1,$$

which implies that $y_j \in Y_1$. If $y_0 \in Y_1^\circ$ and $\varphi(x, y_0) > -\infty$ then the convex function $Y_1^\circ \ni y \mapsto \varphi(x, y)$ is continuous at y_0. Hence

$$\langle x, \eta_0\rangle + \varphi(x, y_0) \le A, \quad x \in F_1, \quad \text{that is, } \Phi(y_0, \eta_0) \le A.$$

If $y_0 \in \overline{Y}_1 \setminus Y_1^\circ$, it follows from Proposition 1.2 that

$$\langle x, \eta_0\rangle + \lim_{\lambda\to+0} \varphi(x, (1-\lambda)y_0 + \lambda y^\circ) \le A, \quad x \in F_1,$$

if $y^\circ \in Y_1^\circ$. By condition (iv) this implies that

$$\langle x, \eta_0\rangle + \varphi(x, y_0) \le A, \quad x \in F_1,$$

which means that $\Phi(y_0, \eta_0) \le A$. This completes the proof that Φ is lower semicontinuous.

The convexity of $Y_1 \ni y \mapsto \varphi(x, y)$ proves that this function is $\equiv -\infty$ in Y_1° if it takes the value $-\infty$. Thus

$$X_1 = \{x \in F_1; \varphi(x, y_1) > -\infty\}$$

is independent of the choice of $y_1 \in Y_1^\circ$. Since $x \mapsto \varphi(x, y_1)$ is concave it is obvious that X_1 is convex, and $X_1 \neq \emptyset$ by condition (ii). If $y_0 \in Y_1 \setminus Y_1^\circ$ and $y_1 \in Y_1^\circ$ the limit $\psi(x) = \lim_{\lambda\to+0} \varphi(x, (1-\lambda)y_0 + \lambda y_1)$ exists, it is a concave function, equal to $-\infty$ in $F_1 \setminus X_1$, and the smallest concave upper semicontinuous majorant is $\varphi(\cdot, y_0)$ by condition (iv). Hence it follows from Proposition 1.2 that $\psi(x) = \varphi(x, y_0)$ if $x \in X_1^\circ$ or $x \in F_1 \setminus \overline{X}_1$. The remaining statements in (v) are consequences of the following lemma:

LEMMA 3.2. *Let v be a real valued nonnegative function in*

$$B = \{(x, y) \in \mathbb{R}^\nu \times \mathbb{R}^\mu ; |x| < a, |y| < b\}$$

such that $v(x, y)$ is a convex (concave) function of x (of y) for fixed y (fixed x). Then v is continuous, and if M is the mean value of v in B then

$$v(x, y) \leq (\mu + 1)(1 - |x/a|)^{-\nu}(1 - |y/b|)^{-\mu} M, \qquad (x, y) \in B, \tag{3.10}$$

$$|v(x, y) - v(0, 0)| \leq (\mu + 1)(2^{\nu+\mu+1}|x/a| + |y/b|)M, \quad (x, y) \in \tfrac{1}{2}B. \tag{3.11}$$

PROOF. Since a real valued convex (concave) function is continuous, and pointwise convergence of a sequence of convex functions implies locally uniform convergence, it is clear that v is continuous. We have $v(0, y) \geq v(0, 0)(1-|y/b|)$ by the concavity in y, and

$$v(0, y) \int_{|x|<a} dx \leq \int_{|x|<a} v(x, y)\, dx$$

since $v(0, y) \leq \frac{1}{2}(v(x, y) + v(-x, y))$ by the convexity in x. Hence

$$v(0, 0) \int_{|x|<a} dx \int_{|y|<b} (1 - |y|/b)\, dy \leq \iint_B v(x, y)\, dx\, dy,$$

and since $\int_0^b (1 - r/b) r^{\mu-1}\, dr = \int_0^b r^{\mu-1}\, dr/(\mu + 1)$ it follows that $v(0, 0) \leq (\mu+1)M$. If we apply this estimate to balls with centers x, y and radii $a - |x|$, $b - |y|$, we obtain (3.10). The concavity in y and the convexity in x give

$$|v(0, y) - v(0, 0)| \leq v(0, 0)||y/b| \leq (\mu + 1)|y/b|M, \quad |y| < b,$$
$$|v(x, y) - v(0, y)| \leq (\mu + 1)2^{\nu+\mu}|2x/a|M, \quad (x, y) \in \tfrac{1}{2}B,$$

since $0 \leq v(x, y) \leq (\mu + 1)2^{\nu+\mu}M$ in $\frac{1}{2}B$. This proves (3.11).

EXAMPLE 3.1. The function φ in $\mathbb{R}^2$ defined by

$$\varphi(x, y) = \begin{cases} \log(x/(x + y)), & \text{if } x > 0, y > 0 \\ -\infty, & \text{if } x \leq 0, y > 0 \text{ or } x < 0, y = 0 \\ 0, & \text{if } x \geq 0, y = 0 \\ +\infty, & \text{if } y < 0 \end{cases}$$

satisfies (A) and (C)†. We have $Y_1 = \{y; y \geq 0\}$, $X_1 = \{x; x > 0\}$. Note that $\varphi(0, y) = -\infty$ when $y \in Y_1^\circ$ but $\varphi(0, y) = 0$ when $y = 0 \in Y_1 \setminus Y_1^\circ$. This proves that regularization had to be taken in condition (iv) and that we had to assume $x \in \overline{X}_1 \setminus X_1^\circ$ in condition (v).

The function φ in Proposition 3.1 is determined by the restriction to $X_1^\circ \times Y_1^\circ$ where it is a continuous real valued concave-convex function. By the remark after Proposition 1.2 it is first determined in $F_1 \times Y_1^\circ$ in view of condition (ii); it is then determined in $F_1 \times \overline{Y}_1$ in view of condition (iv), and finally it is equal to $+\infty$ in $F_1 \times (G_1 \setminus \overline{Y}_1)$. This argument also gives a complete analogue of the remark following Proposition 1.2:

PROPOSITION 3.3. *Let X_1 and Y_1 be relatively open convex subsets of F_1 and of G_1, and let φ be a real valued function in $X_1 \times Y_1$ such that $X_1 \ni x \mapsto \varphi(x, y)$ is concave for every $y \in Y_1$ and $Y_1 \ni y \mapsto \varphi(x, y)$ is convex for every $x \in X_1$. Then φ is locally Lipschitz continuous in $X_1 \times Y_1$, and there is a unique extension of φ to E_1 satisfying the conditions in Proposition* 3.1 *such that $\varphi(x, y) = -\infty$ if $x \in F_1 \setminus \overline{X}_1$ and $y \in Y_1$ and $\varphi(x, y) = +\infty$ if $y \in G_1 \setminus \overline{Y}_1$.*

Summing up, the functions satisfying (A) and $(C)^\dagger$ can either be identified with the lower semicontinuous convex functions in $G_1 \times G_2$ with values in $(-\infty, +\infty]$ or else with the concave-convex functions in a product $X_1 \times Y_1$ of convex relatively open convex subsets of F_1 and G_1 called saddle-elements in [9]. This result was proved in [10].

If φ_j is an increasing (decreasing) sequence of functions satisfying (A) and $(C)^\dagger$, then the corresponding functions Φ_j in $G_1 \times G_2$ defined by (3.5) are also increasing (decreasing) to a convex limit Φ. In the case of increasing sequences the limit Φ is automatically lower semicontinuous, but in the case of decreasing sequences we must take the lower semicontinuous regularization (which may be $\equiv -\infty$). Our next aim is to examine in what sense the functions φ_j converge to the function φ corresponding to (the lower semicontinuous regularization of) Φ.

PROPOSITION 3.4. *If φ_j is an increasing sequence satisfying* (A) *and* $(C)^\dagger$ *then the function φ which for fixed $y \in G_1$ is the smallest upper semicontinuous majorant of* $\lim \varphi_j$ *also satisfies these conditions. If Φ_j and Φ are the corresponding functions in $G_1 \times G_2$ defined by* (3.5), *then $\Phi_j \uparrow \Phi$.*

PROOF. The limit Φ of the increasing sequence Φ_j is convex and lower semicontinuous, and if $\varphi(x, y)$ is defined by (3.6) then φ satisfies (A) and $(C)^\dagger$,

$$\varphi(x, y) \geq \inf_{\eta \in G_2} (\Phi_j(y, \eta) - \langle x, \eta \rangle) = \varphi_j(x, y), \quad \text{hence} \quad \psi = \lim \varphi_j \leq \varphi .$$

On the other hand,

$$\Phi_j(y, \eta) = \sup_{x \in F_1} (\langle x, \eta \rangle + \varphi_j(x, y)) \leq \sup_{x \in F_1} (\langle x, \eta \rangle + \psi(x, y)) ,$$

which means that

$$\Phi(y, \eta) \leq \sup_{x \in F_1} (\langle x, \eta \rangle + \psi(x, y)) = \sup_{x \in F_1} (\langle x, \eta \rangle + \psi_1(x, y))$$

if $\psi_1(\cdot, y)$ is the largest upper semicontinuous concave majorant of $\psi(\cdot, y)$. Thus

$$\psi_1(x, y) \geq \inf_{\eta \in G_2} (\Phi(y, \eta) - \langle x, \eta \rangle) = \varphi(x, y) ,$$

and it follows that there is in fact equality since we have already proved the opposite inequality. The proof is complete.

PROPOSITION 3.5. *If φ_j is a decreasing sequence satisfying* (A) *and* (C)† *then either*

a) *there is a unique function φ satisfying* (A) *and* (C)† *which is equal to* $\lim \varphi_j$ *when y is in the relative interior Y_1 of the set where the limit is not identically $+\infty$ and is equal to $+\infty$ when $y \notin \overline{Y}_1$; or else*
b) $\varphi_j \to -\infty$ *locally uniformly in $F_1 \times Y_1$.*

If Φ_j and Φ are the corresponding functions in $G_1 \times G_2$ defined by (3.5)*, then Φ is the largest lower semicontinuous convex minorant of* $\lim \Phi_j$ *in case* a)*, but $-\infty$ is the only lower semicontinuous convex minorant of* $\lim \Phi_j$ *in case* b).

PROOF. If $\Phi \not\equiv -\infty$ is the largest lower semicontinuous minorant of $\underline{\lim}\, \Phi_j$ and φ is defined by (3.6), then φ satisfies (A) and (C)†, and since $\Phi \leq \Phi_j$ we have $\varphi \leq \varphi_j$. Thus $\varphi \leq \psi = \lim \varphi_j$. The convex set $Y_{1,j}$ where $\varphi_j(\cdot, y) \not\equiv +\infty$ increases with j. The dimension of the affine hull can only increase a finite number of times so we may assume that it is constant. The relative interior Y_1 of $\cup_j Y_{1,j}$ is therefore the union of the relative interiors of the sets $Y_{1,j}$, and ψ is concave and upper semicontinuous with respect to x when $y \in Y_1$.

Since $\varphi(x, y) < +\infty$ when $y \in Y_1$, it follows from (ii) in Proposition 3.1 that $\varphi(\cdot, y_1) \not\equiv -\infty$ when $y_1 \in Y_1$. If $\varphi(x_1, y_1) \neq -\infty$ then $Y_1 \ni y \mapsto \psi(x_1, y)$ is a convex real valued function which is a locally uniform limit of the decreasing sequence of convex functions $\varphi_j(x_0, y)$. Hence it follows from Proposition 3.3 that there is a unique function ψ_1 satisfying (A) and (C)† which is equal to ψ in $F_1 \times Y_1$ and in $F_1 \times (G_2 \setminus \overline{Y}_1)$ (where it is $+\infty$). Since $\psi_1 \leq \varphi_j$ we have $\Psi_1 \leq \Phi_j$ if Ψ_1 is the function corresponding to ψ_1 defined by (3.5). This implies $\Psi_1(y, \eta) \leq \Phi(y, \eta)$, hence $\psi_1 \leq \varphi \leq \psi_1$, which completes the proof that $\varphi = \lim \varphi_j$ in $F_1 \times Y_1$.

If $\varphi_j \to -\infty$ pointwise in $F_1 \times Y_1$ it follows from Lemma 3.2 that the convergence is locally uniform. On the other hand, if $\varphi_j(x_0, y_0)$ has a finite lower bound for some $(x_0, y_0) \in F_1 \times Y_1$ then $\varphi_j(x_0, y)$ also has a finite lower bound for every $y \in Y_1$, and $\psi = \lim \varphi_j$ satisfies (i), (ii), (iii) of Proposition 3.1 in $F_1 \times Y_1$. By Proposition 3.3 we can again extend ψ from $F_1 \times Y_1$ to a function ψ_1 in E_1 satisfying (A) and (C)† with $\psi_1 \leq \varphi_j$ for every j. Hence $\Psi_1 \leq \Phi_j$ so $\lim \Phi_j$ has a lower semicontinuous minorant $\not\equiv -\infty$. This completes the proof.

Our next goal is to prove an analogue of Proposition 1.5 for the modified Legendre transform of the sum φ of two functions φ_1 and φ_2 satisfying (A) and (C)†. However, since φ_j may take both the values $\pm\infty$ the definition of the sum is not obvious. Let Φ_j be the function defined by (3.5) with φ replaced by φ_j and write $Y_k^{\varphi_j}$ for the set Y_k defined by (3.9) with φ replaced by φ_j. The sum $\varphi_1(x, y) + \varphi_2(x, y) \in [-\infty, +\infty)$ is unambiguously defined when $y \in Y_1^{\varphi_1} \cap Y_1^{\varphi_2}$.

DEFINITION 3.6. If φ_1 and φ_2 satisfy conditions (A) and (C)† and $\varphi_1(x, y) + \varphi_2(x, y) > -\infty$ for some $x \in F_1$ and some y in the relative interior Y_1° of the convex set $Y_1^{\varphi_1} \cap Y_1^{\varphi_2}$, then we define $\varphi = \varphi_1 \tilde{+} \varphi_2$ as the function satisfying (A) and (C)† which is equal to $\varphi_1 + \varphi_2$ in $F_1 \times Y_1^\circ$ and is $+\infty$ outside the closure.

Note that it follows from the convexity of $\varphi_j(x, z)$ in $z \in Y_1^\circ$ that $\varphi_j(x, z) \in \mathbb{R}$ for every $z \in Y_1^\circ$. The justification for the definition is given by the following lemma.

LEMMA 3.7. *If $\varphi = \varphi_1 \tilde{+} \varphi_2$ as in Definition 3.6, then $\Phi(y, \eta)$ is the largest lower semicontinuous convex function with*

$$\Phi(y, \eta) \le \inf_{\eta_1+\eta_2=\eta} \big(\Phi_1(y, \eta_1) + \Phi_2(y, \eta_2)\big), \quad \text{if } y \in Y_1^\circ. \tag{3.12}$$

There is equality in the relative interior of M, defined by (3.8), that is, in the relative interior of $\{(y, \eta_1 + \eta_2);\ \Phi_1(y, \eta_1) < +\infty, \Phi_2(y, \eta_2) < +\infty\}$.

PROOF. When $y \in Y_1^\circ$ we have

$$\varphi(\cdot, y) = \varphi_1(\cdot, y) + \varphi_2(\cdot, y),$$

which is a concave function $\not\equiv \pm\infty$. Hence it follows from Proposition 1.5 that $\Phi(y, \eta)$ for fixed $y \in Y_1^\circ$ is the largest minorant of

$$\eta \mapsto \inf_{\eta_1+\eta_2=\eta} (\Phi_1(y, \eta_1) + \Phi_2(y, \eta_2)) \tag{3.13}$$

which is lower semicontinuous with respect to η. By the remark after Proposition 1.2 Φ is the only lower semicontinuous convex function which has this property in $Y_1^\circ \times G_2$ and is equal to $+\infty$ in the complement of the closure, which proves the lemma, for the infimum in (3.13) is convex, hence continuous in the relative interior of the convex set where it is finite. We can therefore also regard Φ as the extension of the right-hand side of (3.13) from M° to $G_1 \oplus G_2$ given in Proposition 1.2 and the remark after it.

When $\eta \in Y_2^\circ$ then the convex set $\{y \in G_1;\ (y, \eta) \in M\}$ has a dense subset where $(y, \eta) \in M^\circ$, for there are such points since Y_2° is the projection of M°, and an open interval in M with one end point in M° is contained in M°. Hence

$$\begin{aligned}
\varphi^\dagger(\xi, \eta) &= \inf_{y\in G_1} (\langle y, \xi\rangle + \Phi(y, \eta)) = \inf_{y;(y,\eta)\in M^\circ} \big(\langle y, \xi\rangle + \Phi(y, \eta)\big) \\
&= \inf_{y;(y,\eta)\in M^\circ} \inf_{\eta_1+\eta_2=\eta} \big(\langle y, \xi\rangle + \Phi_1(y, \eta_1) + \Phi_2(y, \eta_2)\big) \\
&= \inf_{\eta_1+\eta_2=\eta} \varlimsup_{\xi_1+\xi_2\to\xi} \big(\varphi_1^\dagger(\xi_1, \eta_1) + \varphi_2^\dagger(\xi_2, \eta_2)\big), \quad \eta \in Y_2^\circ,
\end{aligned}$$

by Proposition 1.5. This proves the following analogue of Proposition 1.5:

THEOREM 3.8. *Let φ_1 and φ_2 satisfy conditions (A) and (C)†, and assume that $\varphi_1(x, y) + \varphi_2(x, y) > -\infty$ for some $x \in F_1$ and some y in the relative interior Y_1° of the convex set $Y_1^{\varphi_1} \cap Y_1^{\varphi_2}$. With $\varphi = \varphi_1 \tilde{+} \varphi_2$ as in Definition 3.6 it follows that $\varphi^\dagger(\xi, \eta)$ is the function satisfying (A) and (C)† such that*

$$\varphi^\dagger(\xi, \eta) = \inf_{\eta_1+\eta_2=\eta} \varlimsup_{\xi_1+\xi_2\to\xi} \big(\varphi_1^\dagger(\xi_1, \eta_1) + \varphi_2^\dagger(\xi_2, \eta_2)\big), \quad \eta \in Y_2^\circ, \tag{3.14}$$

and $\varphi^\dagger(\xi, \eta) = +\infty$ when $\eta \notin \overline{Y_2^\circ}$. Here Y_2° is the relative interior of the convex set $\{\eta_1 + \eta_2;\ \Phi_1(y, \eta_1) + \Phi(y, \eta_2) < \infty$ for some $y \in G_1\}$.

4. – The Fourier-Laplace transformation

Let φ be a function in $\mathbb{C}^n$ satisfying the conditions (A) and (B)† with respect to the real subspace and the duality in $\mathbb{C}^n$ defined by the bilinear form $(z, \zeta) \mapsto \operatorname{Im}\langle z, \zeta\rangle$, that is,

(A) $\mathbb{R}^n \ni x \mapsto \varphi(x+iy)$ is for fixed $y \in \mathbb{R}^n$ either identically $+\infty$ or else concave and upper semicontinuous with values in $[-\infty, +\infty)$.

(B)† For every $\eta \in \mathbb{R}^n$

$$\Phi(y,\eta) = \sup_x (\langle x, \eta\rangle + \varphi(x+iy)), \quad y, \eta \in \mathbb{R}^n, \tag{4.1}$$

is convex and lower semicontinuous with values in $(-\infty, +\infty]$ as a function of y. (This is trivially true also for $\Phi(y,\eta)$ as a function of η for fixed y.)

If $\varphi^\dagger$ is the Legendre transform defined in (2.11), then

$$\varphi^\dagger(\xi + i\eta) = \inf_y (\langle y, \xi\rangle + \Phi(y,\eta)), \quad \xi, \eta \in \mathbb{R}^n, \tag{4.2}$$

$$\varphi(x+iy) = \inf_\eta (\Phi(y,\eta) - \langle x, \eta\rangle), \quad x, y \in \mathbb{R}^n. \tag{4.3}$$

The following definition was introduced in [5]:

DEFINITION 4.1. If φ satisfies (A) and (B)† we denote by $\mathcal{S}_\varphi^\sharp$ the set of complex valued functions f defined in $\{z \in \mathbb{C}^n; \varphi(z) < \infty\}$ with the following properties:

(i) $\mathbb{R}^n \ni x \mapsto f(x+iy)$ is infinitely differentiable in the domain of definition of $f(x+iy)$, and for arbitrary polynomials P and Q there is a constant C_{PQ} such that

$$|P(x+iy)Q(\partial/\partial x) f(x+iy)| \le C_{PQ} e^{\varphi(x+iy)} \quad \text{when } \varphi(x+iy) < \infty. \tag{4.4}$$

(ii) If $y_0, y_1 \in \mathbb{R}^n$ and $\Phi(y_j, \eta) < \infty$, $j = 0, 1$, for some $\eta \in \mathbb{R}^n$, and if α is any multiindex, then $w \mapsto \partial_x^\alpha f(x + iy_0 + w(y_1 - y_0))$ is continuous in $\{w \in \mathbb{C}; |\operatorname{Im} w| \le 1\}$ and analytic in the interior.

$\mathcal{S}_\varphi^\sharp$ is a locally convex topological vector space with the best constants C_{PQ} in (4.4) as seminorms.

When $f \in \mathcal{S}_\varphi^\sharp$ we define the *Fourier-Laplace transform* $\hat{f}$ of f by

$$\hat{f}(\zeta) = \int_{\mathbb{R}^n} e^{-i\langle x+iy, \zeta\rangle} f(x+iy)\, dx, \tag{4.5}$$
$$\text{when } \varphi^\dagger(\zeta) < \infty \text{ and } \Phi(y,\eta) < \infty, \quad \eta = \operatorname{Im}\zeta.$$

By (4.2) one can find y such that $\Phi(y,\eta) < \infty$, thus $\operatorname{Im}\langle x+iy, \zeta\rangle + \varphi(x+iy) \le \langle y, \operatorname{Re}\zeta\rangle + \Phi(y, \operatorname{Im}\zeta)$, so using (4.4) with $Q = 1$ and $P(z) = z^\alpha$ where $|\alpha| \le n+1$ we conclude using (4.2) that the integral in (4.5) exists and that the infimum over y is bounded by $e^{\varphi^\dagger(\zeta)}$ times a seminorm of f in $\mathcal{S}^\sharp_\varphi$. It follows from condition (ii) and Cauchy's integral formula that the integral is independent of the choice of y. Since the Fourier-Laplace transform of $P(z)Q(\partial/\partial x)f(z)$ is $P(i\partial/\partial\xi)Q(i\zeta)\hat f(\zeta)$ and the order of the factors in (4.4) is irrelevant for the existence of estimates (4.4) for all P and Q, the asserted continuity follows, and $\hat f$ has the property (i) with φ replaced by $\varphi^\dagger$. To verify (ii) we must consider two vectors $\eta_0, \eta_1 \in \mathbb{R}^n$ such that $\Phi(y, \eta_i) < \infty$, $i = 0, 1$, for some $y \in \mathbb{R}^n$. This implies that $\Phi(y, \eta_0 + \lambda(\eta_1 - \eta_0))$ is bounded when $0 \le \lambda \le 1$. Thus

$$\hat f(\xi+i\eta_0+w(\eta_1-\eta_0)) = \int f(x+iy)e^{-i\langle x+iy,\xi+i\eta_0+w(\eta_1-\eta_0)\rangle}\,dx, \quad 0 \le \operatorname{Im} w \le 1,$$

where the integral is locally uniformly convergent since the integrand can be estimated by

$$C(1+|x+iy|)^{-n-1}\exp(\langle y, \xi + \operatorname{Re} w(\eta_1-\eta_0)\rangle + \Phi(y, \eta_0 + \operatorname{Im} w(\eta_1 - \eta_0)))\,.$$

The analyticity follows, for it is an analytic function of w. Since Fourier's inversion formula gives $\hat{\hat f}(z) = (2\pi)^n f(-z)$, and the inversion formula $\varphi^{\dagger\dagger}(z) = \varphi(-z)$ is valid for the modified Legendre transformation, we have now proved:

THEOREM 4.2. *The Laplace transformation is a topological isomorphism of* $\mathcal{S}^\sharp_\varphi$ *on* $\mathcal{S}^\sharp_{\varphi^\dagger}$.

The following example shows that the definition of $\mathcal{S}^\sharp_\varphi$ may not require any analyticity at all even if φ is finite everywhere:

EXAMPLE 4.1. Let $\varphi(x+iy) = xy + \psi(y)$, $x, y \in \mathbb{R}$ (cf. Example 2.5). Then (A) is obviously satisfied and

$$\Phi(y,\eta) = \begin{cases} \psi(y), & \text{when } y+\eta = 0 \\ +\infty & \text{when } y + \eta \ne 0, \end{cases} \qquad \varphi^\dagger(\xi + i\eta) = -\xi\eta + \psi(-\eta)\,.$$

Thus $\mathcal{S}^\sharp_\varphi$ consists of functions in $\mathbb{C}$ which are infinitely differentiable in x and satisfy the condition

$$|\partial_x^j f(x+iy)| \le C_{jk}(1+|x|+|y|)^{-k}e^{xy+\psi(y)}$$

for arbitrary nonnegative integers j and k. We have

$$\hat f(\zeta) = \int e^{-i(x-i\operatorname{Im}\zeta)\zeta} f(x, -\operatorname{Im}\zeta)\,dx$$

so the Fourier-Laplace transform is essentially the Fourier transform of functions in $\mathcal{S}(\mathbb{R})$ depending on a parameter $\operatorname{Im}\zeta$ but otherwise unrelated apart from a decay at infinity after some normalization. This is not very interesting so we shall now introduce a stronger version of the condition (ii) in Definition 4.1 which makes the analyticity conditions relevant.

At first we just assume the following strengthened version of (ii):

(ii)′ $M = \{(y, \eta);\ \Phi(y, \eta) < \infty\}$ is convex and if $(y_j, \eta_j) \in M$, $j = 0, 1$, then $w \mapsto \partial_x^\alpha f(x + iy_0 + w(y_1 - y_0))$ is for every α continuous in the strip $\{w \in \mathbb{C};\ |\operatorname{Im} w| \le 1\}$, analytic in the interior, and bounded by $\exp(C \exp(a|w|))$ for some constants C and a with $a < \pi$.

Since

$$|e^{-i\langle x, i\eta_0 + w(\eta_1 - \eta_0)\rangle} P(x + iy_0 + w(y_1 - y_0)) Q(\partial/\partial x) f(x + iy_0 + w(y_1 - y_0))| \le C_{PQ} e^{\Phi(y_j, \eta_j)}, \quad \text{if } \operatorname{Im} w = j,\ j = 0, 1,$$

it follows from (ii)′ and the three line theorem that the left-hand side is bounded by $C_{PQ} \exp((1-\lambda)\Phi(y_0, \eta_0) + \lambda\Phi(y_1, \eta_1))$ when $\operatorname{Im} w = \lambda \in (0, 1)$. This means that if $(y_\lambda, \eta_\lambda) = (1 - \lambda)(y_0, \eta_0) + \lambda(y_1, \eta_1)$ then

$$|P(x + iy_\lambda) Q(\partial_x) f(x + iy_\lambda)| \le C_{PQ} \exp((1-\lambda)\Phi(y_0, \eta_0) + \lambda\Phi(y_1, \eta_1) - \langle x, \eta_\lambda\rangle).$$

Repeating the argument N times we conclude that

$$|P(x + iy) Q(\partial_x) f(x + iy)| \le C_{PQ} \exp\left(\sum_0^N \lambda_j \Phi(y_j, \eta_j) - \langle x, \eta\rangle\right), \quad \text{if}$$

$$(y, \eta) = \sum_0^N \lambda_j (y_j, \eta_j), \quad \lambda_j > 0,\ \sum_0^N \lambda_j = 1,\ \Phi(y_j, \eta_j) < \infty,\ j = 0, \dots, N.$$

Hence

$$|P(x + iy) Q(\partial_x) f(x + iy)| \le C_{PQ} \exp(\Phi_0(y, \eta) - \langle x, \eta\rangle)$$

if the epigraph $\{(y, \eta, t);\ t \ge \Phi_0(y, \eta)\}$ of Φ_0 is the convex hull of the epigraph of Φ. A mild additional continuity hypothesis on f allows us to take the closure of the epigraph of Φ_0, that is, replace Φ by the largest lower semicontinuous convex minorant Φ_1 of Φ and φ by $\varphi_1(x + iy) = \inf_\eta(\Phi_1(y, \eta) - \langle x, \eta\rangle)$, which is then a function satisfying (A) and

(C)† The function Φ defined by (4.1) is convex and lower semicontinuous in $\mathbb{R}^{n+n}$.

This motivates the following:

Definition 4.3. If φ satisfies (A) and (C)† we denote by $\mathcal{S}_\varphi$ the set of complex valued functions f defined in $\{z \in \mathbb{C}^n;\ \varphi(z) < \infty\}$ with the following properties:

(i) $\mathbb{R}^n \ni x \mapsto f(x + iy)$ is infinitely differentiable in the domain of definition of $f(x + iy)$, and for arbitrary polynomials P and Q there is a constant C_{PQ} such that

$$|P(x + iy) Q(\partial/\partial x) f(x + iy)| \le C_{PQ} e^{\varphi(x+iy)} \quad \text{when } \varphi(x + iy) < \infty. \tag{4.4}$$

(ii) If $y_0, y_1 \in \mathbb{R}^n$ and $\Phi(y_j, \eta_j) < \infty$, $j = 0, 1$, for some $\eta_j \in \mathbb{R}^n$, and if α is any multiindex, then $w \mapsto \partial_x^\alpha f(x + iy_0 + w(y_1 - y_0))$ is continuous in $\{w \in \mathbb{C}; |\operatorname{Im} w| \le 1\}$ and analytic in the interior.

$\mathcal{S}_\varphi$ is a locally convex topological vector space with the best constants C_{PQ} in (4.4) as seminorms.

There is an analogue of Theorem 4.2:

THEOREM 4.4. *If φ satisfies* (A) *and* (C)†, *then the Laplace transformation is a topological isomorphism of $\mathcal{S}_\varphi$ on $\mathcal{S}_{\varphi^\dagger}$.*

PROOF. By Theorem 4.2 we only have to prove that the stronger analyticity property (ii) in Definition 4.3 is inherited by the Laplace transform. To do so we assume that $\Phi(y_j, \eta_j) < \infty$ for $j = 0, 1$. Then we have for $0 \le \operatorname{Im} w \le 1$

$$\begin{aligned}\hat f(\xi + i\eta_0 + w(\eta_1 - \eta_0))\\ = \int_{\mathbb{R}^n} e^{-i\langle x+iy_0+w(y_1-y_0),\xi+i\eta_0+w(\eta_1-\eta_0)\rangle} f(x + iy_0 + w(y_1 - y_0))\, dx\,.\end{aligned}$$

The integral is locally uniformly convergent and the integrand is analytic in w, so the analyticity is obvious.

REMARK. If the interior M° of $M = \{(y, \eta); \Phi(y, \eta) < \infty\}$ is connected and dense in M, then condition (ii) in Definition 4.1 already implies that Φ may be replaced by the largest convex minorant Φ_0 and that the stronger analyticity condition (ii)$'$ is valid for the corresponding function φ_0. This follows from Bochner's theorem. However, since the arguments based on (ii)$'$ were only intended as a motivation for Definition 4.3 we shall not give the details of the proof.

From now on *we assume that φ satisfies* (A) *and* (C)†. We want to examine if φ can be replaced by a smaller function with these properties without changing the space $\mathcal{S}_\varphi$. As in Section 3 we set

$$Y_1 = \{y \in \mathbb{R}^n; \Phi(y, \eta) < \infty \text{ for some } \eta \in \mathbb{R}^n\}\,.$$

By making a translation of φ we can attain that $\operatorname{ah}(Y_1)$ contains the origin, and we shall then denote this vector subspace of $\mathbb{R}^n$ by V_1. If $f \in \mathcal{S}_\varphi$ it follows from condition (ii) in Definition 4.3 and Hartogs' theorem that

$$V_1 \times iY_1^\circ \ni w \mapsto P(x + w)Q(\partial/\partial x)f(x + w)$$

is analytic, and by condition (i) the logarithm of the absolute value of the quotient by C_{PQ} is then in the set $\mathcal{V}_\varphi$ of functions with values in $[-\infty, +\infty)$ defined in $\mathbb{R}^n \times iY_1^\circ$ such that $v \le \varphi$ and

$$V_1 \times iY_1^\circ \ni w \mapsto v(x + w)$$

is plurisubharmonic for every $x \in \mathbb{R}^n$. Of course it may happen that $\mathcal{V}_\varphi$ only contains the function which is $\equiv -\infty$; in that case $\mathcal{S}_\varphi = \{0\}$. Let

$$\varphi_0 = \sup_{v \in \mathcal{V}_\varphi} v, \quad \text{in } \mathbb{R}^n \times iY_1^\circ. \tag{4.6}$$

It is obvious that $\varphi_0 \leq \varphi$, and we claim that $x \mapsto \varphi_0(x+iy)$ is concave when $y \in Y_1^\circ$. To prove this we consider two arbitrary functions $v_j \in \mathcal{V}_\varphi$ and $a_1, a_2 \in \mathbb{R}^n$, $\lambda_1, \lambda_2 \in \mathbb{R}$ non-negative with $\lambda_1 + \lambda_2 = 1$, and observe that

$$\lambda_1 v_1(\cdot + a_1) + \lambda_2 v_2(\cdot + a_2) \leq \lambda_1 \varphi(\cdot + a_1) + \lambda_2 \varphi(\cdot + a_2) \leq \varphi(\cdot + \lambda_1 a_1 + \lambda_2 a_2)$$

by the concavity of φ in the real direction. Hence $\lambda_1 v_1(\cdot + a_1) + \lambda_2 v_2(\cdot + a_2) \in \mathcal{V}_\varphi$ if $\lambda_1 a_1 + \lambda_2 a_2 = 0$, so it follows that

$$\lambda_1 \varphi_0(\cdot + a_1) + \lambda_2 \varphi_0(\cdot + a_2) \leq \varphi_0, \quad \text{if } \lambda_1 a_1 + \lambda_2 a_2 = 0,$$

which proves the concavity of $x \mapsto \varphi_0(x+iy)$. To proceed we need a simple lemma:

LEMMA 4.5. *Let v_ι, $\iota \in I$, be a family of nonpositive subharmonic functions in $\{w \in \mathbb{C}; 0 < \operatorname{Im} w < 1\}$ and set $V = \sup_{\iota \in I} v_\iota$. If $V \not\equiv -\infty$ and $\mathbb{R} \ni x \mapsto V(x+iy)$ is a concave function for $0 < y < 1$, then V is a continuous subharmonic function and $(0,1) \ni y \mapsto V(x+iy)$ is convex when $x \in \mathbb{R}$. The function V has a continuous extension to the closed strip. If v_ι are continuous in the closed strip with values in $[-\infty, +\infty)$ and $x \mapsto V(x+iy)$ is concave also for $y = 0$ and $y = 1$, then V is continuous in the closed strip.*

PROOF. The upper semicontinuous regularisation $\overline{V}$ of V is subharmonic and equal to V almost everywhere, in fact except in a polar set. Since $\partial^2 \overline{V}/\partial x^2 \leq 0$ we have $\partial^2 \overline{V}/\partial y^2 \geq 0$, and it follows from Lemma 3.2 that $\overline{V}$ is continuous, concave in x and convex in y. If $\overline{V}(x_0+iy_0) > V(x_0+iy_0)$ then $V(x+iy_0) < \overline{V}(x+iy_0)$ for all x in a neighborhood of x_0 for the concavity of $V(x+iy_0)$ implies continuity. This is a contradiction since a polar set is a null set on every line. Hence $V = \overline{V}$. From the convexity and the upper bound it follows that $V(x+iy)$ has a limit when $y \to 0$, and it follows from the concavity in x that the limit is a concave function and that the convergence is locally uniform. With $\overline{V}(x)$ defined in this way we have $v_\iota(x) \leq \overline{V}(x)$, $\iota \in I$, if $v_\iota(x+iy)$ is continuous for $0 \leq y < 1$, since this is true in the open strip. Hence $V(x) \leq \overline{V}(x)$. On the other hand, v_ι is bounded above by the Poisson integral of its boundary values, hence by the Poisson integral of $V(x)$ and $V\left(x + \frac{1}{2}i\right)$ in the strip where $0 < \operatorname{Im} z < \frac{1}{2}$. If $V(x_0) = \overline{V}(x_0) - \varepsilon$ with $\varepsilon > 0$, say, then $V(x+iy) = \sup_\iota v_\iota(x+iy) < \overline{V}(x) - \varepsilon/2$ when $|x - x_0| < \delta$ and $0 < y < \delta$, for some $\delta > 0$. This implies that $\overline{V}(x_0) \leq \overline{V}(x_0) - \varepsilon/2$ which is a contradiction completing the proof, for the boundary value at $y = 1$ is handled in the same way.

From Lemma 3.2 it follows now that $\mathbb{R}^n \ni x \mapsto \varphi_0(x+iy)$ is concave if $y \in Y_1^\circ$, that $Y_1^\circ \ni y \mapsto \varphi_0(x+iy)$ is convex if $x \in \mathbb{R}^n$, and that $V_1 \times iY_1^\circ \ni w \mapsto \varphi_0(x+w)$ is plurisubharmonic, possibly $\equiv -\infty$. The set $X_1 = \{x; \varphi_0(x+iy) \neq -\infty$ for some $y \in Y_1^\circ\}$ is convex and invariant under translation in the direction V_1, and this is also true for the relative interior X_1°. Thus φ_0 is a locally Lipschitz continuous function in $X_1^\circ \times Y_1^\circ$ which is concave in the X_1° direction, convex in the Y_1° direction and plurisubharmonic in the complex planes $(\{x\}+V_1) \times iV_1$. Let φ_1 be its unique extension to $\mathbb{C}^n$ satisfying (A) and (C)† which is equal to $-\infty$ in $(\mathbb{R}^n \setminus \overline{X}_1) \times Y_1^\circ$ and $+\infty$ in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \overline{Y}_1)$. Then $\varphi_1 \le \varphi$ and $\mathcal{S}_\varphi = \mathcal{S}_{\varphi_1}$; even the seminorms C_{PQ} are the same in the two spaces. Before stating the result we introduce another definition:

Definition 4.6. By $\mathcal{P}$ we shall denote the set of functions in $\mathbb{C}^n$ with values in $[-\infty, +\infty]$ such that

(A) $\mathbb{R}^n \ni x \mapsto \varphi(x+iy)$ is either identically $+\infty$ or else concave and upper semicontinuous with values in $[-\infty, +\infty)$.

(C)† The function Φ in $\mathbb{R}^{n+n}$ defined by (4.1) is convex and lower semicontinuous; thus $Y_1 = \{y \in \mathbb{R}^n; \Phi(y, \eta) < \infty$ for some $\eta \in \mathbb{R}^n\}$ is convex, with relative interior Y_1°.

(P) For every $x \in \mathbb{R}^n$ the function

$$\mathrm{ah}(Y_1) \times iY_1^\circ \ni w \mapsto \varphi(x+w)$$

is plurisubharmonic.

As already observed it follows that the set X_1 of all x such that $\varphi(x+w) \neq -\infty$ for some $w \in \mathrm{ah}(Y_1) \times iY_1^\circ$ is convex and invariant under translations preserving $\mathrm{ah}(Y_1)$. We have proved:

Theorem 4.7. *If φ satisfies* (A) *and* (C)† *then there is a function $\varphi_1 \le \varphi$ belonging to $\mathcal{P}$ such that $\mathcal{S}_\varphi = \mathcal{S}_{\varphi_1}$ with the same seminorms C_{PQ} in the two spaces.*

Remark. In Definition 4.3 we have required that a function $f \in \mathcal{S}_\varphi$ is defined in $\{z \in \mathbb{C}^n; \operatorname{Im} z \in Y_1\}$. However, it suffices to assume that f is defined in $\{z \in \mathbb{C}^n; \operatorname{Im} z \in Y_1^\circ\}$ with the estimates (4.4), for then it follows at once if $y \in Y_1$ that

$$\lim_{\lambda \to +0} f(x + i(1-\lambda)y + i\lambda y^\circ)$$

exists if $y^\circ \in Y_1^\circ$ and that it is independent of the choice of y°. This gives a unique extension of the definition of f with all the properties required in Definition 4.3.

Theorem 4.7 combined with Theorem 4.4 suggests but does not prove that the modified Legendre transformation is a bijection $\mathcal{P} \to \mathcal{P}$. This will be proved in Section 6 after a preliminary discussion of the case of quadratic forms. In the proof it will be important that $\mathcal{P}$ is closed under increasing or decreasing limits:

LEMMA 4.8. *If $\varphi_j \in \mathcal{P}$ is an increasing sequence then the limit φ defined as in Proposition* 3.4 *is also in $\mathcal{P}$.*

PROOF. If Y_1 and X_1 are defined as in Proposition 3.1 then $\varphi_j \uparrow \varphi$ locally uniformly in $X_1^\circ \times Y_1^\circ$. Hence the vector space defined by X_1° contains that defined by Y_1°, and

$$\mathrm{ah}(Y_1) \times iY_1^\circ \ni w \mapsto \varphi(x+w)$$

is plurisubharmonic when $x + \mathrm{ah}(Y_1) \subset X_1^\circ$ since it is a locally uniform limit of the plurisubharmonic functions $w \mapsto \varphi_j(x+w)$.

LEMMA 4.9. *If $\varphi_j \in \mathcal{P}$ is a decreasing sequence with limit φ as in Proposition* 3.5*, then $\varphi \in \mathcal{P}$.*

PROOF. Since the limit of a decreasing sequence of plurisubharmonic functions is plurisubharmonic, this follows just as Lemma 4.8.

If φ_1 and φ_2 satisfy the conditions (A) and (C)† and $\varphi = \varphi_1 \tilde{+} \varphi_2$ can be defined by Definition 3.6, then $f_1 \in \mathcal{S}_{\varphi_1}$, $f_2 \in \mathcal{S}_{\varphi_2}$ implies $f = f_1 f_2 \in \mathcal{S}_\varphi$. Hence $\hat{f} \in \mathcal{S}_{\varphi^\dagger}$, where $\varphi^\dagger$ is described in Theorem 3.8. If y is in the relative interior Y_1° of $Y_1^{\varphi_1} \cap Y_2^{\varphi_2}$ then $f(x+iy)$ is well defined when $x \in \mathbb{R}^n$, and

$$\hat{f}(\zeta) = \int e^{-i\langle x+iy,\zeta\rangle} f_1(x+iy) f_2(x+iy)\, dx$$

exists if $\operatorname{Im}\zeta = \eta_1 + \eta_2$ and $\Phi_j(y,\eta_j) < \infty$. (We keep the notation used at the end of Section 3.) Since

$$\int e^{-i\langle x+iy,\xi+i\eta_j\rangle} f_j(x+iy)\, dx = \hat{f}_j(\xi + i\eta_j),$$

it follows that $\hat{f}$ is the corresponding convolution,

$$\widehat{f_1 f_2}(\xi + i(\eta_1+\eta_2)) = (2\pi)^{-n} \int \hat{f}_1(\xi - \xi_1 + i\eta_1) \hat{f}_2(\xi_1 + i\eta_2)\, d\xi_1, \tag{4.7}$$
$$\text{if } \Phi_j(y,\eta_j) < \infty, \quad j = 1,2, \quad \text{for some } y \in \mathbb{R}^n .$$

In view of (3.14) this easily confirms that $\widehat{f_1 f_2} \in \mathcal{S}_{\varphi^\dagger}$.

5. – The Legendre transform of a quadratic form

If Q is a real valued nonsingular quadratic form in $\mathbb{R}^n$, then the Legendre transform $\widetilde{Q}$ can be defined using (0.1)–(0.3). If we write $Q(x) = \frac{1}{2}\langle Ax, x\rangle$ where A is a nonsingular symmetric matrix then $\xi = Ax$, $\langle x, \xi\rangle = 2Q(x)$ and $\widetilde{Q}(\xi) = Q(x) = \frac{1}{2}\langle A^{-1}\xi, \xi\rangle$, so $\widetilde{Q}$ has the same signature as Q.

If Q_1 and Q_2 are nonsingular quadratic forms in $\mathbb{R}^n$ with $Q_1 \le Q_2$ and the same signature, then $\widetilde{Q}_1 \ge \widetilde{Q}_2$ (and $Q_1^\dagger \le Q_2^\dagger$). For let F be a maximal subspace where Q_2 is negative definite. Then F is also a maximal subspace where Q_1 is negative definite. Thus

$$\widetilde{Q}_j(\xi) = \sup_{X\in\mathbb{R}^n/F} \big(\inf_{\dot{x}=X} (\langle x, \xi\rangle - Q_j(x))\big), \quad j = 1, 2,$$

which proves that $\widetilde{Q}_1 \ge \widetilde{Q}_2$.

Let us now consider a real valued nonsingular quadratic form Q in $\mathbb{C}^n$ which is plurisubharmonic, that is,

$$\sum_{j,k=1}^n \partial^2 Q(z)/\partial z_j \partial \bar{z}_k w_j \bar{w}_k \ge 0, \quad w \in \mathbb{C}^n.$$

Equivalently, if we write $Q = Q_0 + Q_1$ where

$$(5.1)\quad Q_0(z) = \sum_{j,k=1}^n a_{jk} z_j \bar{z}_k,\ a_{jk} = \overline{a_{kj}};\quad Q_1(z) = \operatorname{Re} \sum_{j,k=1}^n b_{jk} z_j z_k,\ b_{jk} = b_{kj},$$

then the Levi form Q_0 is positive semidefinite. We want to examine the Levi form of $\widetilde{Q}$, defined by the duality $\mathbb{C}^n \times \mathbb{C}^n \ni (z, \zeta) \mapsto \operatorname{Im}\langle z, \zeta\rangle$ where $\langle z, \zeta\rangle = \sum_1^n z_j \zeta_j$. To do so we shall use a well-known normal form for Q. (See e.g. Siegel [11, p. 12].)

LEMMA 5.1. *If the real quadratic form Q in $\mathbb{C}^n$ is strictly plurisubharmonic, that is, the Hermitian matrix A in* (5.1) *is positive definite, then there are new coordinates $(w_1, \dots, w_n)$ in $\mathbb{C}^n$ such that*

$$(5.2)\qquad Q(z) = \sum_1^n (|w_j|^2 + \lambda_j \operatorname{Re}(w_j^2)),$$

where $\lambda_1, \dots, \lambda_n$ are nonnegative.

PROOF. By a complex linear transformation we can diagonalize Q_0, so we may assume that $Q_0(z) = |z|^2$. Set $B = (b_{jk})$. The problem is to reduce $\langle Bz, z\rangle$ by a unitary transformation to diagonal form. The nonnegative Hermitian matrix $B^*B = \overline{B}B$ can be reduced by a unitary transformation to diagonal form, with nonnegative diagonal elements. If $B = B_1 + iB_2$ with B_1, B_2 real and symmetric, then $B^*B = B_1^2 + B_2^2 + i(B_1B_2 - B_2B_1)$ so B_1 and B_2 commute. They can therefore be simultaneously reduced to diagonal form by a real orthogonal, hence unitary, transformation. This gives B diagonal form so

$$Q(z) = \sum_1^n \big(|w_j|^2 + \lambda_j \operatorname{Re}(e^{2i\theta_j} w_j^2)\big)$$

for some $\lambda_j \ge 0$ and real θ_j. Replacing $e^{i\theta_j} w_j$ by w_j we attain the desired form (5.2).

If

$$(5.2)' \quad Q(z)=\sum_1^n(|z_j|^2+\lambda_j \operatorname{Re}(z_j^2))=\sum_1^n\left((1+\lambda_j)(\operatorname{Re} z_j)^2+(1-\lambda_j)(\operatorname{Im} z_j)^2\right),$$

is nondegenerate, then $\lambda_j^2 \neq 1$ and the Legendre transform $\widetilde{Q}$ with respect to the form

$$\mathbb{C}^n \times \mathbb{C}^n \ni (z, \zeta) \mapsto \operatorname{Im}\langle z, \zeta\rangle=\sum_1^n(\operatorname{Re} z_j \operatorname{Im} \zeta_j+\operatorname{Im} z_j \operatorname{Re} \zeta_j)$$

is equal to

$$(5.3) \qquad \frac{1}{4}\sum_1^n\left(\frac{(\operatorname{Im}\zeta_j)^2}{1+\lambda_j}+\frac{(\operatorname{Re}\zeta_j)^2}{1-\lambda_j}\right)=\frac{1}{4}\sum_1^n\left(\frac{|\zeta_j|^2}{1-\lambda_j^2}+\frac{\lambda_j \operatorname{Re}(\zeta_j^2)}{1-\lambda_j^2}\right).$$

Thus $\widetilde{Q}$ is plurisubharmonic if and only if $0 \leq \lambda_j < 1$ for every j, that is, Q is positive definite. In that case $\widetilde{Q}$ is also positive definite of course. It is more interesting for us that $-\widetilde{Q}$ is plurisubharmonic if and only if $\lambda_j > 1$ for every j. This means that the signature of Q is n, n; in fact, Q is negative definite in the n dimensional real subspace defined by $\operatorname{Re} z = 0$. In that case $\widetilde{Q}$ is positive definite in the n dimensional real subspace where $\operatorname{Re}\zeta = 0$. This leads easily to the proof of the following lemma:

LEMMA 5.2. *If the real quadratic form Q in $\mathbb{C}^n$ is plurisubharmonic and nondegenerate, then the signature is $n+k, n-k$ where $0 \leq k \leq n$, and $-\widetilde{Q}$ is plurisubharmonic if and only if the signature of Q is n, n. If $\widetilde{Q}$ is plurisubharmonic then the Levi form has rank k and one can find new complex coordinates $(w_1, \dots, w_n)$ such that*

$$(5.4) \qquad Q(z)=\sum_{j=1}^k(|z_j|^2+\lambda_j \operatorname{Re}(z_j^2))+\sum_{j=k+1}^n \operatorname{Re}(z_j^2),$$

where $0 \leq \lambda_j < 1$ when $j = 1, \dots, k$.

PROOF. If $\varepsilon > 0$ then $Q_\varepsilon(z) = Q(z)+\varepsilon|z|^2$ is strictly plurisubharmonic, and Q_ε has the same signature as Q if ε is small enough. However, Q_ε is of the form (5.2) with suitable coordinates, and the signature is then $n+k, n-k$ where k is the number of $\lambda_j \in [0, 1)$. By (5.3) the Levi form of $\widetilde{Q}_\varepsilon$ has k positive and $n-k$ negative eigenvalues. When $\varepsilon \to 0$ it follows that the Levi form of $\widetilde{Q}$ has at most k positive and $n-k$ negative eigenvalues, so $-\widetilde{Q}$ is plurisubharmonic if $k = 0$. On the other hand, if Q and $-\widetilde{Q}$ are plurisubharmonic, then we know that their signatures are $n+k, n-k$ resp. $n+l, n-l$ for some $k, l \geq 0$. Hence the signature of Q is $n-l, n+l$ which proves that $k = l = 0$.

If Q has signature $n+k, n-k$ where $0<k<n$ and both Q and $\widetilde{Q}$ are plurisubharmonic, then the rank κ of the Levi form of Q is at most equal to k, for Q is the Legendre transform of $\widetilde{Q}$, which also has signature $n+k, n-k$, and there are no negative eigenvalues. We can choose the coordinates so that $Q(z)=|z'|^2+\operatorname{Re}B(z)$ where $z'=(z_1,\dots,z_\kappa)$ and $z''=(z_{\kappa+1},\dots,z_n)$. In the subspace of complex dimension $n-\kappa$ where $z'=0$ the signature of $\operatorname{Re}B$ is ϱ,ϱ where ϱ is the rank of $B((0,z''))$, so $2(n-\kappa)-\varrho\le n-k$, that is,

$$(n-\kappa-\varrho)+(k-\kappa)\le 0.$$

Since the terms are nonnegative it follows that $\kappa=k$ and that $\varrho=n-\kappa$. By a change of z'' coordinates we can therefore attain that $B(z)=\sum_{k+1}^n z_j^2$ when $z'=0$, and by completion of squares we can change them again so that all product terms between z' and z'' coordinates are eliminated. Then Q attains the form (5.4) where $0\le\lambda_j<1$ by the calculation which led to (5.2), now in the z' variables.

The preceding lemma is given a more useful form if one defines the Legendre transform by (2.11):

Lemma 5.3. *If the real quadratic form Q in $\mathbb{C}^n$ is plurisubharmonic then the negative index of inertia is $\le n$. If it is equal to n then Q is nondegenerate with signature n,n, and the critical value $Q^\dagger(\zeta)$ of $z\mapsto \operatorname{Im}\langle z,\zeta\rangle+Q(z)$ is a plurisubharmonic quadratic form with the same signature.*

Proof. If Q is negative definite in a subspace V of $\mathbb{C}^n$ with $\dim_{\mathbb{R}} V>n$, then Q is negative definite in the complex vector space $V\cap(iV)$ of complex dimension $\ge \dim_{\mathbb{R}}V-n>0$, which contradicts that Q is plurisubharmonic. If $\dim_{\mathbb{R}}V=n$ then $V\cap(iV)=\{0\}$, and since the Levi form $z\mapsto\frac12(Q(z)+Q(iz))$ is nonnegative, it follows that Q is positive definite in iV, so Q has signature n,n. Hence $Q^\dagger$ is plurisubharmonic with signature n,n by Lemma 5.2.

Remark. It is easy to prove the last statement in Lemma 5.3 without relying on the normal form in Lemma 5.1. To do so we may assume that Q is negative definite in $\mathbb{R}^n$. The positivity of the Levi form means that $Q(z)\ge -Q(iz)$, $z\in\mathbb{C}^n$, so it follows from an observation at the beginning of this section that $Q^\dagger(\zeta)\ge -Q^\dagger(i\zeta)$, which means that the Levi form of $Q^\dagger$ is non-negative.

Although the preceding argument is very elementary, the conclusion is not quite obvious. Indeed, it would be false if instead we had considered polynomials such that the Levi form is nonnegative just in the space $\mathbb{C}\mathbb{R}^n$ generated by $\mathbb{R}^n$. As an example consider the polynomial

$$Q(x+iy)=-\tfrac12|x|^2+\tfrac12|y|^2+\langle Bx,y\rangle,\quad x,y\in\mathbb{R}^n,$$

where B is a linear map in $\mathbb{R}^n$. It has signature n,n and is harmonic on every complex line in $\mathbb{C}\mathbb{R}^n$, for $Q(x)+Q(ix)=0$ if $x\in\mathbb{R}^n$. However, the Levi form

$$\tfrac12(Q(x+iy)+Q(ix-y))=\tfrac12(\langle Bx,y\rangle-\langle By,x\rangle)=\tfrac12\langle(B-B^*)x,y\rangle$$

is not nonnegative unless B is symmetric. We have

$$\begin{aligned}
Q^\dagger(\xi+i\eta) &= \inf_y \Big(\sup_x (\langle x,\eta\rangle + \langle y,\xi\rangle - \tfrac12|x|^2 + \tfrac12|y|^2 + \langle Bx,y\rangle)\Big)\\
&= \inf_y \Big(\tfrac12|y|^2 + \langle y,\xi\rangle + \tfrac12|\eta + B^*y|^2\Big)\\
&= \inf_y \Big(\tfrac12\langle(\mathrm{Id}+BB^*)y,y\rangle + \langle y,\xi+B\eta\rangle + \tfrac12|\eta|^2\Big)\\
&= -\tfrac12\langle(\mathrm{Id}+BB^*)^{-1}(\xi+B\eta),\xi+B\eta\rangle + \tfrac12|\eta|^2,
\end{aligned}$$

which shows that

$$Q^\dagger(\xi)+Q^\dagger(i\xi) = -\tfrac12\langle(\mathrm{Id}+BB^*)^{-1}\xi,\xi\rangle - \tfrac12\langle(\mathrm{Id}+BB^*)^{-1}B\xi,B\xi\rangle + \tfrac12|\xi|^2.$$

Here $BB^* = A^2$ where A is positive, symmetric and isometric with B^*, so $B^* = OA$ where O is orthogonal. Hence the right-hand side can be written

$$\begin{aligned}
-\tfrac12\langle(\mathrm{Id}+A^2)^{-1}\xi,\xi\rangle - \tfrac12\langle(\mathrm{Id}+A^2)^{-1}AO^*\xi,AO^*\xi\rangle + \tfrac12|\xi|^2&\\
= \tfrac12\langle A^2(\mathrm{Id}+A^2)^{-1}\xi,\xi\rangle - \tfrac12\langle A^2(\mathrm{Id}+A^2)^{-1}O^*\xi,O^*\xi\rangle&.
\end{aligned}$$

If this is nonnegative for all $\xi\in\mathbb{R}^n$ and orthogonal O, then $A^2(\mathrm{Id}+A^2)^{-1} = C\,\mathrm{Id}$ for some constant C, that is, $BB^* = A^2 = C(1-C)^{-1}\,\mathrm{Id}$. This means that B is conformal. Any other B gives a counterexample when multiplied to the right by a suitable orthogonal matrix, so the Levi form of $Q^\dagger$ is not always nonnegative in $\mathbb{CR}^n$ when that of Q is, provided that $n>1$.

Functions in the class $\mathcal{P}$ introduced in Definition 4.3 may be identically $+\infty$ in the complement of an affine subspace in the y variables. To handle them we need a supplement to Lemma 5.3.

LEMMA 5.4. *Let V be a linear subspace of $\mathbb{R}^n$ and let Q be a nondegenerate quadratic form in $\mathbb{R}^n\oplus iV$ such that Q is negative definite in $\mathbb{R}^n$ and the Levi form of the restriction to the complex space $V\oplus iV$ is nonnegative. If*

$$\varphi(z) = \begin{cases} Q(z), & \text{when } \operatorname{Im} z\in V,\\ +\infty, & \text{when } \operatorname{Im} z\notin V,\end{cases}$$

then $\varphi^\dagger(\zeta)$ is a plurisubharmonic quadratic form which is translation invariant in the direction of the orthogonal space of V in $\mathbb{R}^n$.

PROOF. We can choose the coordinates so that $V=\{y\in\mathbb{R}^n;\, y'=0\}$ where $y'=(y_1,\dots,y_\nu)$, and we shall write $z''=(z_{\nu+1},\dots,z_n)$. Regarding Q as a quadratic form in $\mathbb{C}^n$ which is independent of $y'=\operatorname{Im} z'$ we denote by Q_δ the quadratic form $Q_\delta(z) = |\operatorname{Im} z'|^2/\delta + Q(z)$ which converges to φ when $\delta\to0$. The Levi form is equal to the sum of $|z'|^2/2\delta$, the Levi form of $Q(0,z'')$ and the Levi form of $Q(z',z'')-Q(0,z'')$. The latter is $O(|z||z'|)$, so if the Levi form of $Q(0, z'')$ is a positive definite form in z'', then it follows that the Levi form of Q_δ is positive definite when δ is small enough. Since

$$Q_\delta^\dagger(\zeta) = -\tfrac{1}{4}\delta|\operatorname{Re}\zeta'|^2 + \varphi^\dagger(\zeta),$$

it follows when $\delta \to 0$ that $\varphi^\dagger$ is plurisubharmonic. Now $\varphi^\dagger$ is really the modified Legendre transform of the quadratic form Q in $\mathbb{R}^n \oplus iV$ in the dual $(\mathbb{R}^n/V^\perp) \oplus i\mathbb{R}^n$ so it is translation invariant along $V^\perp$ and depends continuously on Q. Since $Q(z)+\varepsilon|z''|^2$ is nondegenerate and strictly plurisubharmonic in the subspace where $z' = 0$ if ε is a sufficiently small positive number, it follows when $\varepsilon \to 0$ that $\varphi^\dagger$ is plurisubharmonic.

6. – The modified Legendre transformation in $\mathcal{P}$

Recall that the space $\mathcal{P}$ of functions in $\mathbb{C}^n$ introduced in Defintion 4.6 is a subset of the set of concave-convex functions satisfying (A) and $(\mathrm{C})^\dagger$, and that this set is invariant under the modified Legendre transformation defined by (4.1), (4.2). This section will be devoted to the proof that $\mathcal{P}$ is also invariant:

THEOREM 6.1. *If $\varphi \in \mathcal{P}$ then $\varphi^\dagger \in \mathcal{P}$ and $\varphi^{\dagger\dagger}(z) = \varphi(-z)$, so $\mathcal{P} \ni \varphi \mapsto \varphi^\dagger \in \mathcal{P}$ is a bijection.*

To make the idea of the proof transparent we shall first prove the special case where $\varphi(z) < \infty$ for every $z \in \mathbb{C}^n$. Then condition (P) requires φ to be plurisubharmonic. If $\varphi \equiv -\infty$ then $\Phi \equiv -\infty$ and $\varphi^\dagger \equiv -\infty$, so we may assume that $\varphi \not\equiv -\infty$.

LEMMA 6.2. *Let $\Omega \subset \mathbb{R}^\nu$ be an open convex set, and let φ be a plurisubharmonic function $\not\equiv -\infty$ in $\{z \in \mathbb{C}^\nu; \operatorname{Im} z \in \Omega\}$ such that $\mathbb{R}^\nu \ni x \mapsto \varphi(x+iy)$ is concave for every $y \in \Omega$. Then it follows that $\Omega \ni y \mapsto \varphi(x+iy)$ is convex, that φ is locally Lipschitz continuous and that there exist everywhere finite convex functions M_1 and M_2 in $\mathbb{R}^\nu$ and in Ω such that*

$$-M_1(x)(1+|y|) \le \varphi(x+iy) \le M_2(y)(1+|x|), \quad x \in \mathbb{R}^\nu,\ y \in \Omega. \tag{6.1}$$

PROOF. Let $0 \le \chi \in C_0^\infty(\mathbb{C}^\nu)$ be a function of $|z|$ only, with integral equal to 1 and support in the unit ball. Set $\chi_\varepsilon(z) = \varepsilon^{-2\nu}\chi(z/\varepsilon)$. Then $\psi_\varepsilon = \varphi * \chi_\varepsilon \in C^\infty$ in $\{z \in \mathbb{C}^\nu; \operatorname{Im} z + y \in \Omega \text{ if } |y| \le \varepsilon\}$, and $\psi_\varepsilon \downarrow \varphi$ as $\varepsilon \to 0$. The convolution ψ_ε is also plurisubharmonic and it is concave in x, thus $\partial^2\psi_\varepsilon(x+iy)/\partial x^2 \le 0$ and $\partial^2\psi_\varepsilon(x+iy)/\partial y^2 \ge 0$, which implies convexity with respect to y. For any compact subset K of $\mathbb{R}^\nu \times i\Omega$ there is a constant C such that $\psi_\varepsilon \le C$ in K for small ε. If K is a ball then the mean value of ψ_ε in K is bounded below by the mean value of φ in K. If we apply Lemma 3.2 to $C - \psi_\varepsilon$ it follows that we have locally uniform bounds for ψ_ε and the first derivatives, and when $\varepsilon \to 0$ it follows that φ is a locally Lipschitz continuous function which is convex as a function of y.

Since $x \mapsto \varphi(x+iy)$ has an affine majorant for fixed $y \in \Omega$, it follows that

$$M_2(y) = \sup_x \varphi(x+iy)/(1+|x|) < \infty, \quad y \in \Omega,$$

and since M_2 is convex as a function of y it is continuous. Similarly

$$-M_1(x) = \inf_{y\in\Omega} \varphi(x+iy)/(1+|y|) > -\infty$$

for fixed $x \in \mathbb{R}^n$, by the convexity in y, and since M_1 is convex this proves the lemma.

If φ is plurisubharmonic in $\mathbb{C}^n$ and $\mathbb{R}^n \ni x \mapsto \varphi(x+iy)$ is concave, it follows from Lemma 6.2 that $\varphi \in \mathcal{P}$. Let ψ_ε be the regularization of φ in the proof of Lemma 6.2, and set with $\gamma, \delta, \varepsilon \geq 0$

$$\varphi_{\gamma,\delta,\varepsilon}(z) = \psi_\gamma(z) - \varepsilon|\operatorname{Re} z|^2 + \delta|\operatorname{Im} z|^2 = \psi_\gamma(z) - \tfrac{1}{2}(\varepsilon+\delta)\operatorname{Re}\langle z, z\rangle + \tfrac{1}{2}(\delta-\varepsilon)|z|^2.$$

This is a C^∞ strictly plurisubharmonic function if $\gamma > 0$ and $\varepsilon < \delta$, and it is strictly concave in the real direction if $\varepsilon > 0$. The supremum

$$\Phi_{\gamma,\delta,\varepsilon}(y,\eta) = \sup_x (\langle x, \eta\rangle + \varphi_{\gamma,\delta,\varepsilon}(x+iy))$$

is attained at a unique point x which is a C^∞ function of y and η, for $\langle x,\eta\rangle + \varphi_{\gamma,\delta,\varepsilon}(x+iy) \to -\infty$ as $x \to \infty$. The supremum is a strictly convex C^∞ function of y and $\partial\Phi_{\gamma,\delta,\varepsilon}(y,\eta)/\partial y = \partial\varphi_{\gamma,\delta,\varepsilon}(x+iy)/\partial y$, so the infimum $\varphi^\dagger_{\gamma,\delta,\varepsilon}(\xi,\eta)$ of $\langle y,\xi\rangle + \Phi_{\gamma,\delta,\varepsilon}(y,\eta)$ is taken at a point y which is a C^∞ function of ξ and η. Thus

$$\varphi^\dagger_{\gamma,\delta,\varepsilon}(\xi,\eta) = \langle x,\eta\rangle + \langle y,\xi\rangle + \varphi_{\gamma,\delta,\varepsilon}(x+iy)$$

where (x,y) is the unique critical point of the right-hand side. It follows from (0.4) that the Hessian of $\varphi^\dagger_{\gamma,\delta,\varepsilon}$ at (ξ,η) is determined by that of $\varphi_{\gamma,\delta,\varepsilon}$ at (x,y). Hence Lemma 5.3 shows that $\varphi^\dagger_{\gamma,\delta,\varepsilon}$ is strictly plurisubharmonic (and strictly concave in the real direction). When $\gamma \to 0$ then $\varphi_{\gamma,\delta,\varepsilon}(z) \downarrow \varphi_{0,\delta,\varepsilon}(z) = \varphi(z) - \varepsilon|\operatorname{Re} z|^2 + \delta|\operatorname{Im} z|^2$, and $\varphi^\dagger_{\gamma,\delta,\varepsilon} \downarrow \varphi^\dagger_{0,\delta,\varepsilon}$ by Proposition 3.5, so it follows from Lemma 4.9 that $\varphi^\dagger_{0,\delta,\varepsilon} \in \mathcal{P}$. When $\delta > 0$ is fixed and $\varepsilon \to 0$ then $\varphi_{0,\delta,\varepsilon}(z) \uparrow \varphi_{0,\delta,0}(z) = \varphi(z) + \delta|\operatorname{Im} z|^2$ and $\varphi^\dagger_{0,\delta,\varepsilon} \uparrow \varphi^\dagger_{0,\delta,0}$ by Proposition 3.4 which is therefore also in $\mathcal{P}$ by Lemma 4.8. Finally, when $\delta \to 0$ then $\varphi^\dagger_{0,\delta,0} \downarrow \varphi^\dagger$ which is therefore in $\mathcal{P}$, so we have proved Theorem 6.1 in this special case. Note that $\varphi^\dagger$ is not necessarily finite in $\mathbb{C}^n$, for if φ is just a convex function of $\operatorname{Im} z$ then $\varphi^\dagger(\zeta) = +\infty$ when $\operatorname{Im}\zeta \neq 0$. Theorem 6.1 could therefore not be valid without the generality in our definition of the class $\mathcal{P}$.

Passing now to the general proof of Theorem 6.1 we begin with a few simple reductions. Since a translation of φ only causes a linear function to be added to $\varphi^\dagger$, we may assume that $0 \in X_1^\circ$ and that $0 \in Y_1^\circ$, with the notation in Proposition 3.1. If $x_n = 0$ in X_1° then $y_n = 0$ in Y_1°, which implies that $\varphi^\dagger(\zeta)$ is independent of ζ_n and as a function of $(\zeta_1, \dots, \zeta_{n-1})$ is the modified Legendre transform of φ considered as a function in $\mathbb{C}^{n-1}$. Hence we may assume without restriction that X_1° is an open convex subset of $\mathbb{R}^n$. We can choose the coordinates so that $\operatorname{ah}(Y_1^\circ)$ is defined by $y'' = (y_{\nu+1}, \dots, y_n) = 0$, and can then consider Y_1° as an open convex subset Ω of $\mathbb{R}^\nu$ while $X_1^\circ = \mathbb{R}^\nu \times \omega$ where ω is an open convex subset of $\mathbb{R}^{n-\nu}$. Condition (P) means that $\varphi(x + i(y', 0))$ is a plurisubharmonic function $\not\equiv -\infty$ in $\mathbb{R}^\nu \times i\Omega$ when $x'' \in \omega$. Here $y' = (y_1, \dots, y_\nu)$.

The proof of Lemma 6.2 is easily extended to the present situation:

LEMMA 6.3. *Let φ be a function in $(\mathbb{R}^\nu \times \omega) \times i\Omega$ where ω is an open convex subset of $\mathbb{R}^{n-\nu}$ and Ω is an open convex subset of $\mathbb{R}^\nu$. Assume that φ is plurisubharmonic $\not\equiv -\infty$ in $\mathbb{R}^\nu \times i\Omega$ when the component in ω is fixed, and that φ is concave in $\mathbb{R}^\nu \times \omega$ when the component in Ω is fixed. Then it follows that $\Omega \ni y \mapsto \varphi(x + iy)$ is convex, that φ is locally Lipschitz continuous and that there exist everywhere finite convex functions M_1 and M_2 in $\mathbb{R}^\nu \times \omega$ and in Ω such that*

$$(6.2)\quad -M_1(x)(1 + |y|) \le \varphi(x + iy) \le M_2(y)(1 + |x|), \quad x \in \mathbb{R}^\nu \times \omega,\ y \in \Omega\,.$$

PROOF. The convexity with respect to y follows it we apply Lemma 6.2 for fixed $x'' \in \omega$, and the other statements are then consequences of Lemma 3.2 as in the proof of Lemma 6.2.

To achieve an approximation by smooth functions we shall choose a nonnegative function $\chi_1 \in C_0^\infty(\mathbb{C}^\nu)$ with integral 1 and support in the unit ball which only depends on $|z'|$, and an even nonnegative function $\chi_2 \in C_0^\infty(\mathbb{R}^{n-\nu})$ with integral 1, and denote by ψ_{γ_1,γ_2} the convolution of φ and $\gamma_1^{-2\nu}\gamma_2^{\nu-n}\chi_1(z'/\gamma_1)\chi_2(x''/\gamma_2)$. If k is a compact subset of ω and K is a compact subset of Ω, then ψ_{γ_1,γ_2} is a C^∞ function in a neighborhood of $(\mathbb{R}^\nu \times k) \times iK$ when γ_1 and γ_2 are small enough. It is clear that ψ_{γ_1,γ_2} is concave with respect to x and plurisubharmonic with respect to $z' = x' + iy'$, and that ψ_{γ_1,γ_2} converges locally uniformly in $(\mathbb{R}^\nu \times \omega) \times i\Omega$ to φ when $\gamma_1, \gamma_2 \to 0$. It is an increasing function of γ_1 by the plurisubharmonicity and a decreasing function of γ_2 by the concavity. To obtain functions in $\mathcal{P}$ for which we can determine the modified Legendre transform by differential calculus we must cut off by adding a strictly convex function of y which becomes $+\infty$ outside a compact subset of Ω and subtracting a strictly convex function of x'' which becomes $+\infty$ outside a compact subset of ω. Such functions are provided by the following lemma.

LEMMA 6.4. *If Ω is an open convex subset of $\mathbb{R}^\nu$, then there is a decreasing sequence of nonnegative convex functions χ_j^Ω such that $\Omega_j = \{x \in \Omega;\ \chi_j^\Omega(x) < \infty\} \Subset \Omega$, $\cup_j \Omega_j = \Omega$, $\chi_{j+1}^\Omega = 0$ in Ω_j, $\chi_j^\Omega \in C^\infty$ in Ω_j, and $\chi_j^\Omega \to +\infty$ at $\partial\Omega_j$.*

PROOF. The sets $K_t = \{x \in \Omega; |x| \le t, |x-y| \ge 1/t$ when $y \notin \Omega\}$ are convex, compact, and increase to Ω when $t \to \infty$. If $0 \in \Omega$, as we may assume, then 0 is an interior point of K_t for $t \ge t_0$, say. Then the distance function d_t which is positively homogeneous of degree one and equal to 1 on ∂K_t is convex in $\mathbb{R}^\nu$. By regularization we can approximate d_t arbitrarily closely by a C^∞ convex function $\tilde{d}_t \ge d_t$. Let g be the C^∞ convex increasing function on $(-\infty, 1)$ defined by

$$g(s) = \begin{cases} 0 & \text{when } s \le 0 \\ (1-s)^{-1}\exp(-2/s), & \text{when } 0 < s < 1, \end{cases}$$

and define $g(s) = +\infty$ when $s \ge 1$. The convex function $g((\tilde{d}_t(x) - 1)/\varepsilon_t)$ is equal to 0 in $K_{t-\frac{1}{2}}$, it is in C^∞ when $\tilde{d}_t(x) < 1+\varepsilon_t$ and equals $+\infty$ otherwise, in particular in $\complement K_{t+\frac{1}{2}}$, if ε_t is sufficiently small and $\tilde{d}_t$ is sufficiently close to d_t. Thus

$$\chi_j^\Omega(x) = g((\tilde{d}_{t_0+j}(x) - 1)/\varepsilon_{t_0+j})$$

has the desired properties, for χ_j^Ω is convex, $\Omega_j \subset K_{t_0+j+\frac{1}{2}}$ and $\chi_{j+1}^\Omega = 0$ there.

We can now present the general proof of Theorem 6.1, with the admissible hypothesis that $X_1^\circ = \mathbb{R}^\nu \times \omega$ and $Y_1^\circ = \{(y', 0); y' \in \Omega\}$ where ω is an open convex set in $\mathbb{R}^{n-\nu}$ containing the origin and Ω is an open convex set in $\mathbb{R}^\nu$ containing the origin. With $0 < \varepsilon < \delta$ and integers j, k we set $\varphi_\theta(x+iy) = +\infty$ if $y \notin Y_1^\circ$ and

$$\varphi_\theta(z) = \psi_{\gamma_1,\gamma_2}(z) + \chi_j^\Omega(y') - \chi_k^\omega(x'') - \varepsilon|\operatorname{Re} z'|^2 + \delta|\operatorname{Im} z'|^2, \quad \theta = (\gamma_1, \gamma_2, \delta, \varepsilon, j, k), \tag{6.3}$$

if $y \in Y_1^\circ$; here γ_1, γ_2 are assumed to be so small that ψ_{γ_1,γ_2} is defined, concave in x and pseudoconvex in z', when y' is in a neighborhood of $\overline{\Omega}_j$ and x'' is in a neighborhood of $\overline{\omega}_k$. In (6.3) $+\infty - (+\infty)$ shall be read as $+\infty$, that is, $\varphi_\theta(z) = +\infty$ if $\chi_j^\Omega(y') = +\infty$ even if $\chi_k^\omega(x'') = +\infty$ too. Since $\varepsilon < \delta$ it is clear that φ_θ is plurisubharmonic in z', so it is a function in $\mathcal{P}$. For the corresponding function $\Phi_\theta(y, \eta) = \sup_x(\langle x, \eta\rangle + \varphi_\theta(x+iy))$ we have $\Phi_\theta(y, \eta) = +\infty$ if $y'' \ne 0$, and

$$\Phi_\theta((y', 0), \eta) = \chi_j^\Omega(y') + \delta|y'|^2 + \sup_x(\langle x, \eta\rangle + \psi_{\gamma_1,\gamma_2}(x + i(y', 0)) - \chi_k^\omega(x'') - \varepsilon|x'|^2).$$

The supremum is attained at some point x with $\chi_k^\omega(x'') < \infty$ which is a C^∞ function of η and y' when y' is in a neighborhood of $\overline{\Omega}_j$. The infimum $\varphi_\theta^\dagger$ of $\langle y', \xi'\rangle + \Phi_\theta((y', 0), \eta)$ with respect to y' is then also attained at a point which is a C^∞ function of ξ and η. Thus

$$\varphi_\theta^\dagger(\xi + i\eta) = \langle x, \eta\rangle + \langle y', \xi'\rangle + \varphi_\theta(x + i(y', 0))$$

at a uniquely defined point where the right-hand side is critical with respect to x and y', so $\varphi_\theta^\dagger(\xi + i\eta)$ is independent of ξ'' and as a function of ξ', η it is the modified Legendre transform of $\varphi_\theta(x + (y', 0))$. By (0.4) we conclude that the Hessian of $\varphi_\theta^\dagger$ at (ξ, η) is uniquely determined by that of φ_θ at the corresponding point (x, y'). Hence it follows from Lemma 5.4 that the Levi form of $\varphi_\theta^\dagger$ is nonnegative at (ξ, η).

We can now start with letting $\gamma_1 \downarrow 0$, which makes φ_θ decrease, then $\gamma_2 \downarrow 0$, which makes φ_θ increase, then $\varepsilon \downarrow 0$ which makes φ_θ increase, then $\delta \downarrow 0$ which makes φ_θ decrease, then $j \to \infty$ which gives a decreasing sequence and finally $k \to \infty$ which gives an increasing sequence. The final limit is φ, so using Propositions 3.4, 3.5 and Lemmas 4.8, 4.9 repeatedly as in the proof of the special case of Theorem 6.1 above, we can conclude that $\varphi^\dagger \in \mathcal{P}$, which completes the proof.

7. – Examples and properties of functions in $\mathcal{P}$

We shall begin by giving an explicit example related to the spaces W_M^Ω of Gelfand and Šilov [3], [4]. At first we shall only discuss the one dimensional case.

Proposition 7.1. *If $1 < p \le 2$ then*

$$\varphi(x + iy) = \operatorname{Re}(|y| + ix)^p/p, \quad x, y \in \mathbb{R}, \tag{7.1}$$

is in $\mathcal{P}$. Here z^p is the continuous branch in the right half plane which is 1 at 1. With $1/p + 1/q = 1$ we have

$$\varphi^\dagger(\xi + i\eta) = \begin{cases} -\operatorname{Re}(|\xi| + i\eta)^q/q, & \text{if } |\eta| \le |\xi| \tan a, \\ (\sin a)^{1-q}|\eta|^q/q & \text{if } |\eta \ge |\xi| \tan a. \end{cases} \tag{7.2}$$

Here $a = (p-1)\pi/2 \in (0, \pi/2]$. We have

$$p\varphi(x + iy) \le (2/\sin(\pi/p))^{p-1}(|y|^p(\sin(\pi/2p))^p - |x|^p(\cos(\pi/2p))^p), \tag{7.3}$$

$$q\varphi^\dagger(\xi + i\eta) \le (2/\sin(\pi/q))^{q-1}(|\eta|^q(\cos(\pi/2q))^q - |\xi|^q(\sin(\pi/2q))^q). \tag{7.4}$$

Proof. φ is harmonic when $y \neq 0$, continuous in $\mathbb{C}$, and $\partial\varphi(x+iy)/\partial y = \pm\cos a|x|^{p-1}$ when $y = \pm 0$, which implies that φ is subharmonic, $\Delta\varphi = 2\cos a|x|^{p-1}\delta(y)$. When $y \neq 0$ we have

$$\partial^2\varphi(x + iy)/\partial x^2 = -(p-1)\operatorname{Re}(|y| + ix)^{p-2} < 0$$

which proves the concavity. To compute $\varphi^\dagger$ we observe that if φ is differentiable at the critical point of $x\eta + y\xi + \varphi(x + iy)$ then $\partial\varphi(x + iy)/\partial x + \eta = 0$ and

$\partial\varphi(x+iy)/\partial y+\xi=0$, hence the critical value is $(1-p)\varphi(x+iy)=-p\varphi(x+iy)/q$. This gives the first case in (7.2), with $(|y|+ix)^{p-1}=|\xi|+i\eta$. In the second case the critical value is attained for $y=0$ so it is a function of $|\eta|$ only. When $|\eta|=|\xi|\tan a$ then $-\operatorname{Re}(|\xi|+i\eta)^q=-|\eta|^q\cos(aq)/(\sin a)^q$, and since $aq=a+\pi/2$ we have $\cos(aq)=-\sin a$ which gives the second case in (7.2). The derivative of $\varphi^\dagger(\xi+i\eta)$ with respect to ξ exists and is equal to 0 when $|\eta|=|\xi|\tan a$, for $(q-1)a=\pi/2$, so $\varphi^\dagger\in C^1$ and $\Delta\varphi=(q-1)(\sin a)^{1-q}|\eta|^{q-2}$ in the second case of (7.2) while $\Delta\varphi=0$ in the first case. Note that the passage between the two definitions in (7.2) takes place at the first lines where passing to a function of η only can lead to a function in C^1. However, $\varphi^\dagger$ is never in C^2 if $q>2$.

It follows from (7.1) that

$$p\varphi(x+iy)\le t|y|^p-a_p(t)|x|^p,\quad t>1, \tag{7.5}$$

where

$$a_p(t)=\min_{x>0}(t-f(x))/x^p,\quad f(x)=\operatorname{Re}(1+ix)^p,$$

is a concave function of t, and (7.5) implies that

$$q\varphi^\dagger(\xi+i\eta)\le a_p(t)^{1-q}|\eta|^q-t^{1-q}|\xi|^q. \tag{7.6}$$

To optimize (7.5) we want to maximize $a_p(t)/t$. If the minimum in the definition of $a_p(t)$ is attained at x then $xf'(x)-p(f(x)-t)=0$. This determines x uniquely as a C^∞ function of t, for

$$\begin{aligned}(xf'(x))'-pf'(x)&=xf''(x)-(p-1)f'(x)\\&=-p(p-1)\operatorname{Re}(i(1+ix)^{p-2})<0,\quad x>0.\end{aligned}$$

Hence $a_p(t)\in C^\infty$, and when the maximum of $a_p(t)/t$ is attained then $ta_p'(t)=a_p(t)$, that is $f(x)=0$, so $x=\tan(\pi/2p)$ and then $a_p(t)=t/x^p$, where $t=-xf'(x)/p=\operatorname{Re}(-ix(1+ix)^{p-1})$, so we obtain (7.3) which implies (7.4).

REMARK. The estimates (7.3) and (7.4) are in a sense optimal, for if φ is subharmonic and $\varphi(x+iy)\le b^p|y|^p-c^p|x|^p$ where $1<p\le 2$ and $c\sin(\pi/2p)>b\cos(\pi/2p)$ then $\varphi\equiv-\infty$. In fact, if we take $z=iw^{1/p}$ where $\operatorname{Re}w\ge 0$ then $\varphi(z)$ as a function of w is bounded above by $b^p|w|$ in the right half plane and is $<-c'|w|$ on the imaginary axis for some $c'>0$, hence not integrable with respect to $|dw|/(1+|w|^2)$. Hence $\varphi\equiv-\infty$ in an open sector, and since φ is subharmonic it follows that $\varphi\equiv-\infty$ in $\mathbb{C}$. Similarly (7.4)$'$ is optimal when $q>2$, which is seen by passing to the Legendre transform.

COROLLARY 7.2. *For every $p\in(1,\infty)$ there is a function $\varphi\in\mathcal{P}(\mathbb{C}^n)$ such that*

$$\varphi(x+iy)\le\begin{cases}|y|^p(\sin(\pi/2p))^p-|x|^p(\cos(\pi/2p))^p, & \text{if } 1<p\le 2,\\ |y|^p(\cos(\pi/2p))^p-|x|^p(\sin(\pi/2p))^p, & \text{if } 2\le p<\infty,\end{cases} \tag{7.7}$$

when $x, y\in\mathbb{R}^n$. Here $|\cdot|$ is the Euclidean norm. Moreover, $\varphi(0)=0$.

PROOF. By Proposition 7.1 there exists a function $\varphi_1 \in \mathcal{P}(\mathbb{C})$ with these properties. We can choose a constant c_p such that

$$|x|^p = c_p \int_{S^{n-1}} |\langle x, \omega\rangle|^p \, d\omega, \quad x \in \mathbb{R}^n,$$

where $d\omega$ is the surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$, for the right-hand side is homogeneous of degree p and orthogonally invariant. Hence

$$\varphi(x+iy) = c_p \int_{S^{n-1}} \varphi_1(\langle x, \omega\rangle + i\langle y, \omega\rangle)\, d\omega$$

is in $\mathcal{P}(\mathbb{C}^n)$ and satisfies (7.7).

Gelfand and Šilov emphasized spaces W_Ω^M of the form $\mathcal{S}_\varphi$ where φ satisfies an estimate of the form (7.7). The following proposition is essentially an observation in [4, Section 1.4]:

PROPOSITION 7.3. *Let $\varphi \not\equiv -\infty$ be a subharmonic function in $\mathbb{C}$ such that $\mathbb{R} \ni x \mapsto \varphi(x+iy)$ is concave, and assume that for some even functions M and Ω on $\mathbb{R}$*

$$\varphi(x+iy) \le \Omega(y) - M(x), \quad x, y \in \mathbb{R}. \tag{7.8}$$

Then it follows that $\sup_x(\Omega(x) - M(x)) = +\infty$ unless $\varphi(x+iy) = a(y^2 - x^2) + c$ for some constants $a > 0$ and c.

PROOF. From (7.8) it follows that

$$\varphi(x+iy) + \varphi(y+ix) \le F(x) + F(y), \quad F = \Omega - M.$$

If F is bounded then the subharmonic function $\varphi(x+iy)+\varphi(y+ix)$ is bounded, hence constant, which implies that each of the subharmonic terms must be harmonic. The harmonic function $\partial^2\varphi(x+iy)/\partial x^2$ is ≤ 0 so it must be a constant $-2a$, where $a \ge 0$, and $\varphi(x+iy) = a(y^2-x^2)+bxy+cx+dy+f$ with constant coefficients. Now the argument above also proves that $\varphi(x+iy)+\varphi(\pm y \pm ix)$ is constant for all combinations of the signs, so the coefficients b, c, d must vanish.

In the positive direction Gelfand and Šilov [4, p. 11] stated that if l is a positive function on $\mathbb{R}$ such that $\log l(x)/\log x \to 0$ as $x \to +\infty$ then there exists for every $p > 0$ an entire analytic function $f \not\equiv 0$ in $\mathbb{C}$ such that

$$|f(x+iy)| \le C \exp(-l(|x|)|x|^p + \gamma l(|y|)|y|^p), \quad x, y \in \mathbb{R},$$

for some constants C and γ. The statement was attributed to B. Ya. Levin. However, it is obviously false when $p = 1$ and $l(x) \equiv 1$, for an entire function of exponential type cannot decrease exponentially on $\mathbb{R}$. It is also false when $p < 1$, for the Phragmén-Lindelöf theorem proves that f must be bounded, hence a constant so $f \equiv 0$. A stronger restriction on l is also required when $p > 1$, for if $l_1 \le l_2$ are two positive functions with $\log l_j(x)/\log x \to 1$ as $x \to \infty$ we could take $l(x) = l_2(x)$ for rational x and $l(x) = l_1(x)$ for irrational x and conclude that

$$|f(x+iy)| \le C \exp(-l_2(|x|)|x|^p + \gamma l_1(|y|)|y|^p)$$

which contradicts Proposition 7.3 if $l_1/l_2 \to 0$ at infinity. It seems likely that Gelfand and Šilov had in mind the stronger conditions in Lewin [7] where it is assumed that l is slowly increasing in the sense that

$$l(s)/l(r) \to 1 \quad \text{if } s, r \to +\infty \text{ while } s/r + r/s \quad \text{is bounded.} \tag{7.9}$$

In that case we shall now prove the existence of appropriate subharmonic functions which will yield the required function f when combined with Theorem 8.3. (Cf. Lewin [7].)

LEMMA 7.4. *If l satisfies* (7.9) *then there exists a function $\gamma \in C^\infty(\mathbb{R})$ such that $l(r)e^{-\gamma(r)} \to 1$ as $r \to +\infty$ and $(rd/dr)^j\gamma(r) \to 0$ as $r \to \infty$ for every $j \ge 1$, hence $\gamma(r)/\log r \to 0$ as $r \to \infty$. For every $\varepsilon > 0$ we have $\underline{\lim}_{y \ge x \to \infty} l(x)x^\varepsilon/(l(y)y^\varepsilon) = 1$.*

PROOF. Let $\chi \in C_0^\infty((1,2))$, $\int \chi(t)\,dt/t = 1$, and set for large $r > 0$

$$\gamma(r) = \int_1^2 \log l(tr)\chi(t)\,dt/t = \int_r^{2r} \log l(t)\chi(t/r)\,dt/t\,.$$

Then $\gamma \in C^\infty$ and we have for $j \ne 0$

$$(-rd/dr)^j\gamma(r) = \int_r^{2r} \log l(t)\chi_j(t/r)\,dt/t = \int_1^2 \log(l(tr)/\log r)\chi_j(t)\,dt/t\,,$$

where $\chi_j(t) = (td/dt)^j\chi(t)$, thus $\int \chi_j(t)\,dt/t = 0$ when $j \ne 0$. For every $\varepsilon > 0$ the derivative of $s \mapsto \gamma(e^s)$ is $\ge -\varepsilon$ for large s which proves that $\gamma(e^t) - \gamma(e^s) \ge -\varepsilon(t-s)$ if $s \le t$ and s is large enough. Hence $\underline{\lim}_{t \ge s \to \infty} \varepsilon(t-s) + \gamma(e^t) - \gamma(e^s) \ge 0$, which proves the last statement in the lemma with l, x, y replaced by e^γ, e^s, e^t.

With $0 < p \le 2$ we shall now compute $\Delta\varphi_0$ when φ_0 is defined in analogy with (7.1) by

$$\varphi_0(re^{i(\theta \pm \pi/2)}) = r^p e^{\gamma(r)}\cos(p\theta), \quad |\theta| \le \pi/2, \ r \ge 0\,.$$

We may assume that $\gamma(r) = 0$ when $r < 1$, say. For $|\theta| < \pi/2$ and $r > 0$ we have

$$r^2\Delta\varphi_0 = ((r\partial/\partial r)^2 + (\partial/\partial\theta)^2)\varphi_0 = \varphi_0 k(r)\,,$$
$$k(r) = 2prd\gamma(r)/dr + (rd\gamma(r)/dr)^2 + (rd/dr)^2\gamma(r)\,.$$

By Lemma 7.3 we know that $k(r) \to 0$ when $r \to \infty$. Since

$$\pm\partial\varphi_0/\partial\theta = \mp pr^p e^{\gamma(r)} \sin(p\theta) \le 0, \quad \text{when } \theta = \pm\pi/2,$$

when $0 < p \le 2$, it follows that $\varphi = \varphi_0 + \varphi_1$ is subharmonic if $\varphi_1 = \varphi_1(r)$ and

$$(rd/dr)^2\varphi_1(r) \ge |k(r)|r^p e^{\gamma(r)}.$$

With $t = \log r$ as a new variable the condition can be written

$$d^2\varphi_1(e^t)/dt^2 \ge |k(e^t)|e^{pt}e^{\gamma(e^t)}$$

so it is satisfied with equality by

$$\tag{7.10} \begin{aligned} \varphi_1(e^t) &= \int_0^t (t-s)|k(e^s)|e^{ps}e^{\gamma(e^s)}\,ds, \quad \text{that is,} \\ \varphi_1(e^t)/(e^{pt}e^{\gamma(e^t)}) &= \int_0^t se^{-ps}|k(e^{t-s})|\exp(\gamma(e^{t-s}) - \gamma(e^t))\,ds \end{aligned}$$

When $0 < s < t/2$ we have $\gamma(e^{t-s}) - \gamma(e^t) < ps/2$ if t is large enough, since $d\gamma(e^t)/dt \to 0$ at infinity. When $t/2 < s < t$ we have $\gamma(e^{t-s}) + \gamma(e^t) < ps/2$ since $\gamma(e^t) = o(t)$. Finally $k \to 0$ at infinity so it follows by dominated convergence that (7.10) converges to 0 when $t \to \infty$.

We shall now prove that the subharmonic function $\varphi = \varphi_0 + \varphi_1$ just constructed is concave with respect to x if $1 < p \le 2$. To do so we introduce polar coordinates $x = r\sin\theta$, $y = r\cos\theta$ and note that $\partial r/\partial x = x/r = \sin\theta$ and that $r\partial\theta/\partial x = \cos\theta$, hence

$$\partial(h(r)\cos(p\theta))/\partial x = h'(r)\sin\theta\cos(p\theta) - h(r)r^{-1}p\cos\theta\sin(p\theta),$$

which after some computation gives

$$\tag{7.11} \begin{aligned} r^2\partial^2(h(r)\cos(p\theta))/\partial x^2 = {} & p\cos((p-2)\theta)(h(r) - rh'(r)) \\ & + \cos(p\theta)\big((rd/dr)^2h(r) - p^2h(r))\sin^2\theta \\ & + (1+p)(rh'(r) - ph(r))\cos(2\theta)\big). \end{aligned}$$

Note that (7.11) reduces to $p(1-p)r^p\cos((p-2)\theta)$ when $h(r) = r^p$ which agrees with a calculation in the proof of Proposition 7.1. To verify (7.11) it suffices to check in addition that the coefficient of $h''(r)$ in both sides is $r^2\cos(p\theta)\sin^2\theta$ and that the coefficient of h is $p\sin(2\theta)\sin(p\theta) - p^2\cos^2\theta$ $\cos(p\theta)$. It follows from (7.11) that

$$\begin{aligned} r^2\partial_x^2\varphi_0 = {} & \big(p\cos((p-2)\theta)((1-p) - r\gamma'(r)) + \cos(p\theta)\big(k(r)\sin^2\theta \\ & + (1+p)r\gamma'(r)\cos(2\theta)\big)\big)r^pe^{\gamma(r)}, \end{aligned}$$

$$\begin{aligned} r^2\partial_x^2\varphi_1 &= r^2d^2\varphi_1/dr^2\sin^2\theta + rd\varphi_1/dr\cos^2\theta \\ &= |k(r)|r^pe^{\gamma(r)}\sin^2\theta + rd\varphi_1/dr\cos(2\theta). \end{aligned}$$

The dominating term in $\partial^2\varphi/\partial x^2$ is $-p(p-1)\cos((p-2)\theta)r^{p-2}e^{\gamma(r)}$; the cosine factor is strictly positive since $0 \le 2-p < 1$. Hence it follows that φ is concave with respect to x for large r. For arbitrarily large R we can choose $\gamma(r)$ constant for $r < R$ while the estimates implied in Lemma 7.4 are independent of R. If R is large enough then $\varphi(x+iy)$ is everywhere concave with respect to x. By (7.3) we have

$$\varphi_0(x+iy) \le (2/\sin(\pi/p))^{p-1}(|y|^p(\sin(\pi/2p))^p - |x|^p(\cos(\pi/2p))^p)e^{\gamma(r)},$$

and $\varphi_1(x+iy)/((|x|^p+|y|^p)e^{\gamma(r)} \to 0$ as $r \to \infty$.

PROPOSITION 7.5. *Let $1 < p \le 2$ and let l be a slowly increasing function as in* (7.9). *If $a > (\tan(\pi/2p))^p$ there exists a subharmonic function $\varphi \not\equiv -\infty$ in $\mathbb{C}$ such that $\mathbb{R} \ni x \mapsto \varphi(x+iy)$ is concave and*

$$\varphi(x+iy) \le (a|y|^p - |x|^p)l(\sqrt{x^2+y^2}), \quad x, y \in \mathbb{R}, \tag{7.12}$$

$$\varphi(x+iy) \le a|y|^p l(|y|) - |x|^p l(|x|), \quad x, y \in \mathbb{R}. \tag{7.13}$$

PROOF. Multiplication of the function $\varphi_0+\varphi_1$ just constructed by a suitable constant gives a function φ satisfying (7.12) for large $x+iy$, and subtraction of a suitable constant gives the estimate in the whole plane. If ε is so small that $a > (1+\varepsilon)^2(\tan(\pi/2p))^p$ we can even choose φ so that

$$\varphi(x+iy) \le \big(a(1+\varepsilon)^{-1}|y|^p - (1+\varepsilon)|x|^p\big)l(\sqrt{x^2+y^2}), \quad x, y \in \mathbb{R}. \tag{7.12$'$}$$

It suffices to prove that (7.12)$'$ implies (7.13) for large $x+iy$, for subtraction of a constant from φ will then give the desired bound in the entire complex plane. If (7.13) is not valid at $x+iy$ then

$$a|y|^p l(|y|) - |x|^p l(|x|) \le \big(a(1+\varepsilon)^{-1}|y|^p - (1+\varepsilon)|x|^p\big)l(\sqrt{x^2+y^2}).$$

Assume first that $|x| \ge |y|$. Then

$$\begin{aligned}((1+\varepsilon)l(\sqrt{x^2+y^2}) - l(|x|))|x|^p &\le a\big((1+\varepsilon)^{-1}l(\sqrt{x^2+y^2}) - l(|y|)\big)|y|^p \\ &\le a(1+\varepsilon)^{-1}|y|^p l(\sqrt{x^2+y^2}).\end{aligned}$$

Since $l(\sqrt{x^2+y^2})/l(|x|) \to 1$ when $x \to \infty$ it follows that $\varepsilon|x|^p \le a|y|^p$ if $|x|$ is large. But $l(|y|)/l(|x|)$ is then also close to 1 which gives a contradiction.

Next assume that $|x| \le |y|$. Then

$$a\big(l(|y|) - (1+\varepsilon)^{-1}l(\sqrt{x^2+y^2})\big)|y|^p \le l(|x|)|x|^p,$$

which for large $|y|$ implies that

$$a\varepsilon(1+2\varepsilon)^{-1}|y|^p l(|y|) \le l(|x|)|x|^p,$$

so $|x|$ must also be large. Hence $|x|^\varepsilon l(|x|) \le (1+\varepsilon)|y|^\varepsilon l(|y|)$ by Lemma 7.4, so

$$a\varepsilon(1+2\varepsilon)^{-1}|y|^p \le (1+\varepsilon)|y|^\varepsilon|x|^{p-\varepsilon}$$

which gives a bound for $|y|/|x|$ and hence a contradiction. This completes the proof.

By passage to the Legendre transform we can get a similar result when $p > 2$, but we leave this for the reader.

We shall now modify (7.2) to a construction of functions in $\mathcal{P}$ which decrease very rapidly on $\mathbb{R}$. To do so we start from any even function ψ on $\{x \in \mathbb{R};\, |x| \geq a\}$ such that when $x \geq a$

$$\begin{gathered}\psi^{(j)}(x) > 0,\ \ 1 \leq j \leq 4, \quad d(\psi'(x)/\psi'''(x))/dx > 0, \\ \psi'(x)/\psi'''(x) \to \infty \quad \text{as } x \to \infty\,.\end{gathered} \tag{7.14}$$

An example is $\psi(x) = \exp(|x|^\gamma)$ where $0 < \gamma < 1$. By (7.14) and the implicit function theorem the equation $\psi'(x) = t\psi'''(x)$ has a unique solution $x = X(t) \geq a$ when $t \geq b = \psi'(a)/\psi'''(a)$, and $X'(t) > 0$. Now define

$$\varphi(x+iy) = \begin{cases} -\psi(x) + (y^2+b)\psi''(x), & \text{when } |x| \geq X(y^2+b)\,, \\ -\psi(X(y^2+b)) & \\ \quad +(y^2+b)\psi''(X(y^2+b)), & \text{when } |x| < X(y^2+b)\,. \end{cases} \tag{7.15}$$

It follows from the definition of X that $\partial\varphi(x+iy)/\partial x$ is continuous, and we have

$$\partial^2\varphi(x+iy)/\partial x^2 = -\psi''(x) + (y^2+b)\psi^{(4)}(x) \quad \text{when } |x| > X(y^2+b)\,.$$

The right-hand side is an increasing function of y^2, and when $x = X(y^2+b)$ it is equal to

$$-\psi''(x) + \psi^{(4)}(x)\psi'(x)/\psi'''(x) = -\psi'''(x)d(\psi'(x)/\psi'''(x))/dx < 0$$

which proves that $\partial^2\varphi(x+iy)/\partial x^2 < 0$ when $|x| > X(y^2+b)$, hence that $x \mapsto \varphi(x+iy)$ is concave. We have $\Delta\varphi(x+iy) = \psi''(x) + (y^2+b)\psi^{(4)}(x) > 0$ when $x > X(y^2+b)$, and $V(y) = -\psi(X(y^2+b)) + (y^2+b)\psi''(X(y^2+b))$ is convex since

$$V'(y) = 2y\psi''(X(y^2+b))$$

is odd and increasing for $y > 0$. Since $\varphi \in C^1$ it follows that φ is subharmonic.

Proposition 7.6. *If $\psi \in C^4$ is even, and satisfies* (7.14) *on $[a, \infty)$, then the function φ defined by* (7.15) *is in $\mathcal{P}(\mathbb{C})$ if $b = \psi'(a)/\psi'''(a)$, and*

$$\varphi(x+iy) \leq -\tfrac{1}{2}\psi(x) + (y^2+b)\psi''(X(2(y^2+b)))\,, \tag{7.16}$$

if in addition $\psi'(x) \geq 0$ when $x \geq 0$.

PROOF. It just remains to prove (7.16). We have

$$\varphi(x+iy)+\tfrac{1}{2}\psi(x)=\tfrac{1}{2}(-\psi(x)+2(y^2+b)\psi''(x)),\qquad |x|>X(y^2+b)\,.$$

The right-hand side is concave and decreasing when $x > X(2(y^2+b))$, so the maximum of the left-hand side for $x\in\mathbb{R}$ is assumed when $X(y^2+b)\le x\le X(2(y^2+b))$, which proves (7.16).

In the example where $\psi(x)=\exp(|x|^\gamma)$ when $|x|>1$, for some $\gamma\in(0,1)$, we have $t(\gamma X(t)^{\gamma-1})^2\sim 1$ and $\psi''(X(t))\sim e^{X(t)^\gamma}/t\sim \exp((\gamma\sqrt{t})^{\gamma/(1-\gamma)})/t$, which means a very fast increase of the second term on the right-hand side of (7.16). This suggests that it is not possible for φ to decrease as fast as $-e^{|x|}$ on $\mathbb{R}$, and this will now be proved.

Let us first recall the explicit formula for the Poisson kernel in a strip. The strip $\Omega=\{z\in\mathbb{C};\,|\operatorname{Im} z|<\frac{1}{2}\pi\}$ is mapped to the right half plane by the map $z\mapsto w=e^z=e^xe^{iy}$. When $z=\xi\pm\pi i/2$ then $e^z=\pm ie^\xi$. Hence the formula for the Poisson kernel in the right half plane gives that the Poisson kernel in the strip is

$$\frac{\operatorname{Re} w}{\pi|w\mp ie^\xi|^2}e^\xi=\frac{e^{x+\xi}\cos y}{\pi|e^xe^{iy}\mp ie^\xi|^2}=\frac{\cos y}{2\pi(\cosh(x-\xi)\mp\sin y)}\,.$$

Green's function at $(z,\zeta)\in\Omega\times\Omega$, $z=x+iy$, $\zeta=\xi+i\eta$, is

$$\frac{1}{2\pi}\log\left|\frac{e^{x+iy}-e^{\xi+i\eta}}{e^{x+iy}+e^{\xi-i\eta}}\right|=\frac{1}{4\pi}\log\left|\frac{\cosh(x-\xi)-\cos(y-\eta)}{\cosh(x-\xi)+\cos(y+\eta)}\right|$$

Hence it follows from Riesz' representation formula that if v is a subharmonic function in Ω and $v(z)\le Ce^{a|\operatorname{Re} z|}$ in Ω for some $a<1$, then

$$\begin{aligned}v(x+iy)=\frac{1}{2\pi}\Bigg(&\int\frac{\cos y}{\cosh(x-\xi)-\sin y}v\big(\xi+\tfrac{1}{2}i\pi\big)\,d\xi\\&+\int\frac{\cos y}{\cosh(x-\xi)+\sin y}v\big(\xi-\tfrac{1}{2}i\pi\big)\,d\xi\\&+\tfrac{1}{2}\iint_\Omega\log\left|\frac{\cosh(x-\xi)-\cos(y-\eta)}{\cosh(x-\xi)+\cos(y+\eta)}\right|\,d\xi\,d\eta\Bigg)\,.\end{aligned}$$

Here $v\big(\xi\pm\frac{1}{2}\pi i\big)\,d\xi$ stands for a measure $\le Ce^{a|\xi|}\,d\xi$, but if $v(x+iy)$ is concave with respect to x it is of course a concave function. The global hypothesis is then automatically fulfilled if the boundary values exist. (See Lemma 4.5.) In particular, the two integrals converge, and we have

$$v(0)\le\frac{1}{2\pi}\int\big(v\big(\xi+\tfrac{1}{2}\pi i\big)+v\big(\xi-\tfrac{1}{2}\pi i\big)\big)\,d\xi/\cosh\xi\,.$$

With $v = \pm 1$ we conclude that $\int d\xi/\cosh\xi = \pi$, and since $v(\xi \pm \frac{1}{2}\pi i) \le v(\pm\frac{1}{2}\pi i) + c_\pm \xi$, it follows that each of the terms in the right-hand side gives a contribution $\le \frac{1}{2} v(\pm\frac{1}{2}\pi i)$, which agrees with the convexity. We also obtain

$$\sum_\pm \frac{1}{\pi} \int |v(\xi \pm \tfrac{1}{2}\pi i) - v(\pm\tfrac{1}{2}\pi i) - c_\pm \xi| \, d\xi/\cosh\xi \le v(\tfrac{1}{2}\pi i) + v(-\tfrac{1}{2}\pi i) - 2v(0). \tag{7.17}$$

Thus a local bound for the convexity in the y variable gives a global bound for the concavity in the x variable. Since the first factor of the integrand is an increasing function of $|\xi|$ on each half axis, and $\int_t^\infty d\xi/\cosh\xi \ge e^{-t}$ when $t > 0$, it follows that

$$|v(\xi \pm \tfrac{1}{2}\pi i) - v(\pm\tfrac{1}{2}\pi i) - c_\pm \xi| \le \pi e^{|\xi|}\big(v(\tfrac{1}{2}\pi i) + v(-\tfrac{1}{2}\pi i) - 2v(0)\big). \tag{7.18}$$

For a concave function f on $\mathbb{R}$ we have, with f'' denoting a negative measure,

$$f(\xi) - f(0) - f'(+0)\xi = \int_0^\xi f''(t)(\xi - t)\, dt, \quad \xi > 0,$$

hence

$$\begin{aligned} \int_0^\infty |f(\xi) - f(0) - c_+\xi| \, d\xi/\cosh\xi &\ge \iint_{0<t<\xi} |f''(t)||\xi - t| e^{-\xi}\, dt\, d\xi \\ &+ \int_0^\infty (c_+ - f'(+0)) e^{-\xi}\, d\xi = \int_{t>0} |f''(t)| e^{-t}\, dt \\ &+ (c_+ - f'(0)), \quad \text{if } c_+ \ge f'(+0), \end{aligned}$$

so we obtain from (7.17)

$$\frac{1}{\pi} \int_{\mathbb{R}} e^{-|\xi|} |dv'_\xi(\xi \pm \tfrac{1}{2}\pi i)| \le v(\tfrac{1}{2}\pi i) + v(-\tfrac{1}{2}\pi i) - 2v(0). \tag{7.19}$$

By a change of variables it follows that if $v \in \mathcal{P}(\mathbb{C})$ is finite when $y \le \operatorname{Im} z \le y + \pi a$ then

$$\begin{aligned} &\frac{1}{\pi} \int_{\mathbb{R}} e^{-|x|/a} |dv'_x(x + iy)| \\ &\le \big(v(i(y + \pi a)) + v(iy) - 2v\big(i\big(y + \tfrac{1}{2}\pi a\big)\big)\big)/a \\ &= \int_y^{y+\pi a} \big(\tfrac{1}{2}\pi a - |\eta - y - \tfrac{1}{2}\pi a|\big) dv'_\eta(0 + i\eta)/a. \end{aligned} \tag{7.20}$$

In particular this means that if $y \mapsto v(iy)$ is affine linear when $y \in I$, then $x \mapsto v(x + iy)$ is linear when $y \in I$ so that $v(x, y) = a(y)x + b(y)$ with b linear. The subharmonicity gives that a is also linear, hence $v(x, y) = axy + bx + cy + d$ when $y \in I$, with a change of notation. It is remarkable that flatness of v in y on a single vertical interval determines v almost completely in the corresponding horizontal strip.

If $v \in \mathcal{P}(\mathbb{C})$ is finite in $\mathbb{C}$ then it follows from (7.20) that $v(x + iy) = O(e^{\varepsilon|x|})$ for every $\varepsilon > 0$ when y is fixed. On the other hand, for arbitrary $\gamma \in (0, 1)$ we have constructed an example where $v(x + iy) \le -e^{|x|^\gamma} + h_\gamma(y)$ for some convex function h_γ.

8. - On the existence of functions in $\mathcal{S}_\varphi$

We recall that by Theorem 4.7 it suffices in principle to examine if $\mathcal{S}_\varphi$ is trivial when $\varphi \in \mathcal{P}$, although as we have seen in Section 7 it is not straightforward to pass from other functions φ to the largest minorant in $\mathcal{P}$. We shall in fact mainly restrict ourselves to the simplest case where φ is a subharmonic function in $\mathbb{C}$ which is concave in the real direction.

PROPOSITION 8.1. *Let φ be a subharmonic function in $\mathbb{C}$ and assume that there is an entire function $f \not\equiv 0$ and a positive number N such that $(1+|z|)^N|f(z)| \le e^{\varphi(z)}$ when $z \in \mathbb{C}$. Then it follows that the total mass of the positive measure $\Delta\varphi/2\pi$ is at least equal to N. If $f \in \mathcal{S}_\varphi$ then the mass is infinite.*

PROOF. There is nothing to prove unless the total mass of the measure $d\mu = \Delta\varphi/2\pi$ is finite. Then we can write

$$\varphi(z) = \int_{|\zeta|<1} \log|z-\zeta|\,d\mu(\zeta) + \int_{|\zeta|\ge 1} \log|1-z/\zeta|\,d\mu(\zeta) + h(z)$$

where h is harmonic, and we have

$$\log|f(z)| - h(z) \le \int_{|\zeta|<1} \log|z-\zeta|\,d\mu(\zeta) + \int_{|\zeta|\ge 1} \log|1-z/\zeta|\,d\mu(\zeta) - N\log(1+|z|)\,.$$

If we take the mean value over the circle $|z| = R$ in both sides it follows that

$$\begin{aligned}\log|f(0)| - h(0) &\le \log R \int_{|\zeta|<1} d\mu(\zeta) + \int_{1\le|\zeta|\le R} \log|R/\zeta|\,d\mu(\zeta) - N\log(1+R)\\ &< \log R\left(\int d\mu(\zeta) - N\right), \quad \text{if } R > 1\,.\end{aligned}$$

Hence it follows that $\log|f(0)| = -\infty$ if $\int d\mu(\zeta) < N$. Application of this to a translation of f gives $\log|f| \equiv -\infty$, so $f = 0$.

When trying to prove a converse we shall assume a polynomial bound for φ,

$$|\varphi(z)| \le C(1+|z|)^\gamma, \quad z \in \mathbb{C}, \tag{8.1}$$

for some positive constants C and γ. The difference quotients $(\varphi(x+iy) - \varphi(x+X+iy))/X$ and $(\varphi(x+iy+iY) - \varphi(x+iy))/Y$ are increasing functions of X and Y by the concavity and convexity in x and in y, if $\varphi \in \mathcal{P}$, so it follows from (8.1) that

$$|\varphi(z+w) - \varphi(z)| \le C'|w|(1+|z|)^{\gamma-1}, \quad z, w \in \mathbb{C}, \; |w| \le 1+|z|\,, \tag{8.2}$$

if φ is subharmonic and concave in the real direction. In what follows we shall only use the subharmonicity and the estimates (8.1) and (8.2).

PROPOSITION 8.2. *If φ is subharmonic in $\mathbb{C}$ and satisfies* (8.1), (8.2) *then one can find an entire function $f \not\equiv 0$ such that $|f(z)|(1+|z|)^N \le e^{\varphi(z)}$ provided that the total mass of $\Delta\varphi/2\pi$ is larger than $N+\gamma$.*

PROOF. For every continuous subharmonic φ and every $a > 0$ one can find an entire function $f \not\equiv 0$ such that

$$\int |f(z)|^2 e^{-2\varphi(z)}(1+|z|^2)^{-a-1}\, d\lambda(z) < \infty,$$

where $d\lambda$ is the Lebesgue measure. This is a special case of [6, Theorem 4.2.7] which is also applicable for n complex variables, with $-a-1$ replaced by $-a-n$. If $r > 0$ it follows that

$$|f(\zeta)| \le \frac{1}{\pi r^2}\int_{|z-\zeta|<r} |f(z)|\, d\lambda(z) \le Cr^{-1}\left(\int_{|z-\zeta|<r} e^{2\varphi(z)}(1+|z|^2)^{a+1}\, d\lambda(z)/r^2\right)^{\frac{1}{2}}.$$

If we choose $r = (1+|\zeta|)^{1-\gamma}$ and use (8.2), it follows that

$$|f(\zeta)| \le Ce^{\varphi(\zeta)}(1+|\zeta|)^{\gamma+a}$$

with a new constant C. To prove the proposition we must reduce the exponent $\gamma+a$ to $-N$. To do so we may assume that $a > 0$ is chosen so small that the total mass of $\Delta\varphi/2\pi$ exceeds $\gamma+a+N$ and can then choose a compact set K where the mass exceeds $\gamma+a+N$. Then

$$\varphi_1(z) = \varphi(z) - \int_K \log|z-\zeta|\, d\mu(\zeta)$$

is subharmonic, and $\varphi(z) \ge \varphi_1(z) + (\gamma+a+N)\log(1+|z|)$ for large $|z|$. It is clear that φ_1 also satisfies (8.1), (8.2) with some other constants outside a compact neighborhood of K. If we choose $f \not\equiv 0$ as above with $|f(z)| \le e^{\varphi_1(z)}(1+|z|)^{\gamma+a}$ when $|z|$ is large, it follows that $|f(z)|(1+|z|)^N \le e^{\varphi(z)}$ when $|z|$ is large. This completes the proof.

THEOREM 8.3. *If φ is subharmonic in $\mathbb{C}$ and satisfies* (8.1), (8.2), *then one can find an entire function $f \not\equiv 0$ such that $|z^j f^{(k)}(z)| \le C_{jk}e^{\varphi(z)}$ for arbitrary nonnegative integers j and k if and only if the total mass of $\Delta\varphi$ is infinite.*

PROOF. The necessity is a part of Proposition 8.1. By Cauchy's inequalities we have if $0 < r < 1+|z|$

$$\begin{aligned}|z^j f^{(k)}(z)|e^{-\varphi(z)}/k! &\le |z|^j \sup_{|z-\zeta|<r} |f(\zeta)|r^{-k}e^{-\varphi(z)} \\ &\le |z|^j \exp(C'(1+|z|)^{\gamma-1}r) \sup_{|z-\zeta|<r} |f(\zeta)|r^{-k}e^{-\varphi(\zeta)}.\end{aligned}$$

Choosing $r = (1+|z|)^{1-\gamma}$ we conclude that this is bounded if $|f(\zeta)|(1+|\zeta|)^{j+k(\gamma-1)}e^{-\varphi(\zeta)}$ is bounded. The theorem will therefore be proved if we can find an entire analytic function f such that $|z^j f(z)|e^{-\varphi(z)}$ is bounded for every j. We shall do so by means of a modification of the proof of Proposition 8.2, but we must now subtract from φ the potential of a measure which does not have compact support which requires a closer look at its continuity and asymptotic behavior.

As before we write $d\mu = \Delta\varphi/2\pi$. It follows from (8.2) that

$$\log 2 \int_{|w|<r/2} d\mu(z+w) \le \int_{|w|<r} \log(r/|w|)\, d\mu(z+w)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} (\varphi(z+re^{i\theta}) - \varphi(z))\, d\theta \le rC'(1+|z|)^{\gamma-1},$$

if $r \le 1+|z|$. We shall choose a positive measure $d\nu \le d\mu$ with $|\zeta| \ge 2$ when $\zeta \in \operatorname{supp} d\nu$ such that the mass in $\{\zeta; |\zeta| < R\}$ tends slowly to infinity when $R \to \infty$, and shall then argue as in the proof of Proposition 8.2 with

$$\varphi_1(z) = \varphi(z) - \int_{\mathbb{C}} \log|1 - z/\zeta|\, d\nu(\zeta)\,. \tag{8.3}$$

We interrupt the proof a moment to prove a lemma on the continuity and asymptotic properties of the potential term in (8.3) which will suggest how the measure $d\nu$ should be chosen.

LEMMA 8.4. *Let $d\nu$ be a positive measure in $\mathbb{C}$ such that $|\zeta| \ge 2$ when $\zeta \in \operatorname{supp} d\nu$, $\int_{\mathbb{C}} |\zeta|^{-1}\, d\nu(\zeta) < \infty$, and*

$$\int_{|w|<r} d\nu(z+w) \le C(1+|z|)^{\gamma-1} r, \quad \text{if } 2r < 1+|z|\,. \tag{8.4}$$

Then

$$v(z) = \int_{\mathbb{C}} \log|1 - z/\zeta|\, d\nu(\zeta), \quad z \in \mathbb{C}, \tag{8.5}$$

is a continuous subharmonic function, and for large $|z|$

$$|v(z+w) - v(z)| \le C'(1+|z|)^{\gamma-1}|w|(\log|z| + |\log|w||), \quad |w| \le \tfrac{1}{3}, \tag{8.6}$$

$$\begin{aligned} v(z) \ge &\int_{|2\zeta|<|z|} \log|z/2\zeta|\, d\nu(\zeta) \\ &- C'\left(1 + \log|z| \int_{|z|/2\le|\zeta|\le 2|z|} d\nu(\zeta) + |z| \int_{|\zeta|>2|z|} \frac{d\nu(\zeta)}{|\zeta|}\right). \end{aligned} \tag{8.7}$$

PROOF. If $0 < \varepsilon < 1$ then

$$\int_{|\zeta-z|<\varepsilon} |\log|\zeta - z||\, d\nu(\zeta) \le \sum_0^\infty \int_{\varepsilon/2\le 2^k|\zeta-z|<\varepsilon} |\log|\zeta - z||\, d\nu(\zeta)$$
$$\le C(1+|z|)^{\gamma-1} \sum_0^\infty (k+1+|\log\varepsilon|)2^{-k}\varepsilon\,,$$

which implies that

$$(8.8)\quad \int_{|\zeta-z|<\varepsilon} \big|\log|\zeta-z|\big|\,d\nu(\zeta) \le C_1(1+|z|)^{\gamma-1}\varepsilon(1+|\log\varepsilon|), \quad 0<\varepsilon<1.$$

This estimate and the fact that $\log|1-z/\zeta| = O(|z/\zeta|)$ as $z/\zeta \to 0$ proves that (8.5) converges to a continuous subharmonic function. To prove (8.6) we take $\varepsilon = 3|w|$ and observe that

$$\begin{aligned}
|v(z+w)-v(z)| &= \Big|\int_{\mathbb{C}} \log|(z+w-\zeta)/(z-\zeta)|\,d\nu(\zeta)\Big| \\
&\le \Big|\int_{|z-\zeta|<2|w|} \big|\log|z-\zeta|\big|\,d\nu(\zeta)\Big| \\
&\quad + \Big|\int_{|z+w-\zeta|<3|w|} \big|\log|z+w-\zeta|\big|\,d\nu(\zeta)\Big| \\
&\quad + \Big|\int_{|z-\zeta|\ge 2|w|} \log|(z+w-\zeta)/(z-\zeta)|\,d\nu(\zeta)\Big|.
\end{aligned}$$

The first two terms in the right-hand side are estimated by (8.8). In the third term we have $|w/(z-\zeta)| \le 1/2$, so it can be estimated by

$$C_2|w|\int_{|z-\zeta|>2|w|} \frac{d\nu(\zeta)}{|z-\zeta|}.$$

The integral when $1/|z-\zeta| \le 3/|\zeta|$ is bounded by hypothesis, and when $|z-\zeta| < |\zeta|/3$ then $2|\zeta|/3 < |z| < 4|\zeta|/3$, which implies $|z-\zeta| < |z|/2$. Now

$$\int_{2|w|<|z-\zeta|<|z|/2} \frac{d\nu(\zeta)}{|z-\zeta|} \le C(1+|z|)^{\gamma-1}\log(|z|/|w|)$$

by the argument which proved (8.8), and this completes the proof of (8.6).

To prove (8.7) we first observe that

$$\begin{aligned}
&\int_{2|\zeta|<|z|} \log|1-z/\zeta|\,d\nu(\zeta) \\
&\qquad = \int_{2|\zeta|<|z|} \log|z/\zeta|\,d\nu(\zeta) + \int_{2|\zeta|<|z|} \log|1-\zeta/z|\,d\nu(\zeta) \\
&\qquad \ge \int_{2|\zeta|<|z|} \log|z/2\zeta|\,d\nu(\zeta),
\end{aligned}$$

for $\log|1+w| = \operatorname{Re}\log(1+w) \ge \log(1/2)$ when $|w| < 1/2$. Similarly we obtain

$$\Big|\int_{|\zeta|>2|z|} \log|1-z/\zeta|\,d\nu(\zeta)\Big| \le |z|\log 4\int_{|\zeta|>2|z|} \frac{d\nu(\zeta)}{|\zeta|}.$$

Finally we have

$$\int_{|z|/2\le|\zeta|\le 2|z|} \log|1-z/\zeta|\,d\nu(\zeta)$$
$$\ge \int_{|\zeta-z|\le 1} \log|\zeta-z|\,d\nu(\zeta) - \int_{|z|/2\le|\zeta|\le 2|z|} \log|\zeta|\,d\nu(\zeta)\,.$$

If we restrict the integration in the first term on the right-hand side to the set where $|\zeta - z| < (1+|z|)^{-\gamma}$ a bound is given by (8.8). In the rest of the integral there is a bound for $\log|\zeta - z|/\log|z|$, and this completes the proof of (8.7).

End of proof of Theorem 8.3. Starting with $R_0 = 1$ we define an increasing sequence $R_0, R_1, \dots$, such that $R_{j+1} \ge 4R_j$ and the mass of $d\mu$ in $\Omega_j = \{\zeta; 4R_j < |\zeta| < R_{j+1}\}$ is at least equal to 1. This is possible since $d\mu$ has infinite mass. Then we choose a positive measure $d\nu \le d\mu$ with mass exactly equal to 1 in each of the annuli Ω_j and no mass elsewhere. Then $\int d\nu(\zeta)/|\zeta| \le \sum 1/4R_j < \infty$ since $R_j \ge 4^j R_0$, and (8.4) is valid for $d\nu$ since it is valid for $d\mu$. It follows from (8.6) that $|v(z+w) - v(z)| \le C$ when $|w| \le (1+|z|)^{-\gamma}$ and $|z|$ is large. As in the proof of Proposition 8.2 we can therefore find an analytic function $f \not\equiv 0$ such that

$$|f(z)| \le Ce^{\varphi_1(z)}(1+|z|)^{\gamma+1+a}, \quad z \in \mathbb{C},$$

where $a > 0$ is fixed. Thus

$$(1+|z|)^N|f(z)|e^{-\varphi(z)} \le C(1+|z|)^{N+\gamma+1+a}e^{-v(z)}\,,$$

so the theorem will be proved if we verify that (8.7) implies

(8.9) $$v(z)/\log|z| \to \infty\,, \quad \text{when } z \to \infty\,.$$

It is clear that

$$\int_{2|\zeta|<|z|} \log|z/2\zeta|\,d\nu(\zeta)/\log|z| \to \infty, \quad \text{when } z \to \infty\,,$$

and $\int_{|z/2|\le|\zeta|\le 2|z|} d\nu(\zeta) \le 1$ since the integration can only be taken over one Ω_j. Given z with $|z|$ large let k be the smallest integer such that $2|z| < R_{k+1}$. Then

$$|z|\int_{\zeta\in\Omega_k,|\zeta|>2|z|} \frac{d\nu(\zeta)}{|\zeta|} \le \tfrac{1}{2}, \quad \sum_{j=k+1}^{\infty}\int_{\Omega_j}\frac{d\nu(\zeta)}{|\zeta|} \le \sum_{j=k+1}^{\infty}\frac{1}{4R_j} \le \frac{1}{3R_{k+1}} < \frac{1}{6|z|}\,.$$

Hence $|z|\int_{|\zeta|>2|z|} d\nu(\zeta)/|\zeta| < 1$ which completes the proof that (8.9) follows from (8.7).

In the case of several variables and functions $\varphi \in \mathcal{P}$ which take the value $+\infty$ also one can still use [6, Theorem 4.2.7] to construct analytic functions bounded by $e^{\varphi(z)}(1+|z|)^M$ for some M. To obtain functions in $\mathcal{S}_\varphi$ it is sufficient to know that there is another function $\varphi_1 \in \mathcal{P}$ such that $\varphi_1(z) + N\log(1+z) \leq C_N + \log\varphi(z)$ for arbitrary N. This is the case for example if φ is one of the functions given by Corollary 7.2.

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