ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

R. DI LISIO
E. GRENIER
M. PULVIRENTI

On the regularization of the pressure field in compressible euler equations

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 24, nº 2 (1997), p. 227-238
<http://www.numdam.org/item?id=ASNSP_1997_4_24_2_227_0>

On the Regularization of the Pressure Field in Compressible Euler Equations

R. DI LISIO - E. GRENIER - M. PULVIRENTI

1. – Introduction

In this paper we want to study the stability of the Euler equations for a generic polytropic fluid, with respect to a regularization of the pressure field. The motivation of this study derives mostly from the problem of the convergence of the so called Smoothed Particle Hydrodynamics (SPH) method, a numerical scheme often used to compute solutions of compressible Euler equations. Indeed in a previous paper (see [2]), the same authors proved the convergence of the empirical measures arising from the SPH scheme to the solutions of a regularized version of the Euler equations for compressible flows (see equations (1.2) below). Therefore to complete the convergence proof one needs to show the convergence of the solutions of equations (1.2) to the corresponding solutions of the genuine system of Euler equations (1.1). This is the aim of the present paper. Besides the motivations related to the SPH method, we believe that the stability problem we takle here has an intrinsic interest.

Let us consider the Euler equations:

$$\begin{cases} \partial_t \bar{\rho} + \nabla \cdot (\bar{\rho}\bar{u}) = 0 \\ \partial_t \bar{u}(\bar{u} \cdot \nabla)\bar{u} + \bar{\rho}^{\alpha} \nabla \bar{\rho} = 0, \end{cases} \tag{1.1}$$

where $\bar{\rho} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $\bar{u} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ are the density and the velocity field respectively. These are the Euler equations for a fluid with the pressure defined by the state law $P = (\alpha + 2)^{-1} \bar{\rho}^{\alpha+2}$ (for $\alpha > -1$).

We regularize the pressure field in equations (1.1) in this way:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t u(u \cdot \nabla)u + \rho_\varepsilon^{\alpha} \nabla \rho_\varepsilon = 0, \end{cases} \tag{1.2}$$

where $\rho_\varepsilon = \rho * \delta_\varepsilon$ and $\delta_\varepsilon = g_\varepsilon * g_\varepsilon$. We assume that

$$g_\varepsilon(x) = \varepsilon^{-3} g_1(x\varepsilon^{-1}) \ \ x \in \mathbb{R}^3 \tag{1.3}$$

Pervenuto alla Redazione il 5 aprile 1996 e in forma definitiva il 8 novembre 1996.

where $g_1 \geq 0$ and $\int g_1 = 1$. g_ε is a mollifier which converges to the delta Dirac measure as $\varepsilon \to 0$ (see [1] for the case $\alpha = 0$, and [2] for the study of this system, and in particular for the existence of global solutions in a kinetic sense, obtained by passing to the Vlasov kinetic picture).

We shall consider periodic solutions of problems (1.1) and (1.2). The following classical Theorem (see [3]), establishes that problem (1.1) has a regular, local solution, if the initial condition is regular enough. Let $\Lambda = [-1, 1]^3$ be the periodic torus.

THEOREM 1.1 ([3]). *Given $\bar{\rho}_0 \in H^s(\Lambda)$, $\bar{u}_0 \in H^s(\Lambda)^3$, $\bar{\rho}_0 \geq m > 0$, $s > 5/2$, there exists a unique classical solution $(\bar{\rho}, \bar{u})$ of the problem* (1.1),

$$(\bar{\rho}, \bar{u}) \in C([0, T]; H^s(\Lambda)^4) \cap C^1([0, T]; H^{s-1}(\Lambda)^4),$$

with $\bar{\rho} > 0$, provided that T is small enough.

Our aim is to state the convergence of the regular solution of problem (1.2) to the solution of (1.1), with the same initial datum, as $\varepsilon \to 0$ (obviously locally in time). Namely we will prove

THEOREM 1.2. *Let $s > 11/2$. If g_1 satisfies*
a) there is $C > 0$ and $\eta > 0$ such that:

$$|D^\beta g_1(x)||x|^4 \leq \frac{C}{1 + |x|^{3/2+\eta}} \tag{1.4}$$

for $|\beta| = 1, 2$,
b) there is $C > 0$ such that

$$|\lambda||D^\beta \tilde{g}_1(\lambda)| < C|\tilde{g}_1(\lambda)|, \tag{1.5}$$

for $|\beta| = 1, 2, 3$, (where $\tilde{f}$ denotes the Fourier transform of the function f), and if the initial data ρ_0 and u_0 belong to $H^s(\Lambda)$ and $\inf_{x\in\Lambda} \rho_0 > 0$, then there exists a positive time T such that, for the solutions $(\bar{\rho}, \bar{u})$ of (1.1) *and (ρ, u) of* (1.2), *the following limit holds:*

$$\lim_{\varepsilon\to 0}(||\bar{\rho} - \rho||_{s'-1} + ||\bar{u} - u||_{s'}) = 0, \qquad \forall t \leq T, \forall s' < s. \tag{1.6}$$

Examples of mollifiers satisfying these conditions are those for which $\tilde{g}(\lambda) = (1 + \lambda^2)^{-p}$ with p large enough.

We mention that L^2 estimates for the regularized problem has been proved by Oelschläger [4] in the case $\alpha = 0$ (see also Caprino *et al.* [1]).

The plan of the paper is the following: first we obtain a-priori bounds in Sobolev norms by studying a suitable norm. Then we prove the convergence in L^2 norm. The appendix is devoted to further considerations involving pseudodifferential operators which have led us to the choice of the right norm.

2. – A-priori estimates in H^s norm

Consider the box $\Lambda = [-1, 1]^3$ (3 is the dimension of the physical space) and two fluids described by equations (1.1) and (1.2). We assume periodic boundary conditions so that Λ can be thought as a 3-dimensional flat torus. The density ρ and the velocity field u can be extended to the whole space $\mathbb{R}^3$ by periodicity. The convolution operator on $\mathbb{R}^3$ is denoted by the symbol $*$.

If $f(x)$ is a function on Λ and $\bar{f}(x)$ is its periodic extension on $\mathbb{R}^3$, we define:

$$\begin{aligned} g * f(x) &= \int_{\mathbb{R}^3} g(x-y)\bar{f}(y)\,dy = \sum_{k\in Z^3} \int_\Lambda g(x+2k-y) f(y)\,dy \\ &= \left[\left(\sum_k g_k\right) *_\Lambda f\right](x), \end{aligned} \tag{2.1}$$

where $*_\Lambda$ means the convolution product on the torus Λ, and $g_k(x) = g(x+2k)$.

Let us introduce the following seminorm:

$$W_s = \frac{1}{2}\int_\Lambda \rho|D^s u|^2 + \frac{1}{2}\int_\Lambda \rho_\varepsilon^\alpha |g_\varepsilon * D^s\rho|^2 + \int_\Lambda |D^{s-1}\rho|^2, \tag{2.2}$$

where

$$D^s = \frac{\partial^{s_1+s_2+s_3}}{\partial^{s_1}x_1\partial^{s_2}x_2\partial^{s_3}x_3} \quad s = (s_1, s_2, s_3) \ \ (|s| = s_1+s_2+s_3). \tag{2.3}$$

In what follows we shall consider only the case when s is an integer. $|\cdot|_p$ and $||\cdot||_s$ denote the $L^p(\Lambda)$ and $H^s(\Lambda)$ Sobolev norms respectively.

The semi-norm W_s controls $|D^\gamma u|_0$ for $|\gamma| = s$, and $|D^\gamma\rho|_0$ for $|\gamma| = s-1$ (see the Appendix).

We will only look for a-priori estimates of W_s on the solution of the regularized problem (1.2). The existence of such a solution can then be deduced by standard arguments (for instance using the Galerkin method).

Notice that, for a fixed initial condition ($\rho_0(x)$,$u_0(x)$), the solution of equations (1.2), $\rho(x,t)$, has a lower and an upper bound. In fact, by the continuity equation we have:

$$\begin{aligned} \rho(x,t) &\geq \rho_0(x)\exp\left(-\int_0^t |\nabla\cdot u|(x)ds\right) \\ &\geq \inf_x \rho_0 \exp\big(-ct \sup_{0\leq s\leq t} ||u||_{\frac{5}{2}}(s)\big). \end{aligned} \tag{2.4}$$

Analogously

$$\rho(x,t) \leq |\rho_0|_\infty \exp\big(ct \sup_{0\leq s\leq t} ||u||_{\frac{5}{2}}(s)\big), \tag{2.5}$$

so that Theorem 1.1 assures the existence of upper and lower bounds for the solution $\bar{\rho}$ of problem (1.1).

To obtain an a priori estimate for W_s (uniform in ε) we consider its time derivative. We have:

$$\begin{aligned}\dot{W}_s =& \frac{1}{2}\int_\Lambda \partial_t\rho|D^s u|^2 \\ & - \int_\Lambda \rho D^s u D^s[(u\cdot\nabla)u] - \int_\Lambda \rho D^s u D^s[\rho_\varepsilon^\alpha \nabla\rho_\varepsilon] \\ & + \frac{1}{2}\int_\Lambda \partial_t\rho_\varepsilon^\alpha |g_\varepsilon * D^s\rho|^2 + \int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho) g_\varepsilon * D^s\partial_t\rho \\ & + \int_\Lambda D^{s-1}\rho D^{s-1}\partial_t\rho \\ =& T_1 + T_2 + T_3 + T_4 + T_5 + T_6.\end{aligned} \tag{2.6}$$

To estimate $\dot{W}_s$, integration by parts, Hölder inequality and Moser type inequalities will be often used (see for example [3] p.43). In particular, we recall:

$$||D^s(fg) - fD^s g||_0 \leq C_s(|Df|_\infty ||D^{s-1}g||_0 + |g|_\infty ||D^s f||_0).$$

In the sequel C_s will denote the generic positive constant possibly depending on the parameter s. We have:

$$\begin{aligned} T_1 + T_2 &= -\int_\Lambda \rho D^s u[D^s(u\cdot\nabla)u - (u\cdot\nabla)D^s u] \\ &\leq ||\rho D^s u||_0 ||D^s(u\cdot\nabla)u - (u\cdot\nabla)D^s u||_0 \\ &\leq C_s|\rho|_\infty^{1/2} W_s^{1/2} |Du|_\infty ||D^s u||_0.\end{aligned} \tag{2.7}$$

Furthermore:

$$T_6 = -\int_\Lambda (D^{s-1}\rho)D^{s-1}(\rho\nabla\cdot u + u\cdot\nabla\rho) = T_{61} + T_{62}, \tag{2.8}$$

$$T_{61} = -\int_\Lambda \nabla\cdot u|D^{s-1}\rho|^2 - \int_\Lambda D^{s-1}\rho[D^{s-1}(\rho\nabla\cdot u) - \nabla\cdot u D^{s-1}\rho]$$

so

$$|T_{61}| \leq |\nabla u|_\infty W_s + C_s||D^{s-1}\rho||_0(|\rho|_\infty ||D^s u||_0 + |\nabla u|_\infty ||D^{s-1}\rho||_0), \tag{2.9}$$

$$T_{62} = \frac{1}{2}\int_\Lambda (D^{s-1}\rho)^2\nabla\cdot u - \int_\Lambda (D^{s-1}\rho)[D^{s-1}(u\cdot\nabla\rho) - u\cdot\nabla(D^{s-1}\rho)]$$

so

$$\begin{aligned}|T_{62}| \leq & W_s|\nabla . u|_\infty \\ & + C_s||D^{s-1}\rho||_0(|Du|_\infty ||D^{s-1}\nabla\rho||_0 + |\nabla\rho|_\infty ||D^{s-1}u||_0).\end{aligned} \tag{2.10}$$

Moreover,

$$T_4 \le |\frac{\partial_t \rho_\varepsilon^\alpha}{\rho_\varepsilon^\alpha}|_\infty W_s \le \frac{\alpha}{2}|\rho_\varepsilon^{-1}|_\infty |\nabla\cdot(\rho u)|_\infty W_s. \tag{2.11}$$

$$\begin{aligned} T_3 = &-\int_\Lambda \rho D^s u \rho_\varepsilon^\alpha D^s \nabla\rho_\varepsilon \\ &-\int_\Lambda \rho D^s u\cdot[D^s(\rho_\varepsilon^\alpha\nabla\rho_\varepsilon) - \rho_\varepsilon^\alpha D^s\nabla\rho_\varepsilon] = T_{31} + T_{32}, \end{aligned} \tag{2.12}$$

$$T_{32} \le ||\rho D^s u||_0 (|\nabla\rho_\varepsilon^\alpha|_\infty ||D^s\rho_\varepsilon||_0 + |\nabla\rho_\varepsilon|_\infty ||D^s\rho_\varepsilon^\alpha||_0). \tag{2.13}$$

It remains to estimate, and we shall do it later

$$T_{31} = -\int_\Lambda \rho D^s u \rho_\varepsilon^\alpha D^s\nabla\rho_\varepsilon. \tag{2.14}$$

Moreover:

$$\begin{aligned} T_5 = &-\int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho) g_\varepsilon * \nabla\cdot(u D^s\rho) \\ &-\int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho) g_\varepsilon * \nabla\cdot(\rho D^s u) \\ &- s\int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho) g_\varepsilon * \nabla\cdot(D^{s-1}\rho D u) \\ &- \sum_{\beta=1}^{s-2} C_s^{\beta+1}\int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)(g_\varepsilon * \nabla\cdot(D^{s-1-\beta}\rho D^{\beta+1}u)) \\ = &\; T_{51} + T_{52} + sT_{53} + T_{54}. \end{aligned} \tag{2.15}$$

By Schwartz inequality, if $s > 7/2$,

$$\begin{aligned} T_{54} &\le C||\rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)||_0 \sum_{\beta=1}^{s-2} ||g_\varepsilon * \nabla\cdot(D^{s-1-\beta}\rho D^{\beta+1}u)||_0 \\ &\le C|\rho^\alpha|_\infty W_s^{1/2} ||\rho||_{s-1} ||u||_{s-1}. \end{aligned} \tag{2.16}$$

$$\begin{aligned} T_{52} &= \int_\Lambda \nabla[\rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)] g_\varepsilon * (\rho D^s u) \\ &= \int_\Lambda \nabla\rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho) g_\varepsilon * (\rho D^s u) \\ &\quad + \int_\Lambda \rho_\varepsilon^\alpha \nabla(g_\varepsilon * D^s\rho)(g_\varepsilon * (\rho D^s u)) = T_{52a} + T_{52b}. \end{aligned}$$

We have

$$|T_{52a}| \le \alpha |\nabla\rho_\varepsilon|_\infty |\rho^{-1}|_\infty |\rho|_\infty^{\alpha/2} W_s \tag{2.17}$$

and, recalling that $\delta_\varepsilon = g_\varepsilon * g_\varepsilon$,

$$
\begin{aligned}
&T_{52b} + T_{31} \\
&= \int_\Lambda g_\varepsilon * (\rho_\varepsilon^\alpha \nabla (g_\varepsilon * D^s \rho))\rho D^s u - \int_\Lambda \rho_\varepsilon^\alpha g_\varepsilon * (\nabla (g_\varepsilon * D^s\rho))\rho D^s u \\
&= \int_\Lambda dx\, \rho(x) D^s u(x) \int_\Lambda dy\, g_\varepsilon (x-y)[\rho_\varepsilon^\alpha(y) - \rho_\varepsilon^\alpha(x)] \nabla (g_\varepsilon * D^s\rho)(y) \\
&= -\int_\Lambda dx\, \rho(x) D^s u(x) \int_\Lambda dy\, \nabla g_\varepsilon (x-y)[\rho_\varepsilon^\alpha(y) - \rho_\varepsilon^\alpha(x)] (g_\varepsilon * D^s\rho)(y) \\
&\quad -\int_\Lambda dx\, \rho(x) D^s u(x) \int_\Lambda dy\, g_\varepsilon (x-y) \nabla \rho_\varepsilon^\alpha(y) (g_\varepsilon * D^s\rho)(y),
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
&|T_{52b} + T_{31}| \\
&\le \int_\Lambda dx\, \rho(x) |D^s u|(x) \int_\Lambda dy |\nabla \rho_\varepsilon^\alpha|_\infty |x-y| |\nabla g_\varepsilon|(x-y) |g_\varepsilon * D^s \rho|(y) \\
&\quad + |\nabla \rho_\varepsilon^\alpha|_\infty ||\rho(D^s u)||_0 ||g_\varepsilon * D^s\rho||_0 \\
&\le |\nabla \rho_\varepsilon^\alpha|_\infty \left(\int_\Lambda \int_\Lambda \rho^2(x) |D^s u|^2(x) |\nabla g_\varepsilon|(x-y)|x-y| dx\ dy \right)^{1/2} \\
&\quad \times \left(\int_\Lambda \int_\Lambda (g_\varepsilon * D^s\rho)^2(y) |\nabla g_\varepsilon|(x-y)|x-y| \right)^{1/2} \\
&\quad + |\nabla \rho_\varepsilon^\alpha|_\infty ||\rho(D^s u)||_0 ||g_\varepsilon * D^s\rho||_0 \\
&\le C |\nabla \rho_\varepsilon^\alpha|_\infty |\rho_\varepsilon^{-\alpha}|_\infty |\rho|_\infty^{1/2} W_s,
\end{aligned} \tag{2.19}
$$

provided that

$$
\int_{\mathbb{R}^3} |\nabla g_1|(x)|x| d^3x \le C,
$$

which is ensured by assumption (a) of Theorem 1.2.

$$
\begin{aligned}
T_{51} &= -\int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)(\nabla \cdot [g_\varepsilon * (u D^s \rho)] - \nabla \cdot [u g_\varepsilon * D^s\rho]) \\
&\quad - \int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho) \nabla \cdot [u g_\varepsilon * D^s \rho] = T_{51a} + T_{51b},
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
T_{51b} &= -\int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)((g_\varepsilon * D^s\rho)\nabla \cdot u + u \cdot \nabla (g_\varepsilon * D^s\rho)) \\
&= -\int_\Lambda \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)^2 \nabla \cdot u + \frac{1}{2} \int_\Lambda [\rho_\varepsilon^\alpha \nabla \cdot u + u \cdot \nabla \rho_\varepsilon^\alpha](g_\varepsilon * D^s\rho)^2
\end{aligned}
$$

so

$$
|T_{51b}| \le C W_s (|\nabla \cdot u|_\infty + |u|_\infty |\nabla \rho|_\infty |\rho^{-1}|_\infty). \tag{2.21}
$$

Finally the terms T_{51a} and T_{53} are the usual terms which can be handled by a Taylor expansion as shown by Oelschläger [4]. To this end, following Caprino *et al.* [1], we need assumptions (a) and (b) of Theorem 1.2 on g_1.

The term T_{51a} can be written as

$$T_{51a} = \int_\Lambda dx\, \rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho) \left[\int_{\mathbb{R}^3} dy\, (\nabla \cdot u(x)) g_\varepsilon(x-y) D^s\rho(y) - \int dy\, (u(x) - u(y)) \cdot \nabla g_\varepsilon(x-y) D^s\rho(y) \right]. \tag{2.22}$$

The first term can be easily estimated by $||\nabla u||_\infty W_s$. So, we shall consider the second one.

We perform a Taylor expansion of u up to the fourth order. The $i-th$ component gives

$$\begin{aligned} &u_i(x) - u_i(y) \\ &= \nabla u_i(x)\cdot(x-y) + \frac{1}{2}\sum_{h,k=1}^{3} \frac{\partial^2}{\partial x_h \partial x_k} u_i(x)(x-y)_h(x-y)_k \\ &+ \frac{1}{3!}\sum_{h,k,r=1}^{3} \frac{\partial^3}{\partial x_h \partial x_k \partial x_r} u_i(x)(x-y)_h(x-y)_k(x-y)_r \\ &+ \frac{1}{3!}\sum_{h,k,r,l=1}^{3} \int_0^1 d\theta (1-\theta)^3 \frac{\partial^4}{\partial x_h \partial x_k \partial x_r \partial x_l} u_i(x+\theta(y-x)) \\ &\times (x-y)_h(x-y)_k(x-y)_r(x-y)_l. \end{aligned} \tag{2.23}$$

Substituting this sum in the second part of (2.22) we have four new terms. The first one is bounded by

$$||\nabla u||_\infty ||\rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)||_0 ||(x\cdot\nabla g_\varepsilon) * D^s\rho||_0. \tag{2.24}$$

By the Plancherel theorem on Λ, using (2.1), assumption b, and the identity $\tilde{g}_\varepsilon(p) = \tilde{g}_1(\varepsilon p)$:

$$\begin{aligned} ||(x\cdot\nabla g_\varepsilon) * D^s\rho||_0^2 &= \sum_{p\in Z^3} |\widetilde{D^s\rho}|^2(p)|p|^2|\nabla_p \tilde{g}_\varepsilon|^2(p) \\ &= \sum_{p\in Z^3} |\widetilde{D^s\rho}|^2(p)|p|^2|\nabla_p \tilde{g}_1|^2(\varepsilon p) \\ &\leq C\sum_{p\in Z^3} |\widetilde{D^s\rho}|^2(p)|\tilde{g}_1|^2(\varepsilon p) = C||g_\varepsilon * D^s\rho||_0^2. \end{aligned} \tag{2.25}$$

The second and third terms can be estimated in a completely analogous way, namely by:

$$C(||D^2u||_\infty + ||D^3u||_\infty)||\rho_\varepsilon^\alpha (g_\varepsilon * D^s\rho)||_0 ||g_\varepsilon * D^s\rho||_0^2 \tag{2.26}$$

after using (1.4) for $|\beta| = 2, 3$.

It remains to estimate

$$
\begin{aligned}
&\int_\Lambda dx\, \rho_\varepsilon^\alpha(x)(g_\varepsilon * D^s\rho)(x) \\
&\times \frac{1}{3!}\sum_{h,k,r,l=1}^{3}\int_0^1 d\theta(1-\theta)^3 \frac{\partial^4}{\partial x_h \partial x_k \partial x_r \partial x_l} u_i(x+\theta(y-x)) \\
&\times (x-y)_h(x-y)_k(x-y)_r(x-y)_l.
\end{aligned} \tag{2.27}
$$

To handle this term we perform an integration by parts. We obtain inside the integral $\int_0^1 d\theta(1-\theta)^3$, a sum of terms of the type:

$$
\begin{aligned}
&C_1 \int_\Lambda dx\, \rho_\varepsilon^\alpha(g_\varepsilon * D^s\rho) \\
&\times \int_{\mathbb{R}^3} dy\, (x-y)_h(x-y)_k(x-y)_r \frac{\partial^4}{\partial x_h \partial x_k \partial x_r \partial x_l} u(x-\theta(y-x)) \\
&\times D^{s-1}\rho(y)\nabla g_\varepsilon(x-y), \\
&C_2 \int_\Lambda dx\, \rho_\varepsilon^\alpha(g_\varepsilon * D^s\rho) \int_{\mathbb{R}^3} dy\, (x-y)_h(x-y)_k(x-y)_r(x-y)_l \\
&\times \frac{\partial^5}{\partial x_h \partial x_k \partial x_r \partial x_l \partial x_j} u(x-\theta(y-x)) D^{s-1}\rho(y)\nabla g_\varepsilon(x-y), \\
&C_3 \int_\Lambda dx\, \rho_\varepsilon^\alpha(g_\varepsilon * D^s\rho) \int_{\mathbb{R}^3} dy\, (x-y)_h(x-y)_k(x-y)_r(x-y)_l \\
&\times \frac{\partial^4}{\partial x_h \partial x_k \partial x_r \partial x_l} u(x-\theta(y-x)) D^{s-1}\rho(y) D^2 g_\varepsilon(x-y).
\end{aligned} \tag{2.28}
$$

We estimate the first one.

$$
\begin{aligned}
&\int_\Lambda dx\, \rho_\varepsilon^\alpha(g_\varepsilon * D^s\rho) \int_{\mathbb{R}^3} dy\, (x-y)_h(x-y)_k(x-y)_r \\
&\times \frac{\partial^4}{\partial x_h \partial x_k \partial x_r \partial x_l} u(x-\theta(y-x)) D^{s-1}\rho(y)\nabla g_\varepsilon(x-y) \\
&\qquad \le ||\rho_\varepsilon^\alpha(g_\varepsilon * D^s\rho)||_0 \times \\
&\left[\int_\Lambda dx \left(\int_{\mathbb{R}^3} dy (x-y)_h(x-y)_k(x-y)_r \frac{\partial^4}{\partial x_h \partial x_k \partial x_r \partial x_l} u(x-\theta(y-x)) \right.\right. \\
&\qquad \left.\left. \times D^{s-1}\rho(y)\nabla g_\varepsilon(x-y)\right)^2\right]^{\frac{1}{2}} \\
&\le C||\rho_\varepsilon^\alpha(g_\varepsilon * D^s\rho)||_0 ||D^{s-1}\rho||_0 ||D^4 u||_\infty |\Lambda|^{\frac{1}{2}} \left(\int dz|z|^6|\nabla g_\varepsilon|(z)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

But

$$\int dz|z|^6|\nabla g_\varepsilon|^2(z) = \int dz|z|^6\varepsilon^{-8}|\nabla g_1|^2\left(\frac{z}{\varepsilon}\right) = \varepsilon\int d\tilde{z}|\tilde{z}|^6|\nabla g_1|^2(\tilde{z}) \le C$$

by assumption a).

The other two terms can be estimated in the same way.

All these estimates lead to

$$T_{51a} \le C W_s. \tag{2.29}$$

All the above estimates yield:

$$\begin{aligned}\dot{W}_s \le f(&W_s, |\rho|_{L^\infty}, |\rho_\varepsilon|_{L^\infty}, |\rho^{-1}|_{L^\infty}, |\rho_\varepsilon^{-1}|_{L^\infty},\\ &|u|_{L^\infty}, |Du|_{L^\infty}, |D^2u|_{L^\infty}, |D^3u|_{L^\infty}, |D^4u|_{L^\infty}),\end{aligned} \tag{2.30}$$

where f is a continuous smooth function, increasing in each of its arguments. Let

$$Z_\sigma = \sum_{s\le\sigma} W_s. \tag{2.31}$$

If $\sigma > 4 + 3/2 = 11/2$, then

$$\dot{Z}_\sigma \le f(Z_\sigma), \tag{2.32}$$

which leads to energy estimates and bounds uniform in ε on the solution of equations (1.2), on a time interval $[0, T]$ independent on ε.

3. – Convergence in L^2 norm

Now we are in position to obtain convergence in Sobolev norms, as $\delta_\varepsilon \to \delta$. By the interpolation inequalities and the H^s bound found in the previous section, it is enough to prove the L^2 convergence of (ρ, u) to $(\bar{\rho}, \bar{u})$.

To this end, let us introduce the following function

$$Q(t) = \frac{1}{2}\int_\Lambda \rho\bar{\rho}^{\alpha-1}(\bar{\rho}-\rho)^2 + \frac{1}{2}\int_\Lambda \bar{\rho}(\bar{u}-u)^2, \tag{3.1}$$

where $(\bar{\rho}, \bar{u})$ is the regular solution of (1.1) and (ρ, u) is the regular one associated to (1.2).

We suppose that there exists a time T^* such that:

i) $0 < m \le \bar{\rho}(x,t),\ \rho(x,t) \le M < +\infty, \qquad \forall x \in \Lambda, \quad \forall t \le T^*;$

ii) $\sup(||\bar{\rho}||_s,\ ||\bar{u}||_s,\ ||\rho||_{s-1},\ ||u||_s) \le C < +\infty, \qquad \forall t \le T^*.$

These hypotheses are satisfied for regular enough initial data ρ_0, u_0.
It is clear that

$$\|\bar{\rho} - \rho\|_0^2 + \|\bar{u} - u\|_0^2 \leq C'Q(t), \tag{3.2}$$

for some $C' < +\infty$. So it is sufficient to show that $Q(t) \to 0$ as $\delta_\varepsilon \to \delta$.

We proceed as before performing the time derivative of $Q(t)$. By straightforward calculations we see that:

$$\frac{1}{2}\frac{d}{dt}\int_\Lambda \rho\bar{\rho}^{\alpha-1}(\bar{\rho} - \rho)^2 \leq CQ(t) - \int_\Lambda \rho\bar{\rho}^\alpha(\bar{\rho} - \rho)\nabla \cdot (\bar{u} - u), \tag{3.3}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Lambda \bar{\rho}(\bar{u} - u)^2 &\leq CQ(t) - \int_\Lambda \rho\bar{\rho}^\alpha(\bar{u} - u)\nabla(\bar{\rho} - \rho_\varepsilon) \\ &\leq C'Q(t) - \int_\Lambda \rho\bar{\rho}^\alpha(\bar{u} - u)\nabla(\bar{\rho} - \rho) - \int_\Lambda \rho\bar{\rho}^\alpha(\bar{u} - u)\nabla(\rho - \rho * \delta_\varepsilon). \end{aligned} \tag{3.4}$$

The second term cancels with the last term of (3.3) and the third is bounded using

$$\int_{\mathbb{R}^3} \delta_\varepsilon(x)|x|\,dx \leq C\varepsilon, \tag{3.5}$$

which gives

$$\frac{dQ}{dt} \leq C(Q(t) + \varepsilon) \ \forall t \leq T^*, \tag{3.6}$$

where C does not depend on ε. So, assuming the same initial conditions for the problems (1.1) and (1.2), the convergence result follows. The end of the proof of Theorem 1.2 is then straightforward.

Appendix: a pseudodifferential approach

The proof of the main Theorem was in fact first done by using the pseudodifferential operators machinery, which leads, in a natural way, to the energy norm (the sum of the seminorms (2.2)) studied in this paper, and which is an approach equivalent to the use of smoothing kernels. In this section we want to give the main ideas.

First, we study the linearized version of (1.2) around a constant state $(\hat{u},\hat{\rho})$ (frozen coefficients)

$$\partial_t u + (\hat{u} \cdot \nabla)u = \hat{\rho}_\varepsilon^\alpha \nabla(\rho * \delta_\varepsilon), \tag{A.1}$$

$$\partial_t \rho + (\hat{u} \cdot \nabla)\rho + \hat{\rho} \text{ div } u = 0. \tag{A.2}$$

The symbol of this system is

$$A^\varepsilon = \begin{pmatrix} i\hat{u}\cdot\xi & 0 & 0 & i\hat{\rho}_\varepsilon^\alpha \xi_1 \tilde{g}_1^2(\varepsilon\xi) \\ 0 & i\hat{u}\cdot\xi & 0 & i\hat{\rho}_\varepsilon^\alpha \xi_2 \tilde{g}_1^2(\varepsilon\xi) \\ 0 & 0 & i\hat{u}\cdot\xi & i\hat{\rho}_\varepsilon^\alpha \xi_3 \tilde{g}_1^2(\varepsilon\xi) \\ i\hat{\rho}\xi_1 & i\hat{\rho}\xi_2 & i\hat{\rho}\xi_3 & i\hat{u}\cdot\xi \end{pmatrix} \tag{A.3}$$

(where ξ is the dual Fourier variable of x), in the sense that

$$\partial_t \begin{pmatrix} u \\ \rho \end{pmatrix} + Op(A^\varepsilon(\xi, \hat{u}, \hat{\rho})) \begin{pmatrix} u \\ \rho \end{pmatrix} = 0 \tag{A.4}$$

is equivalent to (A.1), (A.2) where

$$Op(A^\varepsilon(\xi, \hat{u}, \hat{\rho})) \begin{pmatrix} u \\ \rho \end{pmatrix} = \int_{\mathbb{R}^3} e^{ix\cdot\xi} A^\varepsilon(\xi, \hat{u}, \hat{\rho}) \mathcal{F} \begin{pmatrix} u \\ \rho \end{pmatrix} (\xi) d\xi, \tag{A.5}$$

$\mathcal{F}$ denoting the Fourier transform.

The standard way to study (A.5) is to symmetrize A^ε, that is to look for a positive definite matrix $S^\varepsilon = {}^t P^\varepsilon P^\varepsilon$ such that $S^\varepsilon A^\varepsilon$ is symmetric (see for instance [3]). Here we take

$$P^\varepsilon = \begin{pmatrix} \sqrt{\hat{\rho}_\varepsilon} & 0 & 0 & 0 \\ 0 & \sqrt{\hat{\rho}_\varepsilon} & 0 & 0 \\ 0 & 0 & \sqrt{\hat{\rho}_\varepsilon} & 0 \\ 0 & 0 & 0 & \sqrt{\hat{\rho}_\varepsilon^\alpha \tilde{g}_1^2(\varepsilon\xi)} \end{pmatrix} \tag{A.6}$$

which leads to the norm

$$|||(u, \rho)||| = ||Op(P^\varepsilon(\xi, \hat{u}, \hat{\rho}))(u, \rho)||_{L^2}. \tag{A.7}$$

The second step is to study

$$|||(u, \rho)|||_s = ||Op(P^\varepsilon(\xi, u, \rho))(\Lambda^s u, \Lambda^s \rho)||_{L^2} \tag{A.8}$$

where $\Lambda^s = (1 - \Delta)^{s/2}$, and where the coefficients are no longer frozen. This can be done by differentiating $|||(u, \rho)|||_s$ and using estimates on commutators and adjoints of pseudodifferential operators, or equivalently by using calculus on kernels as in this paper.

Notice that $|||(u, \rho)|||_s$ controls $||u||_{H^s}$ and $||\rho_\varepsilon||_{H^s}$, which leads with (A.2) to a control on $||\rho||_{H^{s-1}}$. So it is natural to consider the norm

$$||\rho||_{H^{s-1}} + ||\rho_\varepsilon||_{H^s} + ||u||_{H^s} \tag{A.9}$$

or, equivalently that one used in this paper.

More heuristically, we see that at high frequencies, the system degenerates in

$$\partial_t u + (u \cdot \nabla) u = 0, \tag{A.10}$$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0. \tag{A.11}$$

If we control $||u||_{H^s}$, we can only control $||\rho||_{H^{s-1}}$ by (A.11). On the contrary, at low frequencies the system behaves like the limit system (Euler equations for compressible fluids), where we can control $||\rho||_{H^s}$. The "transition" between the original system and (A.10), (A.11) occurs when $\varepsilon|\xi| \sim 1$, and is described by $||\rho_\varepsilon||_{H^s}$ which does not take into account wave numbers $|\xi| >> \varepsilon^{-1}$, and behaves like $||\rho||_{H^s}$ for wave numbers $|\xi| << \varepsilon^{-1}$.

REFERENCES

[1] S. Caprino - R. Esposito - R. Marra - M. Pulvirenti, *" Hydrodynamic limits of the Vlasov equation"*, Comm. Partial Differential Equations **18** (1993), 805-820.

[2] R. Di Lisio - E. Grenier - M. Pulvirenti, *On the convergence of the SPH method*, in Special issue devoted to Simulations Methods in Kinetic Theory, C. Cercignani and R. Illner (eds.), Computers and Mathematics with Applications, 1995.

[3] A. Majda, Compressible Fluid Flow and System of Conservations Laws in Several Space Variables, Springer-Verlag, New York, 1984.

[4] K. Oelschläger, *On the connection between Hamiltonian many-particle systems and hydrodynamical equations*, Arch. Rat. Mech. Anal. **115** (1991), 297-310.

R. Di Lisio - M. Pulvirenti:
Dipartimento di Matematica
Università di Roma "La Sapienza"
P.le Aldo Moro, 5
I-00185 Roma

E. Grenier:
Laboratoire d'Analyse Numérique
URA-CNRS 189
Université Paris 6
4 place Jussieu
F-75252 Cedex 05