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On inhomogeneous incompressible fluids and reverse Hölder inequalities

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On Inhomogeneous Incompressible Fluids and Reverse Hölder Inequalities

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1. - On a new proof of the global unique solvability for the two-dimensional problems

In [1] for the system

$$(1.1_1) \quad \rho(v_t + v \cdot \nabla v) - \nu \Delta v + \nabla p = \rho f, \quad \operatorname{div} v = 0, \quad \int_\Omega p(x,t)\, dx = 0,$$

and

$$(1.1_2) \quad \rho_t + v \cdot \nabla \rho = 0,$$

in $Q_T = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, under the first boundary condition

$$(1.2) \quad v\big|_{\partial'' Q_T} = 0, \quad \partial'' Q_T = \partial\Omega \times [0, T],$$

and the initial conditions

$$(1.3_1) \quad v\big|_{t=0} = v^0,$$

$$(1.3_2) \quad \rho\big|_{t=0} = \rho^0, \quad 0 < M_1 \leq \rho^0(x) \leq M_2 < \infty,$$

the theorems on unique solvability, which are similar to the theorems established in [2]-[5] for the case of homogeneous fluids (i.e. for $\rho = \text{const} > 0$) were proved. For the two- dimensional problems it was done in arbitrary interval of time without any smallness restrictions on data. But for the three-dimensional problems the time-interval of existence depended on the values of some norms of data. The solutions $\{v, p, \rho\}$ were found in the spaces $\overset{\circ}{J}{}^{2,1}_q(Q_T) \times W^{1,0}_q(Q_T) \times C^1(\overline{Q}_T) = \overset{\circ}{A}{}^q_T$ with $q > n$. In [1] and in this paper Ω is a bounded domain. For unbounded domains Ω everything can be done using similar techniques. The same results hold for the periodic boundary conditions when Ω is a parallelepiped or a rectangle. In particular, there exists a unique global solution $(v, p, \rho) \in A_T^q = J_q^{2,1}(Q_T) \times W_q^{1,0}(Q_T) \times C^1(\overline{Q}_T)$ of the system (1.1_k), $k = 1, 2$, for $\Omega = (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$, which satisfies the initial condition (1.3_k), $k = 1, 2$, and the periodic boundary conditions

$$(1.4_1) \qquad v\big|_{x_k=-1} = v\big|_{x_k=1}, \quad v_x\big|_{x_k=-1} = v_x\big|_{x_k=1}, \quad p\big|_{x_k=-1} = p\big|_{x_k=1},$$

$$(1.4_2) \qquad \rho\big|_{x_k=-1} = \rho\big|_{x_k=1}, \quad \rho_x\big|_{x_k=-1} = \rho_x\big|_{x_k=1}, \quad k = 1, 2.$$

We suppose in this case that f, v^0 and ρ^0 are periodic functions in x_1 and x_2 with period 2 and $\{f, v^0, \rho^0\} \in L_q(\widetilde{\Omega} \times (0, T)) \times J_q^{2-\frac{2}{q}}(\widetilde{\Omega}) \times C^1(\widetilde{\Omega} \times [0, T])$ for any bounded domain $\widetilde{\Omega} \subset \mathbb{R}^2$. In this paper we use the notations which are close to those from [1] and [4], [6]. Let us remind some of them.

$L_q(Q)$ is the space of functions $v : Q \to \mathbb{R}^m$ for which $| v |^q$ is Lebesque integrable function on Q; $\| \cdot \|_{q,Q}$ is the norm in $L_q(Q)$. If $v \in L_q(Q_T)$, and $Q_T = \Omega \times (0, T)$, then $\|v(t)\|_{q,\Omega} \equiv \big(\int_\Omega | v(x, t) |^q \, dx \big)^{\frac{1}{q}} < \infty$ for almost all $t \in (0, T)$ and $\|v\|_{q,Q_T} = \big(\int_0^T \|v(t)\|_{q,\Omega}^q dt \big)^{\frac{1}{q}}$. $W_q^l(\Omega)$ is the Sobolev space with the norm $\| \cdot \|_{q,\Omega}^{(l)}$, for natural l it is

$$\|u\|_{q,\Omega}^{(l)} = \sum_{0 \le |k| \le l} \|\partial_x^k u\|_{q,\Omega},$$

and for other l the definitions of $W_q^l(\Omega)$ and $\| \cdot \|_{q,\Omega}^{(l)}$ can be found in [1] or [6].

The elements v of $W_q^{2,1}(Q_T)$ have the finite norm

$$\|v\|_{q,Q_T}^{(2,1)} = \|v_{xx}\|_{q,Q_T} + \|v_t\|_{q,Q_T} + \sup_{t \in [0,T]} \|v(t)\|_{q,\Omega}^{\left(2-\frac{2}{q}\right)}$$

and the elements p of $W_q^{1,0}(Q_T)$ have the finite norm

$$\|p\|_{q,Q_T}^{(1,0)} = \|p_x\|_{q,Q_T} + \|p\|_{q,Q_T}.$$

$J_q^{2,1}(Q_T) = \{v \in W_q^{2,1}(Q_T) : \operatorname{div} v = 0\}$ is a subspace of $W_q^{2,1}(Q_T)$. We use also two subspaces of $J_q^{2,1}(Q_T)$:

$$\overset{\circ}{J}{}_q^{2,1}(Q_T) = \left\{ v \in J_q^{2,1}(Q_T) \colon v\big|_{\partial'' Q_T} = 0 \right\}$$

and

$$\hat{J}_q^{2,1}(Q_T) = \left\{ v \in J_q^{2,1}(Q_T) : v \text{ satisfies the boundary conditions from } (1.4_1) \right\},$$

if Ω is the square. $C^1(\overline{Q}_T)$ is the space of continuously differentiable functions $v : \overline{Q}_T \to \mathbb{R}^m$ with the standard norm

$$\|v\|_{C^1(\overline{Q}_T)} = \max_{\overline{Q}_T} | v(x,t) | + \max_{\overline{Q}_T} | v_x(x,t) | + \max_{\overline{Q}_T} | v_t(x,t) | \ .$$

$C^\alpha(\overline{Q}_T)$ is the space of Hölder continuous functions on $\overline{Q}_T$ with the Hölder exponent $\alpha \in (0,1)$ and the standard norm $\|\cdot\|_{C^\alpha(\overline{Q}_T)}$.

We use for the norms in $C(\overline{Q})$ and in $L_\infty(Q)$ the symbol $\|\cdot\|_{\infty,Q}$ and denote parabolic boundary of Q_T as $\partial' Q_T$.

The aim of this paper is to simplify the considerations of [1] for the two-dimensional case. It is attained with the help of the following results about solutions $\{v, p\}$ of linear problem

$$\rho v_t - \nu \Delta v + \nabla p = \Phi, \quad \operatorname{div} v = 0, \quad \int_\Omega p(x,t)\,dx = 0\,, \tag{1.5}$$

where ρ is a known continuous function satisfying the inequalities

$$0 < M_1 \le \rho(x,t) \le M_2\,. \tag{1.6}$$

THEOREM 1.1. *Let $\{v, p\}$ belong to $J_2^{2,1}(Q_T) \times W_2^{1,0}(Q_T)$ and satisfy the system (1.5) with $\Phi \in L_q(Q_T)$, $q > 2$, and ρ belonging to $C^\alpha(\overline{Q}_T)$, $\alpha > 0$, and satisfying (1.6). Then there exists a number $s \in (2, q]$ such that $v_{xx} \in L_s(Q')$ for any $Q' \subset Q_T$ with $\operatorname{dist}(Q', \partial' Q_T) > 0$ and $\|v_{xx}\|_{s,Q'}$ can be majorized by a number determined by $\|v\|^{(2,1)}_{2,Q_T}$, M_1^{-1}, M_2, $\|\Phi\|_{s,Q_T}$, $\|\rho\|_{C^\alpha(\overline{Q}_T)}$ and $[\operatorname{dist}\{Q', \partial' Q_T\}]^{-1}$.*

If, additionally, $v^0 = v\big|_{t=0} \in J_q^{2-\frac{2}{q}}(\Omega)$ and v satisfies a correct homogeneous boundary condition, then $v_{xx} \in L_s(Q_T)$ and

$$\|v_{xx}\|_{s,Q_T} \le c\left[1 + \|\Phi\|_{s,Q_T}\right] \,. \tag{1.7}$$

where constant c is defined by $\|v\|^{(2,1)}_{2,Q_T}$, M_1^{-1}, M_2, $\|\rho\|_{C^\alpha(\overline{Q}_T)}$, $\|v^0\|^{\left(2-\frac{2}{q}\right)}_{q,\Omega}$ and some numerical characteristics of $\partial\Omega$ (these are C^2-norm of $\partial\Omega$ for the boundary condition (1.2) and the side lengths of the rectangle Ω for the periodic boundary conditions).

Note, that the dimension of Ω in Theorem 1.1 can be arbitrary. In Section 2 we prove the first part of Theorem 1.1 and the second part for the conditions (1.4_1). For conditions (1.2) the proof of (1.7) is more complicated.

In this section we explain how a global unique solvability for the two-dimensional problems can be deduced using (1.7) and some results from [1]. According Section 4 of [1], it is enough to get for any solution $\{v, p, \rho\} \in A_T^q$, $q > 2$, of the problem the following apriori estimate

$$\|v\|^{(2,1)}_{q,Q_T} + \|p\|^{(1,0)}_{q,Q_T} + \|\rho\|_{C^1(\overline{Q}_T)} \le \mu_1\left(M_0, M_1^{-1}, M_2, M_3, T\right) \equiv \mu_1\,, \tag{1.8}$$

in which $M_0 = \|f\|_{q,Q_T} + \|v^0\|_{q,\Omega}^{(2-\frac{2}{q})}$, M_1 and M_2 are the constants from (1.6), $M_3 = \|\rho_x^0\|_{\infty,\Omega}$ and μ_1 is a continuous nondecreasing function of indicated arguments. We will denote the majorants of such type depending on the same arguments by μ_k with different k.

We take from [1], Section 4, the estimates

$$\|v\|_{2,Q_T}^{(2,1)} + \|p\|_{2,Q_T}^{(1,0)} \leq \mu_2 \tag{1.9}$$

and

$$\|\rho\|_{C^\alpha(\overline{Q}_T)} \leq \mu_3 \tag{1.10}$$

with some $\alpha > 0$. Their derivation is comparatively easy.

Now we consider $\{v, p\}$ as a solution of the linear system (1.5) with

$$\Phi = \rho(f - v \cdot \nabla v) \tag{1.11}$$

and address (1.7). The norm $\|\Phi\|_{s,Q_T}$ can be estimated with the help of the multiplicative inequality

$$\|u\|_{2m,\Omega} \leq \beta_1 \|u_x\|_{m,\Omega}^{\frac{1}{2}} \|u\|_{2,\Omega}^{\frac{1}{2}} + \beta_2 \|u\|_{2,\Omega}, \quad \forall m < \infty . \tag{1.12}$$

It is true for any $u \in W_m^1(\Omega)$ and $\Omega \subset \mathbb{R}^2$ ([7], Ch. II).

We remind that for $u \in \overset{\circ}{W}{}_m^1(\Omega)$ number $\beta_2 = 0$ and β_1 does not depend on Ω.

Besides (1.12) we use the following consequences of (1.9):

$$\sup_{t\in[0,T]} \|v(t)\|_{2,\Omega}^{(1)} \leq \mu_2, \quad \sup_{t\in[0,T]} \|v(t)\|_{2m,\Omega} \leq c\mu_2 , \quad c = c(2m) . \tag{1.13}$$

Due to (1.12) and (1.13)

$$\begin{aligned} \|v \cdot \nabla v\|_{m,Q_T} &\leq \left(\int_0^T \|v(t)\|_{2m,\Omega}^m \|v_x(t)\|_{2m,\Omega}^m dt \right)^{\frac{1}{m}} \\ &\leq c\mu_2 \left[\int_0^T (\beta_1 \|v_{xx}(t)\|_{m,\Omega}^{\frac{1}{2}} \|v_x(t)\|_{2,\Omega}^{\frac{1}{2}} + \beta_2 \|v_x(t)\|_{2,\Omega})^m dt \right]^{\frac{1}{m}} \\ &\leq c_1\mu_2 \left[\int_0^T \|v_{xx}(t)\|_{m,\Omega}^{\frac{m}{2}} \|v_x(t)\|_{2,\Omega}^{\frac{m}{2}} dt \right]^{\frac{1}{m}} + c_2\mu_2^2 T^{\frac{1}{m}} \\ &\leq \mu_3 \varepsilon \|v_{xx}\|_{m,Q_T} + \mu_4(1 + \varepsilon^{-1}), \quad \forall \varepsilon > 0 . \end{aligned} \tag{1.14}$$

Using (1.14) with $m = s$, we get

$$\|\Phi\|_{s,Q_T} \le M_2 \left[\|f\|_{s,Q_T} + \mu_3 \varepsilon \|v_{xx}\|_{s,Q_T} + \mu_4 \left(1 + \varepsilon^{-1}\right)\right]. \tag{1.15}$$

Taking into account (1.9), (1.10) and (1.15) we derive from (1.7)

$$\|v_{xx}\|_{s,Q_T} \le \mu_5 \left[1 + \|f\|_{s,Q_T} + \mu_3 \varepsilon \|v_{xx}\|_{s,Q_T} + \mu_4 \left(1 + \varepsilon^{-1}\right)\right]. \tag{1.16}$$

This inequality with $\varepsilon = (2\mu_5\mu_3)^{-1}$ gives the estimate

$$\|v_{xx}\|_{s,Q_T} \le \mu_6, \tag{1.17}$$

with some $s \in (2, q]$ and μ_6 under control.
From it the estimate

$$\int_0^T \|v_x(t)\|_{\infty,\Omega} dt \le \mu_7 \tag{1.18}$$

follows. It allows to address Lemma 1.3 of [1] and do the conclusion

$$\|\rho_x\|_{\infty,Q_T} \le \mu_8. \tag{1.19}$$

To get a new information on p_x and v_t we apply the operator div to the system (1.1_1) and obtain

$$\Delta p = \operatorname{div} h + g, \quad x \in \Omega, \tag{1.20}$$

where $h = \rho f$, $g = -\nabla\rho \cdot (v_t + v \cdot \nabla v) - \rho v_{kx_i} v_{ix_k}$ and v_k are the components of vector-function v. It is easy to see that $h(t) \in L_q(\Omega)$ and $g(t) \in L_2(\Omega)$ for almost all $t \in (0, T)$ and

$$\begin{aligned} &\|h(t)\|_{q,\Omega} \le M_2 \|f(t)\|_{q,\Omega}, \\ &\|g(t)\|_{2,\Omega} \le \mu_8 \left[\|v_t(t)\|_{2,\Omega} + \|v(t) \cdot \nabla v(t)\|_{2,\Omega}\right] + M_2 \|v_x(t)\|^2_{4,\Omega}. \end{aligned} \tag{1.21}$$

In the case of boundary conditions (1.4_1) we represent p as a sum $p_1 + p_2$, where p_k are the solutions of following equations

$$\Delta p_1 = \operatorname{div} h, \tag{1.22_1}$$

$$\Delta p_2 = g, \tag{1.22_2}$$

and p_k, p_{kx} are periodic and

$$\int_\Omega p_i(x, t)\, dx = 0, \quad i = 1, 2. \tag{1.23}$$

For p_1 the estimate

$$\|p_{1x}(t)\|_{q,\Omega} \le c\|h(t)\|_{q,\Omega}$$

is well known. It gives

$$\|p_{1x}\|_{q,2,Q_T} \le cM_2\|f\|_{q,2,Q_T}\,. \tag{1.24}$$

For solution p_2 of (1.22_2), (1.23) we have

$$\|p_{2xx}(t)\|_{2,\Omega} \le c\|g(t)\|_{2,\Omega}\,.$$

This estimate and the embedding theorem $W_2^1(\Omega)$ in $L_r(\Omega)$ for all $r < +\infty$ allow to do the conclusion

$$\|p_{2x}(t)\|_{r,\Omega} \le c\|g(t)\|_{2,\Omega}, \quad \forall r < \infty\,. \tag{1.25}$$

(Here, as above, we don't point out the dependence of majorants on the parameters contained in the definitions of norms.)

It follows from (1.25), (1.21) that

$$\|p_{2x}\|_{r,2,Q_T} \le j_1 + j_2\,,$$

where $j_1 = c\mu_8[\|v_t\|_{2,Q_T} + \|v\cdot\nabla v\|_{2,Q_T}]$ and $j_2 = cM_2\|v_x\|^2_{4,Q_T}$. The value of j_1 is estimated by a majorant μ_9 due to (1.14) with $m = 2$ and (1.9). From (1.12) for $u = v_x$, $m = 2$, and estimates (1.9), (1.13) it follows $j_2 \le \mu_{10}$, and therefore

$$\|p_{2x}\|_{r,2,Q_T} \le \mu_{11}, \quad \forall r < \infty\,. \tag{1.27}$$

Now for $p = p_1 + p_2$ we derived from (1.24) and (1.27) an estimate for $\|p_x\|_{q,2,Q_T}$ and also the estimate

$$\|p_x\|_{s,2,Q_T} \le \mu_{12}\,, \tag{1.28}$$

where s is the same as in (1.17).

To estimate $\|v_t\|_{s,2,Q_T}$ we use system (1.1_1) and the fact that norms $\|v\cdot\nabla v\|_{s,2,Q_T}$ and $\|\Delta v\|_{s,2,Q_T}$ can be estimated by some majorants μ_{13} due to (1.14) for $u = v_x$, $m = s$, $\varepsilon = 1$ and (1.17).

Therefore

$$\|v_t\|_{s,2,Q_T} \le \left(1 + \nu M_1^{-1}\right)\mu_{13} + M_1^{-1}\mu_{12} + \|f\|_{s,2,Q_T} \equiv \mu_{14}\,. \tag{1.29}$$

The information (1.17), (1.29) and (1.9) allows to conclude that

$$\|v\|_{\infty,Q_T} \le \mu_{15}\,. \tag{1.30}$$

This fact is known and can be easily derived with the help of fundamental solution of heat equation.

From equation (1.1_2) and (1.30), (1.19) it follows

$$\|\rho_t\|_{\infty,Q_T} \le \mu_{16}\,. \tag{1.31}$$

We use also Theorem 2.1 from [1], in which the global unique solvability in $\overset{\circ}{J}{}^{2,1}_q(Q_T) \times W_q^{1,0}(Q_T)$ of the problem (1.5), (1.2), (1.3_1) with $\rho \in C^1(\overline{Q}_T)$ and zero initial data was proved. With the help of this theorem it is easy to prove the following statement.

THEOREM 1.2. *Let* ρ *in* (1.5) *belong to* $C^1(\overline{Q}_T)$ *and satisfy* (1.6). *Then for any* $\{\Phi, v^0\} \in L_q(Q_T) \times \overset{\circ}{J}{}_q^{2-\frac{2}{q}}(\Omega)$, $q > 1$, *there exists a unique solution* $\{v, p\} \in \overset{\circ}{J}{}_q^{2,1}(Q_T) \times W_q^{1,0}(Q_T)$ *of problem* (1.5), (1.2), *and* (1.3$_1$). *For it the estimate*

$$\|v\|_{q,Q_T}^{(2,1)} + \|p_x\|_{q,Q_T} \le c_1 \left[\|v^0\|_{q,\Omega}^{\left(2-\frac{2}{q}\right)} + \|\Phi\|_{q,Q_T} \right] \tag{1.32}$$

is valid with some constant c_1 *depending only on* T, M_1^{-1}, M_2, $\|\rho_x\|_{\infty,Q_T}$ *and* $\|\rho_t\|_{\infty,Q_T}$.

Analogously the next assertion is proved.

THEOREM 1.2$'$. *Let in problem* (1.5), (1.4$_1$), (1.3$_1$), $\Omega = (-1, 1) \times (-1, 1)$, ρ *belong to* $C^1(\overline{Q}_T)$ *and satisfy* (1.6) *and* (1.4$_2$). *Then for any* $\{f, v^0\} \in L_q(Q_T) \times \hat{J}_q^{2-\frac{2}{q}}(\Omega)$, $q > 1$, *there exists a unique solution* $\{v, p\} \in \hat{J}_q^{2,1}(Q_T) \times W_q^{1,0}(Q_T)$ *of problem* (1.5), (1.4$_1$), (1.3$_1$). *The estimate* (1.32) *is valid for it.*

Let us take the pair $\{v, p\}$ from the solution $\{v, p, \rho\}$ under consideration. This pair $\{v, p\}$ is a solution of (1.5) with Φ defined by (1.11) and we can apply Theorem 1.2 or 1.2$'$ correspondingly. Due to (1.6) and (1.19), (1.31) the constant c_1 in (1.32) is a majorant under our control, therefore we denote it by μ_{17}. The norm $\|\Phi\|_{q,Q_T}$ can be estimated with the help of (1.14) for $m = q$. Namely,

$$\|\Phi\|_{q,Q_T} \le M_2\Big[\|f\|_{q,Q_T} + \|v \cdot \nabla v\|_{q,Q_T}\Big] \le M_2\Big[\|f\|_{q,Q_T} + \mu_3\varepsilon\|v_{xx}\|_{q,Q_T} + \mu_4\Big(1 + \varepsilon^{-1}\Big)\Big].$$

From (1.32) and the last inequality with $\varepsilon \ll 1$ we derive

$$\|v\|_{q,Q_T}^{(2,1)} + \|p_x\|_{q,Q_T} \le \mu_{18}. \tag{1.33}$$

This estimate together with (1.19) and (1.31) supply estimate (1.8) with some majorant μ_1.

As is was said earlier, estimate (1.8) guaranties a global unique solvability of the two-dimensional problems. In this paper we realized our program for the boundary condition (1.4$_k$), $k = 1, 2$.

2. – On the summability of v_{xx} with power $s > 2$

Let $\Omega = (-1, 1) \times (-1, 1)$, $Q_T = \Omega \times (0, T)$, and $\{v, p\} \in J_2^{2,1}(Q_T) \times W_2^{1,0}(Q_T)$ be a solution of

$$\rho v_t - \nu\Delta v + \nabla p = \Phi, \quad \operatorname{div} v = 0, \quad \int_\Omega p(x, t)\, dx = 0, \tag{2.1}$$

under periodic boundary conditions

$$v\big|_{x_k=-1} = v\big|_{x_k=1}, \quad v_x\big|_{x_k=-1} = v_x\big|_{x_k=1}, \tag{2.2}$$
$$p\big|_{x_k=-1} = p\big|_{x_k=1}, \qquad k = 1, 2,$$

and initial condition

$$v\big|_{t=0} = v^0. \tag{2.3}$$

We assume

$$0 < M_1 \le \rho(x,t) \le M_2, \quad \rho \in C^\alpha\left(\overline{Q}_T\right), \quad \alpha > 0, \tag{2.4}$$
$$\rho\big|_{x_k=-1} = \rho\big|_{x_k=1}, \quad k = 1, 2,$$

$$\Phi \in L_q(Q_T), \quad v^0 \in \hat{J}_2^{2-\frac{2}{q}}(\Omega), \ q > 2, \tag{2.5}$$

(2.6) ρ, Φ, v^0 are periodic functions of x_1, x_2 with the period 2; they are determined for all $x \in \mathbb{R}^2$.

Function

$$\hat{v}(x,t) = (4\pi t)^{-1} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}} v^0(y)\, dy$$

is the solution of the heat equation, $\operatorname{div} \hat{v} = 0$ and $\hat{v}\big|_{t=0} = v^0$. Moreover,

$$\|\hat{v}\|^{(2,1)}_{q,Q_T} \le c\|v^0\|^{\left(2-\frac{2}{q}\right)}_{q,\Omega}. \tag{2.7}$$

(see, for example, [6], Ch. IV). We put $v - \hat{v} = u$ and transform (2.1)-(2.3) to the following problem with zero initial condition

$$\rho u_t - \nu\Delta u + \nabla p = F, \qquad \operatorname{div} u = 0, \qquad \int_\Omega p(x,t)\, dx = 0, \tag{2.8}$$

$$u\big|_{x_k=-1} = u\big|_{x_k=1}, \qquad u_x\big|_{x_k=-1} = u_x\big|_{x_k=1}, \tag{2.9}$$
$$p\big|_{x_k=-1} = p\big|_{x_k=1}, \qquad k = 1, 2$$

$$u\big|_{t=0} = 0. \tag{2.10}$$

where $F = \Phi - \rho\hat{v}_t + \nu\Delta\hat{v}$.

We will prove for this problem the following statement.

THEOREM 2.1. *Assume that* $F \in L_q(Q_T)$, $q > 2$, *and* ρ *satisfies* (2.4). *Let* $\{u, p\} \in \hat{J}_2^{2,1}(Q_T) \times W_2^{1,0}(Q_T)$ *be a solution of* (2.8)-(2.10). *Then there exists* $s \in (2, q]$ *such that* $u_{xx} \in L_s(Q_T)$ *and*

$$(2.11) \qquad \|u_{xx}\|_{s,Q_T} \le c\left[\|u_{xx}\|_{2,Q_T} + \langle\rho\rangle^{(\alpha)}_{\overline{Q}_T}\|u_t\|_{2,Q_T} + \|u\|_{2,Q_T} + \|F\|_{s,Q_T}\right],$$

where $\langle\rho\rangle^{(\alpha)}_{\overline{Q}_T}$ *is the Hölder constant of* ρ *in* $\overline{Q}_T$ *and the constant* c *depends on* ν, M_1^{-1}, M_2 *and* T.

Let us prolong functions F, u and p on $\mathbb{R}^2 \times (-\infty, T]$ as 2-periodic functions in $x = (x_1, x_2)$ and as zero for $(x, t) \in \mathbb{R}^2 \times (-\infty, 0]$. The function ρ we prolong on $\mathbb{R}^2 \times (-\infty, T]$ putting $\rho(x, t) = \rho(x, 0)$ for $t \in (-\infty, 0]$ and 2-periodic in x. We preserve the notations for all these functions. It is easy to see that $\{u, p\} \in J_2^{2,1}(\widetilde{Q}) \times W_2^{1,0}(\widetilde{Q})$, $\rho \in C^\alpha(\widetilde{Q})$ and $F \in L_q(\widetilde{Q})$ for any bounded cylinder $\widetilde{Q} \subset \mathbb{R}^2 \times (-\infty, T]$ and $\{u, p\}$ satisfies the system (2.8) for almost all $(x, t) \in \mathbb{R}^2 \times (-\infty, T)$.

We define parabolic cylinders

$$Q_R(z^0) = \left\{z = (x, t) : x \in B_R(x^0),\ t \in \left(t^0 - R^2, t^0\right)\right\},$$

where $B_R(x^0) = \{x \in \mathbb{R}^2 : |x - x^0| < R\}$.

Let $\delta(z^1, z^2) = \max\{|\, x^1 - x^2\,|, |\, t^1 - t^2\,|^{\frac{1}{2}}\}$ be the parabolic metric in $\mathbb{R}^{n+1}$.

Now we fix R_0 in such a way that $Q_T \subset Q_{R_0}(z^*)$, $z^* = (0, T)$, and put $Q' = Q_{R_0}(z^*)$, $Q = Q_{4R_0}(z^*)$.

For the solution $\{u, p\} \in J_2^{2,1}(Q) \times W_2^{1,0}(Q)$ of the system (2.8) we will prove that for some $s \in (2, q]$ $\nabla r(u) \in L_s(Q')$ and

$$(2.12) \qquad \|\nabla r(u)\|_{s,Q'} \le c\left(\|F\|_{s,Q} + \|\nabla r(u)\|_{2,Q} + \langle\rho\rangle^{(\alpha)}_{\overline{Q}_T}\|u_t\|_{2,Q}\right),$$

where $r(u) = \operatorname{rot} u = u_{2x_1} - u_{1x_2}$. The constant c in (2.12) depends on M_1^{-1}, ν and value $R_0 + R_0^{-1}$.

Taking into account the periodicity of the data and the solution, from (2.12) and the known estimate

$$\|u_{xx}\|_{s,\Omega} \le c\left(\|\nabla r(u)\|_{s,\Omega} + \|u\|_{2,\Omega}\right)$$

for any periodic solenoidal field, we deduce (2.11).

So the proof of Theorem 2.1 has been reduced to the proof of estimate (2.12). The last is based on two assertions, which are formulated below as Lemma 2.1 and 2.2.

LEMMA 2.1. *Let* $\{u, p\}$ *be a solution of system* (2.8) *from* $J_2^{2,1}(Q) \times W_2^{1,0}(Q)$, *let* $F \in L_2(Q)$ *and* $\rho \in C^\alpha(\overline{Q})$, $\rho(x, t) \ge M_1 > 0$ *in* $\overline{Q}$. *Then*

$$(2.13) \qquad \begin{aligned} \|\nabla r(u)\|^2_{2,Q_R(z^0)} \le \theta\|\nabla r(u)\|^2_{2,Q_{4R}(z^0)} + c\Big[&\theta^{-1}R^{-4}\|\nabla r(u)\|^2_{1,Q_{4R}(z^0)} \\ &+ \|F\|^2_{2,Q_{4R}(z^0)} + \left(\langle\rho\rangle^{(\alpha)}_{\overline{Q}}R^\alpha\|u_t\|_{2,Q_{4R}(z^0)}\right)^2\Big], \end{aligned}$$

for any cylinder $Q_{4R}(z^0) \subset Q$ *and any* $\theta > 0$; *the constant* c *depends on* ν, M_1^{-1}.

PROOF. Let $\zeta = \zeta(x)$ be a cut-off function for $B_1(0)$, $\zeta(x) = 1$ in $B_{\frac{1}{2}}(0)$, $|\nabla\zeta(x)| \leq 4$. Put $\varkappa_\zeta = \int_{B_1} \zeta^2(x)\,dx$. Then function $\zeta_R(x) = \zeta\left(\frac{x}{R}\right)$ is the cut-off function for $B_R(0)$ and

$$\int_{\mathbb{R}^2} \zeta_R^2(x)\,dx = \int_{B_R(0)} \zeta^2\left(\frac{x}{R}\right)dx = \varkappa_\zeta R^2, \quad |\nabla\zeta_R(x)| \leq \frac{4}{R}.$$

For any function $v \in L_2(B_R(x^0))$ and its "weighted" mean-value

$$\overline{v}_{R,x^0} = \int_{B_R(x^0)} v(x)\zeta_R^2(x-x^0)\,dx \cdot \left(\int_{B_R(x^0)} \zeta_R^2(x-x^0)\,dx\right)^{-1}$$

we have

$$\|v - \overline{v}_{R,x^0}\|_{2,B_R(x^0)} \leq \left(1 + \left(\frac{|B_1|}{\varkappa_\zeta}\right)^{\frac{1}{2}}\right)\|v-\mu\|_{2,B_R(x^0)} \tag{2.14}$$

for any constant μ, in particular for $\mu = \frac{1}{|B_R|}\int_{B_R(x^0)} v\,dx$.

We will also use some well-known relations for a vector-function $w = (w_1, w_2)$ and a scalar function φ:

$$\int_\Omega r(w)\,\varphi\,dx = \int_\Omega w\cdot r^*(\varphi)\,dx + \int_{\partial\Omega} \varphi(v\cdot\tau)\,ds\,, \tag{2.15}$$

where $r^*(\varphi) = \text{rot}^*\varphi = (\varphi_{x_2}, -\varphi_{x_1})$, τ is a tangent to $\partial\Omega$ vector with $|\tau| = 1$;

$$r(\varphi w) = \varphi r(w) - w r^*(\varphi)\,. \tag{2.16}$$

Now we fix $z^0 \in Q$ and a cylinder $Q_R(z^0)$ such that $Q_{4R}(z^0) \subset Q$.

We rewrite system (2.8) as

$$\overline{\rho}u_t - \nu\Delta u + \nabla p = F + (\overline{\rho}-\rho)u_t, \quad \text{div}\,u = 0\,, \tag{2.17}$$

where $\overline{\rho} = \rho(z^0)$, and note that

$$\max_{Q_{2R}(z^0)} |\rho(z) - \overline{\rho}| \leq 2^\alpha L R^\alpha, \quad L = \langle\rho\rangle_Q^{(\alpha)}\,. \tag{2.18}$$

Put

$$\overline{r}_{2R}(t) = \int_{B_{2R}(x^0)} r(u(x,t))\,\zeta_{2R}^2(x-x^0)\,dx\left(\int_{B_{2R}(x^0)} \zeta_{2R}^2(x-x^0)\,dx\right)^{-1},$$

multiply system (2.17) by $r^*[(r(u(x,t)) - \overline{r}_{2R}(t))\zeta_{2R}^2(x-x^0)]$, and integrate the result over $B_{2R}(x^0)$.

The first term in the derived equality can be written in the form of

$$\overline{\rho}\int_{B_{2R}(x^0)} u_t\cdot r^*\left[(r(u(x,t))-\overline{r}_{2R}(t))\zeta_{2R}^2(x-x^0)\right]dx$$
$$=\overline{\rho}\int_{B_{2R}(x^0)}\frac{\partial}{\partial t}r(u(x,t))\cdot[r(u(x,t))-\overline{r}_{2R}(t)]\,\zeta_{2R}^2(x-x^0)\,dx$$
$$=\frac{\overline{\rho}}{2}\frac{d}{dt}\|\,[r(u(\cdot,t)-\overline{r}_{2R}(t)]\,\zeta_{2R}\|^2\,.$$

Next term we transform with the help of twice integrating by parts

$$\left(-\nu\Delta u, r^*\left[(r(u)-\overline{r}_{2R})\,\zeta_{2R}^2\right]\right)$$
$$=\nu\left(r_{x_k}(u),\frac{\partial}{\partial x_k}\left[(r(u)-\overline{r}_{2R})\,\zeta_{2R}^2\right]\right)$$
$$=\nu\|\zeta_{2R}\nabla r(u)\|^2+\nu(r_{x_k}(u),(r(u)-\overline{r}_{2R})2\zeta(\zeta_{2R})_{x_k})\,.$$

The integral with ∇p will dissappear. All other terms we do not change. As a result we get

$$\frac{\overline{\rho}}{2}\frac{d}{dt}\|(r(u)-\overline{r}_{2R})\zeta_{2R}\|^2+\nu\|\zeta_{2R}\nabla r(u)\|^2$$
$$=-\nu(r_{x_k}(u),(r(u)-\overline{r}_{2R})2\zeta_{2R}(\zeta_{2R})_{x_k})$$
$$+(F+(\overline{\rho}-\rho)u_t),r^*[(r(u)-\overline{r}_{2R})\zeta_{2R}^2])\,.$$

Using the Cauchy inequality to estimate the right-hand side of the last relation we come to the inequality

$$\begin{aligned}&\frac{\overline{\rho}}{2}\frac{d}{dt}\|(r(u)-\overline{r}_{2R})\zeta_{2R}\|^2\\ &+\nu\|\zeta_{2R}\nabla r(u)\|^2\le c\|\zeta_{2R}\nabla(r(u))\|R^{-1}\|r(u)-\overline{r}_{2R}\|\\ &+(\|F\|+L2^\alpha R^\alpha\|u_t\|)\left(\|\zeta_{2R}\nabla r(u)\|+\|r(u)-\overline{r}_{2R}\|R^{-1}\right),\end{aligned}\tag{2.19}$$

with $\|\cdot\|=\|\cdot\|_{2,B_{2R}(x^0)}$.

Applying to the right-hand side of (2.19) the Cauchy inequality with some small $\varepsilon>0$ we derive

$$\begin{aligned}&\overline{\rho}\frac{d}{dt}\|(r(u)-\overline{r}_{2R})\zeta_{2R}\|^2+\nu\|\zeta_{2R}\nabla r(u)\|^2\\ &\le\frac{c_1}{R^2}\|r(u)-\overline{r}_{2R}\|^2+c_2\|F\|^2+c_2L^2R^{2\alpha}\|u_t\|^2\,.\end{aligned}\tag{2.20}$$

To define a cut-off function in t we consider a smooth nondecreasing function $\chi:\mathbb{R}^1\to[0,1]$, $\chi(t)=0$ for $t\le-1$ and $\chi(t)=1$ for $t\ge-\frac{1}{4}$. We put $\chi_R(t,t_0)=\chi\left(\frac{t-t_0}{R^2}\right)$ and multiply (2.20) by $\chi_{2R}(t,t_0)$. After integration of the result over $t\in(t^0-4R^2,\hat{t})$, where $\hat{t}\in(t^0-R^2,t^0]$ we have

$$\begin{aligned}&\overline{\rho}\|r(u)-\overline{r}_{2R})\zeta_{2R}\|^2_{t=\hat{t}}+\nu\int_{t^0-4R^2}^{\hat{t}}\chi_{2R}\|\zeta_{2R}\nabla r(u)\|^2dt\\ &\le\frac{c_1}{R^2}\|r(u)-\overline{r}_{2R}\|^2_{2,Q_{2R}(z^0)}+c_2\|F\|^2_{2,Q_{2R}(z^0)}\\ &+c_2L^2R^{2\alpha}\|u_t\|^2_{2,Q_{2R}(z^0)}\equiv j(2R)\,.\end{aligned}\tag{2.21}$$

It follows from (2.21) that

$$\|\nabla r(u)\|^2_{2,Q_R(z^0)}\le\frac{1}{\nu}j(2R)\tag{2.22}$$

and

$$\begin{aligned}&\overline{\rho}\sup_{t\in I_R=[t^0-R^2,t^0]}\|r(u(\cdot,t))-\overline{r}_{2R}(t)\|^2_{2,B_R}\le j(2R)\\ &\le c\left(\|\nabla r(u)\|^2_{2,Q_{2R}(z^0)}+\|F\|^2_{2,Q_{2R}(z^0)}+L^2R^{2\alpha}\|u_t\|^2_{2,Q_{2R}(z^0)}\right).\end{aligned}\tag{2.23}$$

For estimating $\|r(u)-\overline{r}_{2R}\|^2_{2,B_{2R}}$ we have used here the inequality

$$\int_{B_{2R}}|w-\overline{w}_{2R}|^2dx\le cR^2\int_{B_{2R}}|w_x|^2dx,\quad\forall w\in W_2^1(B_{2R}),$$

which is the consequence of (2.14) and Poincare inequality. Besides in the case of two variables the imbedding theorem of $W_1^1(B_{2R})$ in $L_2(B_{2R})$ guarantees the inequality

$$\|r(u)-\hat{r}_{2R}\|\le\beta\|\nabla r(u)\|_{1,B_{2R}},\tag{2.24}$$

where $\hat{r}_{2R}$ is the mean of $r(u)$ over B_{2R} and constant β does not depend on R.

From (2.14) and (2.24) it follows

$$\|r(u)-\overline{r}_{2R}\|\le c\|\nabla r(u)\|_{1,B_{2R}(x^0)}\,.\tag{2.25}$$

Let c_1 be the constant from (2.21). The next chain of inequalities holds:

$$\begin{aligned}&\frac{c_1}{\nu R^2}\|r(u)-\overline{r}_{2R}\|^2_{2,Q_{2R}}\le\frac{c_1}{\nu R^2}\sup_{t\in I_{2R}}\|r(u)-\overline{r}_{2R}\|_{2,B_{2R}}\\ &\times\int_{t^0-4R^2}^{t^0}\|r(u)-\overline{r}_{2R}\|_{2,B_{2R}}dt\underset{(a)}{\le}\frac{c_2}{R^2}\{\|\nabla r(u)\|_{2,Q_{4R}}\\ &+\|F\|_{2,Q_{4R}}+LR^{\alpha}\|u_t\|_{2,Q_{4R}}\}\int_{t^0-4R^2}^{t^0}\|\nabla r(u)\|_{1,B_{2R}}dt\\ &\underset{(b)}{\le}\theta\|\nabla r(u)\|^2_{2,Q_{4R}}+c_3\theta^{-1}R^{-4}\|\nabla r(u)\|^2_{1,Q_{4R}}\\ &+c_4\left(\|F\|^2_{2,Q_{4R}}+L^2R^{2\alpha}\|u_t\|^2_{2,Q_{4R}}\right).\end{aligned}$$

Here to establish (a) we have used (2.23) for the couple of cylinders $Q_{2R}(z^0)$, $Q_{4R}(z^0)$ and (2.25). To install (b) we have used the Cauchy inequality with a parameter $\theta > 0$.

Now from (2.22) it follows (2.13) in the couple of cylinders $Q_R(z^0)$ and $Q_{4R}(z^0)$.

Let us denote $|\nabla r(u)|$ by g and rewrite (2.13) in the form of

$$\begin{aligned}\fint_{Q_R(z^0)} g^2 dz \leq \widetilde{\theta} \fint_{Q_{4R}(z^0)} g^2 dz + c\widetilde{\theta}^{-1}\left(\fint_{Q_{4R}(z^0)} g\, dz\right)^2 \\ + \fint_{Q_{4R}(z^0)} f^2 dz + (4R)^\beta \fint_{Q_{4R}(z^0)} \psi dz, \quad \forall\, Q_{4R}(z^0) \subset Q,\end{aligned} \tag{2.26}$$

where

$$\fint_{Q_r} h\, dz = \frac{1}{|Q_r|}\int_{Q_r} h dz,$$

$$\widetilde{\theta} = 2^8\theta, \; \beta = 2\alpha, \; f = cF, \; \psi = cL^2|u_t|^2, \; L = \langle\rho\rangle_{Q}^{(\alpha)}.$$

Inequalities (2.26) are the reverse Hölder inequalities for function g in the parabolic cylinders with additional terms given by functions f and ψ. Under the conditions of Theorem 2.1 the norms $\|g\|_{2,Q}$, $\|f\|_{q,Q}$, $q > 2$, and $\|\psi\|_{1,Q}$ are finite. In the case when g satisfies (2.26) with $\psi \equiv 0$ or $\psi \in L_m(Q)$, $m > 1$, the Gehring lemma ([8]) and its modifications ([9-12]) allow to state a higher integrability of g in $\forall\, Q' \subset Q$, $\delta(Q', \partial' Q) > 0$, and to get an estimate for $\|g\|_{s,Q'}$ with some $s > 2$.

In our case we know about ψ only that $\psi \in L_1(Q)$. Nevertheless it is appeared that if the integral $\fint_{Q_{4R}} \psi\, dz$ in (2.26) is multiplied by R^β with a $\beta > 0$, then the statement about the higher integrability of g on $Q' \subset Q$, $\delta(Q', \partial' Q) > 0$, also holds and $\|g\|_{s,Q'}$, $s > 2$, can be estimated by a majorant depending only on $\|g\|_{2,Q}$, $\|f\|_{s,Q}$, $\|\psi\|_{1,Q}$, and $[\delta(Q', \partial' Q)]^{-1}$.

Now we formulate the corresponding result in a suitable form.

Let $x = (x_1 \dots x_n) \in \mathbb{R}^n$, $n \geq 1$, $\mathbb{P}_a^- = \{(x,t) : |x_i| < a, i = 1, \dots, n, -a^2 < t < 0\}$ be a half of parabolic cube in $\mathbb{R}^{n+1}$ with parabolic boundary $\partial'\mathbb{P}_a^-$.

LEMMA 2.2. *Let $\mathbb{P} = \mathbb{P}_a^-$ with an $a > 0$ and for nonnegative functions $g \in L_l(\mathbb{P})$, $l > 1$, $f \in L_m(\mathbb{P})$, $m > l$, and $\psi \in L_1(\mathbb{P})$ for all $z^0 \in \mathbb{P}$ the inequalities*

$$\begin{aligned}\fint_{Q_R(z^0)} g^l dz \leq \theta \fint_{Q_{bR}(z^0)} g^l dz + \mathbb{B}\left(\fint_{Q_{bR}(z^0)} g\, dz\right)^l + \fint_{Q_{bR}(z^0)} f^l dz \\ + (bR)^\beta \fint_{Q_{bR}(z^0)} \psi\, dz, \quad \forall\, R < \frac{1}{b}\min\left\{\delta(z^0, \partial'\mathbb{P}), R_1\right\},\end{aligned} \tag{2.27}$$

hold with some parameters $R_1 > 0$, $\theta \geq 0$, $\mathbb{B} > 1$, $b \geq 2$, $\beta > 0$. There exists a number $\theta_0 = \theta_0(l, b) \in (0, 1)$ such that if (2.27) is true for $\theta \leq \theta_0$ then $g \in L_s(\widetilde{\mathbb{P}})$

for any $\widetilde{\mathbb{P}} \subset \mathbb{P}$, $\delta(\widetilde{\mathbb{P}}, \partial'\mathbb{P}) > 0$, *with any* $s \in [l, l+\varepsilon)$

$$\|g\|_{s,\tilde{\mathbb{P}}} \leq c\left\{\|g\|_{l,\mathbb{P}} + \|f\|_{s,\mathbb{P}} + \|\psi\|_{1,\mathbb{P}}^{\frac{1}{l}}\right\}. \tag{2.28}$$

Here $\varepsilon \leq \min\left\{m-l, \frac{\beta l}{n+2}\right\}$, *the constants* c *and* ε *depend on* n, l, m, a, b, $\mathbb{B}$, β, *and* c *also depends on* $[\delta(\widetilde{\mathbb{P}}, \partial'\mathbb{P})]^{-1}$.

Now we describe our proof of Lemma 2.2. First of all we prolong g, f, ψ as even functions from $\mathbb{P} = \mathbb{P}_a^-$ on parabolic cube $\mathbb{P}_a = \{(x,t) : |x_i| < a, i = 1, \dots, n, |t| < a^2\}$. Note, that for prolonged functions g, f, ψ from (2.27) the similar inequalities follow in the full cylinders $\hat{Q}_R(z^0) = \{z = (x,t) : x \in B_R(x^0), |t - t^0| < R^2\}$. More exactly, there exist numbers $b_1 > b$ and $c_1 = c_1(l, b) > 1$ such that for all $z^0 \in \mathbb{P}_a$ inequalities

$$\begin{aligned} \fint_{\hat{Q}_R(z^0)} g^l\,dz &\leq \varkappa \fint_{\hat{Q}_{b_1R}(z^0)} g^l\,dz + \mathbb{B}_1\left(\fint_{\hat{Q}_{b_1R}(z^0)} g\,dz\right)^l \\ &+ \fint_{\hat{Q}_{b_1R}(z^0)} f_1^l\,dz + (b_1R)^\beta \fint_{\hat{Q}_{b_1R}(z^0)} \psi_1\,dz, \\ &\forall R < \frac{1}{b_1}\min\left\{\delta\left(z^0, \partial\mathbb{P}_a\right), R_1\right\}, \end{aligned} \tag{2.29}$$

hold with $\varkappa = \theta \cdot c_1$, $\mathbb{B}_1 = c_1\mathbb{B}$, $f_1 = c_1^{\frac{1}{l}} f$, $\psi_1 = c_1\psi$.

Put $\theta_0 = \frac{1}{2c_1}$ and fix $\theta \leq \theta_0$ in (2.27), then $\varkappa \leq \frac{1}{2}$ in (2.29). According to well-known scheme we rewrite (2.29) for some normed functions G, $\hat{F}$, Ψ. Namely, put $M = \|g\|_{l,\mathbb{P}_a} + \|f_1\|_{l,\mathbb{P}_a} + \|\psi_1\|_{1,\mathbb{P}_a}^{\frac{1}{l}}$ and define normed functions by equalities

$$G = \frac{\varkappa^{\frac{1}{2l}} g}{M}, \quad \hat{F} = \frac{\varkappa^{\frac{1}{2l}} f_1}{M}, \quad \Psi = \frac{\varkappa^{\frac{1}{2}}\psi_1}{M^l}, \quad \text{if } \varkappa > 0,$$

$$G = \frac{g}{M}, \quad \hat{F} = \frac{f_1}{M}, \quad \Psi = \frac{\psi_1}{M^l} \quad \text{if } \varkappa = 0.$$

For them we have

$$\max\{\|G\|_{l,\mathbb{P}_a}^l, \|\hat{F}\|_{l,\mathbb{P}_a}^l, \|\Psi\|_{1,\mathbb{P}_a}\} \leq \begin{cases} \sqrt{\varkappa}, & \varkappa > 0, \\ 1, & \varkappa = 0, \end{cases}$$

and

$$\begin{aligned} \fint_{\hat{Q}_R(z^0)} G^l\,dz &\leq \varkappa \fint_{\hat{Q}_{b_1R}(z^0)} G^l\,dz + \mathbb{B}_1\left(\fint_{\hat{Q}_{b_1R}(z^0)} G\,dz\right)^l \\ &+ \fint_{\hat{Q}_{b_1R}(z^0)} \hat{F}^l\,dz + (b_1R)^\beta \fint_{Q_{b_1R}} \Psi\,dz, \\ &\forall R < \frac{1}{b_1}\min\left\{\delta\left(z^0, \partial\mathbb{P}_a\right), R_1\right\}. \end{aligned} \tag{2.30}$$

These inequalities are valid for a $\varkappa \in \left[0, \frac{1}{2}\right]$ and all $z^0 \in \mathbb{P}_a$.

We introduce functions $G_0 = G\zeta^{\frac{1}{l}}$, where $\zeta(z) = \delta(z, \partial\mathbb{P}_a)^{n+2}$ and

$$h(\tau) = \int_{E(G_0,\tau)} G_0\, dz\,,$$

where $E(\Phi, \tau) = \{z \in \mathbb{P}_a : \Phi(z) > \tau\}$. Using (2.30) we prove for h the inequalities

$$(2.31) \quad -\int_\tau^\infty \xi^{l-1} dh(\xi) \le c_0 \left[\tau^{l-1}h(\tau) + H_1(\tau) + H_2(\tau)\right], \quad \forall \tau \ge \tau_0 > 0\,,$$

where $H_1(\tau) = \int_{E(F,\tau)} \hat{F}^l\, dz$, $H_2(\tau) = \|\Psi\|_{1,\mathbb{P}_a} \cdot \tau^{-\frac{\beta l}{n+2}}$. Constants $c_0 > 1$ and $\tau_0 > 0$ are defined by the parameters of (2.30). Functions h, H_1, H_2: $[\tau_0, \infty) \to [0, \infty)$ are nonincreasing and tend to zero for $\tau \to +\infty$. The proof of (2.31) demands a place and we omit it here. Due to lemma on the Stieltjes integral proved by Gehring in [8], from (2.31) with $l = \alpha + 1$ it follows

$$(2.32) \quad \begin{aligned} -\int_{\tau_0}^\infty \tau^\gamma dh(\gamma) \le & \frac{\alpha}{c_0\alpha - (c_0 - 1)\gamma} \left(-\int_{\tau_0}^\infty \tau^\alpha dh(\alpha)\right) \\ & - \frac{c_0\gamma}{c_0\alpha - (c_0 - 1)\gamma} \left(\int_{\tau_0}^\infty \tau^{\gamma-\alpha} dH_1(\tau) + \int_{\tau_0}^\infty \tau^{\gamma-\alpha} dH_2(\tau)\right) \end{aligned}$$

with a $\gamma \in \left[\alpha, \frac{c_0\alpha}{c_0-1}\right)$. Taking into account the equality

$$\int_{E(G_0,\tau)} G_0^s(z)\, dz = -\int_\tau^\infty \zeta^{s-1} dh(\zeta), \quad \forall s > 1\,,$$

one can deduce from (2.32) that G_0 belongs to $L_s(\mathbb{P}_a)$ for some $s > l$ ($s = \gamma+1$, $l = \alpha + 1$) and the estimate holds

$$(2.33) \qquad \|G\|^s_{s,\mathbb{P}'} \le c\left\{\|G\|^l_{l,\mathbb{P}_a} + \|\hat{F}\|^s_{s,\mathbb{P}_a} + \|\Psi\|_{1,\mathbb{P}_a}\right\}, \quad \forall \mathbb{P}' \subset\subset \mathbb{P}_a\,,$$

with the constant $c > 0$ depending on c_0, a and $\delta(\mathbb{P}', \partial\mathbb{P}_a) > 0$.

Remark that inequality $\gamma < \frac{c_0\alpha}{c_0-1}$ in (2.32) provides the restriction $s < l + \frac{l-1}{c_0-1}$ with c_0 from (2.31). Moreover, to guarantee in (2.32) the finiteness of integrals containing H_1 and H_2 we need to require: $s \le m$ and $s < l + \frac{\beta l}{n+2}$.

As G, $\hat{F}$, Ψ are normed functions g, f_1 and ψ_1, we get from (2.33)

$$(2.34) \qquad \|g\|_{s,\mathbb{P}'} \le c\left\{\|g\|_{l,\mathbb{P}_a} + \|f_1\|_{s,\mathbb{P}_a} + \|\psi_1\|^{\frac{1}{l}}_{1,\mathbb{P}_a}\right\}$$

with constant c depending on $\delta(\mathbb{P}', \partial\mathbb{P}_a)$.

Recalling that $f_1 = c_1^{\frac{1}{T}} f$,$\psi_1 = c_1\psi$ and g, f, ψ are the even function in $\mathbb{P}_a$ we deduce from (2.34) the desirable inequalities (2.28).

Complete proof of Lemma 2.2 and some of its generalizations will be published in an other paper.

According to Lemma 2.1 function $g = | \nabla r(u) |$ satisfies the reverse Hölder inequalities (2.27) in Q with $l = 2$, $n = 2$, $f = cF$, $\psi = cL^2 \mid u_t \mid^2$, $m = q > 2$, $\beta = 2\alpha$, $b = 4$. From this fact and Lemma 2.2 it follows the estimate

$$\|\nabla r(u)\|_{s,Q'} \leq c\Big\{\|\nabla r(u)\|_{2,Q} + \|F\|_{s,Q} + \langle\rho\rangle_{\overline{Q}_T}^{(\alpha)}\|u_t\|_{2,Q}\Big\}$$

with some $s \in (2, q]$, $s < 2+\alpha$, and constant c depending on the value $R_0+R_0^{-1}$. This estimates coincides with (2.12). As it was pointed above, inequality (2.12) yields (2.11).

Theorem 2.1 has been proved for the function $u = v - \hat{v}$, where v is a solution of (2.1)-(2.3). As a consequence of this theorem we get estimate (1.7) for v. It means that the second part of Theorem 1.1 is stated in the case of the periodic boundary conditions.

Besides, in reality, we have also proved the first part of Theorem 1.1 about local estimate of v_{xx} in L_s-norm. More exactly, if $\{v, p\} \in J_2^{2,1}(Q_T)\times W_2^{1,0}(Q_T)$ is a solution of system (2.1), where ρ satisfies (2.4) and $\Phi \in L_q(Q_T)$, $q > 2$, then there exists a number $s \in (2, q]$ such that $v_{xx} \in L_s(Q')$; $\forall Q' \subset Q_T$, $\delta(Q', \partial' Q_T) > 0$, and

$$\|v_{xx}\|_{s,Q'} \leq c\left\{\|v_{xx}\|_{2,Q_T} + \langle\rho\rangle_{\overline{Q}_T}^{(\alpha)}\|v_t\|_{2,Q_T} + \|v\|_{2,Q_T} + \|\Phi\|_{s,Q_T}\right\} \tag{2.35}$$

with a constant $c > 0$ depending on ν, M_1^{-1}, T and $[\delta(Q', \partial' Q_T)]^{-1}$.

To prove this result we note that for v the assertion close to Lemma 2.1 is valid.

LEMMA 2.1$'$. *For the solution* $\{v, p\}$ *of* (2.1) *the following inequality holds*

$$\begin{aligned}\|\nabla r(v)\|^2_{2,Q_R(z^0)} \leq \theta\|\nabla r(v)\|^2_{2,Q_{4R}(z^0)} &+ c\left\{\theta^{-1}r^{-4}\|\nabla r(v)\|^2_{1,Q_{4R}}\right.\\ &\left.+ \|\Phi\|^2_{2,Q_{4R}(z^0)} + \left(\langle\rho\rangle_{\overline{Q}_T}^{(\alpha)}\|v_t\|_{2,Q_{4R}}R^\alpha\right)^2\right\}\end{aligned} \tag{2.36}$$

for cylinders $Q_R(z^0)$, $Q_{4R}(z^0) \subset Q_T$ *and* $\forall\theta > 0$.

If to attract Lemma 2.2 then from (2.36) it is easy to deduce the estimate of $\|\nabla r(v)\|_{s,Q'}$ which is similar to (2.12). From that estimate (2.35) follows.

REFERENCES

[1] O. A. LADYZHENSKAYA – V. A. SOLONNIKOV, *On unique solvability of initial-boundary value problem for viscous incompressible nonhomogeneous fluids*, Zapiski Nauchn. Semin. LOMI **52** (1975), 52-109.

[2] A. A. KISELEV – O. A. LADYZHENSKAYA, *On existence and uniqueness of the solution of the nonstationary problem for a viscous incompressible fluids*, Izv. Akad. Nauk USSR, Ser. Mat. **21** (1957), 655-680.

[3] O. A. LADYZHENSKAYA, *Solution "in the large" of the boundary value problem for the Navier-Stokes equations for the case of two space variables*, Dokl. Akad. Nauk USSR **123** (1958), 427-433.

[4] O. A. LADYZHENSKAYA, "Mathematical Problems of the Dynamics for Viscous Incompressible Fluids", Moscow, 1961 (the first Russian edition); Moscow 1970 (the second Russian edition). English translation of the first Russian Edition: "The Mathematical Theory of Viscous Incompressible Flow", Gordon and Breach, 1963.

[5] V. A. SOLONNIKOV, *On differential properties for the nonstationary Navier-Stokes equations*, Trudy MIAN USSR **73** (1964), 221-291.

[6] O. A. LADYZHENSKAYA – V. A. SOLONNIKOV – N. N. URALTSEVA, "Linear and Quasilinear Equations of the Parabolic Type", Moscow, Nauka, 1967.

[7] O. A. LADYZHENSKAYA – N. N. URALTSEVA, "Linear and Quasilinear Equations of the Elliptic Type", Moscow, 1973 (the second Russian edition).

[8] F. W. GEHRING, *L_p-integrability of the partial derivatives of a quasi-conformal mapping*, Acta Math. **130** (1973), 265-277.

[9] M. GIAQUINTA – G. MODICA , *Regularity results for some classes of higher order nonlinear elliptic systems*, J. Reine Angew. Math. **311/312** (1979), 145-169.

[10] E. W. STREDULINSKY, *Higher integrability from reverse Hölder inequalities*, Indiana Univ. Math. J. **29** (1980), 408-417.

[11] M. GIAQUINTA – M. STRUWE, *On the partial regularity of weak solutions of nonlinear parabolic systems*, Math. Z. **179** (1982), 437-451.

[12] A. ARKHIPOVA, *Reverse Hölder inequalities with boundary integrals and L_p-estimates for solutions of nonlinear elliptic and parabolic boundary-value problems*, Amer. Math. Soc. Transl. **164** (1995), 15-42.

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