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Partial regularity of free discontinuity sets, I

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Partial Regularity of Free Discontinuity Sets, I

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1. - Introduction

In the last few years new variational problems have been studied, where the functionals involved depend on a variable closed set K and a function which is smooth outside K (see e.g. [3], [4], [7], [13], [17], [18], [24]): such problems have been called *free discontinuity problems* (notice that K is not necessarily a boundary). The canonical example of free discontinuity problem is the minimization of the so-called Mumford-Shah functional

$$(1.1) \qquad G(u,K) = \int_{\Omega\setminus K} [|\nabla u|^2 + (u-g)^2]dx + \mathcal{H}^{n-1}(K),$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $g \in L^\infty(\Omega)$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. In the two dimensional case, this functional has been proposed (see [24], [23]) as a variational model of image segmentation: g is the grey level function giving the image to be segmented, K is the competing segmentation, and $u \in C^1(\Omega \setminus K)$ is the smoothed approximation of g. The proof of the existence of minimizing pairs (K,u) is due to De Giorgi, Carriero and Leaci (see [19]): they relax the problem in the space SBV of special functions with bounded variation introduced in [18], where a weak solution is known to exist (see [3]), and then prove that from any weak minimizer in SBV a minimizing pair for the original functional can be constructed. A different proof in the planar case is also given in [14].

The singular sets K of minimizers are known only to be rectifiable, but are expected to be piecewise regular (see [17]). The main aim of this and of the subsequent paper [6] is to prove some results in this direction (see [15], [16] and [8] for some results in the case $n=2$ and [25], where the main results of our two papers are summarized). More generally, we study a class of perturbations of the functional

$$(1.2) \qquad F(u,K) = \int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathcal{H}^{n-1}(K)$$

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which is suitable to encompass G, and we prove that if (K,u) is a quasi minimizing pair in Ω and $|\nabla u|$ belongs to the Morrey space $L^{2,\lambda}(\Omega)$ for some $\lambda > n-1$, then K is a $C^{1,\alpha}$ hypersurface outside a relatively closed, $\mathcal{H}^{n-1}$-negligible singular set (see Theorem 3.1 and the subsequent remarks).

The main difficulty in facing the regularity problem for (quasi) minimizers of F relies in the necessity of dealing at the same time with an elliptic term and a surface integral; this can perhaps be more easily seen by looking at the first variation formula (see [12]):

$$\int_{\Omega} \left[|\nabla u|^2 \mathrm{div}\eta - 2\langle \nabla u, \nabla u \nabla \eta \rangle \right] dx + \int_K \mathrm{div}^\tau \eta \, d\mathcal{H}^{n-1} = 0 \tag{1.3}$$

(where div^τ is the tangential divergence along K) which holds for any vector field $\eta \in C_0^1(\Omega, \mathbb{R}^n)$ and any local minimizer u of (1.2). In fact, the surface integral in (1.3) is the first variation of area, hence (1.3) shows that the mean curvature of K is related to the Dirichlet integral of u. Thus, even though we have taken as a starting point the regularity theory of minimal surfaces, as it has been developed in the context of varifolds by Allard in [1], it has been necessary to adapt these ideas to the peculiar form of the term controlling the mean curvature of K. In particular, we borrow many ideas from Brakke's book [9], where the properties of varifolds with mean curvature in L^1 are analyzed. However, to make the paper more self contained, we will not directly use results from the theory of varifolds.

In this paper we do not treat the regularity problem in full generality, because of the a priori assumption $|\nabla u| \in L^{2,\lambda}(\Omega)$ (see Remark 3.3): this hypothesis will be removed in [6] by means of a suitable decay lemma for the Dirichlet integral which will be the natural counterpart of the flatness improvement Theorem 6.2 of this paper. Looking for decay properties of the Dirichlet integral is a natural approach, because u solves a Neumann problem in the domain $\Omega \setminus K$, which has a highly irregular (*a priori* only rectifiable) boundary, hence no apriori estimates on the Dirichlet integral of u near K are available. A similar approach to the the regularity of free interface problems has been developed by Lin in [22].

Before passing to a brief description of the paper, we point out that the techniques developed here are suitable to be further exploited and applied in a large class of free discontinuity problems (see also Remark 3.4), leading likely to a general scheme for the proof of similar regularity results.

The content of the paper is the following.

In § 2 we collect a few notions and notations which are needed in the sequel, and give the definition of quasi minimizer; the main density properties of quasi minimizers are studied in § 4 and pave the way to the main geometric results of § 5, namely the Lipschitz approximation Theorem 5.2 and Theorem 5.3, which gives a sufficient condition in order that a portion of K be a $C^{1,\alpha}$ graph.

The main point in the proof of Theorem 3.1, presented in § 7, is checking the hypotheses of Theorem 5.3, and this will pass as usual through an iterative procedure applied to the flatness improvement theorem of § 6.

2. – Notations and statement of the problem

2.1. – Rectifiable sets, approximate tangent space

The dimension $n \geq 2$ of the ambient space $\mathbb{R}^n$ will be fixed throughout the paper; since we will often deal with $(n-1)$-dimensional sets, we will use m for $n-1$. We indicate by ω_p the measure of the unit ball in $\mathbb{R}^p$.

In this paper $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure in $\mathbb{R}^n$, $\mathcal{H}^m \llcorner E$ the measure $\mathcal{H}^m$ restricted to the set E, $\theta_m(E,x)$ the m-dimensional density of E at x and $\mathbb{G}_{n,m}$ the Grassmann manifold of m-planes in $\mathbb{R}^n$. Each $T \in \mathbb{G}_{n,m}$ will be identified with the matrix (T_{ij}) representing the orthogonal projection onto T with respect to the canonical basis $\{e_1, \dots, e_n\}$; the distance $\|S-T\|$ between two m-planes is the euclidean norm of the matrix $S-T$.

Definition 2.1 (rectifiable sets) (see [20, 3.2.14]). *We say that $E \subset \mathbb{R}^n$ is countably $(\mathcal{H}^m, m)$-rectifiable if $\mathcal{H}^m$-almost all of E can be covered with a sequence of C^1 hypersurfaces Γ_i*, i.e.

$$\mathcal{H}^m\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0\,.$$

We say that E is $(\mathcal{H}^m, m)$-rectifiable if E is countably $(\mathcal{H}^m, m)$-rectifiable and $\mathcal{H}^m(E) < +\infty$.

The approximate tangent space $\mathrm{Tan}^m(E,x)$ of a $(\mathcal{H}^m, m)$-rectifiable set E at x is the m-plane S such that, setting $E_\rho = \rho^{-1}(E-x)$, we have

$$\lim_{\rho\to 0} \int_{E_\rho} \phi(y)\, d\mathcal{H}^m(y) = \int_S \phi(y)\, d\mathcal{H}^m(y) \qquad \forall \phi \in C_0^1(\mathbb{R}^n)\,.$$

The map $x \mapsto \mathrm{Tan}^m(E,x)$ is defined $\mathcal{H}^m$-a.e. on E and is $\mathcal{H}^m$-measurable (see [20, 3.2.25]). When integrations are involved and no confusion is possible we shorten $\mathrm{Tan}^m(E,x)$ to S_x; for instance

$$\int_E \|S_x - T\|^2\, d\mathcal{H}^m$$

denotes the mean square deviation of the approximate tangent space from a given m-plane T (this quantity, which is usual to call *tilt*, will be crucial in the Lipschitz approximation Theorem 5.2).

Finally, we recall that any Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ maps $(\mathcal{H}^m, m)$-rectifiable sets into $(\mathcal{H}^m, m)$-rectifiable sets. If f is one-to-one then $\mathcal{H}^m\big(f(E)\big)$ can be computed using the area formula (see e.g. [26 § 8]]):

$$\mathcal{H}^m\big(f(E)\big) = \int_E \mathbb{J}_m(f, S_x)\, d\mathcal{H}^m\,, \tag{2.1}$$

where by definition $\mathbb{J}_m^2(f, S) = \det\big((df|_S)^* \circ (df|_S)\big)$ for any m-plane S. We will often use (2.1) with $f(x) = \Phi_\varepsilon(x) = x + \varepsilon\phi(x)$, $\phi \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$ and ε small; in this case

$$\frac{d}{d\varepsilon}\mathbb{J}_m(\Phi_\varepsilon, S)\Big|_{\varepsilon=0} = \operatorname{div}_S\phi = \sum_{i,j=1}^n S_{ij}\frac{\partial\phi_i}{\partial x_j} \tag{2.2}$$

for any m-plane S.

2.2. – SBV functions and free discontinuity problems

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u \in BV(\Omega)$, i.e. a function with bounded variation in Ω. We denote by S_u the jump set of u, defined as the complement of the Lebesgue set of u:

$$x \notin S_u \quad \Longleftrightarrow \quad \exists z \in \mathbb{R} \text{ s.t. } \lim_{\rho\to 0} \rho^{-n} \int_{B_\rho(x)} |u(y) - z|\, dy = 0\,.$$

It is well known (see for instance [20, 4.5.9(16)]) that S_u is countably $(\mathcal{H}^m, m)$-rectifiable. The vector measure Du representing the distributional derivative of u can be decomposed into absolutely continuous part ∇u and singular part $D^s u$ with respect to the Lebesgue measure $\mathcal{L}^n$. It is clear that u belongs to the Sobolev space $W^{1,1}(\Omega)$ if and only if $Du = \nabla u\mathcal{L}^n$, or $D^s u = 0$.

Definition 2.2. (SBV functions) We say that u is a special function with bounded variation in Ω, and we write $u \in SBV(\Omega)$, if $D^s u$ is supported in S_u, i.e., $|D^s u|(\Omega \setminus S_u) = 0$.

The space SBV has been introduced in [18] to give a rigorous mathematical formulation to several variational problems involving both a "volume" energy and a "surface" energy. The main feature of such problems, also called free discontinuity problems, see [17], is that the surface energy is supported on a set which is not fixed *a priori* and is not necessarily a boundary.

The model problem has been suggested by Mumford and Shah in [24]. Given $g \in L^\infty(\Omega)$ and α, $\beta > 0$, we look for minimizers of the functional

$$G(u, K) = \int_{\Omega\setminus K} \big[|\nabla u|^2 + \alpha(u - g)^2\big]\, dx + \beta\mathcal{H}^m\big(K \cap \Omega\big) \tag{2.3}$$

where K is a closed subset of $\mathbb{R}^n$ and $u \in C^1(\Omega \setminus K)$. We refer to [7], [24] for the relevance of this problem in computer vision theory and some properties of minimizers; here, we want only to emphasize that, by minimizing G, one looks for a "piecewise C^1 approximation" of g.

Existence of minimizers is achieved passing through a weak formulation of the problem in $SBV(\Omega)$, namely, minimizing the functional

$$F(u) = \int_\Omega \left[|\nabla u|^2 + \alpha(u-g)^2\right] dx + \beta\mathcal{H}^m(S_u) .$$

The equivalence between the strong and weak formulation of the problem has been proved in [19], whereas the existence of minimizers of F easily follows from a compactness property of SBV first proved in [2] (see also [5]).

THEOREM 2.3. (compactness theorem) *Let* $(u_h) \subset SBV(\Omega) \cap L^\infty(\Omega)$, $p > 1$ *and assume that*

$$\sup_{h\in\mathbb{N}} \left\{ \int_\Omega |\nabla u_h|^p \, dx + \mathcal{H}^m(S_{u_h}) + \|u_h\|_\infty \right\} < +\infty .$$

Then, there exists a subsequence (u_{h_k}) *converging in* $L^1(\Omega)$ *to* $u \in SBV(\Omega)$. *Moreover,* ∇u_{h_k} *weakly converges to* ∇u *in* $L^p(\Omega, \mathbb{R}^n)$ *and* $\mathcal{H}^m \llcorner S_{u_{h_k}}$ *weakly converges in* Ω *to a measure* μ *such that* $\mu \geq \mathcal{H}^m \llcorner S_u$.

2.3. – Quasi minimizers

We are interested in the interface regularity of solutions $u \in SBV_{\text{loc}}(\Omega)$ of variational problems whose leading term is

$$\int_\Omega |\nabla u|^2 \, dx + \beta\mathcal{H}^m(S_u \cap \Omega) .$$

Our model is the Mumford-Shah functional described above. By a scaling argument, it is not restrictive to assume $\beta = 1$.

DEFINITION 2.4. (local minimizers) *We say that* $u \in SBV_{\text{loc}}(\Omega)$ *is a local minimizer in* Ω *if*

$$\int_A |\nabla u|^2 \, dx + \mathcal{H}^m(S_u \cap A) < +\infty \qquad \forall A \subset\subset \Omega \tag{2.4}$$

and

$$\int_A |\nabla u|^2 \, dx + \mathcal{H}^m(S_u \cap A) \leq \int_A |\nabla v|^2 \, dx + \mathcal{H}^m(S_v \cap A)$$

whenever $v \in SBV_{\text{loc}}(\Omega)$ *and* $\{v \neq u\} \subset\subset A \subset\subset \Omega$.

DEFINITION 2.5 (quasi minimizers). *We will call deviation from minimality* $\mathrm{Dev}(u,\Omega)$ *of a function* $u \in SBV_{\mathrm{loc}}(\Omega)$ *satisfying* (2.4) *the smallest* $\lambda \in [0,+\infty]$ *such that*

$$\int_A |\nabla u|^2\,dx + \mathcal{H}^m(S_u \cap A) \le \int_A |\nabla v|^2\,dx + \mathcal{H}^m(S_v \cap A) + \lambda$$

for any $v \in SBV_{\mathrm{loc}}(\Omega)$ *such that* $\{v \ne u\} \subset\subset A \subset\subset \Omega$. *Clearly,* $\mathrm{Dev}(u,\Omega) = 0$ *if and only if* u *is a local minimizer in* Ω. *Moreover, we say that* u *is a quasi minimizer in* Ω *if there exists a nondecreasing function* $\omega(t) : (0,+\infty) \to [0,+\infty)$ *such that* $\omega(t) \downarrow 0$ *as* $t \downarrow 0$ *and*

$$\mathrm{Dev}(u, B_\rho(x)) \le \rho^m \omega(\rho) \tag{2.5}$$

for any ball $B_\rho(x) \subset \Omega$. *We denote by* $\mathcal{M}_\omega(\Omega)$ *the class of functions satisfying* (2.5).

REMARK 2.6. Let $u \in SBV(\Omega)$ be a solution of the Mumford-Shah problem of minimizing (2.3). Then, if $g \in L^\infty(\Omega)$, the function u is a quasi minimizer. Indeed, since

$$F(v) \ge F(-M \vee v \wedge M) \qquad \forall v \in SBV_{\mathrm{loc}}(\Omega)$$

with $M = \|g\|_\infty$, we have $\|u\|_\infty \le M$ and, for any competing function v, the inequality $F(u) \le F(-M \vee v \wedge M)$ implies

$$\mathrm{Dev}(u, B_\rho(x)) \le \alpha \int_{B_\rho(x)} |-M \vee v \wedge M - g|^2\,dy \le 4\alpha \|g\|_\infty^2 \omega_n \rho^n$$

for any ball $B_\rho(x) \subset \Omega$, hence we can take

$$\omega(t) = 4\alpha \|g\|_\infty^2 \omega_n t\,.$$

The following fundamental density estimate on the jump set of quasi minimizers was proved in [19], [10] (see also [4] for two-dimensional minimizers of (2.3)).

THEOREM 2.6 (density lower bound). *There exist constants* ρ_ω *and* $\theta_\omega > 0$ *depending only on* n *and* ω *with the following property: if* $u \in \mathcal{M}_\omega(\Omega)$, *we have*

$$\mathcal{H}^m(S_u \cap B_\rho(x)) \ge \theta_\omega \rho^m$$

for any ball $B_\rho(x) \subset \Omega$ *centered at* $x \in \overline{S}_u$ *with* $\rho \le \rho_\omega$.

PROOF. By approximation, we need only to prove that the inequality is valid for balls centered at points $x \in S_u$. The proof can be achieved by a decay lemma (see [10], Proposition 1.2 and Remark 3.13). Setting

$$F(u, B_\rho(x)) = \int_{B_\rho(x)} |\nabla u|^2 \, dy + \mathcal{H}^m\big(B_\rho(x) \cap S_u\big)$$

it has been proved that for $\alpha \in (0,1)$ and $\varepsilon(n, \alpha, \omega)$ small enough, the conditions

$$\mathcal{H}^m\big(B_\rho(x) \cap S_u\big) < \varepsilon \rho^m, \qquad \rho < \varepsilon$$

imply $F(u, B_{\alpha\rho}(x)) \leq (\alpha)^{m+1/2} F(u, B_\rho(x))$. Now, let $\alpha \in (0,1)$ and choose $\alpha' \in (0,1)$ so small that

$$2\sqrt{\alpha'} n \omega_n < \varepsilon(n, \alpha, \omega)\,.$$

We define $\theta_\omega = \varepsilon(n, \alpha', \omega)$ and ρ_ω satisfying the conditions

$$\rho_\omega < \min\{\varepsilon(n, \alpha, \omega), \varepsilon(n, \alpha', \omega)\}, \qquad \omega(\rho_\omega) \leq n\omega_n\,.$$

If $\rho < \rho_\omega$ and $\mathcal{H}^m(S_u \cap B_\rho(x)) < \theta_\omega \rho^m$, using the energy upper bound (see Remark 4.1) and the decay property with α', we have

$$F(u, B_{\rho\alpha'}(x)) \leq (\alpha')^{m+1/2} F(u, B_\rho(x)) \leq 2\sqrt{\alpha'} n\omega_n (\rho\alpha')^m < \varepsilon(n, \alpha, \omega)(\rho\alpha')^m\,.$$

Applying now k times the decay property with α we get

$$F(u, B_{\rho\alpha^k\alpha'}(x)) \leq \alpha^{k(m+1/2)} F(u, B_{\rho\alpha'}(x))\,.$$

It follows that $\eta^{-m} F(u, B_\eta(x))$ converges to 0, as $\eta \downarrow 0$, hence (see [19]) $x \notin S_u$. □

A first consequence of Theorem 2.7 is the following

PROPOSITION 2.8. *Let $u \in SBV_{\rm loc}(\Omega)$ be a quasi minimizer in Ω. Then*

$$\mathcal{H}^m\big(\overline{S}_u \cap \Omega \setminus S_u\big) = 0\,.$$

PROOF. Let μ be the restriction of $\mathcal{H}^m$ to S_u and $B = \overline{S}_u \cap \Omega \setminus S_u$. Since

$$\liminf_{\rho \to 0^+} \frac{\mu(B_\rho(x))}{\omega_m \rho^m} \geq \frac{\theta}{\omega_m} \qquad \forall x \in \overline{S}_u$$

by [26, Th.3.2] we infer that $\mu \geq (\theta/\omega_m)\mathcal{H}^m \llcorner \overline{S}_u$, hence

$$0 = \mu(B) \geq \frac{\theta}{\omega_m} \mathcal{H}^m\big(B \cap \overline{S}_u\big) = \frac{\theta}{\omega_m} \mathcal{H}^m(B)\,. \qquad \square$$

REMARK 2.9. Let $u \in \mathcal{M}_\omega(\Omega)$ and let us denote by $K(u)$ the set $\overline{S}_u$. Thanks to Proposition 2.8, the quasi minimality of u implies:

$$\int_{B_\rho(x)} |\nabla u|^2\,dy + \mathcal{H}^m\big(K(u) \cap B_\rho(x)\big) \le \int_{B_\rho(x)} |\nabla v|^2\,dy + \mathcal{H}^m\big(K(v) \cap B_\rho(x)\big) + \rho^m\omega(\rho) \tag{2.6}$$

for any ball $B_\rho(x) \subset \Omega$ and any function $v \in SBV(\Omega)$ such that $\{v \neq u\} \subset\subset B_\rho(x)$. In the following, we will always work with pairs $(u, K(u))$, using the (weaker) minimality condition (2.6) and avoiding the SBV theory (except in Step 5 of Theorem 4.3).

From now on, we will use the notation $\mu(u)$ for $\mathcal{H}^m \llcorner K(u)$ and we denote by div^τ the tangential divergence along the approximate tangent space to $K(u)$.

3. – Statement of the main result

The main result of this paper is the following (here $L^{p,\lambda}$ are the classical Morrey spaces, see e.g. [21]):

THEOREM 3.1. *Let $u \in SBV(\Omega)$, assume that $|\nabla u| \in L^{2,\lambda}(\Omega)$ for some $\lambda > m$ and that constants $c_0 > 0$, $s \in (0, 2)$ exist such that*

$$\operatorname{Dev}(u, B_\rho(x)) \le c_0\rho^{2s} \tag{3.1}$$

for any ball $B_\rho(x) \subset \Omega$. Then, setting $\alpha = \min\{\lambda - m, s\}$, there is a constant $\varepsilon_0(n, c_0, \alpha, \|\nabla u\|_{L^{2,\lambda}})$ such that for any $x \in K(u)$ and any ball $B_\rho(x) \subset \Omega$, the conditions

$$\rho < \varepsilon_0, \qquad \min_{T \in \mathbb{G}_{n,m}} \int_{B_\rho(x)} |T^\perp(y - x)|^2\,d\mu(u) < \varepsilon_0\rho^{m+2} \tag{3.2}$$

imply that $B_{\rho/2}(x) \cap K(u)$ is a $C^{1,\alpha/(m+2)}$ hypersurface.

REMARK 3.2. Let $R(u)$ be the set of regular points of $K(u)$, i.e. the set of those points $x \in K(u)$ such that (3.2) holds for a sufficiently small $\rho > 0$. It is easy to see that $R(u)$ is relatively open in $K(u)$. Moreover, using in the definition of approximate tangent space a function $\phi(y) = |T^\perp y|\zeta(y)$ with a cut-off function ζ, it is easy to check that $R(u)$ contains the set of the points where the approximate tangent space exists. In particular,

$$\mathcal{H}^m\big(K(u) \setminus R(u)\big) = 0\,.$$

Theorem 3.1 applies in particular to minimizers u of the Mumford-Shah functional (2.3) under the assumption $|\nabla u| \in L^{2,\lambda}(\Omega)$ for some $\lambda > m$. Indeed, arguing as in Remark 2.6, we see that (3.1) holds with $s = 1/2$.

REMARK 3.3. We remark that an easy comparison argument (see Remark 4.1 below) and (3.1) imply that $|\nabla u| \in L^{2,m}(\Omega)$. Our regularity theorem requires the stronger assumption $|\nabla u| \in L^{2,\lambda}(\Omega)$ with $\lambda > m$; by Hölder's inequality, this assumption is satisfied if $|\nabla u| \in L^p(\Omega)$ for some $p > 2n$.

At the moment, no nontrivial example of local minimizer of (1.2) is known. However, there is some evidence (see [12], [24]) that the function (in polar coordinates) $u(\rho, \theta) = \sqrt{2\rho/\pi}\sin(\theta/2)$ is a local minimizer in $\mathbb{R}^2$ and that it presents the typical behaviour of a local minimizer near the tip of the singular set. Elementary computations show that $|\nabla u| \in L^{2,1}(\mathbb{R}^2)$, hence the hypothesis $|\nabla u| \in L^{2,\lambda}$, with $\lambda = m$, rather than $\lambda > m$, is suitable to cover the general case. This question will be tackled in a successive paper [6].

REMARK 3.4. The techniques of this paper apply, more generally, to quasi minimizers of functionals

$$F(u) = \int_\Omega H(\nabla u)\, dx + \mathcal{H}^m(S_u) \qquad u \in SBV_{\text{loc}}(\Omega, \mathbb{R}^k)$$

assuming that the following three condition hold:

1) H is continuous, p-homogeneous for some $p > 1$ and $H(z) > 0$ for $z \neq 0$;

2) $|\nabla u| \in L^{p,\lambda}(\Omega)$ for some $\lambda > m$;

3) the density lower bound of Theorem 2.7 holds.

Up to now, using the blow-up method of [19], [10], [11] the density lower bound has been proved only in the cases $H(z) = |z|^p$ for $k \geq 1$, even under the constraint $|u| = 1$. Its general validity is still an open problem.

4. – Some properties of quasi minimizers

We start by recalling an elementary upper bound and the natural scaling properties of quasi minimizers.

REMARK 4.1 (energy upper bound). For every $u \in SBV(\Omega)$ and every ball $B_\rho(x_0) \subset \Omega$, comparing with functions constant inside $B_{\rho'}(x_0) \subset\subset B_\rho(x_0)$ and letting $\rho' \uparrow \rho$ we get

$$\int_{B_\rho(x_0)} |\nabla u|^2\, dx + \mathcal{H}^m(K(u) \cap B_\rho(x_0)) \leq n\omega_n \rho^m + \text{Dev}(u, B_\rho(x_0))\,.$$

REMARK 4.2 (scaling). Let $u \in SBV(\Omega)$ and let $B_\rho(x_0) \subset \Omega$. Then, it is easy to check that

$$u_\rho(y) = \frac{1}{\sqrt{\rho}} u(x_0 + \rho y)$$

belongs to $SBV(\Omega_\rho)$ with $\Omega_\rho = \rho^{-1}(\Omega - x_0)$. Moreover, for any ball $B_\eta(y) \subset \Omega_\rho$ we have

$$\int_{B_\eta(y)} |\nabla u_\rho|^2 \, dx = \rho^{-m} \int_{B_{\eta\rho}(x_0+\rho y)} |\nabla u|^2 \, dx,$$

$$\mathcal{H}^m\left(S_{u_\rho} \cap B_\eta(y)\right) = \rho^{-m} \mathcal{H}^m\left(S_u \cap B_{\eta\rho}(x_0 + \rho y)\right)$$

$$\mathrm{Dev}(u_\rho, B_\eta(y)) = \rho^{-m} \mathrm{Dev}(u, B_{\eta\rho}(x_0 + \rho y)).$$

In particular, if $\rho \leq 1$, the monotonicity of $\omega(\rho)$ shows that

$$u \in \mathcal{M}_\omega(\Omega) \qquad \Longrightarrow \qquad u_\rho \in \mathcal{M}_\omega(\Omega_\rho).$$

The following theorem shows the asymptotic behaviour of $K(u)$ in regions where both the Dirichlet energy and the tilt of tangent planes are small. Using essentially a first variation argument we show that $\mu(u)$ is close to a locally finite sum of measures μ_i supported in parallel planes T_i. Then, lower semicontinuity of the energy and the minimality imply that $\mu_i = \mathcal{H}^m \llcorner T_i$.

THEOREM 4.3. *Let $(u_h) \subset \mathcal{M}_\omega(\Omega)$ be a sequence and let us assume that*

$$\lim_{h \to +\infty} \int_A |\nabla u_h|^2 \, dx + \int_A \|S_x - T\|^2 \, d\mu(u_h) + \mathrm{Dev}(u_h, A) = 0$$

for any open set $A \subset\subset \Omega$. Then, a subsequence of $\mu(u_h)$ weakly converges in Ω to a measure μ which is locally sum of affine m-planes parallel to T with multiplicity 1.

PROOF. STEP 1. Let $\mathbb{G}_m(\Omega)$ be the product of Ω and $\mathbb{G}_{n,m}$; the so-called varifold measures V_h associated to $\mu(u_h)$ are defined by

$$\int_{\mathbb{G}_m(\Omega)} \phi(x, S) \, dV_h(x, S) = \int_\Omega \phi(x, S_x) \, d\mu(u_h)$$

for any bounded Borel function $\phi(x, S)$ with compact support in x. Since V_h are locally equibounded in $\mathbb{G}_m(\Omega)$, we can assume that V_h are weakly converging in $\mathbb{G}_m(\Omega)$ to a measure V.

STEP 2. Now we prove that V is stationary in the varifold sense, i.e.,

$$\int_{\mathbb{G}_m(\Omega)} S_{ij} \frac{\partial \varphi_i}{\partial x_j} \, dV(x, S) = 0 \qquad \forall \varphi \in C_0^1(\Omega, \mathbb{R}^n). \tag{4.1}$$

Indeed, we fix an open set $A \subset\subset \Omega$ containing the support of φ such that $V(\partial A \times \mathbb{G}_{n,m}) = 0$. For ε sufficiently small, the map $\Phi_\varepsilon(x) = x + \varepsilon\varphi(x)$ is a C^1 diffeomorphism of Ω; comparing u_h with $u_h \circ \Phi_\varepsilon^{-1}$, taking into account the strong convergence of the gradients and the identities (see (2.1))

$$\mathcal{H}^m\big(A \cap K(u_h \circ \Phi_\varepsilon^{-1})\big) = \mathcal{H}^m\big(A \cap \Phi_\varepsilon(K(u_h))\big) = \int_A \mathbb{J}_m(\Phi_\varepsilon, S_x)\, d\mu(u_h)(x)$$

we find

$$\eta_h + \mathrm{Dev}(u_h, A) + \int_{A\times\mathbb{G}_{n,m}} \mathbb{J}_m(\Phi_\varepsilon, S)\, dV_h(x,S) \geq V_h\big(A \times \mathbb{G}_{n,m}\big),$$

for a suitable infinitesimal η_h. Passing to the limit with respect to h we obtain

$$\int_{A\times\mathbb{G}_{n,m}} \mathbb{J}_m(\Phi_\varepsilon, S)\, dV(x,S) \geq V\big(A \times \mathbb{G}_{n,m}\big).$$

Differentiation with respect to ε and (2.2) yield (4.1). Moreover, denoting by $\mu(C) = V(C \times \mathbb{G}_{n,m})$ the projection of V on Ω, the measures $\mu(u_h)$ weakly converge to μ and the density lower bound implies $\mu(B_\rho(x)) \geq \theta_\omega \rho^m$ for any $x \in \mathrm{spt}\mu$ and any ball $B_\rho(x) \subset \Omega$ with $\rho < \rho_\omega$. Hence, by [26, Th.3.2], the support of μ is a set with locally $\mathcal{H}^m$-finite measure in Ω. The density lower bound also implies that the supports of $\mu(u_h)$ converge to the support of μ in the sense of Kuratowski. Finally, by lower semicontinuity we have

$$\int_{\mathbb{G}_m(\Omega)} \|S - T\|^2\, dV(x,S) = 0$$

that means that V is supported in $\Omega \times \{T\}$.

STEP 3. Our next step is to prove that V is invariant by translations along T. For, let $b \in T$, and $\tau_t(x) = x + tb$, $V_t = (\tau_t)_\# V$; for any $\phi \in C_0^1(\Omega)$ we have

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{G}_m(\Omega)} \phi(x)\, dV_t(x) &= \frac{d}{dt}\int_{\mathbb{G}_m(\Omega)} \phi(x - tb)\, dV(x,S) \\
&= -\int_{\mathbb{G}_m(\Omega)} \langle \nabla\phi(x - tb), b\rangle\, dV(x,S) \\
&= -\int_{\mathbb{G}_m(\Omega)} \frac{\partial\phi}{\partial x_i}(x - tb) T_{ij} b_j\, dV(x,S) \\
&= -b_j \int_{\mathbb{G}_m(\Omega)} S_{ij} \frac{\partial\phi}{\partial x_i}(x - tb)\, dV(x,S) = 0
\end{aligned}$$

hence μ is invariant by translations along T and then $\operatorname{spt}\mu$ is contained in a locally finite union of affine m-planes parallel to T.

STEP 4. It remains to prove that the density of μ is equal to 1. To this end, assume that $T = e_n^\perp$ and consider a m-plane P parallel to T such that $P \cap \operatorname{spt}\mu \neq \emptyset$. For every $x \in P \cap \operatorname{spt}\mu$, we choose $\rho_0 \in (0, \rho_\omega)$ such that for every $\rho \in (0, \rho_0)$ the inclusion $Q_\rho \cap \operatorname{spt}\mu \subset P$ holds, where Q_ρ is a cube symmetric with respect to P with centre x and side ρ. Fix $\rho \in (0, \rho_0/2)$ and let $w_h(y) = \rho^{-1/2}u_h(x + \rho y)$; then, for $Q = (-1, 1)^n$, $H = Q \cap \{x_n = 0\}$ the sequence (w_h) satisfies

$$\int_Q |\nabla w_h|^2 dx + \operatorname{Dev}(w_h, 2Q) \to 0 \tag{4.2}$$

and $K(w_h) \to H$ in the sense of Kuratowski. For $a \in (0, 2^{-n}\theta_\omega/m)$, set $Q_a^+ = Q \cap \{x_n > a\}$, $Q_a^- = Q \cap \{x_n < -a\}$ and $Q_a = Q_a^+ \cup Q_a^-$. If h is large enough we have $w_h \in W^{1,2}(Q_a)$; furthermore, translating w_h if necessary, we can suppose that

$$\frac{1}{\mathcal{H}^n(Q_a^+)} \int_{Q_a^+} w_h(x)\, dx = 0 \qquad \forall h \in \mathbb{N}, \tag{4.3}$$

which implies $w_h \to 0$ strongly in $W^{1,2}(Q_a^+)$. Similarly, setting

$$c_h = \frac{1}{\mathcal{H}^n(Q_a^-)} \int_{Q_a^-} w_h(x)\, dx \qquad \forall h \in \mathbb{N},$$

we have $u_h - c_h \to 0$ strongly in $W^{1,2}(Q_a^-)$. Up to subsequences and a change of sign we can suppose $c_h \to c \in [0, +\infty]$. We first show that $c = 0$ is impossible. In fact, for a as above and $b \in (a, 2^{-n}\theta_\omega/m)$ we can find $a_1, a_2 \in (a, b)$ such that (here $Q = Q' \times (-1, 1)$)

$$\begin{cases} \displaystyle\int_{Q'} \left[|\nabla w_h|^2(x', a_1) + |\nabla w_h|^2(x', -a_2)\right] dx' \to 0 \\ \displaystyle\int_{Q'} \left[|w_h|^2(x', a_1) + |w_h|^2(x', -a_2)\right] dx' \to 0 \end{cases} \tag{4.4}$$

where $x = (x', x_n)$. Let us modify w_h in Q, setting

$$v_h(x) = \begin{cases} w_h(x) & \text{if } x \in Q_a\,, \\ w_h(x', a_1) - \dfrac{x_n - a_1}{a_1 + a_2}\left[w_h(x', -a_2) - w_h(x', a_1)\right] & \text{otherwise.} \end{cases}$$

Taking into account that (4.4) implies

$$\lim_{h\to+\infty} \int_{-a_2}^{a_1} dx_n \int_{Q'} |\nabla v_h|^2\, dx' = 0,$$

and that the area of the subset of ∂Q where v_h can be different from w_h does not exceed $m2^m(a_1+a_2)$, by the quasi minimality of w_h and (4.2) we get

$$\begin{aligned} 0 \le \limsup_{h\to+\infty} & \int_Q |\nabla v_h|^2 + \mathcal{H}^m\big(K(v_h)\cap \overline{Q}\big) \\ & - \int_Q |\nabla w_h|^2 - \mathcal{H}^m\big(K(w_h)\cap Q\big) + \mathrm{Dev}(w_h, 2Q) \\ & \le m2^m(a_1+a_2) - \liminf_{h\to+\infty} \mathcal{H}^m(K(w_h)\cap Q) \le m2^n b - \theta_\omega, \end{aligned}$$

a contradiction. Hence, $c > 0$.

STEP 5. Consider the sequence $h \mapsto \bar{w}_h = (-1)\vee(w_h\wedge 1)$, which is compact in $SBV(Q)$, hence (up to subsequences) converges to a function, say v, in $L^1(Q)$ and a.e. in Q. From (4.2) and (4.3) it follows that $v \equiv 0$ in $Q\cap\{x_n>0\}$, whereas

$$\int_{Q_a^-} |w_h - c_h|\, dx \ge \int_{Q_a^-} \big|\bar{w}_h - (-1)\vee(c_h\wedge 1)\big|\, dx$$

implies

$$\int_{Q_a^-} \big|v - (-1)\vee(c\wedge 1)\big|\, dx = 0;$$

since $c>0$, it holds $(-1)\vee(c\wedge 1)\not\equiv 0$, whence $v\not\equiv 0$ in Q. On the other hand, by (4.2) again, v is constant in $Q\cap\{x_n<0\}$. Therefore

$$\liminf_{h\to+\infty} \mathcal{H}^m\big(K(w_h)\cap Q\big) \ge \liminf_{h\to+\infty} \mathcal{H}^m\big(K(\bar{w}_h)\cap Q\big) \ge \mathcal{H}^m\big(K(v)\cap Q\big) = 2^m.$$

Coming back to the sequence u_h, the previous inequality shows that

$$\mu(\overline{Q}_\rho) \ge \liminf_{h\to+\infty} \mathcal{H}^m\big(Q_\rho\cap K(u_h)\big) \ge 2^m\rho^m,$$

for every $\rho\in(0,\rho_0/2)$, i.e. that the density of μ at x is ≥ 1 for every $x\in \operatorname{spt}\mu$. Let us prove the opposite inequality. For, suppose for simplicity $x=0$, consider two parameters $a, b\in(0,\rho_0)$, and the neighbourhood of 0 given by $Q=(-b/2,b/2)^m\times(-a,a)$. Choose then $a_1, a_2\in(0,a)$ such that

$$\int_{Q\cap\{x_n=a_1\}} |\nabla_{x'} u_h|^2 dx' + \int_{Q\cap\{x_n=-a_2\}} |\nabla_{x'} u_h|^2 dx' \to 0,$$

(where as above $x = (x', x_n)$) and define the comparison functions

$$v_h(x) = \begin{cases} u_h(x) & \text{if } x_n \geq a_1 \text{ or } x_n \leq -a_2\,, \\ u_h(x', a_1) & \text{if } 0 < x_n < a_1 \\ u_h(x', -a_2) & \text{if } -a_2 < x_n < 0\,. \end{cases}$$

By comparing the energies of v_h and u_h we get

$$\begin{aligned} \mu\big((-b/2, b/2)^n\big) = \mu(Q) &\leq \liminf_{h\to+\infty} \mathcal{H}^m\big(K(u_h)\cap Q\big) \\ &\leq b^m + 2mb^{m-1}(a_1+a_2) \leq b^m + 4amb^{m-1}\,, \end{aligned}$$

which gives the desired estimate letting $a \downarrow 0$. □

Using Theorem 4.3 we can show in Corollary 4.4 below that the m-dimensional density of $K(u)$ in $B_R(x)$ is less than $\tau > 1$ if $K(u)$ is sufficiently flat in some direction T and both the Dirichlet energy and the deviation from minimality are small in a larger ball.

Similarly, we can show in Corollary 4.5 below that the m-dimensional density of $K(u)$ in $B_R(x)$ is greater than $\tau < 1$ if $x \in K(u)$ and the tilt of tangent planes, the Dirichlet energy and the deviation from minimality are small in a larger ball. In this corollary, for later use in Theorem 5.2, we allow a small translation in the direction T.

Corollary 4.4. *For any $\tau > 1$ and any $\beta \in (0,1)$ there exists a constant $\gamma(\beta,\tau,\omega) \in (0,1)$ such that, for any m-plane T, any $R \in (0,1]$ and any $u \in \mathcal{M}_\omega(B_R)$, the condition*

$$\int_{B_R} |\nabla u|^2\,dx + R^{-2}\int_{B_R} |T^\perp x|^2\,d\mu(u) + \mathrm{Dev}(u, B_R) < \gamma R^m$$

implies

$$\mathcal{H}^m\big(K(u)\cap B_{\beta R}\big) < \tau\omega_m(\beta R)^m\,.$$

Proof. We argue by contradiction. Suppose that $\beta \in (0,1)$, $\tau > 1$ and sequences R_h, (u_h) exist such that, for any $h \in \mathbb{N}$,

$$\int_{B_{R_h}} |\nabla u_h|^2\,dx + R_h^{-2}\int_{B_{R_h}} |T^\perp x|^2\,d\mu(u_h) + \mathrm{Dev}(u_h, B_{R_h}) < \frac{R_h^m}{h} \tag{4.5}$$

and

$$\mathcal{H}^m\big(K(u_h)\cap B_{\beta R_h}\big) \geq \tau\omega_m(\beta R_h)^m\,. \tag{4.6}$$

Using Remark 4.2, it is not restrictive to assume that $R_h = 1$ for any h. By compactness, (up to subsequences) $\mu(u_h)$ weakly converges in B_1 to a measure μ supported in $T \cap B_1$ by (4.5). In order to get a contradiction, we need only to show that $\mu \le \mathcal{H}^m \llcorner T$. The inequality follows by the same argument used in the final part of the proof of Theorem 4.3. Indeed, for any box $(x_0-b/2, x_0+b/2)^n$ contained in B_1 and any $a < b/2$ we have

$$\mu\big((x_0-b/2, x_0+b/2)^n\big) = \mu\big((x_0-b/2, x_0+b/2)^m \times (-a, a)\big) \le b^m + 4amb^{m-1}.$$

Since a and x_0 are arbitrary, $\mu \le \mathcal{H}^m \llcorner T$. □

COROLLARY 4.5. *For any $\tau \in (0, 1)$ there exists a constant $\gamma_1(\tau, \omega) > 0$ such that, for any m-plane T, $R \in (0, 1]$, the conditions*

$$u \in \mathcal{M}_\omega(B_{2R}), \quad 0 \in K(u), \quad b \in T,\ |b| \le R$$

$$\int_{B_{2R}} |\nabla u|^2\, dx + \int_{B_{2R}} \|S_x - T\|^2\, d\mu(u) + \mathrm{Dev}(u, B_{2R}) < \gamma_1 (2R)^m$$

imply

$$\mathcal{H}^m\big(K(u) \cap B_R(b)\big) \ge \tau\omega_m R^m.$$

PROOF. If the statement were false, $\lambda \in (0, 1)$ and sequences $b_h \in T_h$, R_h and $u_h \in \mathcal{M}_\omega(B_{2R_h})$ would exist such that for any $h \in \mathbb{N}$ both

$$\text{(4.7)} \qquad \begin{aligned} &\int_{B_{2R_h}} |\nabla u_h|^2\, dx + \int_{B_{2R_h}} \|S_x - T\|^2\, d\mu(u_h) + \mathrm{Dev}(u_h, B_{2R_h}) < \frac{(2R_h)^m}{h} \\ &\mathcal{H}^m\big(K(u_h) \cap B_{R_h}(b_h)\big) < \tau\omega_m R_h^m \end{aligned}$$

hold. Using Remark 4.2 and the invariance under rotations, we can assume that $R_h = 1$ and $T_h = T$ for any h. Then, up to subsequences, b_h converges to some $b \in T \cap \overline{B}_1$ and $\mu(u_h)$ weakly converges in B_2 to a measure μ satisfying

$$\mu\big(B_1(b)\big) \le \tau\omega_m$$

because of (4.7). On the other hand, since $0 \in \operatorname{spt}\mu$, Theorem 4.3 implies that $\mu \ge \mathcal{H}^m \llcorner T$ in B_2, hence $\mu\big(B_1(b)\big) \ge \omega_m$ holds. □

The next proposition is essentially an adaptation to our problem of the multilayer monotonicity lemma of [9, 5.3]: we consider two points x, $x' \in K(u) \cap B_R$ where $K(u)$ has density 1; given their vertical separation and a parameter $\lambda \in (1/2, 1)$, Proposition 4.6 shows that a good control on the Dirichlet energy and on the tilt implies that $\mathcal{H}^m \llcorner K(u)$ satisfies

$$\text{(4.8)} \qquad \mathcal{H}^m\big(K(u) \cap \big[B_R(b+x) \cup B_R(b+x')\big]\big) \ge 2\lambda\omega_m R^m$$

where $b \in T \cap \overline{B}_R$. This proposition will be the main tool in the proof of the Lipschitz approximation Theorem 5.2, showing that in regions where the m-dimensional density of $K(u)$ is close to 1 the vertical separation of a large portion of $K(u)$ can be controlled.

The proof of the proposition follows by Theorem 4.3 after a suitable blow-up argument.

PROPOSITION 4.6. *Given* $\lambda \in (1/2, 1)$ *and* $L \in (0, 1]$, *there exists a constant* $\gamma_2(\lambda, L, \omega) \in (0, 1)$ *such that, for any m-plane* T, $b \in T \cap \overline{B}_R$, x, $x' \in B_R$ *the conditions*

$$\theta_m(K(u), x) = \theta_m(K(u), x') = 1, \quad |T^{\perp}(x - x')| > L|T(x - x')| \tag{4.9}$$

$$R \leq 1, \quad u \in \mathcal{M}_{\omega}(B_{3R}), \quad \omega(3R) \leq \gamma_2 \tag{4.10}$$

$$\int_{B_\rho(x)\cup B_\rho(x')} |\nabla u|^2 \, dy < \gamma_2 \rho^m \qquad \forall \rho \in (0, 2R) \tag{4.11}$$

$$\int_{B_\rho(x)\cup B_\rho(x')} \|S_y - T\|^2 d\mu(u) < \gamma_2 \rho^m \qquad \forall \rho \in (0, 2R) \tag{4.12}$$

imply (4.8).

PROOF. Let us argue by contradiction. Let T_k, b_k, R_k, u_k, x_k, x'_k be satisfying (4.9), (4.10), (4.11), (4.12) with $\gamma_k = 1/k$ and

$$\mathcal{H}^m\left(K(u_k) \cap \left[B_{R_k}(b_k + x_k) \cup B_{R_k}(b_k + x'_k)\right]\right) < 2\lambda\omega_m R_k^m .$$

By a rotation we can assume that $T_k = T = e_n^{\perp}$. We will denote $x \in \mathbb{R}^n$ by (z, t) with $z = Tx$ and $t = T^{\perp}x$. Let

$$r_k = \sup\Big\{r < R_k : \mu(u_k)\big(B_s(x_k + sb_k/R_k) \cup B_s(x'_k + sb_k/R_k)\big) \geq 2\lambda\omega_m s^m \ \forall s \in (0, r)\Big\}.$$

Since the density of $K(u_k)$ is 1 at x_k and at x'_k, we have $0 < r_k < R_k \leq 1$. Now we rescale all by a factor r_k and we translate x'_k to the origin, defining

$$\bar{x}_k = (\bar{z}_k, \bar{t}_k) = \frac{x_k - x'_k}{r_k}, \qquad \bar{u}_k(x) = \frac{1}{\sqrt{r_k}} u\big(r_k x + x'_k\big).$$

Then, by Remark 4.2, $\bar{u}_k \in \mathcal{M}_{\omega}(\Omega_k)$, where $\Omega_k = r_k^{-1}\big(B_{3R_k} - x'_k\big)$. Moreover, the monotonicity of $\omega(\rho)$ yields

$$\begin{aligned} \mathrm{Dev}(\bar{u}_k, B_\rho) = \frac{\mathrm{Dev}(u_k, B_{\rho r_k}(x'_k))}{r_k^m} &\leq \frac{(\rho r_k)^m \omega(\rho r_k)}{r_k^m} \\ &\leq \rho^m \omega(3R_k) \leq \frac{\rho^m}{k} \end{aligned} \tag{4.13}$$

for any $\rho \in (0, 3R_k/r_k)$. Similarly, we have

$$\mathrm{Dev}(\bar{u}_k, B_\rho(\bar{x}_k)) \leq \frac{\rho^m}{k} \tag{4.14}$$

for any $\rho \in (0, 3R_k/r_k)$. Notice also that $\bar{x}_k$ and 0 belong to $K(\bar{u}_k)$ and (4.9), (4.11), (4.12) together with the definition of r_k yield

$$|\bar{t}_k| \geq L|\bar{z}_k| \tag{4.15}$$

$$\int_{B_\rho(\bar{x}_k)\cup B_\rho} |\nabla \bar{u}_k|^2\, dx < \frac{1}{k}\rho^m \qquad \forall\, \rho \in (0, 2R_k/r_k) \tag{4.16}$$

$$\int_{B_\rho(\bar{x}_k)\cup B_\rho} \|S_x - T\|^2\, d\mu(\bar{u}_k) < \frac{1}{k}\rho^m \qquad \forall\, \rho \in (0, 2R_k/r_k) \tag{4.17}$$

$$\mathcal{H}^m\big(K(\bar{u}_k) \cap \big[B_1(\bar{x}_k + b_k/R_k) \cup B_1(b_k/R_k)\big]\big) = 2\lambda\omega_m \tag{4.18}$$

$$\mathcal{H}^m\big(K(\bar{u}_k) \cap \big[B_\rho(\bar{x}_k + b_k/R_k) \cup B_\rho(b_k/R_k)\big]\big) \geq 2\lambda\omega_m \quad \forall\, \rho \in (0, 1). \tag{4.19}$$

It is easy to see that $|\bar{x}_k| \leq 2$ for k large enough. Indeed, if $|\bar{x}_k| > 2$ the balls $B_1(\bar{x}_k + b_k/R_k)$ and $B_1(b_k/R_k)$ are disjoint; since, by Corollary 4.5 and (4.13), (4.14), (4.16), (4.17)

$$\limsup_{k\to+\infty}\big[\mathcal{H}^m\big(K(\bar{u}_k) \cap B_1(\bar{x}_k + b_k/R_k)\big) + \mathcal{H}^m\big(K(\bar{u}_k) \cap B_1(b_k/R_k)\big)\big] \geq 2\omega_m$$

this contradicts (4.18) for k large enough. Possibly passing to a subsequence we can assume that $\bar{x}_k$ converges to $x = (z, t)$ and b_k/R_k converges to $b \in T \cap \overline{B}_1$ as $k \to +\infty$. We note that $|t| \geq L|z|$ because of (4.15).

Since $B_1(\bar{x}_k + b_k/R_k) \cup B_1(b_k/R_k) \subset \Omega_k$ by Theorem 4.3 and (4.13)–(4.17) we can assume that $\mu(\bar{u}_k)$ weakly converge in $B_1(x+b) \cup B_1(b)$ to a measure μ which is locally sum of m-dimensional Hausdorff measures associated to m-planes parallel to T. By (4.18) we get

$$\mu\big(B_1(x+b) \cup B_1(b)\big) \leq 2\lambda\omega_m\,. \tag{4.20}$$

On the other hand, the density lower bound implies that x and 0 belong to the support of μ; we need only to show that $x \neq 0$ (hence $t \neq 0$) to contradict (4.20). If $\bar{x}_k$ were converging to 0, for any $\tau > 0$ we would have

$$B_\rho(\bar{x}_k + b_k/R_k) \cup B_\rho(b_k/R_k) \subset B_{\rho+\tau}(b_k/R_k)$$

for k large enough, so that (4.19) would imply $\mu(\overline{B}_{\rho+\tau}(b)) \geq 2\lambda\rho^m$. Letting first $\tau \downarrow 0$ then $\rho \downarrow 0$ we find that the density of μ in 0 is $2\lambda > 1$, a contradiction. □

5. – Lipschitz approximation

In this section we show that the set $K(u)$ of a quasi minimizer u can be approximated by the graph $\Gamma(f)$ of a Lipschitz function f in small balls (depending on ω and the Lipschitz constant of f). Related results are proved in [15] in the two dimensional case, for minimizers of the Mumford-Shah functional. We also estimate the measure of the symmetric difference $K(u)\Delta\Gamma(f)$; this estimate will be useful in the proof of Theorem 6.2.

The following proposition shows that an integral estimate of $|T^{\perp}(x-x_0)|$ in $B_R(x_0)$ leads to a pointwise estimate of the same quantity in a smaller ball, via the density lower bound.

PROPOSITION 5.1 (height bound). *Let $u \in \mathcal{M}_\omega(B_{2R}(x_0))$. Denoting by ρ_ω and θ_ω the constants given by Theorem 2.7, we have*

$$\sup_{x\in B_R(x_0)\cap K(u)} |T^{\perp}(x-x_0)|^{m+2} \le \frac{2^{m+2}}{\theta_\omega} \int_{B_{2R}(x_0)} |T^{\perp}(x-x_0)|^2\,d\mu(u)$$

provided $R \le 2\rho_\omega$.

PROOF. Let for simplicity $x_0 = 0$, assume

$$\sigma = \left[\frac{2^{m+2}}{\theta_\omega}\int_{B_{2R}} |T^{\perp}x|^2\,d\mu(u)\right]^{\frac{1}{m+2}} < R$$

(otherwise the thesis is trivial), and suppose that a point $x_1 \in K(u)$ with $|x_1| \le R$ exists such that $|T^{\perp}x_1| > \sigma$; then $|T^{\perp}x| > \sigma/2$ in $B_{\sigma/2}(x_1)$, whence, using the density lower bound, we deduce

$$\begin{aligned}\int_{B_{2R}} |T^{\perp}x|^2\,d\mu(u) &\ge \int_{B_{\sigma/2}(x_1)} |T^{\perp}x|^2\,d\mu(u)\\ &> \theta_\omega\Big(\frac{\sigma}{2}\Big)^{m+2} = \int_{B_{2R}} |T^{\perp}x|^2\,d\mu(u),\end{aligned}$$

which gives a contradiction. □

Choice of the constants. Now we fix all the constants of the previous density bounds. All these constants depend on n and possibly on ω, L.

1. Let $r \in (1,2)$ such that $r^m < 2$ and $\lambda \in (1/2,1)$ such that $r^m < 2\lambda$. Now we fix $\delta > 0$ so small that

$$1+\delta < r, \qquad \sqrt{\frac{1}{16}+\delta^2} < \frac{5}{8}. \tag{5.1}$$

2. Now, let $\gamma(r/2, 2\lambda/r^m, \omega)$, $\gamma_1(1/2, \omega)$, $\gamma_2(\lambda, L, \omega)$ be given by Corollary 4.4, Corollary 4.5 and Proposition 4.6 respectively. We will impose a smallness condition on R, denoting by $R_{n,\omega,L}$ the largest constant $R \le \inf\{2\rho_\omega, 1/2\}$ such that

$$\omega(6\rho) \le \min\left\{\frac{\gamma}{2}, \frac{\gamma_1}{3}, \gamma_2, n\omega_n\right\} \qquad \forall \rho \in (0, R]. \tag{5.2}$$

3. Since

$$\omega_m (2r)^m < \omega_m 2^{m+1}\lambda$$

Corollary 4.4 (with $\beta = r/2$ and $\tau = 2\lambda/r^m$) and our choice of γ and of $R_{n,\omega,L}$ guarantee that the condition

$$R^{-m} \int_{B_{4R}} |\nabla u|^2 \, dx + R^{-m-2} \int_{B_{4R}} |T^\perp x|^2 \, d\mu(u) < \frac{\gamma}{2}$$

implies

$$\mathcal{H}^m\big(K(u) \cap B_{2rR}\big) < 2^{m+1}\lambda\omega_m R^m$$

provided $u \in \mathcal{M}_\omega(\dot{B}_{4R})$ and $R \le R_{n,\omega,L}$.

Now, we can state the following Lipschitz approximation theorem (similar to [9, 5.4], see also [1] and [26]). The proof is based on the density estimates of the previous section, the height bound and a covering argument.

THEOREM 5.2 (Lipschitz approximation). *Let $L \in (0, 1]$, $u \in \mathcal{M}_\omega(B_{7R})$ for some $R \le R_{n,\omega,L}$. Suppose also that $K(u) \cap B_{R/16} \ne \emptyset$ and*

$$R^{-m-2} \int_{B_{4R}} |T^\perp x|^2 \, d\mu(u) < \min\left\{\theta_\omega \left(\frac{\delta L}{4}\right)^{m+2}, \frac{\gamma}{4}\right\} \tag{5.3}$$

for some m-plane T. Then, there exists a Lipschitz function $f : T \to T^\perp$ with Lipschitz constant less than L such that

$$\sup |f|^{m+2} \le \frac{2^{m+2}}{\theta_\omega} \int_{B_{2R}} |T^\perp x|^2 \, d\mu \tag{5.4}$$

and, denoting by $\Gamma(f)$ the graph of f, we have

$$\begin{aligned} &\mathcal{H}^m\big(B_R \cap K(u) \setminus \Gamma(f)\big) + \mathcal{H}^m\big(B^m_{R/4} \cap T[\Gamma(f) \setminus K(u)]\big) \\ &\le P(n, L, \omega)\left[\int_{B_{5R}} |\nabla u|^2 \, dx + \int_{B_{5R}} \|S_x - T\|^2 \, d\mu(u)\right]. \end{aligned} \tag{5.5}$$

PROOF. It is not restrictive to assume that T is spanned by $e_1, \ldots, e_m$, and set $x = (z, t)$, $z = Tx$, $t = T^\perp x$. Since the function u will be fixed throughout the proof we set $K = K(u)$, $\mu = \mu(u)$ and denote by K^* the set of points of density 1 of K.

Since $L \leq 1$ and $R \leq 2\rho_\omega$, by the height bound and (5.3) we get

$$|T^\perp x| \leq 2 \left(\frac{1}{\theta_\omega} \int_{B_{2R}} |T^\perp x|^2 \, d\mu \right)^{1/(m+2)} < \delta R \tag{5.6}$$

for any $x \in K \cap B_R$. We can also assume that

$$\mathcal{H}^m \left(K \cap B_{2rR}\right) < 2^{m+1} \lambda \omega_m R^m \, . \tag{5.7}$$

Indeed, because of our choice of γ and (5.3), if (5.7) does not hold then

$$R^{-m} \int_{B_{4R}} |\nabla u|^2 \, dx \geq \frac{\gamma}{4}$$

and we can choose the constant P so large that (5.5) trivially holds with $f \equiv 0$. Now, let $\sigma = \min\{\gamma_1, \gamma_2\}$ and define

$$A = \Big\{x \in B_R \cap K^* : \int_{B_\rho(x)} |\nabla u|^2 \, dy + \int_{B_\rho(x)} \|S_y - T\|^2 \, d\mu < \frac{\sigma}{2} \rho^m \ \forall \rho \in (0, 4R)\Big\} ,$$

Let

$$m(z) = \mathrm{card}\big\{t : (z, t) \in A\big\}$$

be the multiplicity function of A with respect to T.

STEP 1. We claim that $m(z) \leq 1$ on B^m_R. Indeed, let $z \in B^m_R$ and let $x = (z, t)$, $x' = (z, t') \in A$. Recalling our choice of $R_{n,\omega,L}$ and the definition of A, we can apply Proposition 4.6 with $b = -z$, to get

$$\mathcal{H}^m \left(K \cap \left[B_{2R}((0, t)) \cup B_{2R}((0, t'))\right]\right) \geq 2^{m+1} \lambda \omega_m R^m \, .$$

Since t, $t' \in (-\delta R, \delta R)$ and $1 + \delta/2 < r$ we have

$$B_{2R}((0, t)) \cup B_{2R}((0, t')) \subset B_{2rR}$$

hence $\mathcal{H}^m (K \cap B_{2rR}) \geq 2^{m+1} \lambda \omega_m R^m$ and this contradicts (5.7).

STEP 2. Let $E = \{z \in B^m_R : m(z) = 1\}$ and let $\phi : E \to \mathbb{R}$ such that $(z, \phi(z)) \in A$ for any $z \in E$.

We now claim that ϕ satisfies the following condition:

$$z, \, z' \in E \quad \Longrightarrow \quad |\phi(z) - \phi(z')| \leq L |z - z'| \, . \tag{5.8}$$

Indeed, let z, $z' \in E$; let us assume that $|\phi(z)-\phi(z')| > L|z-z'|$; if $|z-z'| > \delta R$ we get by Proposition 5.1

$$(\delta R)^{m+2} < 2^{m+2} \sup_{x\in K\cap B_R} |T^{\perp}x|^{m+2} \le \frac{4^{m+2}}{\theta_\omega} \int_{B_{2R}} |T^{\perp}x|^2 \, d\mu ,$$

contradicting assumption (5.3). Hence $|z - z'| \le \delta R$. Let $x = (z, \phi(z))$, $x' = (z', \phi(z'))$; by applying Proposition 4.6 with $b = -z$ we get

$$\mathcal{H}^m\left(K\cap\left[B_{2R}(0, \phi(z)) \cup B_{2R}(z' - z, \phi(z))\right]\right) \ge 2^{m+1}\lambda\omega_m R^m .$$

By (5.6) and the inequalities $|z - z'| \le \delta R$, $1 + \delta/\sqrt{2} < r$ we infer

$$B_{2R}(0, \phi(z)) \subset B_{2rR}, \qquad B_{2R}(z' - z, \phi(z)) \subset B_{2rR}$$

and we find a contradiction with (5.7) as in Step 1.

STEP 3. Now we estimate $\mathcal{H}^m(K \cap B_R \setminus A)$: if $x \in K^* \cap B_R$ does not belong to A we can find $\rho(x) \in (0, 4R)$ such that

$$\int_{B_{\rho(x)}(x)} |\nabla u|^2 \, dy + \int_{B_{\rho(x)}(x)} \|S_y - T\|^2 \, d\mu \ge \frac{\sigma}{2}\rho^m(x) ,$$

hence, by using the density upper bound and (5.2)

$$\begin{aligned}\mathcal{H}^m\left(K^* \cap B_{\rho(x)}(x)\right) &\le n\omega_n\rho^m(x) + \rho^m(x)\omega(\rho(x))\\ &\le \left(n\omega_n + \omega(4R)\right)\rho^m(x) \le 2n\omega_n\rho^m(x)\end{aligned}$$

we get

$$\int_{B_{\rho(x)}(x)} |\nabla u|^2 \, dy + \int_{B_{\rho(x)}(x)} \|S_y - T\|^2 \, d\mu \ge \frac{\sigma}{4n\omega_n}\mathcal{H}^m\left(K^* \cap B_{\rho(x)}(x)\right) .$$

Using a standard covering argument and $\mathcal{H}^m\left(K \setminus K^*\right) = 0$, we get

$$\mathcal{H}^m\left(B_R \cap K \setminus A\right) \le \frac{4n\omega_n\xi(n)}{\sigma}\left[\int_{B_{5R}} |\nabla u|^2 \, dx + \int_{B_{5R}} \|S_x - T\|^2 \, d\mu\right] \tag{5.9}$$

where $\xi(n)$ is the constant of Besicovitch's theorem. In particular we can choose the constant P in (5.5) so large that

$$\frac{4n\omega_n\xi(n)}{\sigma}\left[\int_{B_{5R}} |\nabla u|^2 \, dx + \int_{B_{5R}} \|S_x - T\|^2 \, d\mu\right] \ge \theta_\omega(R/16)^m$$

implies (5.5) with $f \equiv 0$. Hence, in the following we can assume

$$\frac{4n\omega_n\xi(n)}{\sigma}\left[\int_{B_{5R}} |\nabla u|^2\,dx + \int_{B_{5R}} \|S_x - T\|^2\,d\mu\right] < \theta_\omega(R/16)^m$$

and Theorem 2.7, (5.9) and the assumption $K \cap B_{R/16} \neq \emptyset$ yield

$$A \cap B_{R/8} \neq \emptyset\,. \tag{5.10}$$

Step 4. Because of (5.6), (5.8) we can find a Lipschitz extension f of ϕ satisfying (5.4). Now we claim that f satisfies (5.5). Indeed, since

$$B_R \cap K \setminus \Gamma(f) \subset B_R \cap K \setminus A\,,$$

by (5.9) we get

$$\begin{aligned} &\mathcal{H}^m\big(B_R \cap K \setminus \Gamma(f)\big) \\ &\leq \frac{4n\omega_n\xi(n)}{\sigma}\left[\int_{B_{5R}} |\nabla u|^2\,dx + \int_{B_{5R}} \|S_x - T\|^2\,d\mu\right]. \end{aligned} \tag{5.11}$$

Now we estimate

$$\mathcal{H}^m\big(B^m_{R/4} \cap T(\Gamma(f) \setminus K)\big)\,.$$

Recalling that the multiplicity of A does not exceed 1 on B^m_R, we have

$$B^m_{R/4} \cap T(\Gamma(f) \setminus K) \subset B^m_{R/4} \cap T(\Gamma(f) \setminus A) \subset B^m_{R/4} \setminus E$$

so that we need only to estimate the measure of $F = B^m_{R/4} \setminus E$. To this aim, we fix a compact set $G \subset A$ such that $G \cap B_{R/8} \neq \emptyset$ (see (5.10)). For any $z \in F$ let $C_r(z) = B^m_r(z) \times \mathbb{R}$ be the largest cylinder which does not intersect G. Since $G \cap B_{R/8} \neq \emptyset$ we have $r \leq 3R/8$ and we can find $x = (z', f(z')) \in \partial C_r(z) \cap G$. By applying Corollary 4.5 with $b = z - z'$ we get

$$\mathcal{H}^m\big(K \cap B_r((z, f(z'))\big) \geq \frac{\omega_m}{2} r^m\,.$$

Since (see (5.6) and (5.1))

$$R^{-1}\big|(z, f(z'))\big| \leq \sqrt{\frac{1}{16} + \delta^2} < \frac{5}{8}$$

we have the inclusion $B_r((z, f(z')) \subset B_R \cap C_r(z)$, hence

$$\mathcal{H}^m\big(B_R \cap K \cap C_r(z) \setminus G\big) \geq \frac{\omega_m}{2} r^m\,.$$

By Besicovitch's theorem, we can find a disjoint collection $\{B^m_{r_i}(z_i)\}_{i\in I}$ such that

$$\xi(n)\mathcal{H}^m\Big(\bigcup_{i\in I} B^m_{r_i}(z_i)\Big) \geq \mathcal{H}^m(F)$$

and

$$\mathcal{H}^m\big(B_R \cap K \cap C_{r_i}(z_i) \setminus G\big) \geq \frac{\omega_m}{2} r_i^m \qquad \forall i \in I\,.$$

By summing with respect to i, since the cylinders $C_{r_i}(z_i)$ are disjoint, we get

$$\mathcal{H}^m(F) \leq 2\xi(n)\mathcal{H}^m\big(B_R \cap K \setminus G\big)\,.$$

Since $G \subset A$ is arbitrary, by (5.11) we get the estimate

$$\begin{aligned} &\mathcal{H}^m\big(B^m_{R/4} \cap T(\Gamma(f) \setminus K)\big) \\ &\leq \frac{8n\omega_n\xi^2(n)}{\sigma}\left[\int_{B_{5R}} |\nabla u|^2\,dx + \int_{B_{5R}} \|S_x - T\|^2\,d\mu\right]. \end{aligned} \tag{5.12}$$

The statement now follows from (5.11) and (5.12). □

A first consequence of the Lipschitz approximation theorem is the following $C^{1,\alpha}$ regularity criterion for $K(u)$. Basically, we need to control with a power strictly greater than ρ^m the deviation from minimality and the Dirichlet energy in $B_\rho(x)$; moreover, we need a power strictly greater than ρ^{m+2} to control the quantity

$$\mathbb{T}(x,\rho) = \min_{T\in\mathbb{G}(n,m)} \int_{B_\rho(x)} |T^\perp(y-x)|^2\,d\mu(u) \tag{5.13}$$

which measures the flatness of $K(u)$. The tilt lemma (see § 6) guarantees also a control on the oscillation of tangent planes.

THEOREM 5.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in \mathcal{M}_\omega(\Omega)$. Assume that, for some constants $C > 0$, $s > 0$, $\omega(t) \leq Ct^{2s}$ and*

$$\int_{B_\rho(x)} |\nabla u|^2\,dy + \rho^{-2}\mathbb{T}(x,\rho) \leq C\rho^{m+s}$$

for any ball $B_\rho(x) \subset \Omega$ and any $x \in K(u)$. Then $\Omega \cap K(u)$ is a $C^{1,s/2}$-hypersurface.

PROOF. Let $K = K(u)$, $\mu = \mu(u)$ and $\alpha = s/2$. By the tilt Lemma 6.1 of § 6 with $\beta = 1/2$ we have also

$$\sup \left\{ \rho^{-m-\alpha} \min_T \int_{B_\rho(x)} \|S_y - T\|^2 \, d\mu(y) : B_{2\rho}(x) \subset \Omega \right\} < +\infty .$$

We fix an open set $D \subset\subset \Omega$ and we prove the regularity of K in D.

Using the Hölder inequality and Remark 4.1 we find a constant $M > 0$ such that

$$\min_T \int_{B_\rho(x)} \|S_y - T\| \, d\mu(y) \le M\rho^{m+\alpha}$$

for any ball $B_{2\rho}(x) \subset \Omega$ with $\rho < 1$. We denote by T_ρ^x any m-plane such that

$$\min_T \int_{B_\rho(x)} \|S_y - T\| \, d\mu(y) = \int_{B_\rho(x)} \|S_y - T_\rho^x\| \, d\mu(y) .$$

STEP 1. We claim that T_ρ^x converges as $\rho \to 0^+$ to some T^x and $x \mapsto T^x$ is α-Hölder continuous in $K \cap D$. Indeed, let $\rho_0 < \min\{2\rho_\omega, 1\}$ be less than the distance of D from $\partial\Omega$ and $x \in K \cap D$, $\rho \in (0, \rho_0/2)$; we have

$$\begin{aligned} \|T_\rho^x - T_{\rho/2}^x\| &\le \frac{2^m}{\theta_\omega \rho^m} \int_{B_{\rho/2}(x)} \|T_\rho^x - T_{\rho/2}^x\| \, d\mu(y) \\ &\le \frac{2^m}{\theta_\omega \rho^m} \left[\int_{B_\rho(x)} \|T_\rho^x - S_y\| \, d\mu(y) + \int_{B_{\rho/2}(x)} \|T_{\rho/2}^x - S_y\| \, d\mu(y) \right] \qquad (5.14) \\ &\le \frac{2^m M}{\theta_\omega \rho^m} 2\rho^{m+\alpha} \le \frac{2^n M}{\theta_\omega} \rho^\alpha . \end{aligned}$$

Using (5.14) it is easy to see that $T_{2^{-k}}^x$ converges as $k \to +\infty$ to some T^x. Moreover, a similar argument shows that T_ρ^x converges to T^x as $\rho \downarrow 0$ and

$$\|T_\rho^x - T^x\| \le c\rho^\alpha \qquad \forall x \in K \cap D, \ \rho \in (0, \rho_0/2)$$

with c depending only on n, θ_ω, M, α. In order to check the Hölder continuity of T^x we choose $x, z \in D \cap K$ with $\rho = |x - z| \le \rho_0/4$ and we estimate

$$\begin{aligned} \|T_\rho^x - T_\rho^z\| &\le \frac{1}{\theta_\omega \rho^m} \int_{B_\rho(x)} \|T_\rho^x - T_\rho^z\| \, d\mu(y) \\ &\le \frac{1}{\theta_\omega \rho^m} \left[\int_{B_\rho(x)} \|T_\rho^x - S_y\| \, d\mu(y) + \int_{B_{2\rho}(z)} \|T_\rho^z - S_y\| \, d\mu(y) \right] \qquad (5.15) \\ &\le \frac{M}{\theta_\omega \rho^m} \left[(\rho)^{m+\alpha} + (2\rho)^{m+\alpha} \right] \le \frac{2^{m+1+\alpha} M}{\theta_\omega} |x - z|^\alpha . \end{aligned}$$

Using (5.14) and (5.15) the Hölder continuity of $x \mapsto T^x$ in $K \cap D$ follows.

STEP 2. For $\mathcal{H}^m$-almost every $x \in K \cap D$ the m-plane T^x of Step 1 coincides with the approximate tangent space to K. Indeed, let $x_0 \in K \cap D$ be a Lebesgue point for the approximate tangent space map S_x; for any $\rho \in (0, \rho_0/2)$ we have

$$\begin{aligned} \|T^x_\rho - S_{x_0}\| &\le \frac{1}{\theta_\omega \rho^m} \int_{B_\rho(x_0)} \|T^x_\rho - S_{x_0}\| \, d\mu(y) \\ &\le \frac{1}{\theta_\omega \rho^m} \left[\int_{B_\rho(x_0)} \|T^x_\rho - S_y\| \, d\mu(y) + \int_{B_\rho(x_0)} \|S_y - S_{x_0}\| \, d\mu(y) \right] \\ &\le \frac{M}{\theta_\omega} \rho^\alpha + \frac{1}{\theta_\omega \rho^m} \int_{B_\rho(x_0)} \|S_y - S_{x_0}\| \, d\mu(y) \end{aligned}$$

and letting $\rho \downarrow 0$ we find $S_{x_0} = T^{x_0}$.

STEP 3. Now we see that $K \cap D$ is locally the graph of a Lipschitz continuous function. To this aim, we remark that Corollary 4.4 and Corollary 4.5 imply that K has density 1 everywhere. We define σ as in the proof of Theorem 5.2.

Let $x_0 \in K \cap D$ and let $\rho_1 \le \min\{\rho_0/7, R_{n,\omega,1}\}$ so small that

$$A = \left\{ x \in B_{\rho_1}(x_0) \cap K : \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + \int_{B_\rho(x_0)} \|S_x - T\|^2 \, d\mu < \frac{\sigma}{2} \rho^m \ \forall \rho \in (0, 4\rho_1) \right\}$$

coincides with $B_{\rho_1}(x_0) \cap K$ and

$$\rho_1^{-m-2} \int_{B_{4\rho_1}(x_0)} |T^\perp(x - x_0)|^2 \, d\mu < \min\left\{ \theta_\omega \left(\frac{\delta}{4}\right)^{m+2}, \frac{\gamma}{4} \right\}.$$

Let $f : T \to T^\perp$ be the 1-Lipschitz function given by Theorem 5.2. By construction, the graph of f contains A, hence $B_{\rho_1}(x_0) \cap K$ is contained in the graph of f. We now claim that $\overline{B}_{\rho_1/2}(x_0) \cap K$ *contains* the graph of f on $B^m_{\rho_1/4}(T(x_0))$: indeed if there were some $z \in B^m_{\rho_1/4}(T(x_0))$ such that $(z, f(z)) \notin \overline{B}_{\rho_1/2}(x_0) \cap K$ we could take the largest cylinder $C_r = B^m_r(z) \times \mathbb{R}$ that does not intersect $\overline{B}_{\rho_1/2}(x_0) \cap K$ and some point $x_1 \in \partial C_r \cap K \cap \overline{B}_{\rho_1/2}(x_0)$. Since $|z - T(x_0)| < \rho_1/4$ we have $r < \rho_1/4$, hence $x_1 \in B_{\rho_1/2}(x_0)$ because the Lipschitz constant of f does not exceed 1.

Since $K \cap B_{\rho_1}(x_0)$ is contained in the graph of f and since C_r does not intersect $K \cap B_{\rho_1/2}(x_0)$, the density of K at x_1 can be at most $\sqrt{2}/2 < 1$, a contradiction.

STEP 4. In order to show that the function f of Step 3 is actually $C^{1,\alpha}$ we need only to remark that at $\mathcal{L}^m$-almost any differentiability point $z \in B^m_{\rho_1/4}(T(x_0))$ of f the approximate tangent space to K (i.e., the graph of f) exists and is related in the classical way to $\nabla f(x_0)$. In particular, Step 1 and Step 2 imply that the map ∇f (defined at differentiability points of f) has a α-Hölder continuous extension to $B^m_{\rho_1/4}(T(x_0))$. It is easy to check, e.g. using mollifiers, that any Lipschitz map with this property is actually in $C^{1,\alpha}$. □

REMARK 5.4. Let $\mathbb{A}_{n,m}$ be the set of *affine* m-planes in $\mathbb{R}^n$ and define, in analogy with (5.13)

$$\mathbb{A}(x,\rho) = \min_{A\in\mathbb{A}_{n,m}} \int_{B_\rho(x)} |A^\perp y|^2\, d\mu(u) \tag{5.16}$$

where $|A^\perp y|$ is the distance of y from A. Then, clearly $\mathbb{A} \le \mathbb{T}$. However, it is not hard to see that Theorem 5.3 still holds if $\mathbb{A}$ instead of $\mathbb{T}$ is controlled. Indeed, the following inequality holds

$$\mathbb{T}(x,\rho) \le 2\mathbb{A}(x,\rho) + 16n\omega_n\rho^m \left(\frac{\mathbb{A}(x,\rho)}{\theta_\omega}\right)^{2/(m+2)} \tag{5.17}$$

for any $\rho < 2\rho_\omega$ such that $\omega(\rho) \le n\omega_n$. In particular, if $\mathbb{A}(x,\rho) \le C\rho^{m+2+\alpha}$ for some $\alpha \le s$, the assumptions of Theorem 5.3 are satisfied with $s' = 2\alpha/(m+2)$.

To check (5.17), let $A \in \mathbb{A}_{n,m}$ be satisfying

$$\int_{B_\rho(x)} |A^\perp y|^2\, d\mu(u) = \mathbb{A}(x,\rho)\,.$$

Denoting by $T \in \mathbb{G}_{n,m}$ the m-plane parallel to A and by δ the distance of A and x, by the energy upper bound, we have

$$\mathbb{T}(x,\rho) \le 2\mathbb{A}(x,\rho) + 2\delta^2\mu(u)(B_\rho(x)) \le 2\mathbb{A}(x,\rho) + 4\delta^2 n\omega_n\rho^m\,.$$

Hence, we need only to estimate δ/ρ; using the density lower bound we get

$$\int_{B_\rho(x)} |A^\perp y|^2\, d\mu(u) \ge \left(\frac{\delta}{2}\right)^2 \mu(u)\big(B_{\delta/2}(x)\big) \ge \theta_\omega\left(\frac{\delta}{2}\right)^{m+2}$$

and this implies (5.17).

6. – Flatness improvement

We begin this section proving a classical (compare [1], [26], [9]) estimate on the tilt of tangent planes. The proof is achieved by a deformation argument.

LEMMA 6.1 (tilt estimate). *For any $\beta \in (0,1)$ there exist $c(\beta) > 0$, $\varepsilon_\beta > 0$ such that for any $u \in SBV(B_R)$ and any m-plane T we have*

$$\int_{B_{\beta R}\cap S_u} \|S_x - T\|^2 \, d\mathcal{H}^m(x) \tag{6.1}$$
$$\le c(\beta)\left[\int_{B_R} |\nabla u|^2\,dx + R^{-2}\int_{B_R\cap S_u} |T^\perp x|^2\,d\mathcal{H}^m(x) + R^m\sqrt{\frac{\mathrm{Dev}(u,B_R)}{R^m}}\right]$$

provided $\mathrm{Dev}(u, B_R) < \varepsilon_\beta^2 R^m$.

PROOF. Let $\mu = \mathcal{H}^m \llcorner S_u$, $R = 1$, and let $\zeta \in C_0^1(B_1)$ be a function such that $0 \le \zeta \le 1$, $\zeta = 1$ in B_β and $|\nabla\zeta| \le 2/(1-\beta)$. Consider the vector field $\phi(x) = \zeta^2(x)T^\perp(x)$, set $\tau_\varepsilon(x) = x + \varepsilon\phi(x)$, and remark that positive constants ε_β, c_1 exist such that for $0 < |\varepsilon| < \varepsilon_\beta$ the inequality

$$\left|\mathbb{J}_m(\tau_\varepsilon, S) - 1 - \varepsilon\,\mathrm{div}_S\phi\right| \le c_1\varepsilon^2 \tag{6.2}$$

holds uniformly with respect to $S \in \mathbb{G}_{n,m}$ and τ_ε is a C^1 diffeomorphism of B_1. Moreover, for $u_\varepsilon(x) = u(\tau_\varepsilon(x))$, the definition of $\mathrm{Dev}(u, B_1)$ implies

$$\int_{B_1} |\nabla u_\varepsilon|^2\,dx - \int_{B_1} |\nabla u|^2\,dx + \mathcal{H}^m(S_{u_\varepsilon}) - \mathcal{H}^m(S_u) + \mathrm{Dev}(u, B_1) \ge 0\,. \tag{6.3}$$

Now compute (cf. [26 § 22])

$$\mathrm{div}_S T^\perp = \frac{1}{2}\|S - T\|^2$$
$$\mathrm{div}_S\phi = 2\zeta\langle\nabla_S\zeta, T^\perp\rangle + \zeta^2\mathrm{div}_S T^\perp = 2\zeta\langle\nabla_S\zeta, T^\perp\rangle + \frac{\zeta^2}{2}\|S - T\|^2\,,$$

(where div_S and ∇_S are the tangential operators along S) and remark that

$$\begin{aligned} 2\zeta\langle\nabla_S\zeta, T^\perp\rangle &\ge -2\zeta\|S - T\|\,|T^\perp x|\,|\nabla\zeta| \\ &\ge -\frac{1}{4}\zeta^2\|S - T\|^2 - 4|\nabla\zeta|^2\,|T^\perp x|^2\,. \end{aligned} \tag{6.4}$$

To check (6.4), suppose that $T = \mathrm{span}\{e_1, \ldots, e_{n-1}\}$, and compute

$$|\langle\nabla_S\zeta, T^\perp(x)\rangle| = \left|x_n S_{nj}\frac{\partial\zeta}{\partial x_j}\right| = \left|x_n(S_{nj} - T_{nj})\frac{\partial\zeta}{\partial x_j}\right|,$$

whence (6.4) readily follows. Notice also that

$$\int_{B_1} |\nabla u_\varepsilon|^2\,dx = \int_{B_1} |\nabla u(x)\nabla\tau_\varepsilon^{-1}(x)|^2 |\det\nabla\tau_\varepsilon(x)|dx\,,$$

whence

$$\left|\frac{1}{\varepsilon}\left[\int_{B_1} |\nabla u_\varepsilon|^2\,dx - \int_{B_1}|\nabla u|^2\,dx\right]\right| \le c_2 \int_{B_1} |\nabla u|^2\,dx,\quad 0<|\varepsilon|<\varepsilon_\beta \tag{6.5}$$

(where c_2 depends only on ϕ and ε_β). Since

$$\mathcal{H}^m(S_{u_\varepsilon}) = \int_{B_1} \mathbb{J}_m(\tau_\varepsilon, S_x)\,d\mu\,,$$

from (6.2) the inequality

$$\int_{B_1} |\nabla u_\varepsilon|^2\,dx - \int_{B_1} |\nabla u|^2\,dx + \varepsilon\int_{S_u} \mathrm{div}^\tau\phi\,d\mathcal{H}^m + \mathrm{Dev}(u,B_1) \ge -c_1\varepsilon^2$$

follows. Using (6.4), (6.5) we obtain for $-\varepsilon_\beta < \varepsilon < 0$:

$$\begin{aligned}\frac{1}{4}\int_{B_\beta} \|S_x - T\|^2\,d\mu \le \frac{16}{(1-\beta)^2}\int_{B_1} |T^\perp x|^2\,d\mu + c_2\int_{B_1}|\nabla u|^2\,dx \\ - \frac{\mathrm{Dev}(u,B_1)}{\varepsilon} + c_1|\varepsilon|\,.\end{aligned} \tag{6.6}$$

Finally, assuming $\mathrm{Dev}(u,B_1)\le\varepsilon_\beta^2$ we choose $\varepsilon = -\sqrt{\mathrm{Dev}(u,B_1)}$ and use (6.6) to get

$$\begin{aligned}\int_{B_\beta} \|S_x - T\|^2\,d\mu \le \frac{64}{(1-\beta)^2}\int_{B_1} |T^\perp x|^2\,d\mu \\ + 4c_2\int_{B_1}|\nabla u|^2\,dx + 4(1+c_1)\sqrt{\mathrm{Dev}(u,B_1)}\,,\end{aligned} \tag{6.7}$$

and the thesis follows for $R = 1$. To recover the general case, exploit the scaling properties of the various quantities in (6.7) and conclude. □

The following theorem (whose statement is similar to [9, 5.6]) shows a decay property of the quantity $\mathbb{A}(x,\rho)$ defined in (5.16), assuming that $\mathbb{A}(x,\rho)/\rho^{m+2}$ is sufficiently small and that the Dirichlet energy and the deviation from minimality are comparable with $\mathbb{A}(x,\rho)$. The proof is obtained by the classical harmonic comparison argument of De Giorgi (see also [26]), i.e., we approximate $K(u)$ by the graph of a Lipschitz function f and show that a rescaled function g is close to a harmonic function.

THEOREM 6.2. *For any $\beta \in (0, 1/112)$ and $M > 0$ there exists $\varepsilon(\beta, M, \omega) > 0$ such that, for any $u \in \mathcal{M}_\omega(B_\rho(x))$, the conditions*

$$x \in K(u), \quad \mathbb{A}(x,\rho) \le \varepsilon\rho^{m+2}, \quad M\mathbb{A}(x,\rho) \ge \rho^2\left[\int_{B_\rho(x)} |\nabla u|^2 + \rho^m\sqrt{\omega(\rho)}\right]$$

imply

$$\mathbb{A}(x,\beta\rho) \le C\beta^{m+4}\mathbb{A}(x,\rho) \tag{6.8}$$

with C depending only on n and ω.

PROOF. Let us fix β, M, ω, and assume for convenience that $\omega(\rho) > 0$ for any $\rho > 0$ (see Remark 6.3); let also be $R = R_{n,\omega,1}$, where $R_{n,\omega,1}$ is the constant defined before Theorem 5.2. During the proof, we shall denote by c a positive constant (which can change from a line to another) depending only on n, M, ω, R and the constants given by Theorem 5.2.

Arguing by contradiction, sequences x_h, ρ_h, $v_h \in \mathcal{M}_\omega(B_{\rho_h}(x_h))$ and A_h (affine m-planes) exist such that

$$\rho_h^{-m-2}\int_{B_{\rho_h}(x_h)} |A_h^\perp x|^2\,d\mu(v_h) = \eta_h'^2 \to 0, \tag{6.9}$$

$$M\rho_h^{m+2}\eta_h'^2 \ge \rho_h^2\left[\int_{B_{\rho_h}(x_h)} |\nabla v_h|^2\,dx + \rho_h^m\sqrt{\omega(\rho_h)}\right] \tag{6.10}$$

and for every affine m-plane S and $h \in \mathbb{N}$

$$\int_{B_{\beta\rho_h}(x_h)} |S^\perp x|^2\,d\mu(v_h) \ge C\beta^{m+4}\eta_h'^2\rho_h^{m+2}, \tag{6.11}$$

where the constant C will be specified later (see (6.30) below).

By a rotation and a translation we can assume that $A_h = T = e_n^\perp$ for any h. Moreover, translating x_h in a direction parallel to T we can assume that $T(x_h) = 0$.

By (6.9), (6.10) we infer that $\rho_h \to 0$. In particular, $\rho_h \le 8R$ for h large enough and using Remark 4.2 with $\rho = \rho_h/8R$ and $x_0 = 0$ we have a new sequence $u_h \in \mathcal{M}_\omega(B_{8R}(y_h))$ such that, for $\eta_h^2 = \eta_h'^2(8R)^{m+2}$

$$\int_{B_{8R}(y_h)} |T^\perp x|^2\,d\mu(u_h) = \eta_h^2 \to 0, \tag{6.12}$$

$$M\eta_h^2 \ge \left[\int_{B_{8R}(y_h)} |\nabla u_h|^2\,dx + (8R)^m\sqrt{\omega(\rho_h)}\right] \tag{6.13}$$

and for every affine m-plane S and $h \in \mathbb{N}$

$$\int_{B_{\beta 8R}(y_h)} |S^{\perp}x|^2 \, d\mu(u_h) \geq C\beta^{m+4}\eta_h^2 . \tag{6.14}$$

By the tilt estimate (6.1) and (6.10) we obtain

$$\rho_h^{-m} \int_{B_{\tau\rho_h}(x_h)} \|S_x - T\|^2 \, d\mu(v_h) \leq c(M,\tau)\eta_h'^2 \qquad \forall \tau \in (0,1)$$

hence

$$\int_{B_{\tau 8R}(y_h)} \|S_x - T\|^2 \, d\mu(u_h) \leq c(M,\tau)\eta_h^2 \qquad \forall \tau \in (0,1) . \tag{6.15}$$

Finally, using (6.13) we can estimate the deviation from minimality of u_h in $B_{8R}(y_h)$ as follows:

$$\mathrm{Dev}(u_h, B_{8R}(y_h)) = \frac{(8R)^m}{\rho_h^m} \mathrm{Dev}(v_h, B_{\rho_h}(x_h)) \leq c\eta_h^4 \to 0 \tag{6.16}$$

and we observe that the density lower bound of Theorem 2.7 and (6.12) easily imply that $y_h = T^{\perp}y_h$ tends to 0 as $h \to +\infty$ (recall that $y_h \in K(u_h)$).

Now, we shall construct for every h a Lipschitz continuous function f_h whose graph approximates $K(u_h)$ according to Theorem 5.2 (Step 1) and prove (Step 2) that a subsequence of $g_h = \eta_h^{-1} f_h$ converges to a harmonic (Step 3) function g. Using estimates on g we shall find (Step 4) for sufficiently large h a m-plane S_h violating (6.14).

STEP 1. For h large enough y_h belongs to $B_{R/16}$, hence $B_{7R} \subset B_{8R}(y_h)$. Using (6.12) we see that for h large enough (5.3) holds and Theorem 5.2 gives a 1-Lipschitz function $f_h : T \to T^{\perp}$ such that

$$\sup |f_h| \leq c\eta_h^{2/(m+2)} \tag{6.17}$$

and, setting

$$X_h = B_R \cap \left[K(u_h) \setminus \Gamma(f_h)\right] \qquad F_h = B_R \cap \left[\Gamma(f_h) \cap K(u_h)\right]$$

$$Q_h = B^m_{R/4} \cap T\left[\Gamma(f_h) \setminus K(u_h)\right] \qquad E_h = B^m_{R/4} \setminus Q_h = T(F_h) \cap B^m_{R/4} ,$$

the inequality

$$\begin{aligned} \mathcal{H}^m(Q_h) + \mathcal{H}^m(X_h) &\leq P\left[\int_{B_{5R}} |\nabla u_h|^2 \, dx + \int_{B_{5R}} \|S_x - T\|^2 \, d\mu(u_h)\right] \\ &\leq c\eta_h^2 \end{aligned} \tag{6.18}$$

follows from the tilt estimate (6.15); moreover, (6.13) implies

$$\int_{B_{7R}} |\nabla u_h|^2 \, dx = o(\eta_h) . \tag{6.19}$$

STEP 2. Let us show that (up to subsequences) $g_h = \eta_h^{-1} f_h$ weakly converges in $W^{1,2}(B^m_{R/4})$. For, recall that $|\nabla f_h| \le 1$ a.e. by Rademacher's theorem. Thus, by (6.17), (6.12) and (6.18):

$$\begin{aligned}\int_{B^m_{R/4}} |f_h(z)|^2 \, dz &\le c\eta_h^{4/(m+2)} \mathcal{H}^m(Q_h) + \int_{E_h} |f_h|^2 \, dz \\ &\le c\eta_h^{2+4/(m+2)} + \int_{B_R} |T^\perp x|^2 \, d\mu(u_h) \le o(\eta_h^2) + \eta_h^2 ,\end{aligned}$$

whence in particular

$$\limsup_{h \to +\infty} \int_{B^m_{R/4}} |g_h(z)|^2 \, dz \le 1 . \tag{6.20}$$

Regarding the gradients, first remark that

$$\int_{Q_h} |\nabla f_h(z)|^2 \, dz \le c\eta_h^2 \tag{6.21}$$

by (6.18), and then notice that for $\mathcal{H}^m$-a.e. $x = \big(z, f_h(z)\big) \in F_h$ the equality

$$S_x = \operatorname{span}\Big\{e_i + \frac{\partial f_h}{\partial z_i} e_n : \ i = 1, \dots , m\Big\}$$

holds. A simple computations shows that

$$\|S_x - T\|^2 = 2\big(1 - \nu_n\big) = \frac{2}{1 + \nu_n}(1 - \nu_n^2) \ge \frac{|\nabla f_h(z)|^2}{1 + |\nabla f_h(z)|^2} \ge \frac{1}{2} |\nabla f_h(z)|^2$$

where ν is the normal to the graph of f at x. We can then infer that (see (6.15))

$$\begin{aligned}\int_{E_h} |\nabla f_h(z)|^2 \, dz &\le 2 \int_{E_h} \|S_{(z, f_h(z))} - T\|^2 \, dz \\ &\le 2 \int_{B_R} \|S_x - T\|^2 \, d\mu(u_h) \le c\eta_h^2 .\end{aligned} \tag{6.22}$$

Therefore, the sequence g_h is bounded in $W^{1,2}(B^m_{R/4})$ and we can assume that it is weakly convergent to a function $g \in W^{1,2}(B^m_{R/4})$.

STEP 3. In order to see that $g = \lim \eta_h^{-1} f_h$ is harmonic in $B^m_{R/4}$, we shall show that

$$\lim_{h\to+\infty} \frac{1}{\eta_h} \int_{B^m_{R/4}} \langle \nabla f_h, \nabla\psi\rangle \, dz = 0 \qquad \forall\, \psi \in C^1_0\big(B^m_{R/4}\big)\,. \tag{6.23}$$

Let us consider the vector field $\phi = (0, \dots, 0, \psi)$; since $\psi \in C^1_0(B^m_{R/4})$ and (by Proposition 5.1) the maximum of $|T^\perp x|$ on $\overline{B}_R \cap K(u_h)$ is infinitesimal, we can find for sufficiently large h a vector field $\hat\phi$ with compact support in $B_{R/4}$ and coinciding with ϕ on $K(u_h) \cap B_R$ (just take $\hat\phi(z, y) = \phi(z)\chi(y)$ for a suitable χ).

We use comparison functions $w_h(\tau_h(x)) = u_h(x)$, where $\tau_h(x) = x - \eta_h^2\hat\phi$. By the definition of $\mathrm{Dev}(u_h, B_{R/4})$ we get (see (6.3))

$$\begin{aligned}\int_{B_{R/4}} |\nabla w_h|^2\, dx - \int_{B_{R/4}} |\nabla u_h|^2\, dx + \mathcal{H}^m\big(K(w_h)\big) - \mathcal{H}^m\big(K(u_h)\big)&\\ + \mathrm{Dev}(u_h, B_{R/4}) \geq 0\,,&\end{aligned} \tag{6.24}$$

whence arguing as in the proof of Lemma 6.1

$$\begin{aligned}\int_{B_{R/4}} \mathrm{div}^\tau \hat\phi \, d\mu(u_h) \leq{}& \frac{\mathrm{Dev}(u_h, B_{R/4})}{\eta_h^2}\\ &+ \frac{1}{\eta_h^2}\left[\int_{B_{R/4}} |\nabla w_h|^2\, dx - \int_{B_{R/4}} |\nabla u_h|^2\, dx\right] + c_1\eta_h^2\end{aligned}$$

(where c_1 has the same meaning as in (6.2)). By (6.16), (6.5) and (6.19) we deduce

$$\limsup_{h\to+\infty} \eta_h^{-1} \int_{B_{R/4}} \mathrm{div}^\tau \phi \, d\mu(u_h) \leq 0\,.$$

From this inequality and the analogous one obtained taking $\tau_h(x) = x + \eta_h^2\hat\phi(x)$ we get

$$\lim_{h\to+\infty} \eta_h^{-1} \int_{B_{R/4}} \mathrm{div}^\tau \phi \, d\mu(u_h) = 0\,. \tag{6.25}$$

Now, since by (6.18)

$$\int_{Q_h} \langle \nabla f_h, \nabla\psi\rangle\, dz = o(\eta_h), \qquad \int_{X_h} \mathrm{div}^\tau \phi\, d\mu(u_h) = o(\eta_h)$$

for every $\psi \in C_0^1(B^m_{R/4})$, (6.23) can be deduced from (6.25) and

$$\lim_{h\to+\infty}\frac{1}{\eta_h}\left[\int_{F_h}\operatorname{div}^\tau\phi\,d\mu(u_h)-\int_{E_h}\langle\nabla f_h,\nabla\psi\rangle\,dz\right]=0. \tag{6.26}$$

To prove (6.26), notice that $\mathcal{H}^m$-a.e. in F_h, denoting by ν the normal unit vector to $\Gamma(f_h)$

$$\operatorname{div}^\tau\phi=\delta_n\psi=\frac{\partial\psi}{\partial x_n}-\langle\nabla\psi,\nu\rangle\nu_n=\frac{\langle\nabla\psi,\nabla f_h\rangle}{1+|\nabla f_h|^2}. \tag{6.27}$$

Inserting (6.27) in (6.26) and using the equality $T(F_h)\cap B^m_{R/4}=E_h$ we obtain

$$\begin{aligned}&\frac{1}{\eta_h}\left|\int_{E_h}\langle\nabla f_h,\nabla\psi\rangle\big(1-(1+|\nabla f_h|^2)^{-1/2}\big)\,dz\right|\\&\le\frac{1}{2\eta_h}\int_{E_h}|\nabla f_h|^2|\nabla\psi|\,dz\le\frac{1}{2\eta_h}\|\nabla\psi\|_\infty\int_{B^m_{R/4}}|\nabla f_h|^2\,dz\end{aligned}$$

which is infinitesimal by Step 2.

STEP 4. Let $\Gamma(n,R)$ be the constant given by [20, 5.2.5] such that

$$\max\{|g(0)|,R|\nabla g(0)|\}\le\Gamma\left(\int_{B^m_{R/4}}|g(z)|^2\,dz\right)^{1/2} \tag{6.28}$$

and

$$|g(z)-g(0)-\langle\nabla g(0),z\rangle|\le\Gamma\left(\int_{B^m_{R/4}}|g(w)|^2\,dw\right)^{1/2}|z|^2 \tag{6.29}$$

for any $z\in B^m_{R/8}$. We can now define the constant C by

$$C=2^{m+6}\Gamma^2\int_{B^m_{7R}}|z|^4\,dz. \tag{6.30}$$

Define also

$$Z_h=\{x\in B_{7\beta R}:|T^\perp x|>\eta_h^{1/(m+2)}\}\qquad L_h(z)=z+\eta_h\langle\nabla g(0),z\rangle e_n$$

$$K_h(x)=L_h(Tx)+\eta_h g(0)e_n\qquad S_h=\text{ image of }K_h\,,$$

and remark that

$$Z_h \cap \operatorname{spt}\mu(u_h) = \emptyset \qquad \text{for } h \text{ large enough}. \tag{6.31}$$

Indeed, let $x_0 \in Z_h \cap \operatorname{spt}\mu(u_h)$ and $\tau = |T^\perp x_0|/2$; then $\mu(u_h)(B_\tau(x_0)) \geq \theta_\omega \tau^m$, whence arguing as in the proof of Lemma 5.1

$$\eta_h^2 \geq \tau^2 \theta_\omega \tau^m = \frac{\theta_\omega}{2^{m+2}} |T^\perp x_0|^{m+2} > \frac{\theta_\omega}{2^{m+2}} \eta_h ,$$

a contradiction for h large enough. Let $\gamma = 2\beta$; we now want to estimate

$$\int_{B_{7\gamma R}} |S_h^\perp x|^2 \, d\mu(u_h)$$

to get a contradiction with (6.14). For every $x \in B_R$ we have

$$x - K_h(x) = x - Tx - \eta_h \big[g(0) + \langle \nabla g(0), Tx\rangle\big] e_n$$

whence by (6.20) and (6.28)

$$\begin{aligned} |S_h^\perp x| &\leq |x - K_h(x)| \leq |T^\perp x| + \eta_h \big[|g(0)| + |\langle Tx, \nabla g(0)\rangle|\big] \\ &\leq |T^\perp x| + 2\Gamma \eta_h \Bigg(\int_{B^m_{R/4}} |g(z)|^2 \, dz \Bigg)^{1/2} \\ &\leq |T^\perp x| + 2\Gamma \eta_h . \end{aligned} \tag{6.32}$$

Moreover, if in particular $x = (z, f_h(z))$, then

$$\begin{aligned} |x - K_h(x)| &= |f_h(z) - \eta_h \big[g(0) + \langle \nabla g(0), z\rangle\big]| \\ &= |f_h(z) - \eta_h g(z) + \eta_h \big[g(z) - g(0) - \langle \nabla g(0), z\rangle\big]| \end{aligned}$$

whence, using (6.29) and (6.20), for $|z| \leq R/8$ and $x = (z, f_h(z))$ we obtain

$$|S_h^\perp x| \leq \eta_h \big[|g_h(z) - g(z)| + \Gamma |z|^2\big] . \tag{6.33}$$

Using (6.32) and (6.33) (recall that $7\gamma R < R/8$ because $\gamma < 1/56$) we get

$$\begin{aligned} \int_{B_{7\gamma R}} |S_h^\perp x|^2 \, d\mu(u_h) &\leq 2 \int_{B_{7\gamma R} \cap X_h} |T^\perp x|^2 \, d\mu(u_h) + 8\Gamma^2 \eta_h^2 \mathcal{H}^m(X_h) \\ &+ 2\eta_h^2 \int_{B_{7\gamma R} \cap E_h} \sqrt{1 + |\nabla f_h|^2(z)} \, \big[|g_h(z) - g(z)|^2 + \Gamma^2 |z|^4\big] \, dz . \end{aligned}$$

By virtue of $|\nabla f_h| \le 1$ and (6.31)

$$\int_{B_{7\gamma R}} |S_h^{\perp} x|^2 \, d\mu(u_h) \le \left[2\eta_h^{2/(m+2)} + 8\Gamma^2\eta_h^2\right]\mathcal{H}^m(X_h)$$

$$+ 2\sqrt{2}\eta_h^2 \int_{B^m_{R/4}} |g_h(z) - g(z)|^2 \, dz + 2\sqrt{2}\eta_h^2\Gamma^2 \int_{B^m_{7\gamma R}} |z|^4 \, dz \, .$$

Finally, since the first two terms in the above inequality are $o(\eta_h^2)$, we get

$$\limsup_{h\to+\infty} \beta^{-m-4}\eta_h^{-2} \int_{B_{7\gamma R}} |S_h x|^2 \, d\mu(u_h) \le 2^{m+5}\sqrt{2}\Gamma^2 \int_{B^m_{7R}} |z|^4 \, dz < C \, ,$$

and this contradicts (6.14) because $B_{7\gamma R} = B_{14\beta R}$ contains $B_{8\beta R}(y_h)$ for h large enough. □

REMARK 6.3. If we assume that $\omega(t)$ vanishes on some interval $(0, b) \subset (0, +\infty)$, then Theorem 6.2 continues to hold with the additional assumption $\rho < b$. In this case u is a local minimizer, according to Definition 2.4, in small balls.

Notice also that we won't use the full generality of Theorem 6.2 to prove Theorem 3.1 (see the next section). In [6], coupling Theorem 6.2 with a decay lemma for the Dirichlet integral we will prove that the $L^{2,\lambda}$ assumption on $|\nabla u|$ can be removed. The proof is based on a suitable choice of M and on the independence of the constant C in (6.8) on M.

7. – Proof of partial regularity

In this section we prove Theorem 3.1. By (3.1) and the $L^{2,\lambda}$ assumption on $|\nabla u|$ we can find $N > 0$ such that

$$\int_{B_\rho(x)} |\nabla u|^2 + \rho^m \sqrt{\omega(\rho)} \le N\rho^{m+\alpha} \tag{7.1}$$

for any ball $B_\rho(x) \subset \Omega$ with $\rho \le 1$ (recall that $\alpha = \min\{\lambda - m, s\}$).

Since $\alpha \le s < 2$, we can choose $\beta \in (0, 1/112)$ such that $C\beta^2 \le \beta^\alpha$ (where C is defined in (6.30)). By (7.1) and Theorem 6.2 with $M = 1$ we infer the existence of $\varepsilon > 0$ such that the function $A(\rho) = \mathbb{A}(x, \rho)$ satisfies the property

$$A(\rho) \le \varepsilon\rho^{m+2}, \; A(\rho) \ge N\rho^{m+2+\alpha} \quad \Longrightarrow \quad A(\beta\rho) \le \beta^{m+2+\alpha}A(\rho) \tag{7.2}$$

for any $x \in K(u)$ and any ball $B_\rho(x) \subset \Omega$.

LEMMA 7.1 (iteration). *Let $A(\rho)$ be a nondecreasing function satisfying* (7.2). *Then, the conditions*

$$\frac{N\rho^\alpha}{\beta^{m+2+\alpha}} \le \varepsilon \qquad A(\rho) \le \varepsilon\rho^{m+2}$$

imply

$$A(\tau) \le \beta^{-(m+2+\alpha)} \max\left\{ \frac{N\tau^{m+2+\alpha}}{\beta^{m+2+\alpha}}, \varepsilon\rho^{m+2}\left(\frac{\tau}{\rho}\right)^{m+2+\alpha} \right\} \tag{7.3}$$

for every $\tau \in (0, \rho]$.

PROOF. Let $\rho_k = \rho\beta^k$; by the monotonicity of A, we need only to show that

$$A(\rho_k) \le \max\left\{ \frac{N\rho_k^{m+2+\alpha}}{\beta^{m+2+\alpha}}, \varepsilon\rho^{m+2}\left(\frac{\rho_k}{\rho}\right)^{m+2+\alpha} \right\} \tag{7.4}$$

for every $k \ge 0$. We shall prove (7.4) by induction on k. remark that the inequality holds trivially for $k = 0$, and assume that (7.4) holds; then, our assumption implies

$$\max\left\{ \frac{N\rho_k^{m+2+\alpha}}{\beta^{m+2+\alpha}}, \varepsilon\rho^{m+2}\left(\frac{\rho_k}{\rho}\right)^{m+2+\alpha} \right\} \le \varepsilon\rho_k^{m+2},$$

hence $A(\rho_k) \le \varepsilon\rho_k^{m+2}$. If $A(\rho_{k+1}) \le N\rho_{k+1}^{m+2+\alpha}\beta^{-m-2-\alpha}$ we are done; otherwise

$$A(\rho_k) \ge A(\rho_{k+1}) \ge \frac{N\rho_{k+1}^{m+2+\alpha}}{\beta^{m+2+\alpha}} = N\rho_k^{m+2+\alpha}$$

and we are in a position to apply (7.2), getting

$$A(\rho_{k+1}) \le \beta^{m+2+\alpha} A(\rho_k)$$

which shows that (7.4) holds with $k+1$ in place of k. □

By Lemma 7.1 we obtain (for a suitable $\varepsilon_0 < 1$ depending on ε, N, n, α) that the ratio

$$\frac{\mathbb{A}(y, \eta)}{\eta^{m+2+\alpha}} \qquad y \in B_{\rho/2}(x) \cap K(u),\ 0 < \eta < \rho/2$$

is bounded for any x satisfying (3.2). The $C^{1,\alpha/(m+2)}$ regularity of $K(u) \cap B_{\rho/2}(x)$ follows by Theorem 5.3 and Remark 5.4.

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