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On Pseudosymmetric Hyperbolic Systems

PIERO D'ANCONA - SERGIO SPAGNOLO

In memory of Ennio De Giorgi

1. - Introduction

At the end of seventies, under the impulse of De Giorgi, a current of research was started in Pisa in the field of weakly hyperbolic equations[(1)]. The fundamental work by E.E. Levi dates back at the beginning of this century, however only after the sixties the theory of weakly hyperbolic equations has been extensively investigated by many mathematicians like Leray, Hörmander, Mizohata, Oleinik, Ivrii, among the others. In [CDS] the Cauchy problem

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} = 0 \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{1.2}$$

was studied both in the strictly hyperbolic case

$$\sum a_{ij}\xi_i\xi_j \geq \lambda_0 \langle \xi \rangle^2, \qquad \lambda_0 > 0,$$

but with non smooth coefficients, and in the weakly hyperbolic case

$$\sum a_{ij}\xi_i\xi_j \geq 0. \tag{1.3}$$

Under these assumptions, (1.1)-(1.2) is not well posed, in general, in Sobolev spaces or in $\mathcal{C}^\infty$, and the natural classes are the Gevrey or analytic functions.

In the weakly hyperbolic case (1.3), one can find equations of type (1.1), even with $\mathcal{C}^\infty$ coefficients, for which the Cauchy problem is not well posed in $\mathcal{C}^\infty$ ([CS]); however it was proved:

(1) Actually the interest of De Giorgi in these problems was much older; in his 1955 paper [DG] he gave the first example of non-uniqueness for a weakly hyperbolic Cauchy problem with smooth coefficients.

THEOREM ([CJS]). *Consider the Cauchy problem* (1.1)-(1.2) *under the assumption* (1.3).

i) *If the coefficients $a_{ij}(t)$ are real analytic, the problem is well posed in C^∞.*

ii) *If the $a_{ij}(t)$ are C^∞, the problem is well posed in all Gevrey classes γ^s, $s \geq 1$. More precisely, if $a_{ij} \in C^k$ for some $k \geq 1$,* (1.1)-(1.2) *is well posed in γ^s for $1 \leq s < 1 + k/2$.*

We recall that a C^∞ function $\varphi(x)$ belongs to the Gevrey class $\gamma^s = \gamma^s(\mathbb{R}^n)$, $s \geq 1$, if for all compact $K \subset \mathbb{R}^n$

$$|D^\alpha \varphi(x)| \leq C_K^{1+|\alpha|} \alpha!^s, \quad \forall \alpha \in \mathbb{N}^n, \quad \forall x \in K.$$

In this paper we shall use the following uniform versions of Gevrey classes:

$$\gamma^s_{L^2} = \gamma^s_{L^2}(\mathbb{R}^n) = \left\{ \varphi \in H^\infty(\mathbb{R}^n) : \|D^\alpha \varphi\|_{L^2(\mathbb{R}^n)} \leq C^{|\alpha|+1} \alpha!^s, \ \forall \alpha \in \mathbb{N}^n \right\},$$

where $H^\infty = \bigcap_{r \in \mathbb{R}} H^r$, H^r being the usual Sobolev spaces on $\mathbb{R}^n$.

The above result is quite unstable, for instance it may fail after perturbation by first order terms: the equation $u_{tt} - u_x = 0$ is well posed in γ^s only for $1 \leq s < 2$. The situation is not better for a complete homogeneous second order equation; for instance, the hyperbolic equation

$$u_{tt} + 2tu_{tx} + t^2 u_{xx} = 0 \tag{1.4}$$

is equivalent to $v_{tt} - v_x = 0$ via the transformation $v(t,x) = u(t, x + t^2/2)$.

Also the special form of the coefficients, depending only on time, is essential for the proof of the above Theorem, the general case being still an open problem.

Despite of this, it is natural to ask whether such a result can be extended from the special class of second order equations (1.1) to more general classes of hyperbolic equations or systems. Here we investigate the first order system

$$u_t = \sum_{j=1}^n A_j(t) u_{x_j} \tag{1.5}$$

$$u(0,x) = u_0(x), \tag{1.6}$$

where $u = u(t,x) \in \mathbb{C}^N$, under the *hyperbolicity* assumption: for all real t, ξ, the $N \times N$ matrix

$$A(t,\xi) = \sum_{j=1}^n \xi_j A_j(t) \quad \text{has only real eigenvalues.} \tag{1.7}$$

Even for very smooth coefficients, condition (1.7) alone does not ensure the well-posedness in C^∞ and not even in the high order Gevrey classes (we recall

that each problem of type (1.5)-(1.7) is well posed in γ^s for $1 \le s < N/(N-1)$).
For instance, the hyperbolic system

$$u_t = \begin{pmatrix} -t & 1 \\ -t^2 & t \end{pmatrix} u_x \tag{1.8}$$

is well posed in γ^s only for $1 \le s < 2$; indeed, by the transformation

$$u = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} v,$$

it can be reduced to the system

$$v_t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v_x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} v$$

which is equivalent to the scalar equation $v^1_{tt} - v^1_x = 0$.

A class of systems for which the conclusions of the above Theorem are trivially satisfied, are the symmetric systems (i.e., $A(t,\xi)$ is Hermitian), which are well posed in each Sobolev class; the same holds true for the smoothly symmetrizable systems, a class including the strictly hyperbolic systems. Another special class of 2×2 systems was discovered by S. Tarama ([T]):

$$u_t = \begin{pmatrix} 0 & a(t) \\ b(t) & 0 \end{pmatrix} u_x \tag{1.9}$$

where $a(t), b(t)$ are C^∞ functions satisfying

$$a(t) \ge 0, \quad b(t) \ge 0, \tag{1.10}$$

$$C_1 a(t) \le b(t) \le C_2 a(t) \tag{1.11}$$

for some $C_1, C_2 > 0$. Tarama proved that these systems are well posed in all Gevrey classes. It should be noticed that the quotient $a(t)/b(t)$, although bounded by (1.11), may be non-smooth; as a consequence, system (1.9) is symmetrizable for all t, but not smoothly with respect to t.

In the present paper, we propose a new class of $N \times N$ first order systems with coefficients depending only on time, to which we can extend the above results. This class includes in particular any system of type (1.9) with

$$a(t) \cdot b(t) \ge 0,$$

hence also the scalar equations (1.1)-(1.3), which can be reduced to a system of this type (see Remark 6 below).

Definition 1. Let $A = [a_{ij}]$ be an $N \times N$ complex matrix. We say that A is *pseudosymmetric* if the following conditions are fulfilled for all $i, j, h_1, \dots, h_\nu \in \{1, \dots, N\}$:

$$a_{ij} \cdot a_{ji} \ \ge 0 \tag{1.12}$$

$$a_{h_1h_2} \cdot a_{h_2h_3} \cdot \dots \cdot a_{h_\nu h_1} = \overline{a}_{h_1h_\nu} \cdot \dots \cdot \overline{a}_{h_3h_2} \cdot \overline{a}_{h_2h_1}. \tag{1.13}$$

[It is sufficient to check (1.13) for all ν-tuples of distinct indices $h_1, \dots, h_\nu$ only, $1 \le \nu \le N$].

If $A(t, \xi) = \sum \xi_j A_j(t)$ is pseudosymmetric for all real t, ξ, we say that system (1.5) is pseudosymmetric.

REMARKS.

1) Any Hermitian matrix is pseudosymmetric.

2) Condition (1.12) implies, in particular, that the elements a_{jj} on the diagonal are real. We notice also that, writing $\alpha_{ij} = |a_{ij}|$ and taking (1.12) into account, we can replace (1.13) by

$$\alpha_{h_1 h_2} \cdot \ldots \cdot \alpha_{h_\nu h_1} = \alpha_{h_1 h_\nu} \cdot \ldots \cdot \alpha_{h_2 h_1} \qquad (\nu \ge 3).$$

3) The 2×2 matrix

$$A = \begin{pmatrix} c & a \\ b & d \end{pmatrix}$$

is pseudosymmetric if and only if $c, d \in \mathbb{R}$ and $a \cdot b \ge 0$.

4) In the case $N = 3$, the pseudosymmetricity conditions reduce to:

$$\begin{aligned} a_{ij} \cdot a_{ji} &\ge 0 \\ a_{12} \cdot a_{23} \cdot a_{32} &= \overline{a}_{32} \cdot \overline{a}_{23} \cdot \overline{a}_{12}. \end{aligned}$$

5) A matrix A is pseudosymmetric if and only if it is *diagonally quasi-symmetrizable*, i.e., for all $\epsilon > 0$ there exists some diagonal matrix $P_\epsilon = P_\epsilon^* > 0$ such that

$$\text{(1.14)} \qquad R_\epsilon \equiv P_\epsilon A P_\epsilon^{-1} - P_\epsilon^{-1} A^* P_\epsilon \longrightarrow 0 \qquad \text{as} \quad \epsilon \to 0.$$

One implication is easy to prove: let $A = [a_{ij}]$ and assume that (1.14) holds for some $P_\epsilon = diag[\mu_1^\epsilon, \dots, \mu_N^\epsilon]$, $\mu_j^\epsilon > 0$, i.e. $|\mu_i^\epsilon a_{ij} (\mu_j^\epsilon)^{-1} - \mu_j^\epsilon \overline{a}_{ji} (\mu_i^\epsilon)^{-1}| \to 0$ $(\epsilon \to 0)$. Passing to a subsequence, we can assume that $(\mu_i^\epsilon / \mu_j^\epsilon) \to \delta_{ij} \in [0, +\infty]$, for all $i, j = 1, \dots, N$; hence we find the equality $\delta_{ij} a_{ij} = \delta_{ji} \overline{a}_{ji}$, from which (1.12) follows, since $\delta_{ji} = 1/\delta_{ij}$. By a similar computation we can prove (1.13). The converse implication (for any A pseudosymmetric it is possible to find some diagonal matrix P_ϵ satisfying (1.14)) needs a more delicate proof, and in fact it will follow from the proof of Theorem 1 (see Section 2).

Using this characterization, we easily see in particular that any pseudosymmetric matrix is hyperbolic: let λ be an eigenvalue of A and $v \ne 0$ a corresponding eigenvector, let $v_\epsilon = P_\epsilon v$. Then we have

$$|\lambda - \bar{\lambda}| \cdot |v_\epsilon|^2 = |(R_\epsilon v_\epsilon, v_\epsilon)| \le \|R_\epsilon\| \cdot |v_\epsilon|^2,$$

hence, for $\epsilon \to 0$, we get $\lambda = \bar{\lambda}$.

6) The higher order scalar equation

$$\partial_t^m v = \sum_{j=1}^{m} H_j(t, D)\langle D\rangle^j \partial_t^{m-j} v \tag{1.15}$$

where $H_j(t, \xi)$ are homogeneous symbols of order 0 in ξ, $D = i^{-1}\partial_x$ and $\langle D\rangle$ is the operator with symbol $\langle\xi\rangle = (1 + \langle\xi\rangle^2)^{1/2}$, can be reduced to a first order pseudo-differential system via the usual transformation

$$u = \begin{pmatrix} \langle D\rangle^{m-1} v \\ \langle D\rangle^{m-2}\partial_t v \\ \vdots \\ \partial_t^{m-1} u \end{pmatrix}.$$

We then obtain the system $u_t = iA(t, D)u$ where $A(t, \xi)$ is in the Sylvester form

$$A = \frac{\langle\xi\rangle}{i} \cdot \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ H_m & H_{m-1} & H_{m-2} & \dots & H_1 \end{pmatrix}.$$

Unfortunately, such a matrix is pseudosymmetric if and only if

$$H_m \equiv H_{m-1} \equiv \cdots \equiv H_3 \equiv 0, \qquad H_2 \le 0, \qquad H_1 \in i\mathbb{R},$$

which means that the equation (1.15) has order $m = 2$ or, more generally, $\partial_t^{m-2} u$ satisfies a second order equation. Hence, the theory of pseudosymmetric systems cannot be applied to scalar equations of order ≥ 3, but, essentially, only to the second order equations of the form

$$v_{tt} = \sum a_{ij}(t) v_{x_i x_j} + \sum b_j(t) v_{x_j t},$$

with

$$\sum a_{ij}(t)\xi_i\xi_j \ge 0, \qquad b_j(t) \in \mathbb{R}. \tag{1.16}$$

We notice that the last condition is stronger than the hyperbolicity condition

$$4\sum a_{ij}(t)\xi_i\xi_j + \Big(\sum b_j(t)\xi_j\Big)^2 \ge 0; \tag{1.17}$$

for instance, equation (1.4) satisfies (1.17) but not (1.16).

We can now state our results.

Consider a $N \times N$ system of type (1.5), and we assume that the matrix $A(t, \xi) = \sum_j \xi_j A_j(t)$ is pseudosymmetric for all real t, ξ (Definition 1); then we prove:

THEOREM 1. *Let $A(t, \xi)$ be real analytic in t. Then the Cauchy problem* (1.5)-(1.6) *is well posed in H^∞*, i.e., *for all $T > 0$ there exists some $k = k(T)$ such that, for any $u_0 \in H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, there is a unique solution $u \in C^\infty([-T, T], H^{r-k}(\mathbb{R}^n))$.*

THEOREM 2. *Let* $N = 2$ *and* $A(t,\xi)$ *be* C^∞ *in* t. *Then the Cauchy problem* (1.5)-(1.6) *is well posed in all Gevrey classes,* i.e., *for any* $u_0 \in \gamma^s_{L^2}(\mathbb{R}^n)$, $s \geq 1$, *there is a unique solution* $u \in C^\infty([-T,T], \gamma^s_{L^2}(\mathbb{R}^n))$.

REMARKS. More generally, Theorem 1 holds true (without change of proof) for any matrix $A(t,\xi) = [a^0_{hk}(t,\xi) \cdot a^1_{hk}(t,\xi)]_{h,k=1,\dots,N}$, with $A^0(t,\xi) \equiv [a^0_{hk}]$ Hermitian and C^∞ in t, $A^1(t,\xi) \equiv [a^1_{hk}]$ pseudosymmetric and analytic in t.

If $N = 2$, the above results hold true if we consider, in place of the differential system (1.5), any pseudodifferential system with symbol $A(t,\xi)$ homogeneous of order 1 in ξ.

THEOREM 3. *Let* $N = 3$ *and* $A(t,\xi)$ *be* C^∞ *in* t. *Assume moreover that an inequality like*

$$|a_{pq}(t,\xi)| \leq C|a_{rs}(t,\xi)|^\theta, \quad \textit{for some } \theta > 0 \tag{1.18}$$

holds for a pair of distinct elements a_{pq}, a_{rs}, *with* $p \neq q, r \neq s$ *and* $(r,s) \neq (q,p)$. *Then* (1.5)-(1.6) *is well posed in all Gevrey classes* $\gamma^s_{L^2}(\mathbb{R}^n)$, $s \geq 1$.

In the case $n = 1$, (1.18) *holds when at least one of the* $a_{hk}(t)$*'s with* $h \neq k$ *is analytic.*

The crucial idea of the proof is to construct a diagonal *quasi-symmetrizer* for $A(t,\xi)$, i.e., a smooth diagonal matrix $P_\epsilon(t,\xi)$ such that $P^*_\epsilon = P_\epsilon > 0$ and

$$\|P_\epsilon A P_\epsilon^{-1} - P_\epsilon^{-1} A^* P_\epsilon\| \leq C\epsilon^\sigma \langle \xi \rangle, \qquad \sigma > 0.$$

We notice that any hyperbolic matrix A admits some (possibly nondiagonal) quasi-symmetrizer P_ϵ, at least pointwise (see [DS]), whereas the diagonality of P_ϵ leads necessarily to the pseudosymmetry condition on A, as observed above. On the other hand, a diagonal quasi-symmetrizer is essential in order to prove the well posedness in H^∞ or in γ^s for high s. Indeed, when we estimate the energy function

$$E_\epsilon(t,\xi) = |P_\epsilon(t,\xi)\hat{u}|^2,$$

we need for the matrix $K_\epsilon = \partial_t P_\epsilon \cdot P_\epsilon^{-1}$ a bound such as $\|K_\epsilon\|_{L^1} \leq C \cdot log(1/\epsilon)$ or, at least, $\|K_\epsilon\|_{L^1} \leq C(\delta)\epsilon^{-\delta}$, $\forall\delta > 0$. Now these bounds can be found only for diagonal P_ϵ, when they reduce to an estimate for scalar functions like $\mu'_\epsilon(t)/\mu_\epsilon(t)$ (see Lemma 2, Section 3).

We conclude the Introduction by the following

EXAMPLE. After Theorem 1, the Cauchy problem for the system

$$u_t = \begin{pmatrix} c_1(t) & t^\alpha & t^{\beta'} \\ t^{\alpha'} & c_2(t) & t^\gamma \\ t^\beta & t^{\gamma'} & c_3(t) \end{pmatrix} u_x, \qquad t \geq 0,$$

is well posed in H^∞ as soon as the functions $c_j(t)$ are real valued and bounded, and the powers $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \geq 0$ satisfy the relation

$$\alpha + \beta + \gamma = \alpha' + \beta' + \gamma'.$$

2. – Well posedness in C^∞.

We consider the first order, homogeneous $N \times N$ system

$$u_t = \sum_{j=1}^{n} A_j(t) u_{x_j} \tag{2.1}$$

$$u(0, x) = u_0(x). \tag{2.2}$$

THEOREM 1. *Assume that* $A(t, \xi) = \sum \xi_j A_j(t)$ *is pseudosymmetric for all real* t, ξ, *and real analytic in* t.

Then the Cauchy problem (2.1)-(2.2) *is well posed in* H^∞, i.e., *for all* $T > 0$ *there is some* $k = k(T)$ *such that, for all* $u_0 \in H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, *it has a unique solution* $u(t, x)$ *in* $C^\infty([-T, T], H^{r-k}(\mathbb{R}^n))$.

PROOF. Throughout this proof, we shall write $f \not\equiv 0$, $f \equiv 0$, to mean that the function f is not identically zero, resp. is identically zero. Writing $A(t, \xi) = [a_{pq}(t, \xi)]_{p,q=1,\dots,N}$, we set

$$\alpha_{pq}(t, \xi) = |a_{pq}(t, \xi)|.$$

2.1. – Some relations on the indices

We define an equivalence relation $\sim$ on the set of indices $\{1, \dots, N\}$ as follows: we say that $p \sim q$ if one of the following condition holds: either $p = q$, or both α_{pq} and α_{qp} are not identically zero, or else there exist $h_1, \dots, h_\nu$, $\nu \geq 1$, such that

$$\alpha_{ph_1},\ \alpha_{h_1h_2}, \dots,\ \alpha_{h_\nu q},\ \alpha_{qh_\nu}, \dots,\ \alpha_{h_2h_1},\ \alpha_{h_1p} \not\equiv 0$$

Thanks to the pseudosymmetry condition (1.13), it is sufficient to take $h_1, \dots, h_\nu$ all distinct and $\nu \leq N - 2$. Clearly, $\sim$ is an equivalence relation.

Now we define a relation on the quotient set $S = \{1, \dots, N\}/\sim$: given two equivalence classes $[p]$, $[q]$, we write $[p] < [q]$ if there exist some $p' \in [p]$ and some $q' \in [q]$ such that $\alpha_{p'q'} \equiv 0$ but $\alpha_{q'p'} \not\equiv 0$.

Notice that the relation $<$ is not a true partial order on S, since it is not transitive; however, it is antisymmetric and moreover it does enjoy the following property for all $\nu \geq 1$:

for any $p_1, \dots, p_\nu$ *the relation* $[p_1] < [p_2] < \dots < [p_\nu] < [p_1]$ *is impossible.*

Indeed, if $\nu = 1$, we have only to prove that, if $p' \sim p$ and $\alpha_{p'p} \equiv 0$, then also $\alpha_{pp'} \equiv 0$ and this follows directly by (1.13) and the definition of $<$. Let us now consider the case $\nu = 2$, the case $\nu > 2$ being completely analogous. Assume by contradiction that for some $p_1' \sim p_1$ and $p_2' \sim p_2$ we have

$$\alpha_{p_1'p_2} \equiv 0,\ \alpha_{p_2p_1'} \not\equiv 0,\ \alpha_{p_2'p_1} \equiv 0,\ \alpha_{p_1p_2'} \not\equiv 0. \tag{2.3}$$

By definition we can find $h_1, \dots, h_r$ and $k_1, \dots, k_s$ such that

$$\alpha_{p_1 h_1}, \dots, \alpha_{h_r p_1'}, \alpha_{p_1' h_r}, \dots, \alpha_{h_1 p_1}, \alpha_{p_2 k_1}, \dots, \alpha_{k_s p_2'}, \alpha_{p_2' k_s}, \dots, \alpha_{k_1 p_2} \not\equiv 0.$$

By (1.13) and (2.3) we have then

$$\begin{aligned} 0 &\equiv \alpha_{p_1 h_1} \cdot \dots \cdot \alpha_{h_r p_1'} \cdot \alpha_{p_1' p_2} \cdot \alpha_{p_2 k_1} \cdot \dots \cdot \alpha_{k_s p_2'} \cdot \alpha_{p_2' p_1} \\ &\equiv \alpha_{p_1 p_2'} \cdot \alpha_{p_2' k_s} \cdot \dots \cdot \alpha_{k_1 p_2} \cdot \alpha_{p_2 p_1'} \cdot \alpha_{p_1' h_r} \cdot \dots \cdot \alpha_{h_1 p_1} \not\equiv 0 \end{aligned}$$

which is absurd.

Later, we shall need the following elementary result.

LEMMA 1. *Let S be a finite set, endowed with a relation $<$ such that there do not exist any $s_1, \dots, s_\nu \in S$, $\nu \geq 1$, for which $s_1 < s_2 < \cdots < s_\nu < s_1$.*

Then there exists a function $\gamma : S \to \mathbb{N}$ such that $s < s'$ implies $\gamma(s) < \gamma(s')$.

PROOF OF LEMMA 1. We observe that S has a maximal element, i.e., an element $\bar{s}$ such that it does not exist any $s' \in S$ with $\bar{s} < s'$; otherwise, starting from any $s_1 \in S$ and recalling that S is finite, we could find a cycle $s_1 < s_2 < \cdots < s_\nu < s_1$. To conclude the proof, it is sufficient to argue by induction on the cardinality of S, by defining first the function γ on $S \setminus \{\bar{s}\}$.

As stated in the Introduction, we shall now construct a diagonal quasi-symmetrizer for $\sum \xi_j A_j(t) = [a_{pq}(t, \xi)]$. This amounts to constructing N functions $\lambda_p^\epsilon(t, \xi)$, $p = 1, \dots, N$, smooth on $t \neq T_j$, $j = 1, \dots, m$, in such a way that, for some $C_1, C_2, C, \sigma > 0$,

$$C_1 \epsilon^\sigma \leq \lambda_p^\epsilon(t, \xi) \leq C_2 \tag{2.4}$$

$$\left| \sqrt{\frac{\lambda_p^\epsilon}{\lambda_q^\epsilon}}\, a_{pq} - \bar{a}_{qp} \sqrt{\frac{\lambda_q^\epsilon}{\lambda_p^\epsilon}} \right| \leq C \langle \xi \rangle \epsilon^{1/2} \tag{2.5}$$

$$\sum_j \int_{T_j}^{T_{j+1}} \left| \frac{\partial_t \lambda_p^\epsilon}{\lambda_p^\epsilon} \right| dt \leq C \log\left(\frac{1}{\epsilon}\right). \tag{2.6}$$

To this end, we first approximate all the functions $\alpha_{pq}(t, \xi) = |a_{pq}(t, \xi)|$, except those which are identically zero, with strictly positive functions $\alpha_{pq}^\epsilon(t, \xi)$ which still satisfy the cyclic condition (1.13) and also the properties

$$\frac{|\alpha_{pq}^\epsilon - \alpha_{pq}|}{\sqrt{\alpha_{pq}^\epsilon}} \leq C \langle \xi \rangle^{1/2} \epsilon^{1/2} \tag{2.7}$$

$$C_1 \langle \xi \rangle \epsilon^\rho \leq \alpha_{pq}^\epsilon \leq C_2 \langle \xi \rangle, \qquad \rho \geq 0, \ C_i > 0. \tag{2.8}$$

If $\alpha_{pq} \equiv 0$ we simply set $\alpha_{pq}^\epsilon \equiv 0$, so that condition (1.13) for the α_{ij}^ϵ's will be always satisfied.

2.2. – Construction of the functions α^{ϵ}_{pq}.

Denoting by $[a^j_{pq}]$ the coefficients of the matrix $A_j(t)$, we write

$$a_{pq}(t, \xi) = \sum_{j=1}^{n} a^j_{pq}(t)\xi_j.$$

Thanks to the analyticity, for any fixed $t_0 \in [0, T]$ each function $a_{pq}(t, \xi)$ (unless $a_{pq} \equiv 0$) can be factored as

$$a_{pq}(t, \xi) = (t - t_0)^k b_{pq}(t, \xi) = (t - t_0)^k \sum_{j=1}^{n} b^j_{pq}(t)\xi_j, \tag{2.9}$$

for some integer $k = k(p, q) \geq 0$ and for a suitable analytic function $b_{pq}(t, \xi)$ with $b_{pq}(t_0, \xi) \not\equiv 0$. Thus, using the analyticity in ξ, we can find a unit vector $\eta_0 \in \mathbb{R}^n$ such that

$$b_{pq}(t_0, \eta_0) \neq 0 \tag{2.10}$$

for all p, q (such that $a_{pq} \not\equiv 0$). This implies

$$|b_{pq}(t, \eta_0)| \geq C > 0 \tag{2.11}$$

on some neighborhood I of t_0. We can cover $[-T, T]$ with a finite number of such intervals $I_1, \ldots, I_k$; thus, by possibly splitting $[-T, T]$, it is not restrictive to assume that that (2.9) and (2.11) hold on a neighborhood of the whole interval $[-T, T]$ and that $t_0 = 0$. In conclusion, we can assume that there exist η_0, $|\eta_0| = 1$, and $k = k(p, q) \geq 0$ such that for each $p, q = 1, \ldots, N$ with $a_{pq} \not\equiv 0$, we can write

$$a_{pq}(t, \xi) = t^k b_{pq}(t, \xi) = t^k \sum_{j=1}^{n} b^j_{pq}(t)\xi_j \tag{2.12}$$

with

$$|b_{pq}(t, \eta_0)| \geq C > 0, \tag{2.13}$$

for all $t \in [-T, T]$. When $a_{pq} \equiv 0$ we set $b_{pq} \equiv 0$ and $k(p, q) = 0$.

Using (2.12) and (2.13) it is easy to see that, for all $h_1, \ldots, h_\nu$,

$$b_{h_1h_2} \cdot \ldots \cdot b_{h_\nu h_1} = b_{h_1h_\nu} \cdot \ldots \cdot b_{h_2h_1}. \tag{2.14}$$

Moreover, if $h_1, \ldots, h_\nu$ are such that

$$\alpha_{h_1h_2} \cdot \ldots \cdot \alpha_{h_\nu h_1} \equiv \alpha_{h_1h_\nu} \cdot \ldots \cdot \alpha_{h_2h_1} \not\equiv 0,$$

we have also

$$k(h_1, h_2) + \cdots + k(h_\nu, h_1) = k(h_1, h_\nu) + \cdots + k(h_2, h_1). \tag{2.15}$$

We now define the approximating functions α^ϵ_{pq} by setting

$$\begin{aligned} \alpha^\epsilon_{pq}(t, \xi) &= (t + \epsilon)^{k(p,q)} \cdot |b_{pq}(t, \xi + i\epsilon\eta_0\langle\xi\rangle)|, \quad \text{if} \quad t \in [0, T] \\ \alpha^\epsilon_{pq}(t, \xi) &= (t - \epsilon)^{k(p,q)} \cdot |b_{pq}(t, \xi + i\epsilon\eta_0\langle\xi\rangle)|, \quad \text{if} \quad t \in [-T, 0] \end{aligned} \tag{2.16}$$

and we prove that (2.7), (2.8) are fulfilled. We shall consider in detail only the case $t \in [0, T]$, the case $t \in [-T, 0]$ being completely analogous. First of all we remark that the α^ϵ_{pq} satisfy the cyclic condition (1.13), i.e.,

$$\alpha^\epsilon_{h_1 h_2} \cdot \ldots \cdot \alpha^\epsilon_{h_\nu h_1} \equiv \alpha^\epsilon_{h_1 h_\nu} \cdot \ldots \cdot \alpha^\epsilon_{h_2 h_1}, \tag{2.17}$$

as it follows directly from (2.14), (2.15). To prove (2.8), we observe that, if $a_{pq} \neq 0$,

$$\begin{aligned} \alpha^\epsilon_{pq} &= (t + \epsilon)^{k(p,q)} \big| b_{pq}(t, \xi) + i\epsilon\langle\xi\rangle b_{pq}(t, \eta_0) \big| \\ &\geq (t + \epsilon)^{k(p,q)} \epsilon |\xi| \big| b_{pq}(t, \eta_0) \big| \geq C\langle\xi\rangle \epsilon^{k(p,q)+1}. \end{aligned} \tag{2.18}$$

Thus we can choose $\rho = 1 + \max_{p,q} k(p, q)$ in (2.8). Finally, to prove (2.7) we write

$$|\alpha^\epsilon_{pq} - \alpha_{pq}| = \Big| \big| (t + \epsilon)^k b_{pq}(t, \xi) + i\epsilon|\xi| b_{pq}(t, \eta_0) \big| - t^k \big| b_{pq}(t, \xi) \big| \Big|$$

whence

$$|\alpha^\epsilon_{pq} - \alpha_{pq}| \leq \left((t + \epsilon)^k - t^k \right) |b_{pq}(t, \xi)| + (t + \epsilon)^k \epsilon \langle\xi\rangle |b_{pq}(t, \eta_0)|,$$

while, be (2.18), we have

$$\alpha^\epsilon_{pq} \geq (t + \epsilon)^k |b_{pq}(t, \xi)|, \qquad \alpha^\epsilon_{pq} \geq (t + \epsilon)^k \epsilon \langle\xi\rangle |b_{pq}(t, \eta_0)|.$$

Hence we deduce that

$$\frac{|\alpha^\epsilon_{pq} - \alpha_{pq}|}{\sqrt{\alpha^\epsilon_{pq}}} \leq \frac{(t + \epsilon)^k - t^k}{(t + \epsilon)^{k/2}} \big| b_{pq}(t, \xi) \big|^{1/2} + \left((t + \epsilon)^k \epsilon \langle\xi\rangle |b_{pq}(t, \eta_0)| \right)^{1/2}.$$

Using the inequality

$$\frac{(t + \epsilon)^k - t^k}{(t + \epsilon)^{k/2}} \leq \sqrt{(t + \epsilon)^k - t^k} \leq C(T)\epsilon^{1/2}, \qquad 0 \leq t \leq T,$$

we finally obtain (2.7):

$$\frac{|\alpha^\epsilon_{pq} - \alpha_{pq}|}{\sqrt{\alpha^\epsilon_{pq}}} \leq C\langle\xi\rangle^{1/2}\epsilon^{1/2}. \tag{2.19}$$

We notice that, in proving (2.7) and (2.8), we have used that the symbol $A(t, \xi)$ is linear in ξ.

2.3. – Definition of the functions λ_p^ϵ.

Starting from any family of approximate coefficients $\alpha_{pq}^\epsilon(t,\xi)$ satisfying (2.7), (2.8) and the cyclic condition (2.17), we are now in the position to define the functions $\lambda_p^\epsilon(t,\xi)$, $p=1,\dots,N$, satisfying (2.4)-(2.6). By Lemma 1, we can fix an integer valued function γ, defined on the equivalence classes $[p]\in\{1,\dots,N\}/\sim$, such that $[p]<[q]$ implies $\gamma([p])<\gamma([q])$. For any class $[p]$, we shall write briefly

$$p_0=\min\{p' \;:\; p'\in[p]\}. \tag{2.20}$$

Then, for each $p\sim p_0$ we have: i) either $p=p_0$, or ii) $\alpha_{pp_0}\not\equiv 0$, $\alpha_{p_0p}\not\equiv 0$, or else iii) there exist $h_1,\dots,h_\nu$ $(1\le\nu\le N-2)$ such that

$$\alpha_{p_0h_1},\dots,\alpha_{h_\nu p}\not\equiv 0 \quad\text{and}\quad \alpha_{ph_\nu},\dots,\alpha_{h_1p_0}\not\equiv 0. \tag{2.21}$$

We then define: in case i),

$$\lambda_p^\epsilon(t,\xi)=\epsilon^{\gamma([p])\tau} \tag{2.22}$$

where the constant $\tau>0$ will be chosen later; in case ii)

$$\lambda_p^\epsilon(t,\xi)=\epsilon^{\gamma([p])\tau}\cdot\frac{\alpha_{p_0p}^\epsilon}{\alpha_{pp_0}^\epsilon}; \tag{2.23}$$

finally, in case iii), we set

$$\lambda_p^\epsilon(t,\xi)=\epsilon^{\gamma([p])\tau}\cdot\frac{\alpha_{p_0h_1}^\epsilon\cdot\dots\dots\cdot\alpha_{h_\nu p}^\epsilon}{\alpha_{ph_\nu}^\epsilon\cdot\dots\cdot\alpha_{h_1p_0}^\epsilon} \tag{2.24}$$

where the h_j are such that (2.21) holds. Note that definition (2.24) does not depend on the choice of the h_j, as long as (2.21) holds, thanks to the cyclic relations (2.17).

We can now prove (2.4)-(2.6). Inequality (2.4) follows immediately from (2.8) and the definition of λ_p^ϵ, which give

$$\epsilon^{\gamma([p])\tau}\cdot C\epsilon^{(N-1)\rho}\le\lambda_p^\epsilon(t,\xi)\le\epsilon^{\gamma([p])\tau}\cdot C^{-1}\epsilon^{-(N-1)\rho};$$

thus, if we choose τ so large that

$$\tau\cdot\min\gamma\ge(N-1)\rho,$$

we obtain (2.4) with $\sigma=\tau\cdot\max\gamma+(N-1)\rho$. Let us now consider (2.5), which can be written

$$I_{pq}^\epsilon(t,\xi)\le C\langle\xi\rangle\epsilon^{1/2} \tag{2.25}$$

where, recalling that $a_{pq} \cdot a_{qp} \geq 0$, and $\alpha_{pq} = |a_{pq}|$,

$$I^{\epsilon}_{pq} = \left| \sqrt{\frac{\lambda^{\epsilon}_p}{\lambda^{\epsilon}_q}} a_{pq} - \overline{a}_{qp} \sqrt{\frac{\lambda^{\epsilon}_q}{\lambda^{\epsilon}_p}} \right| \equiv \left| \sqrt{\frac{\lambda^{\epsilon}_p}{\lambda^{\epsilon}_q}} \alpha_{pq} - \alpha_{qp} \sqrt{\frac{\lambda^{\epsilon}_q}{\lambda^{\epsilon}_p}} \right| . \tag{2.26}$$

We can distinguish three possibilities:

1) $\alpha_{pq} \equiv \alpha_{qp} \equiv 0$. In this case (2.25) is trivial.

2) $\alpha_{pq} \equiv 0$ but $\alpha_{qp} \not\equiv 0$. This means $[p] < [q]$, hence $\gamma([p]) \leq \gamma([q]) - 1$. Thus

$$I^{\epsilon}_{pq} = \sqrt{\frac{\lambda^{\epsilon}_q}{\lambda^{\epsilon}_p}} \alpha_{qp} = \epsilon^{\{\gamma([q]) - \gamma([p])\}\tau/2} \alpha_{qp} \cdot \left(\frac{\alpha^{\epsilon}_{q_0 k_1} \cdot \dots \cdot \alpha^{\epsilon}_{k_\mu q} \cdot \alpha^{\epsilon}_{p h_\nu} \cdot \dots \cdot \alpha^{\epsilon}_{h_1 p_0}}{\alpha^{\epsilon}_{p_0 h_1} \cdot \dots \cdot \alpha^{\epsilon}_{h_\nu p} \cdot \alpha^{\epsilon}_{q k_\mu} \cdot \dots \cdot \alpha^{\epsilon}_{k_1 q_0}} \right)^{1/2}$$

whence, using (2.8), we get

$$I^{\epsilon}_{pq} \leq \epsilon^{\tau/2} \cdot C \epsilon^{-(\mu+\nu+2)\rho/2} \langle \xi \rangle \leq C \langle \xi \rangle \epsilon^{\tau/2 - (N-1)\rho}.$$

But this gives (2.25), as soon as τ is chosen large enough, i.e.,

$$\tau \geq 2(N-1)\rho + 1.$$

3) $\alpha_{pq} \not\equiv 0$, $\alpha_{qp} \not\equiv 0$. In particular this implies $p \sim q$. We have then

$$\begin{aligned} I^{\epsilon}_{pq} = \Bigg| \alpha_{pq} & \left(\frac{\alpha^{\epsilon}_{p_0 h_1} \cdots \alpha^{\epsilon}_{h_\nu p} \cdot \alpha^{\epsilon}_{q k_\mu} \cdots \alpha^{\epsilon}_{k_1 q_0}}{\alpha^{\epsilon}_{q_0 k_1} \cdots \alpha^{\epsilon}_{k_\mu q} \cdot \alpha^{\epsilon}_{p h_\nu} \cdots \alpha^{\epsilon}_{h_1 p_0}} \right)^{1/2} \\ & - \alpha_{qp} \left(\frac{\alpha^{\epsilon}_{q_0 k_1} \cdots \alpha^{\epsilon}_{k_\mu q} \cdot \alpha^{\epsilon}_{p h_\nu} \cdots \alpha^{\epsilon}_{h_1 p_0}}{\alpha^{\epsilon}_{p_0 h_1} \cdots \alpha^{\epsilon}_{h_\nu p} \cdot \alpha^{\epsilon}_{q k_\mu} \cdots \alpha^{\epsilon}_{k_1 q_0}} \right)^{1/2} \Bigg| \end{aligned}$$

which, using (2.17), can be simplified to

$$I^{\epsilon}_{pq} = \frac{\left| \alpha^{\epsilon}_{qp} \cdot \alpha_{pq} - \alpha_{qp} \cdot \alpha^{\epsilon}_{pq} \right|}{\sqrt{\alpha^{\epsilon}_{pq} \cdot \alpha^{\epsilon}_{qp}}}.$$

Hence we find, by (2.17), (2.19),

$$I^{\epsilon}_{pq} \leq \frac{|\alpha^{\epsilon}_{pq} - \alpha_{pq}|}{\sqrt{\alpha^{\epsilon}_{pq}}} \sqrt{\alpha^{\epsilon}_{qp}} + \frac{|\alpha^{\epsilon}_{qp} - \alpha_{qp}|}{\sqrt{\alpha^{\epsilon}_{qp}}} \sqrt{\alpha^{\epsilon}_{pq}} \leq C \langle \xi \rangle \epsilon^{1/2}.$$

It remains to prove (2.6). This follows from an elementary property of the analytic functions (cf. [CJS]). Indeed, if $\lambda(t, \xi)$ is real analytic and strictly positive on a neighborhood of $[-T, T] \times K$, $K \subset \mathbb{R}^n$ compact, then the number $N(\xi)$ of oscillations of $\lambda(\cdot, \xi)$ on $[-T, T]$ (i.e., the minimum number of intervals we can split $[-T, T]$ into, such that $\lambda(\cdot, \xi)$ is monotone on each) is clearly finite for each $\xi \in K$. As a consequence of the Weierstrass preparation theorem, applied to $\partial_t \lambda$, it is easy to see that $N(\xi)$ is also locally bounded, so that we have, for some N_0,

$$N(\xi) \leq N_0 < \infty, \qquad \xi \in K.$$

It follows that

$$\int_{-T}^{T} \frac{|\partial_t \lambda(t,\xi)|}{\lambda(t,\xi)} dt \leq N_0 \log \frac{M}{m} \tag{2.27}$$

where

$$M = \max_{[-T,T]\times K} \lambda(t,\xi), \qquad m = \min_{[-T,T]\times K} \lambda(t,\xi). \tag{2.28}$$

Using (2.4) we then obtain

$$\int_{-T}^{T} \frac{|\partial_t \lambda^{\epsilon}(t,\xi)|}{\lambda^{\epsilon}(t,\xi)} dt \leq C \log \frac{1}{\epsilon}. \tag{2.29}$$

(Recalling that we have split $[-T,T]$ in a finite number of subintervals in an earlier step, this implies formula (2.6)).

2.4. – Conclusion of the proof

We now define the following energy function of $\widehat{u} = v(t,\xi) = (v_1, \ldots, v_N)$:

$$E_{\epsilon}(t,\xi) = \sum_{p=1}^{N} \lambda_p^{\epsilon}(t,\xi) |v_p(t,\xi)|^2. \tag{2.30}$$

Differentiating E_{ϵ} with respect to time and recalling that v solves the system

$$\partial_t v_p = i \sum_{q=1}^{N} a_{pq}(t,\xi) v_q$$

we find

$$E'_{\epsilon}(t,\xi) = \sum_{p=1}^{N} \partial_t \lambda_p^{\epsilon} \cdot |v_p|^2 + 2 \sum_{p,q} \operatorname{Im}\left(\lambda_p^{\epsilon} \overline{a}_{pq} \overline{v}_q v_p\right)$$

whence

$$E'_{\epsilon}(t,\xi) \leq \sum_{p=1}^{N} \partial_t \lambda_p^{\epsilon} \cdot |v_p|^2 + 2 \sum_{p<q} \operatorname{Im}\left[\left(\lambda_p^{\epsilon} \overline{a}_{pq} - \lambda_q^{\epsilon} a_{qp}\right) v_p \overline{v}_q\right]$$

and recalling the definition (2.26) of the I^{ϵ}_{pq},

$$E'_{\epsilon}(t,\xi) \le E_{\epsilon}(t,\xi)\left[\sum_{p=1}^{n}\frac{|\partial_t\lambda^{\epsilon}_p|}{\lambda^{\epsilon}_p} + \sum_{p<q} I^{\epsilon}_{pq}(t,\xi)\right]. \tag{2.31}$$

Thus, by Gronwall's inequality and (2.25) we get

$$E_{\epsilon}(t,\xi) \le E_{\epsilon}(0,\xi)\exp\left[C_0\left(\log\frac{1}{\epsilon} + \epsilon^{1/2}\langle\xi\rangle\right)\right] \tag{2.32}$$

for some $C_0 = C_0(T)$ independent of ϵ, ξ, t. Finally, we choose $\epsilon = \langle\xi\rangle^{-2}$ and we define

$$E(t,\xi) = E_{\epsilon}(t,\xi)|_{\epsilon=\langle\xi\rangle^{-2}} \tag{2.33}$$

so that $E(t,\xi)$ satisfies the a priori estimate

$$E(t,\xi) \le E(0,\xi)\langle\xi\rangle^{2C_0}\cdot e^{C_0}. \tag{2.34}$$

Since by (2.4)

$$C_1\langle\xi\rangle^{-2\sigma}|v|^2 \le E(t,\xi) \le C_2^{-1}|v|^2,$$

estimate (2.34) implies the well posedness in Sobolev classes, with a loss of $k = \sigma + C_0$ derivatives.

3. – Well posedness in all Gevrey classes

Let us now consider the case of a pseudosymmetric matrix $A(t,\xi) = \sum \xi_j A_j(t)$ with coefficients of class C^{∞} in t. We expect in general a well-posedness result in all Gevrey classes; unfortunately, we are only able to prove a few partial results in this direction, in particular for low dimensions $N = 2, 3, 4$.

Let us first consider the case of a 2×2 homogeneous system

$$u_t = A(t,\partial)u \tag{3.1}$$

$$u(0,x) = u_0(x); \tag{3.2}$$

actually we can handle any homogeneous pseudodifferential system of first order. Thus, applying Fourier transform with respect to space variables, we obtain a system of the form

$$v'(t,\xi) = i\begin{pmatrix} c(t,\xi) & a(t,\xi) \\ b(t,\xi) & d(t,\xi)\end{pmatrix} v(t,\xi) \tag{3.3}$$

$$v(0,\xi) = v_0(\xi) \tag{3.4}$$

where a, b, c, d are any homogeneous functions of order 1 in ξ.

THEOREM 2. *Assume the system is pseudosymmetric,* i.e., *for all* $t \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^n$,

$$c(t,\xi), d(t,\xi) \text{ are real valued} \tag{3.5}$$

$$a(t,\xi)\cdot b(t,\xi) \geq 0 \tag{3.6}$$

and that $a(t,\xi)$, $b(t,\xi)$ *are* C^∞ *in* t.

Then the Cauchy problem (3.1)-(3.2) *is well posed in all Gevrey classes,* i.e., *for any* $u_0 \in \gamma^s_{L^2}(\mathbb{R}^n)$ *there exists a unique solution* $u \in C^\infty([-T,T], \gamma^s_{L^2}(\mathbb{R}^n))$.

PROOF (sketch). Let us define

$$\alpha_\epsilon(t,\xi) = \left(|a(t,\xi)|^2 + \epsilon^2\langle\xi\rangle^2\right)^{1/2}$$
$$\beta_\epsilon(t,\xi) = \left(|b(t,\xi)|^2 + \epsilon^2\langle\xi\rangle^2\right)^{1/2}$$

and the energy of $v = (v_1, v_2)$:

$$E_\epsilon(t,\xi) = \beta_\epsilon|v_1|^2 + \alpha_\epsilon|v_2|^2. \tag{3.7}$$

We have

$$E'_\epsilon = \beta'_\epsilon|v_1|^2 + \alpha'_\epsilon|v_2|^2 + 2\,\mathrm{Re}\left[i(\beta_\epsilon\overline{a} - \alpha_\epsilon b)v_1\overline{v}_2\right]. \tag{3.8}$$

Thanks to (3.6) we can write

$$|\beta_\epsilon\overline{a} - \alpha_\epsilon b| = |\beta_\epsilon\alpha - \alpha_\epsilon\beta|$$

with

$$\alpha(t,\xi) = |a(t,\xi)|, \quad \beta(t,\xi) = |b(t,\xi)|.$$

Hence

$$E'_\epsilon \leq \left(\left|\frac{\alpha'_\epsilon}{\alpha_\epsilon}\right| + \left|\frac{\beta'_\epsilon}{\beta_\epsilon}\right| + 2\frac{|\beta_\epsilon\alpha - \alpha_\epsilon\beta|}{\sqrt{\alpha_\epsilon\beta_\epsilon}}\right)\cdot E_\epsilon. \tag{3.9}$$

Since $0 \leq \beta_\epsilon - \beta = (\sqrt{\beta_\epsilon - \beta})^2 \leq \sqrt{\beta_\epsilon}\cdot C(\langle\xi\rangle\epsilon)^{1/2}$, and analogously for α, we have

$$\frac{|\beta_\epsilon\alpha - \alpha_\epsilon\beta|}{\sqrt{\alpha_\epsilon\beta_\epsilon}} \leq \frac{(\beta_\epsilon - \beta)}{\sqrt{\beta_\epsilon}}\frac{\alpha}{\sqrt{\alpha_\epsilon}} + \frac{(\alpha_\epsilon - \alpha)}{\sqrt{\alpha_\epsilon}}\frac{\beta}{\sqrt{\beta_\epsilon}} \leq C\langle\xi\rangle\epsilon^{1/2}. \tag{3.10}$$

To estimate the other terms, we shall resort to the following result of real analysis:

LEMMA 2 ([CJS]). *If $f(t) \geq 0$ is of class C^k on $[-T,T]$, $k \geq 1$, then $\sqrt[k]{f}$ is an absolutely continuous function, and*

$$\left\| \frac{d}{dt} \sqrt[k]{f} \right\|_{L^1([-T,T])} \leq c(k,T) \|f\|^{1/k}_{C^k([-T,T])}.$$

Now, if $a(t,\xi)$, $b(t,\xi)$ are of class C^k, we write

$$\left|\frac{\alpha'_\epsilon}{\alpha_\epsilon}\right| = \frac{1}{2}\left|\frac{(\alpha_\epsilon^2)'}{\alpha_\epsilon^2}\right| = \frac{1}{2}\left|\frac{(\alpha_\epsilon^2)'}{(\alpha_\epsilon^2)^{1-1/k}}\right|(\alpha_\epsilon^2)^{-1/k} \leq \frac{1}{2}\left|\frac{d}{dt}(\alpha_\epsilon^2)^{1/k}\right|(\epsilon\langle\xi\rangle)^{-2/k} \tag{3.11}$$

since $\alpha_\epsilon^2 \geq \epsilon^2\langle\xi\rangle^2$. We then apply Lemma 2 to the C^k function α_ϵ^2, noticing that

$$\left\|\alpha_\epsilon^2\right\|_{C^k} = \left\|\alpha^2 + \epsilon^2\langle\xi\rangle^2\right\|_{C^k} \leq C_0\langle\xi\rangle^2 \tag{3.12}$$

with C_0 independent of ξ (recall that α is homogeneous in ξ). Hence we get

$$\int_0^T \left|\frac{\alpha'_\epsilon}{\alpha_\epsilon}\right| dt \leq C_1(k,T)\langle\xi\rangle^{2/k}(\epsilon\langle\xi\rangle)^{-2/k} \leq C_1(k,T)\epsilon^{-2/k} \tag{3.13}$$

and an identical inequality for β_ϵ.

In conclusion, by (3.9), (3.10) and (3.13), we obtain, using Gronwall's inequality,

$$E_\epsilon(t,\xi) \leq E_\epsilon(0,\xi)\exp\left[C_2(k,T)\left(\epsilon^{-2/k} + \langle\xi\rangle\epsilon^{1/2}\right)\right]. \tag{3.14}$$

We can now define

$$\epsilon = \langle\xi\rangle^{-2k/(k+4)} \tag{3.15}$$

and set

$$E(t,\xi) = E_\epsilon(t,\xi)|_{\epsilon=\langle\xi\rangle^{-2k/(k+4)}},$$

thus obtaining the apriori estimate

$$E(t,\xi) \leq E(0,\xi)\cdot\exp\left[C(k,T)\,\langle\xi\rangle^{4/(k+4)}\right]. \tag{3.16}$$

This estimate implies the well-posedness in $\gamma^s_{L^2}$ for

$$1 \leq s < \frac{k+4}{4} = 1 + \frac{k}{4}; \tag{3.17}$$

indeed, $\langle\xi\rangle^{(4-k)/(4+k)}|v|^2 \leq E(t,\xi) \leq C_0\langle\xi\rangle|v|^2$.

We now consider the case of a 3×3 homogeneous differential system (3.1); in Fourier transform it can be written

$$v' = iA(t,\xi)v$$

and we shall denote as usual with $a_{pq}(t,\xi)$ the coefficients of $A(t,\xi)$.

THEOREM 3. *Assume* (3.1) *is a* 3×3 *pseudosymmetric system with* C^∞ *coefficients,* i.e.,

$$a_{pq}(t,\xi) \cdot a_{qp}(t,\xi) \geq 0 \qquad p,q = 1,2,3, \tag{3.18}$$

$$a_{12} \cdot a_{23} \cdot a_{31} = \overline{a}_{13} \cdot \overline{a}_{32} \cdot \overline{a}_{21}. \tag{3.19}$$

Moreover, assume that there exist $\theta > 0$, *and two distinct coefficients* a_{pq}, a_{rs}, *with* $p \neq q, r \neq s$ *and* $(r,s) \neq (q,p)$, *such that*

$$|a_{pq}(t,\xi)| \leq C|a_{rs}(t,\xi)|^\theta \tag{3.20}$$

(*or, more generally, that* $[0,T]$ *can be split in a finite number of intervals such that an inequality like* (3.20) *holds on each of them*).

Then Problem (3.1), (3.2) *is well posed in all Gevrey classes* $\gamma^s_{L^2}(\mathbb{R}^n)$, $s \geq 1$.

REMARK. Assumption (3.20) is satisfied in the following two special cases:

i) there is some coefficient $a_{pq}(t,\xi)$, with $p \neq q$, which is identically 0;

ii) the space dimension is $n = 1$, and there is some coefficient $a_{pq}(t,\xi) \equiv a_{pq}(t)\xi$, with $p \neq q$, which is analytic in t.

The first assertion is obvious. To prove the second, we first notice the following combinatorial property. Given two ordered k-tuples $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$, with $k \geq 3$, and an equivalence relation on $S = X \cup Y$, we have only two possibilities: either all the elements of S are equivalent, or we can find $i \neq j$ such that $x_i \in X$ and $y_j \in Y$ are not equivalent (indeed, if the second case does not hold, we see that $y_2, \dots, y_k$ must be equivalent to x_1, hence by transitivity also $x_2, \dots, x_k$ are equivalent to x_1, and finally y_1 is equivalent to x_2 hence to x_1).

Consider now the following equivalence relation on $C^\infty([-T,T])$: fixed $t_0 \in]-T,T[$, we say $f(t)$ and $g(t)$ are equivalent if they have the same order ν at $t = t_0$. By *order* of $f(t)$ at $t = t_0$ we mean the smallest integer $j \geq 0$ such that $f^{(j)}(t_0) \neq 0$, while we define $\nu = +\infty$ if all derivatives vanish. We can apply the preceding combinatorial remark to the triplets of functions

$$X = \left\{a_{12}(t), a_{23}(t), a_{31}(t)\right\} \quad \text{and} \quad Y = \left\{a_{13}(t), a_{32}(t), a_{21}(t)\right\};$$

we obtain that either all the functions have the same order at $t = t_0$, which can be finite or infinite, or else there exist $a_{pq} \in X$ and $a_{rs} \in Y$ with different order. Since we have assumed that one of the functions a_{ij} $(i \neq j)$ is analytic, in the case of all the functions with infinite order we must have $a_{ij} \equiv 0$, so that (3.20) is trivially satisfied; in all the other cases, it is easy to prove that (3.20) holds in a neighborhood of t_0, for a suitable pair of coefficients.

PROOF (sketch). Without loss of generality we can assume that (3.20) holds for $(p,q) = (2,3)$ and $(r,s) = (2,1)$, i.e.,

$$|a_{23}(t,\xi)| \leq C|a_{21}(t,\xi)|^\theta \tag{3.21}$$

for all t, ξ (in the other cases the proof is the same, after rearranging the indices).

Let us define, for $i \neq j$,

$$\alpha_{ij}^{\epsilon} = \left(|a_{ij}(t,\xi)|^2 + \epsilon^{2\nu(i,j)}\langle\xi\rangle^2\right)^{1/2}, \tag{3.22}$$

with $\nu(i, j) > 0$ integers to be chosen, and set

$$\lambda_1^{\epsilon} = \alpha_{21}^{\epsilon} \cdot \alpha_{31}^{\epsilon}, \quad \lambda_2^{\epsilon} = \alpha_{31}^{\epsilon} \cdot \alpha_{12}^{\epsilon}, \quad \lambda_3^{\epsilon} = \alpha_{13}^{\epsilon} \cdot \alpha_{21}^{\epsilon}. \tag{3.23}$$

The energy of $v = (v_1, v_2, v_3)$ will be defined as

$$E_{\epsilon}(t,\xi) = \lambda_1^{\epsilon}|v_1|^2 + \lambda_2^{\epsilon}|v_2|^2 + \lambda_3^{\epsilon}|v_3|^2. \tag{3.24}$$

Proceeding as above we obtain

$$E'_{\epsilon}(t,\xi) \leq \left[\sum_{j=1}^{3} \frac{|\partial_t \lambda_j^{\epsilon}|}{\lambda_j^{\epsilon}} + I_1^{\epsilon} + I_2^{\epsilon} + I_3^{\epsilon}\right] \cdot E_{\epsilon}(t,\xi)$$

where ($\alpha_{pq} = |a_{pq}|$; recall also (3.18))

$$I_1^{\epsilon} = \left|\sqrt{\frac{\lambda_2^{\epsilon}}{\lambda_3^{\epsilon}}}a_{23} - \bar{a}_{32}\sqrt{\frac{\lambda_3^{\epsilon}}{\lambda_2^{\epsilon}}}\right| = \frac{|\alpha_{31}^{\epsilon}\alpha_{12}^{\epsilon}\alpha_{23} - \alpha_{32}\alpha_{21}^{\epsilon}\alpha_{31}^{\epsilon}|}{\sqrt{\alpha_{13}^{\epsilon}\alpha_{31}^{\epsilon}\alpha_{21}^{\epsilon}\alpha_{12}^{\epsilon}}}$$

$$I_2^{\epsilon} = \left|\sqrt{\frac{\lambda_1^{\epsilon}}{\lambda_3^{\epsilon}}}a_{13} - \bar{a}_{31}\sqrt{\frac{\lambda_3^{\epsilon}}{\lambda_1^{\epsilon}}}\right| = \frac{|\alpha_{31}^{\epsilon}\alpha_{13} - \alpha_{13}^{\epsilon}\alpha_{31}|}{\sqrt{\alpha_{13}^{\epsilon}\alpha_{31}^{\epsilon}}}$$

$$I_3^{\epsilon} = \left|\sqrt{\frac{\lambda_1^{\epsilon}}{\lambda_2^{\epsilon}}}a_{12} - \bar{a}_{21}\sqrt{\frac{\lambda_2^{\epsilon}}{\lambda_1^{\epsilon}}}\right| = \frac{|\alpha_{12}^{\epsilon}\alpha_{21} - \alpha_{21}^{\epsilon}\alpha_{12}|}{\sqrt{\alpha_{12}^{\epsilon}\alpha_{21}^{\epsilon}}}.$$

As in the proof of Theorem 2, we have, for all $k \geq 1$,

$$\int_0^T \frac{|\partial_t \lambda_j^{\epsilon}|}{\lambda_j^{\epsilon}} dt \leq C(k,T)\epsilon^{-2\nu_0/k} \tag{3.25}$$

where $\nu_0 = \max_{i,j} \nu(i, j)$. The quantities I_2^{ϵ} and I_3^{ϵ} can be easily estimated as before, giving

$$I_2^{\epsilon} \leq C\langle\xi\rangle\left[\epsilon^{\nu(1,3)/2} + \epsilon^{\nu(3,1)/2}\right], \quad I_3^{\epsilon} \leq C\langle\xi\rangle\left[\epsilon^{\nu(1,2)/2} + \epsilon^{\nu(2,1)/2}\right]. \tag{3.26}$$

Let us now estimate the quantity I_1^{ϵ}. Using (3.19) we find

$$I_1^{\epsilon} \leq \frac{|\alpha_{31}^{\epsilon}\alpha_{12}^{\epsilon} - \alpha_{31}\alpha_{12}|}{\sqrt{\alpha_{31}^{\epsilon}\alpha_{12}^{\epsilon}}} \frac{\alpha_{23}}{\sqrt{\alpha_{21}^{\epsilon}\alpha_{13}^{\epsilon}}} + \frac{|\alpha_{21}^{\epsilon}\alpha_{13}^{\epsilon} - \alpha_{21}\alpha_{13}|}{\sqrt{\alpha_{21}^{\epsilon}\alpha_{13}^{\epsilon}}} \frac{\alpha_{32}}{\sqrt{\alpha_{31}^{\epsilon}\alpha_{12}^{\epsilon}}};$$

since $0 \leq \alpha_{ij}^{\epsilon} - \alpha_{ij} \leq \epsilon^{\nu(i,j)}$, this implies

$$I_1^{\epsilon} \leq C\epsilon^{[\nu(3,1)+\nu(1,2)]/2} \frac{\alpha_{23}}{\sqrt{\alpha_{21}^{\epsilon}\alpha_{13}^{\epsilon}}} + C\epsilon^{[\nu(2,1)+\nu(1,3)]/2} \frac{\alpha_{32}}{\sqrt{\alpha_{31}^{\epsilon}\alpha_{12}^{\epsilon}}}. \tag{3.27}$$

Recalling assumption (3.20), i.e.,

$$\alpha_{23} \leq C\alpha_{21}^{\theta}$$

we have

$$\frac{\alpha_{23}}{\sqrt{\alpha_{21}^{\epsilon}\alpha_{13}^{\epsilon}}} \leq C\frac{\alpha_{21}^{\theta}}{\sqrt{\alpha_{21}^{\epsilon}\alpha_{13}^{\epsilon}}} \leq C\frac{(\alpha_{21}^{\epsilon})^{\theta}}{\sqrt{\alpha_{21}^{\epsilon}\alpha_{13}^{\epsilon}}} \leq C\epsilon^{(\theta-1/2)\nu(2,1)-\nu(1,3)/2} \tag{3.28}$$

(we have assumed $\theta < 1/2$, which is no loss in generality), while

$$\frac{\alpha_{32}}{\sqrt{\alpha_{31}^{\epsilon}\alpha_{12}^{\epsilon}}} \leq C\epsilon^{[\nu(3,1)+\nu(1,2)]/2}. \tag{3.29}$$

In conclusion, (3.27) gives

$$I_1^{\epsilon} \leq C\langle\xi\rangle\left[\epsilon^{\sigma_1} + \epsilon^{\sigma_2}\right] \tag{3.30}$$

where

$$\sigma_1 = \frac{1}{2}[\nu(3,1) - \nu(1,3)] + \frac{1}{2}[\nu(1,2) - (1-2\theta)\nu(2,1)],$$

$$\sigma_2 = \frac{1}{2}[\nu(1,3) - \nu(3,1)] + \frac{1}{2}[\nu(2,1) - \nu(1,2)].$$

Thus, if we choose

$$\nu(1,3) = \nu(3,1) = \nu(3,2) = \nu(2,3) = \nu(2,1) = 1, \quad \nu(1,2) = 1-\theta, \tag{3.31}$$

we obtain the required estimate

$$I_1^{\epsilon} \leq C\epsilon^{\theta/2}. \tag{3.32}$$

Finally by Gronwall's lemma we have

$$E_{\epsilon}(t,\xi) \leq E_{\epsilon}(0,\xi)\exp C(k,T)[\epsilon^{-2/k} + \langle\xi\rangle\epsilon^{\theta/2}]. \tag{3.33}$$

for all $k \geq 1$. Choosing

$$\epsilon = \langle\xi\rangle^{-2k/(4+k\theta)}, \quad E(t,\xi) = E_{\epsilon}(t,\xi)|_{\epsilon=\langle\xi\rangle^{-2k/(4+k\theta)}},$$

we obtain the a priori estimate

$$E(t,\xi) \le E(0,\xi)\exp[C\langle\xi\rangle^{4/(4+k\theta)}] \tag{3.34}$$

which implies the well-posedness in $\gamma^s_{L^2}(\mathbb{R}^n)$, for

$$1 \le s < 1 + \frac{\theta}{4}k. \tag{3.35}$$

REMARK. We do not know if the pseudosymmetry condition alone, without the additional assumption (3.20), is sufficient to prove the well posedness in $\gamma^s_{L^2}$. However, we believe that some assumption like (3.20) is necessary, in order to avoid that the quotients a_{pq}/a_{rs} be unbounded near $t = 0$.

REMARK. The previous theory can be extended to the cases $N \ge 4$, but we need additional assumptions like (3.20). For instance, in the case $N = 4$ we can prove the well posedness in $\gamma^s_{L^2}$ for any pseudosymmetric system which satisfies also the assumption

$$|a_{21}|^2 \le C|a_{24}\cdot a_{41}|, \quad |a_{31}|^2 \le C|a_{34}\cdot a_{41}|,$$

$$|a_{32}|^2 \le C|a_{34}\cdot a_{42}|, \quad |a_{23}|^2 \le C|a_{24}\cdot a_{43}|,$$

or one of the similar conditions which can be obtained by suitable rearrangements of the indices.

ADDED IN PROOF. After the conclusion of the present paper, we had knowledge of an interesting paper of T. Nishitani [N] where he gives necessary and sufficient conditions for the C^∞ well-posedness of a 2×2 system with analytic coefficients, in one space variable. In particular, he proves the well-posedness for the weakly hyperbolic systems

$$u_t = \begin{pmatrix} c(t,x) & a(t,x) \\ b(t,x) & d(t,x) \end{pmatrix} u_x$$

under the condition

$$K \cdot \left(\frac{c-d}{2}\right)^2 + ab \ge 0$$

for some constant $K < 1$. This improves our Theorem 1, at least in the case $N = 2$, $n = 1$.

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