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DORIN BUCUR
JEAN-PAUL ZOLÉSIO

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1. – Introduction

In the field of shape optimization, we are concerned with problems of the type: minimize the shape functional

$$\Omega \rightarrow J(\Omega)$$

in the family of open (or closed, or measurable) subsets of a fixed bounded hold all. Some-times, a measure constraint can be given. The functional J can depend also on Ω via the solution of some p.d.e. defined on an open set related to Ω. The usual method to prove the existence of minima for such a functional is a compactness-lower semi-continuity result or a Γ-convergence method, which supposes the existence of a sequence of functionals which have minimal points and which Γ-converges to J.

Obviously, a very important role in this study is played by the topology in the space of domains and the associated compact families. There are two classical topologies which are usually used, like the Hausdorff topology in the family of closed sets and the characteristic topology in the family of measurable sets, each one having some advantages and drawbacks. Anyway, there is no direct link between the two topologies.

Our first idea was to find a constraint which could be imposed on a family of sets and which could insure some relation between the two topologies. Therefore we introduce the (γ, H)-density perimeter, where $\gamma > 0$ is a positive number which in a way represents a scale which we impose in the problem, and $H : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function which will play a corrector role. If we shall impose a boundedness constraint on the (γ, H)-density perimeter we shall obtain for example that the Hausdorff convergence implies the char-convergence, or moreover that the measure of the limit set boundary is equal to zero. An important remark is that chosing H we can handle some topological invariants of the sets, as for example the number of connected components in 2 D, or the number of holes of torus in 3 D. All these results are tied to a pseudo curvature

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concept, the maximal density curvature $c_{\partial,H}$ which will be important also in the study of the Γ-convergence of the (γ, H)-density perimeter for $\gamma \to 0$.

These results will be applied for existence of minima of functionals defined in the family of open sets of the type

$$\Omega \to J(\Omega) + P_{\gamma,H}(\partial\Omega)$$

which arise in free boundary problems, or defined on boundary sets of the type

$$\Gamma \to f(\Gamma) + \mathcal{M}(\Gamma) + c_{\partial,H}(\Gamma)$$

where $\mathcal{M}(\Gamma)$ is the Minkowski content of Γ, which arise in computer vision.

The Γ convergence result, means that one can approximate the continuous problem by some discretizations given in terms of γ. In fact, the parameter γ is in a way a scale on which the problem is considered.

2. – Topology in the space of domains

In this section we recall the concepts of Hausdorff and char convergence of a sequence of domains and some basic properties involving the generalized perimeter of a measurable set. We shall also give some examples comparing the two concepts. For the simplicity of the talk we shall introduce some special classes of domains which will be used all over the paper. Let $D \subseteq \mathbb{R}^N$ be a bounded open smooth domain. Let's denote by

$$\mathcal{C}(D) = \{\Gamma \subseteq \mathrm{cl}\,(\mathrm{D}), \quad \Gamma = \mathrm{cl}\,(\Gamma)\}$$

the family of closed subsets of cl (D) and by

$$\mathcal{C}_0(D) = \mathcal{C}(D) \cap \{\Gamma \,|\, m(\Gamma) = 0\}$$

the family of closed subsets of cl (D) of zero Lebesgue measure. We can also denote the family of open subsets of D by

$$\mathcal{O}(D) = \{\Omega \subseteq D | \Omega = \mathrm{int}\,(\Omega)\}.$$

For any closed sets F_1, $F_2 \in \mathcal{C}(D)$ the Hausdorff metric is defined by:

$$d_{H^d}(F_1, F_2) = \sup_{x \in \mathbb{R}^N} |d_{F_1}(x) - d_{F_2}(x)| \tag{1}$$

where

$$d_{F_1}(x) = \inf_{y \in F_1} \|x - y\|. \tag{2}$$

By complementarity one can define a topology in the space of open sets namely, the Hausdorff complementary topology, denoted H^c and given by the metric

$$d_{H^c}(\Omega_1, \Omega_2) = d_{H^d}(\Omega_1^c, \Omega_2^c)$$

where Ω_1, $\Omega_2 \in \mathcal{O}(D)$ and Ω^c is the complementary set of Ω.

We recall the main properties of this topology.

PROPOSITION 2.1. *The family $\mathcal{C}(D)$ is compact in the H^d-topology.*

The proof of this proposition is an easy consequence of the Ascoli-Arzela theorem. A necessary remark is that the family $\mathcal{C}_0(D)$ it is not compact. Other interesting properties of the Hausdorff topology are the following.

PROPOSITION 2.2. *Denoting by $\#(\Gamma)$ the number of connected components of Γ, the mapping*

$$\Gamma \to \#(\Gamma)$$

is lower semi-continuous in the H^d-topology.

PROPOSITION 2.3. *The family $\mathcal{O}(D)$ is compact in the H^c-topology.*

PROPOSITION 2.4. *Let $\Omega_n \xrightarrow{H^c} \Omega$. Then $\forall K \subset\subset \Omega$, there exists $n_K \in N$ such that $\forall n \geq n_K$ we have $K \subseteq \Omega_n$.*

The char-topology is defined on the family of measurable subsets of $\mathbb{R}^N$ by the L^2-metric:

$$d_{\text{char}}(A_1, A_2) = \int_{\mathbb{R}^N} |\chi_{A_1} - \chi_{A_2}| dx \tag{3}$$

where A_1, A_2 are measurable subsets of $\mathbb{R}^N$ and χ_{A_1}, χ_{A_1} are their characteristic functions. A classical result states that the $L^2(\mathbb{R}^N)$-unity ball is weak compact in the L^2 topology. This statement does not give the compactness in the space of domains, but it is used in the proof of relaxed problems, and in homogenization theory. To obtain a strong compactness result it is defined the generalized perimeter of a measurable set A in the ball D by

$$P_D(A) = \sup_{\bar{\varphi}\mathcal{D}(D,\mathbb{R}^N), \|\bar{\varphi}\| \leq 1} - \int_A \operatorname{div} \bar{\varphi}\, dx\,. \tag{4}$$

The main result concerning the generalized perimeter is the next one.

PROPOSITION 2.5. *Let $\{\Omega_n\}$ be sequence of measurable sets in D, with $P_D(\Omega_n) \leq c$. Then there exists a subsequence $\{\Omega_{n_k}\}$ and a measurable set Ω in D, such that $P_D(\Omega) \leq \liminf_{n\to\infty} P_D(\Omega_{n_k})$ and $\Omega_{n_k} \xrightarrow{\text{char}} \Omega$.*

Since the generalized perimeter is associated to measurable sets, the char convergence has no special behavior for the family of open sets where usually p.d.e. are defined. So it is possible that a sequence of open sets with uniformly bounded generalized perimeter to converge in the char topology to a measurable set which has not open representative. For a simple example, constructed by finite unions of balls centred in the points of rational coordinates of the unity ball, see [6].

In oposition with the char topology, the H^c-topology has a good compactness property in the family of open sets, but there is no direct link between the H^c convergence and the char-convergence, namely if $\Omega_n \xrightarrow{H^c} A$ and

$\Omega_n \xrightarrow{\text{char}} B$ we do not have $\chi_A = \chi_B$. A very simple example is to consider $\Omega_n = B(0,1)\setminus\{x_1, \dots, x_n\}$, x_i being the first n points of rational coordinates in the unity ball. Then $\Omega_n \xrightarrow{H^c} \emptyset$ and $\Omega_n \xrightarrow{\text{char}} B(0,1)$.

These two topologies can be compared by introducing the (γ, H)-density perimeter. One of the purposes of the paper is to find a possible constraint on the boundaries, of the generalized perimeter-type, such to obtain the two convergence properties in the same time. More exactly, a sequence of open sets which converges in the H^c-topology to an open set, has also to converge in the char-topology to the same set. In the same time we shall also have a control on the measure of the boundary of the limit term, since in free boundary problems it is quite natural to ask the measure of the boundary to be equal to zero.

3. – Density perimeter $P_{\gamma,H}$

We define a new concept which could take the place of the perimeter in free boundary problems, but which has more specific properties in our desired direction than the generalized perimeter of De Giorgi [9].

Let $A \subseteq \mathbb{R}^n$ an arbitrary set, and $\varepsilon > 0$.

DEFINITION 3.1. *The ε-dilation of the set A is*

$$A^\varepsilon = \bigcup_{x \in A} B(x, \varepsilon)\,. \tag{5}$$

Let $H : [0, \infty) \to \mathbb{R}$ be a given continuous function with $H(0) = 0$ which will be a "corrector" of the perimeter and $\gamma > 0$ a fixed number which represents a maximal scale in the problem. We shall introduce the following new concept for the "perimeter", which is an extension of the density perimeter defined in [5].

DEFINITION 3.2. *Let $\gamma > 0$. The (γ, H)-density perimeter of the set A is*

$$P_{\gamma,H}(A) = \sup_{\varepsilon \in (0,\gamma)} \left[\frac{m(A^\varepsilon)}{2\varepsilon} + H(\varepsilon) \right]. \tag{6}$$

In the first term we have expanded the boundary ∂A with balls of the ray ε, and we computed the "area" of the boundary which intuitively is the quotient of the volume and the "high" 2ε. The default which may appear is corrected by the function H. A typical example is in 2 D for the density perimeter of a segment. If we take $H(\varepsilon) = -\frac{\pi\varepsilon}{2}$ then for any $\gamma > 0$ we have that the (γ, H)-density perimeter is equal to the length of the segment and in fact the function H corrects the default - the area of the two half-disks formed by the two extremal points of the segment (one can see in fact that $H(\varepsilon) \le -\frac{\pi\varepsilon}{2}$ is also convenient). If we want to compute the same quantity for a circle, one can see that no corrector is necessary, since for γ small we have that $\frac{m(A^{\varepsilon})}{2\varepsilon} = lenght\ of\ the\ circle.$

In a way, this construction means that we introduced a larger scale, given by γ, on which we are looking at the problem. From the numerical viewpoint this is a natural constraint. For γ vanishing to zero, we shall study the Γ-convergence of the density perimeter, and under some supplementary constraints we shall deduce its convergence towards classical length measures, like the Minkowski content or the Hausdorff measure.

A first question which naturally holds is: "When the density perimeter it is equal to the *classical* perimeter?".

Following the Steiner theorem (see [3]), if Γ is an $(N-1)$-dimensional C^{∞}-manifold without boundary, then for ε enough small we have that

$$m(\Gamma^{\varepsilon}) = \sum_{i=1}^{\left[\frac{N-1}{2}\right]} a_{2i}\varepsilon^{2i+1}$$

where a_{2i} are real numbers depending on the geometry of the manifold. In two dimensions, if Γ is a C^{∞}-manifold without boundary then chosing $H(\varepsilon) = 0$, $P_{\gamma,0}$ is equal to the length of Γ for γ small. For the case $N = 3$ we have

$$m(\Gamma^{\varepsilon}) = 2\varepsilon\, \text{surface}\,(\Gamma) + 4\pi\frac{\varepsilon^3}{3}\chi(\Gamma)$$

where $\chi(\Gamma)$ is the Euler-Poicaré characteristic (see [2]). It is surprising that $m(\Gamma^{\varepsilon})$ depends only on the surface of Γ and its Euler-Poicaré characteristic (which in three dimensions is very well controled). Finally, we can remark that if we chose in three dimensions the corrector function

$$H(\varepsilon) = -4\pi\frac{\varepsilon^2}{3}$$

then for ε enough small the density perimeter is equal to the surface for any connected C^{∞}-manifold without boundary of 2-dimensions. This is true, since the Euler-Poincaré characteristic for such a manifold in three dimensions is less or equal to two. For example, for a tours with γ holes we have

$$m(\Gamma^{\varepsilon}) = 2\varepsilon\, \text{surface}\,(\Gamma)\varepsilon + \frac{8\pi(1-\gamma)}{3}\varepsilon^3\,.$$

The problem we are concerned with is linked in fact to the value of ε for which the previous formulae are not anymore valid. In this context we shall introduce the maximal density curvature, which we shall study to find some Γ-convergence properties of the density perimeter.

A main difference between the generalized perimeter of De Giorgi and the density perimeter is that the first one is defined in a class of sets, namely it is constant in the family of sets with the same L^2 characteristic function, and hence it does not "see" a crack, or a point. In opposition with this behavior, for the density perimeter each point is important. One can also ask which is the link between the density perimeter and other type of length measures. We recall the definition of the Minkowski contents and of the Hausdorff measure which are related to the density perimeter. The upper and lower Minkowski contents are defined in [7] for $S \subseteq \mathbb{R}^N$ respectively as

$$\mathcal{M}^*(S) = \limsup_{\varepsilon \to 0_+} \frac{m(S^\varepsilon)}{2\varepsilon}, \tag{7}$$

$$\mathcal{M}_*(S) = \liminf_{\varepsilon \to 0_+} \frac{m(S^\varepsilon)}{2\varepsilon}. \tag{8}$$

If the upper and the lower Minkowski contents coincide, the common value is called the Minkowski content of the set.

Another essential length measure is the Hausdorff s-dimensional measure, given by the following relation. For $s > 0$ and $\delta > 0$ we put

$$\mathcal{H}^s_\delta(S) = 2^{-s} \frac{\Gamma\left(\frac{1}{2}\right)^s}{\Gamma\left(\frac{s}{2}+1\right)} \inf\left\{ \sum_{i=1}^\infty |U_i|^s : S \subseteq \bigcup_{i=1}^\infty U_i, \quad \operatorname{diam}(U_i) \le \delta \right\}$$

where $\Gamma(\cdot)$ is the usual Gamma function. The Hausdorff s-dimensional measure of S is given by

$$\mathcal{H}^s(S) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(S).$$

A first relation between the density perimeter and classical length measures is contained in the following

PROPOSITION 3.1. *Let $A \subseteq \mathbb{R}^N$. The map $\gamma \to P_{\gamma,H}(A)$ is non-increasing, and*

$$\lim_{\gamma \to 0} P_{\gamma,H}(A) = \mathcal{M}^*(A)$$

where $\mathcal{M}^*(A)$ is the upper Minkowski content of A.

For a sequence of open sets which have the (γ, H)-density perimeter uniformly bounded, we shall prove the existence of a subsequence which converges simultanously in the H^c-topology and in the char-topology to the same set.

LEMMA 3.1. *Let $\{\Omega_n\}_n$ be sequence of open subsets of D, $\Omega_n \xrightarrow{H^c} \Omega$. Then $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$, $\forall n > n_\varepsilon$ we have $\partial\Omega \subseteq (\partial\Omega_n)^\varepsilon$.*

PROOF. The proof of this lemma is classical. Suppose that $\exists \varepsilon > 0$, a subsequence of $\{\Omega_n\}_{n\in\mathbb{N}}$ still denote with the same index and some points $x_n \in \partial\Omega$, $x_n \notin (\partial\Omega_n)^\varepsilon$. Since $\partial\Omega$ is compact we can assume that $x_n \to x$, $x \in \partial\Omega$. From the H^c-convergence there exists $n_{\varepsilon/4} \in N$ such that $\forall n > n_{\varepsilon/4}$ we have $x \in (\Omega_n^c)^{\varepsilon/4}$. So $x \in (\Omega_n^c)^{\varepsilon/4}$, but $x \notin (\partial\Omega_n)^{\varepsilon/2}$, and hence $B(x, \frac{\varepsilon}{4}) \subseteq \Omega_n^c$, $\forall n > n_{\varepsilon/4}$. We get $B(x, \frac{\varepsilon}{4}) \subseteq \Omega^c$, which implies x is an interior point of (Ω^c) in contradiction with the choice $x \in \partial\Omega$. □

One can give the following compactness result which makes the link between the H^c-topology and the char-convergence.

THEOREM 3.1. *Let $k > 0$ be fixed. Then the family of open subsets of D for which the density perimeter of the boundary is upper bounded by k*

$$\mathcal{F}_{\gamma,k}(D) = \{\Omega \subseteq D | \Omega \quad open, \quad P_{\gamma,H}(\partial\Omega) \leq k\}$$

is compact in the H^c-topology. Moreover, if $\{\Omega_n\}_n \subseteq \mathcal{F}_{\gamma,k}(D)$, $\Omega_n \xrightarrow{H^c} \Omega$ then $\Omega_n \xrightarrow{\text{char}} \Omega$.

PROOF. For compactness in the H^c topology it suffices to prove the closedness since this space is compact. For that we shall prove the following lemma which asserts the lower semi-continuity of the (γ, H) density perimeter.

LEMMA 3.2. *The mapping $\Omega \to P_{\gamma,H}(\partial\Omega)$ is lower semi-continuous in the H^c-topology.*

For fixed $\varepsilon \in (0, \gamma)$ and δ chosen such that $\varepsilon + \delta < \gamma$ we have that

$$\begin{aligned}
&\frac{m((\partial\Omega)^\varepsilon)}{2\varepsilon} + H(\varepsilon) \leq \frac{m(\partial\Omega_n)^{\varepsilon+\delta}}{2(\varepsilon+\delta)} \frac{\varepsilon+\delta}{\varepsilon} + H(\varepsilon) \\
&= \left[\frac{m(\partial\Omega_n)^{\varepsilon+\delta}}{2(\varepsilon+\delta)} + H(\varepsilon+\delta)\right] \frac{\varepsilon+\delta}{\varepsilon} - \frac{\varepsilon+\delta}{\varepsilon} H(\varepsilon+\delta) + H(\varepsilon).
\end{aligned}$$

Making $\delta \to 0$ and $n \to \infty$ we get

$$\frac{m((\partial\Omega)^\varepsilon)}{2\varepsilon} + H(\varepsilon) \leq \liminf_{n\to\infty} P_{\gamma,H}(\partial\Omega_n)$$

The lower semi-continuity of the density perimeter proves that $\mathcal{F}_{\gamma,k}(D)$ is compact in the H^c topology. Moreover, by the definition of the H^c-convergence we have that $\forall\, \varepsilon > 0$, $\exists\, n_\varepsilon \in N$, $\forall\, n > n_\varepsilon$, $\Omega_n^c \subseteq (\Omega^c)^\varepsilon$ and $\Omega^c \subseteq (\Omega_n^c)^\varepsilon$. Remark that in general $A^\varepsilon \backslash A \subseteq (\partial A)^\varepsilon$ and so we can write

$$\begin{aligned}
&m(\Omega_n^c \Delta \Omega^c) = m(\Omega_n^c \backslash \Omega^c) + m(\Omega^c \backslash \Omega_n^c) \leq m((\Omega^c)^\varepsilon \backslash \Omega^c) + m((\Omega_n^c)^\varepsilon \backslash \Omega_n^c) \\
&\leq m((\partial\Omega^c)^\varepsilon) + m((\partial\Omega_n^c)^\varepsilon) \leq 2k\varepsilon - 2\varepsilon H(\varepsilon)
\end{aligned}$$

Making $\varepsilon \to 0$ the conclusion follows. □

COROLLARY 3.1. *The family $\mathcal{F}_{\gamma,k}(D)$ is compact in the* char *topology.*

The measure of a set of finite density perimeter is equal to zero. Indeed, suppose that A is an arbitrary subset of D such that $P_{\gamma,H}(A) = k < \infty$. Then $m(A) = 0$ since $m(A^\varepsilon) \leq 2k\varepsilon - 2H(\varepsilon)\varepsilon$. We conclude that if an arbitrary set O has the (γ, H)-density perimeter of the boundary finite, then O has an open L^2-representative in the same class, since the set $O \backslash \partial O$ is open.

For measuring lengths, various notions can be defined in relation with the density perimeter, all having the same lower semi-continuity property. For example if one denote by $\alpha(k)$ the $k - L^2$ measure of the unity ball of $\mathbb{R}^k$, one can define the k-dimensional (γ, H)-density perimeter of a set A by

$$P^k_{\gamma,H}(A) = \sup_{\varepsilon\in(0,\gamma)} \left[\frac{m(A^\varepsilon)}{\alpha(N-k)\varepsilon^{N-k}} + H(\varepsilon)\right] .$$

Since as a general fact the $(N-1)$-density perimeter measures the area of the boundary of an open set in the H^c-topology, for the m-density perimeter it is sufficient to consider only closed sets and the Hausdorff topology (denoted H^d). Then one can prove the following:

PROPOSITION 3.2. *For $k \in \{1, \dots, N-1\}$ the mapping $A \to P^k_{\gamma,H}(A)$ is lower semi-continuous in the family of closed sets with the H^d topology.*

For Neumann problems, when we are interested in measuring the boundaries of the cracks, a more suitable measure can be defined using the Sobolev capacity, but which preservs the lower semi-continuity property. So one can replace the $(N-2)$-density perimeter in the following way. Let $g : [0, \infty) \to \mathbb{R}_+$ continuous with $g(x) = 0$ if and only if $x = 0$ and define

$$C_{\gamma,H}(A) = \sup_{\varepsilon\in(0,\gamma)} \left[\frac{C(A^\varepsilon)}{g(\varepsilon)} + H(\varepsilon)\right] .$$

For an application of this type of constraint see [4].

4. – Pseudo curvature and correctors

We observe that for some sets and for γ small we may have $P_{\gamma,H}(A) = \mathcal{M}^*(A)$. To study the situations when this holds, we shall extend the definition of the density curvature for the (γ, H)-density perimeter.

DEFINITION 4.1. *Let $A \subseteq \mathbb{R}^N$. The maximal density curvature of a set A is*

(9)
$$c_{\partial,H}(A) = \left[\sup\left\{\gamma > 0 | \forall 0 < \varepsilon < \delta < \gamma \frac{m(\text{cl } A^\varepsilon)}{2\varepsilon} + H(\varepsilon) \geq \frac{m(A^\delta)}{2\delta} + H(\delta)\right\}\right]^{-1} .$$

Any finite value of $c_{\partial,H}(A)$ is reached. Indeed, if $\delta_n < \frac{1}{c_{\partial,H}(A)}$ and $\delta_n \to \frac{1}{c_{\partial,H}(A)}$ for $n \to \infty$, then for $0 < \varepsilon < \frac{1}{c_{\partial,H}(A)}$ there exists a rank n_ε such that $\forall n \geq n_\varepsilon$ we have

$$\frac{m(\text{cl } A^\varepsilon)}{2\varepsilon} + H(\varepsilon) \geq \frac{m(A^{\delta_n})}{2\delta_n} + H(\delta_n) . \tag{10}$$

Using the continuity of the measure on increasing sequences we obtain the inequality for $\frac{1}{c_{\partial,H}(A)}$. On the other hand, if for fixed δ and for any $0 < \varepsilon < \delta$ we have the inequality

$$\frac{m(\mathrm{cl}\ A^{\varepsilon})}{2\varepsilon} + H(\varepsilon) \geq \frac{m(A^{\delta})}{2\delta} + H(\delta) \tag{11}$$

we also have

$$\frac{m(A^{\varepsilon})}{2\varepsilon} + H(\varepsilon) \geq \frac{m(A^{\delta})}{2\delta} + H(\delta) \,. \tag{12}$$

Indeed, we shall take an increasing sequence $\{\varepsilon_n\}$ which converges to ε and write (11) for ε_n. By the same argument of the continuity of the measure on increasing sequences we get (12). Thus, in Definition 4.1 can be considered inequality (12).

If $c_{\partial,H}(A) < \infty$, then for all $\gamma \leq \frac{1}{c_{\partial,H}(A)}$ we have $P_{\gamma,H}(A) = \mathcal{M}^*(A)$ and moreover the upper and the lower Minkowski contents coincide.

There exists a connection between the density curvature and the mean curvature. Indeed, let's suppose that Γ is the boundary of a smooth open set Ω. To obtain that $c_{\partial,H}(\Gamma) \leq k$ it would be sufficient that on $(0, k^{-1})$ to have

$$\frac{d}{d\varepsilon}\left[\frac{m(\Gamma^{\varepsilon})}{2\varepsilon} + H(\varepsilon)\right] \leq 0$$

or

$$\frac{2\varepsilon\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon}) - 2\int_0^{\varepsilon}\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon})d\varepsilon}{4\varepsilon^2} + H'(\varepsilon) \leq 0$$

or

$$2\varepsilon\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon}) - 2\int_0^{\varepsilon}\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon})d\varepsilon + 4\varepsilon^2 H'(\varepsilon) \leq 0 \,.$$

Since this inequality is true for $\varepsilon = 0$ it would be sufficient to have on $(0, k^{-1})$

$$2\varepsilon\frac{d}{d\varepsilon}\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon}) + 2\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon}) - 2\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon}) + 8\varepsilon H'(\varepsilon) + 4\varepsilon^2 H''(\varepsilon) \leq 0$$

equivalent to

$$\frac{d}{d\varepsilon}\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon}) \leq -[4\varepsilon H'(\varepsilon) + 2\varepsilon^2 H''(\varepsilon)] \,.$$

But following [9], $\frac{d}{d\varepsilon}\mathcal{H}^{N-1}(\partial\Gamma^{\varepsilon})$ is the sum on $\partial\Gamma^{\varepsilon}$ of the mean curvature of $\partial\Gamma^{\varepsilon}$ and this remark makes the connection between the boundedness of the mean curvature and the density curvature.

LEMMA 4.1. *Let* $\{\Gamma_n\}$ *be a sequence of closed subsets of* cl (D) *such that* $\Gamma_n \xrightarrow{H^d} \Gamma$. *Then*

$$c_{\partial,H}(\Gamma) \leq \liminf_{n\to\infty} c_{\partial,H}(\Gamma_n).$$

PROOF. For a sequence $\Gamma_n \xrightarrow{H^d} \Gamma$ suffices to prove that $\frac{1}{c_{\partial,H}(\Gamma)} \geq \limsup_{n\to\infty} \cdot \frac{1}{c_{\partial,H}(\Gamma_n)}$. Let's fix $0 < \varepsilon < \delta < \limsup_{n\to\infty} \frac{1}{c_{\partial,H}(\Gamma_n)}$. For $\mu > 0$ it exists $n_{\mu/2} \in \mathbb{N}$, such that $\forall n > n_{\mu/2}$, $\Gamma_n \subseteq \Gamma^{\mu/2}$. So $\Gamma_n^\varepsilon \subseteq \Gamma^{\mu/2+\varepsilon}$ and $\overline{\Gamma_n^\varepsilon} \subseteq \overline{\Gamma^{\mu/2+\varepsilon}} \subseteq \Gamma^{\mu+\varepsilon}$ and

$$\frac{m(\overline{\Gamma_n^\varepsilon})}{2\varepsilon} + H(\varepsilon) \leq \frac{m(\Gamma^{\mu+\varepsilon})}{2\varepsilon} + H(\varepsilon).$$

But for subsequence $\{n_k\}$ we get

$$\frac{m(\Gamma_{n_k}^\delta)}{2\delta} + H(\delta) \leq \frac{m(\overline{\Gamma_{n_k}^\varepsilon})}{2\varepsilon} + H(\varepsilon) \leq \frac{m(\Gamma^{\mu+\varepsilon})}{2\varepsilon} + H(\varepsilon).$$

For all $\theta < \delta$ and for all $n > n_{\delta-\theta}$ the H^c convergence gives $\Gamma^\theta \subseteq \Gamma_n^\delta$. So

$$\frac{m(\Gamma^\theta)}{2\delta} + H(\delta) \leq \frac{m(\Gamma^{\mu+\varepsilon})}{2\varepsilon} + H(\varepsilon).$$

Since $\cup_{0<\theta<\delta}\Gamma^\theta = \Gamma^\delta$ and $\cap_{\mu>0}\Gamma^{\mu+\varepsilon} = \overline{\Gamma^\varepsilon}$, the continuity of the measure gives

$$\frac{m(\Gamma^\delta)}{2\delta} + H(\delta) \leq \frac{m(\overline{\Gamma^\varepsilon})}{2\varepsilon} + H(\varepsilon).$$

Since our hypothesis was $0 < \varepsilon < \delta < \limsup_{n\to\infty} \frac{1}{c_{\partial,H}(\Gamma_n)}$ the previous relations conclude $\frac{1}{c_{\partial,H}(\Gamma)} \geq \limsup_{n\to\infty} \frac{1}{c_{\partial,H}(\Gamma_n)}$. □

In the further considerations of this section we shall study the density curvature of a compact connected set for $H(\varepsilon) = -\frac{\pi\varepsilon}{2}$ and we shall see that the density curvature is zero.

Firstly we give the following lemma.

LEMMA 4.2. *Let* $K \subseteq \mathbb{R}^2$ *be a compact connected set consisting of a finite union of segments. Then* $c_{\partial,H}(K) = 0$.

PROOF. In a first step we shall suppose that K doesn't contain parallel segments. Then, for given $\delta > 0$, the set K^δ has the boundary consisting of vortex, arcs of circle and segments. Any vertex can join two segments or two arcs or an arc and a segment, but always the exterior angle in the vertex is inferior or equal to π.

Since there are no parallel segments in K, and since the segments of the boundary ∂K^δ are parallel with the segments of K, then the mapping $\delta \to \mathcal{H}^1(\partial K^\delta)$ is continuous on $[0, \infty)$. From [10] we have that

$$m(K^\delta) = \int_0^\delta \mathcal{H}^1(\partial K^\mu) d\mu$$

and so the mapping $\delta \to m(K^\delta)$ is derivable on $[0,\infty)$ and

$$\frac{d}{d\mu} m(K^\mu)_{|\mu=\delta} = \mathcal{H}^1(\partial K^\delta)\,.$$

We intend to prove that the map

$$\delta \to \frac{m(K^\delta)}{2\delta} - \frac{\pi\delta}{2} = G(\delta)$$

is non-increasing on $[0,\infty)$. Since $G(\cdot)$ is derivable we compute its derivative

$$G'(\delta) = \frac{2\delta\mathcal{H}^1(\partial K^\delta) - 2m(K^\delta)}{4\delta^2} - \frac{\pi}{2}\,.$$

To prove that $G'(\delta) \le 0$ it is equivalent to prove that

$$\delta\mathcal{H}^1(\partial K^\delta) - m(K^\delta) - \pi\,\delta^2 \le 0\,.$$

This is a continuous function which vanishes in $\delta = 0$. So it would be sufficient to prove that the right hand derivative of this function is negative on $[0,\infty)$, i.e.

$$\delta\bar{d}\mathcal{H}^1(\partial\, K^\delta) + \mathcal{H}^1(\partial\, K^\delta) - \mathcal{H}^1(\partial\, K^\delta) - 2\pi\,\delta \le 0$$

where by $\bar{d}$ we denote the right hand derivative. Hence, it is sufficient to prove that the right hand derivative of the mapping $\delta \to \mathcal{H}^1(\partial\, K^\delta)$ exits and is less that 2π.

For the computation of $\bar{d}\mathcal{H}^1(\partial\, K^\delta)$ we see that the contribution of a segment is zero, the contribution of an arc is the measure of the arc, and the contribution of a vertex is $-2\,\mathrm{cotan}\,\frac{\alpha}{2}$ (the same for any of the three possible situations) where α is the exterior angle. So we can write

$$\bar{d}\mathcal{H}^1(\partial\, K^\delta) = \sum_i U_i - 2\sum_i \mathrm{cotan}\,\frac{\alpha_i}{2}$$

where U_i are the measures of the arcs, and α the measures of angles. Hence

$$\bar{d}\mathcal{H}^1(\partial\, K^\delta) = \sum_i U_i - 2\sum_j \tan\frac{\pi - \alpha_j}{2}\,.$$

Since for $x \ge 0$ we have $\tan(x) \ge x$ then

$$\bar{d}\mathcal{H}^1(\partial\, K^\delta) \le \sum_i U_i - 2\sum_j \frac{\pi - \alpha_j}{2} = \sum_i (\pi + U_i) + \sum_J \alpha_j - N\pi$$

where N is the number of vertices. The sum $\sum_{i,j}[(\pi + U_i) + \alpha_j]$ is in fact the sum of exterior angles to K^δ. Since K is connected then $\partial\, K^\delta$ may have more connected components, but only one exterior envelope whose sum of angles is $(N_e+2)\pi$, and more but finite number of interior envelopes whose exterior sum of angles is $(N_i^p - 2)\pi$. So

$$\bar{d}\mathcal{H}^1(\partial K^\delta) \leq (N_e + 2)\pi + \sum_p (N_i^p - 2)\pi - N\pi$$

where $N = N_e + \sum_p N_i^p$ and we get

$$\bar{d}\mathcal{H}^1(\partial K^\delta) \leq 2\pi .$$

The passage to any K, even with parallel segments, is now trivial since in the H^d topology we can find a sequence of K_n with no parallel segments such that $K_n \xrightarrow{H^d} K$, and from the semi-continuity of $c_{\partial,H}$ (Lemma 4.1) we conclude the proof. □

We can state the following assertion.

THEOREM 4.1. *If K is a compact connected set in $\mathbb{R}^2$ then $c_{\partial,H}(K) = 0$.*

PROOF. From the lower semi-continuity of $c_{\partial,H}$ in the H^d topology it is sufficient to approach K by a sequence of sets such as in Lemma 4.2. We cover K by open balls of ray $\frac{\delta}{2}$ and since K is compact there is a finite number of balls which cover K. Moreover, since K is connected this family of open balls can be considered connected. We shall consider the union of segments constructed in the following way: we link two centers of two balls if their intersection is non-empty. We get a set denoted Γ_δ which consists of a union of segments, is connected and $d_{H^d}(K, \Gamma_\delta) \leq \delta$. Therefore $\Gamma_\delta \xrightarrow{H^d} K$ for $\delta \to 0$ and since $c_{\partial,H}(\Gamma_n) = 0$ we get $c_{\partial,H}(K) = 0$. □

According to [8], for a compact set with a finite number of connected components in two dimensions we have that $\mathcal{H}^1(K) = \mathcal{M}(K)$. Then we derive as a consequence the following result which can be found in [7].

COROLLARY 4.1. *If K_n is a sequence of compact connected sets which H^d-converges to a compact connected set K then*

$$\mathcal{H}^1(K) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n) .$$

PROOF. Since K is a compact connected set in 2-D we have $\mathcal{H}^1(K) = P_{1,\pi\delta/2}(K)$ and the mapping

$$K \to P_{1,\pi\delta/2}(K)$$

is lower semi-continuous in the H^d topology. □

THEOREM 4.2. *If K is a compact set with k connected components then for all $l \geq k$ we have $c_{\partial,-l\pi\delta/2}(K) = 0$.*

PROOF. For the proof, we only remark that we can suppose that the connected components of K are as in Lemma 4.2 with no parallel segments. Then by the same considerations as in Lemma 4.2 we have that the right-hand derivative of the Hausdorff measure of the boundary of K^δ is less than $2\pi k$. □

5. – Γ-convergence of the density perimeter

We recall the definition of the Γ-convergence and its most important property in a simplified form. Let $F_n : K \to \mathbb{R}$ be a sequence of functionals defined on a compact space K which have minimizer points y_n, and $F : K \to \mathbb{R}$.

DEFINITION 5.1. *It is said that F_n Γ-converges to F for $n \to \infty$ if*
1. $\forall x_n \in K$, $x_n \to x$ *we have* $F(x) \le \liminf_{n\to\infty} F_n(x_n)$
2. $\forall x \in K$, $\exists \{x_n\} \in K$, $x_n \to x$ *and* $F(x) \ge \limsup_{n\to\infty} F_n(x_n)$

Let's suppose that $y_n \to y$. Then y is a minimizer for F. Indeed, let $x \in K$. Then there exists $x_n \in K$, $x_n \to x$ and $F(x) \ge \limsup_{n\to\infty} F_n(x_n) \ge \limsup_{n\to\infty} F_n(y_n) \ge \liminf_{n\to\infty} F_n(y_n) \ge F(y)$.

We have the following result which proves that a numerical algorithm computing a minimizer for energy functionals penalized with the density perimeter leads to minimizers for the energy functional penalized with the Minkowski's content length measure.

THEOREM 5.1. *The density perimeter $P_{\gamma,H}$ Γ-converges to the Minkowski's content for $\gamma \to 0$ in the family of closed subset of* cl(D) *of zero measure and with uniformly bounded density curvature.*

PROOF. We have to verify the two conditions for Γ-convergence.
1. For any $\Gamma_n \xrightarrow{H^d} \Gamma$ with $c_{\partial,H}(\Gamma_n) \le K$ we have to prove

$$\mathcal{M}(\Gamma) \le \liminf_{n\to\infty} P_{\gamma_n,H}(\Gamma_n)\,.$$

If $\liminf_{n\to\infty} P_{\gamma_n,H}(\Gamma_n) = \infty$ the inequality above is true. If this limit is finite, then for a subsequence we shall have $P_{\gamma_n,H}(\Gamma_n) \le M < \infty$ and so $m(\Gamma_n) = 0$. For any $\mu > 0$, $\exists n_\mu \in N$ such that $\forall n > n_\mu$, $\Gamma \subseteq \Gamma_n^\mu$. So

$$\begin{aligned}\frac{m(\Gamma^\varepsilon)}{2\varepsilon} + H(\varepsilon) &\le \frac{m(\Gamma_n^{\varepsilon+\mu})}{2\varepsilon} + H(\varepsilon)\\ &= \left(\frac{m(\Gamma_n^{\varepsilon+\mu})}{2(\varepsilon+\mu)} + H(\varepsilon+\mu)\right)\frac{\varepsilon+\mu}{\varepsilon} - H(\varepsilon+\mu)\frac{\varepsilon+\mu}{\varepsilon} + H(\varepsilon)\,.\end{aligned}$$

Since $c_{\partial,H}(\Gamma_n) \le K$ then for $\gamma_n \le \frac{1}{K}$ we have

$$\frac{m(\Gamma_n^{\varepsilon+\mu})}{2\varepsilon} + H(\varepsilon) \le P_{1/K,H}(\Gamma_n) = P_{\gamma_n,H}(\Gamma_n)\,.$$

Making $n \to \infty$ and $\mu \to 0$ we get

$$\frac{m(\Gamma^\varepsilon)}{2\varepsilon} + H(\varepsilon) \le \liminf_{n\to\infty} P_{1/K,H}(\Gamma_n) = \liminf_{n\to\infty} P_{\gamma_n,H}(\Gamma_n)$$

and that inequality is true for all $\frac{1}{2K}\varepsilon < 0$. Making $\varepsilon \to 0$ the proof is finished.

2. For any Γ in the class, there exists a sequence of Γ_n such that $\Gamma_n \xrightarrow{H^d} \Gamma$ and

$$\mathcal{M}(\Gamma) \geq \limsup_{n \to \infty} P_{\gamma_n}(\Gamma_n)$$

We shall chose $\Gamma_n = \Gamma$. Then for $\gamma_n \leq \frac{1}{2K}$ we get that $P_{\gamma_n}(\Gamma) = \mathcal{M}(\Gamma)$. □

In 2-D we have the following Γ-convergence result which takes into consideration the connected components instead of the maximal density curvature.

THEOREM 5.2. *$P_{\gamma,H}$ Γ-converges to the Minkowski's content for $\gamma \to 0$ in the family of compact sets with at most k connected components.*

PROOF. We can apply Theorem 5.1 for $c_{\partial,-k\pi\delta/2}(K) = 0$ and we get

$$\mathcal{M}^1(K) \leq \liminf_{n \to \infty} P_{\gamma_n,H}(K_n)$$

and so the first condition for Γ-convergence is verified. The second one is trivial as in Theorem 5.1. □

6. – Existence of optimal domain

According to the physical problem which we solve, there are some typical classes of shape functionals to minimize. A first type which is characteristic for free boundary problems consists in minimizing a shape functional

$$\mathcal{O}(D) \ni \Omega \to f(\Omega) \in \mathbb{R}\,.$$

The usual method used in [11] consists in a relaxation of f to measurable sets an to penalize with the generalized perimeter. In such a way it is made a deviation from the initial problem previously defined on open sets.

Using a penalty term involving the density perimeter, we can recover in the same time the char-convergence and the openness of the limiting term. We can state

THEOREM 6.1. *If f is lower semi-continuous in the* char*-topology then for any γ and H satisfying the usual conditions the functional*

$$E(\Omega) = f(\Omega) + P_{\gamma,H}(\partial\,\Omega)$$

has a minimizer in the class $\mathcal{O}(D)$.

A similar result holds also if the lower semi-continuity of the functional f is given in the H^c topology, but the general frame of free boundary problems is the char-convergence.

Another important class of functionals is of the type

$$\mathcal{C}_0(D) \ni \Gamma \to f(\Gamma) \in \mathbb{R}$$

which we want to minimize in the family $\mathcal{C}_0(D)$ and which is supposed to be lower semi-continuous in the H^d topology. Since the set $\mathcal{C}_0(D)$ is not compact in the H^d topology there is no an immediate existence result. We shall add a penalty term of the type $\mathcal{M}(\Gamma) + c_{\partial,H}(\Gamma)$ and minimize

$$E(\Gamma) = f(\Gamma) + \mathcal{M}(\Gamma) + c_{\partial,H}(\Gamma).$$

In other to apply the Γ-convergence result we give the following:

LEMMA 6.1. *For any $\gamma > 0$ the functional*

$$E_\gamma(\Gamma) = f(\Gamma) + P_{\gamma,H}(\Gamma) + c_{\partial,H}(\Gamma)$$

has a minimizing term in $\mathcal{C}_0(D)$.

PROOF. Let consider a minimizing sequence $\{\Gamma_n\}_{n\in\mathbb{N}} \in \mathcal{C}_0(D)$. From the compactness of $\mathcal{C}(D)$, we can suppose that $\Gamma_n \xrightarrow{H^d} \Gamma$. Since for a minimizing sequence $\{P_{\gamma,H}(\Gamma_n)\}_{n\in\mathbb{N}}$ is bounded, we get that $P_{\gamma,H}(\Gamma) < \infty$ and hence we have $\Gamma \in \mathcal{C}_0(D)$. From the lower semi-continuity of the functional f and of the density perimeter we obtain that Γ is a minimizer. □

THEOREM 6.2. *The functional $\Gamma \to E(\Gamma)$ has a minimizing term in $\mathcal{C}_0(D)$.*

PROOF. From Theorem 5.1 we get that the density perimeter $P_{\gamma,H}$ Γ-converges for $\gamma \to 0$ to the Minkowski's content in the family of closed sets with bounded density curvature and from the properties of Γ-convergence and Lemma 6.1 we get that $E(\cdot)$ has a minimizer. □

Following Theorem 5.2 in 2-D we can give:

THEOREM 6.3. *In two dimensions, the functional*

$$E(\Gamma) = f(\Gamma) + \mathcal{H}^1(\Gamma) + \#(\Gamma)$$

has a minimizer in $\mathcal{C}_0(D) \subseteq \mathbb{R}^2$.

7. – Minimal energy functionals

The main field of applications of the previous considerations are the minimal energy functionals derived from modelizing process of physical problems.

The first example considered in Theorem 6.1 is characteristic in free boundary problems. Usually, a penalty term of the generalized perimeter is considered to modelize the superficial tension which appear on the free boundary. Unfortunately, to solve such a problem we have to relax it to measurable sets, and the free boundary might have even a positive measure, which is completely different from the physical reality. Nevertheless, in some specific examples one can deduce some regularity results (see [1]).

We shall give here a simple solution to a Bernoulli like free boundary problem considered also in [11]. The author was constrained to relax the original problem to measurable sets, and to define special Sobolev-like spaces. Using the density perimeter, we give a simple existence result which will also furnish the openness of the optimal domain as well as the measure of the boundary equal to zero.

Let $D \subseteq \mathbb{R}^N$ be an open set, the hold-all for our moving domain and C a compact subset of D. Let $f \in L^2(D)$ with $essup\, f \subseteq C$ and for any open Ω with $C \subseteq \Omega \subseteq D$ we consider the energy

$$E(\Omega) = \min_{u \in H^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 + u^2\, dx - \int_\Omega u\, f\, dx \right\}.$$

Remark that for fixed smooth Ω there is a unique element in $H^1(\Omega)$ which minimizes $E(\Omega)$, namely the solution of the Neumann problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Given $a \in \mathbb{R}_+$ with $m(C) < a < m(D)$ we shall minimize in the family of open sets the following problem

$$\min_{C \subseteq \Omega \subseteq D, m(\Omega) = a} E(\Omega) + P_{\gamma,H}(\partial\Omega). \tag{14}$$

We give:

THEOREM 7.1. *Problem* (14) *has a solution in the family of open sets with zero boundary measure.*

PROOF. Indeed, let $\{\Omega_n\}$ be a minimizing sequence. From the compactness property of the H^c-topology one can suppose that $\Omega_n \xrightarrow{H^c} \Omega$ and $C \subseteq \Omega \subseteq D$ is an open set. Moreover, from Proposition 3.2 we have that

$$P_{\gamma,H}(\partial\Omega) \leq \liminf_{n\to\infty} P_{\gamma,H}(\partial\, \Omega_n) < \infty$$

and hence $m(\partial\, \Omega) = 0$.

It is sufficient to prove that $E(\Omega) \leq \liminf_{n\to\infty} E(\Omega_n)$. Let $u_n \in H^1(\Omega_n)$ be the minimizer of $E(\Omega_n)$. Then a simple calculus give the existence of $M \in \mathbb{R}$ such that

$$\|u_n\|_{H^1(\Omega_n)} \leq M \leq \infty .$$

Denote by $\tilde{u}_n$ and $\tilde{\nabla} u_n$ the extensions by 0 on D of the functions u_n and ∇u_n to elements of $L^2(D)$ respectively $L^2(D, \mathbb{R}^N)$. Then

$$\|\tilde{u}_n\|_{L^2(D)} \leq M \quad \text{and} \quad \|\tilde{\nabla} u_n\|_{L^2(D,\mathbb{R}^N)} \leq M .$$

From the classical weak compactness result, one can suppose that

$$\tilde{u}_n \overset{L^2(D)}{\rightharpoonup} u \quad \text{and} \quad \tilde{\nabla} u_n \overset{L^2(D,\mathbb{R}^N)}{\rightharpoonup} \bar{v} .$$

But, $\Omega_n \xrightarrow{H^c} \Omega$ and using Proposition 2.4 we get that

$$\forall\, i = 1, N \quad \frac{\partial u}{\partial x_i} = v_i \quad \text{in} \quad H^1(\Omega) .$$

Indeed, let $\varphi \in \mathcal{D}(\Omega)$, $\operatorname{supp} \varphi = K$ compactly contained in Ω. From Proposition 2.4 one can write

$$\int_D (\tilde{\nabla} u_n)_i \varphi \, dx = - \int_D \tilde{u}_n \frac{\partial \varphi}{\partial x_i} dx .$$

Making $n \to \infty$, by the weak convergence we get

$$\int_D v_i \varphi \, dx = - \int_D u \frac{\partial \varphi}{\partial x_i} dx .$$

Hence $u_{|\Omega} \in H^1(\Omega)$ and $\nabla(u_{|\Omega}) = \bar{v}_{|\Omega}$.

But, also $(1 - \chi_{\Omega_n}) u_n =$ and since $P_{\gamma,H}(\partial\Omega_n)$ is uniformly bounded one can suppose (eventual substracting a sequence) that $\Omega_n \xrightarrow{\text{char}} \Omega$. Then making $n \to \infty$ one gets

$$(1 - \chi_\Omega) u = 0$$

and hence $u = 0$ a.e. on Ω^c. Similarly $\bar{v} = 0$ a.e. on Ω^c.

To conclude the proof we have

$$E(\Omega) \leq \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2) dx - \int_\Omega f\, u \, dx = \frac{1}{2} \int_D (|\nabla u|^2 + u^2) dx - \int_D f\, u \, dx$$
$$\leq \liminf_{n\to\infty} \frac{1}{2} \int_\Omega |\tilde{\nabla} u_n|^2 + \tilde{u}_n^2 \, dx - \int_\Omega f\, \tilde{u}_n \, dx = \liminf_{n\to\infty} E(\Omega_n) .$$

Hence

$$E(\Omega) + P_{\gamma,H}(\partial\Omega) \leq \liminf_{n\to\infty} E(\Omega_n) + P_{\gamma,H}(\partial\Omega_n) .$$

Thus Ω is a minimizer, and from the char-convergence we have $m(\Omega) = a$. □

Comparing this result to that of [11], where the minimal set was only measurable with finite generalized perimeter, in our statements one can also ask $f \in H^{-1}(D)$ with the existence of an open set U satisfying a uniform cone property with $\operatorname{supp} f \subseteq U \subseteq C$. Then the proof follows in the same way, but in the energy functional instead of the term $\int_\Omega u\, f\, dx$ we consider the dual pairing form $< f, P_D u_{|U} >_{H^{-1}(D)\times H_0^1(D)}$ where $P_D : H^1(U) \to H_0^1(D)$ is an extension operator. This situation is without sense if the set Ω is only measurable, since the support of f might be of zero measure.

We also remark that under the assumption of the uniform boundedness of the density perimeter, the convergence of the energies implies the convergence of the states. Indeed, if $\forall n \in \mathbb{N}$, $P_{\gamma,H}(\partial\Omega_n) \leq M$, $\Omega_n \overset{H^c,\text{char}}{\longrightarrow} \Omega$, and $E(\Omega_n) \to E(\Omega)$, then using the previous notations we obtain that

$$\frac{1}{2}\int_D (|\nabla u|^2 + u^2) dx - \int_D f\, u\, dx = E(\Omega)\,.$$

From the uniqueness of the minimizer w get $u_{|\Omega} = u_\Omega$, and hence

$$\tilde{u}_n \overset{L^2(D)}{\rightharpoonup} \tilde{u}_\Omega \quad \text{and} \quad \tilde{\nabla} u_n \overset{L^2(D,\mathbb{R}^N)}{\rightharpoonup} \tilde{\nabla} u_\Omega\,.$$

The second example is the Mumford-Shah functional which appears in image segmentation (Lemma 6.1 and Theorems 6.2, 6.3). In this case, the choice of γ is like a maximal scale introduced in the problem, and following the Γ-convergence result, we have that for γ small, the optimal solution of E_γ, approaches the optimal image, for which the length of the contours is given in terms of the Hausdorff measure of the boundary or Minkowski content.

Let D be a bounded open subset of $\mathbb{R}^N$ and $g \in L^2(D)$ the image to treat. Taking the energy

$$E(\Gamma) = \min_{u\in H^1(D\setminus\Gamma)} \alpha_1 \int_\Omega (u-g)^2 dx + \alpha_2 \int_{\Omega\setminus\Gamma} |\nabla u|^2 dx$$

one can apply Theorems 6.2 and 6.3, and we obtain existence results for the image segmentation.

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CNRS-Equipe de Mathématiques
Université de Franche-Comté
16, route de Gray, 25030 Besançon, France

CNRS-INLN
1361 Route des Lucioles
06902 Valbonne, France
and
Centre de Mathématiques Appliquées
INRIA,
2004 Route des Lucioles
06902 Sophia Antipolis, France